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AN INERTIAL TSENG'S EXTRAGRADIENT METHOD FOR SOLVING MULTI-VALUED VARIATIONAL INEQUALITIES WITH ONE PROJECTION*

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Abstract: In this paper, we introduce an inertial Tseng's extragradient method for solving multi-valued variational inequalities, in which only one projection is needed at each iteration. We also obtain the strong convergence results of the proposed algorithm, provided that the multi-valued mapping is continuous and pseudomonotone with nonempty compact convex values. Moreover, numerical simulation results illustrate the efficiency of our method when compared to existing methods.

Key words: *inertial method, Tseng's extragradient method, multi-valued variational inequalities, pseudomonotone, convergence*

Mathematics Subject Classification: *65K15, 47H04, 47H10*

1 Introduction

In this paper, we consider the following multi-valued variational inequality, denoted by $MVI(A, C)$: to find $x^* \in C$ and $w^* \in A(x^*)$ such that

$$\langle w^*, y - x^* \rangle \geq 0 \qquad \forall\, y \in C, \tag{1.1}$$

where C is a nonempty closed convex set in $\mathbb{R}^n$, A is a multi-valued mapping from $\mathbb{R}^n$ into $2^{\mathbb{R}^n}$ with nonempty values, $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the usual inner product and norm in $\mathbb{R}^n$, respectively. If A is a single-valued mapping, then $MVI(A, C)$ reduces to the classic variational inequality problem.

It is well known that many convex optimization problems in data processing related to machine learning and image processing can be formulated as equivalent variational inequality ones. Thus, there has been an increasing interest in studying numerical algorithms for solving variational inequality problems; see [12–14, 17, 27, 28, 30, 33] and the references therein.

In order to explore relevant convergent results and analyze error estimates, many methods for solving variational inequality problem (1.1) have been proposed, in which the most popular method is the projection-type one. An important projection algorithm for solving variational inequalities is the Extragradient Method proposed by Korpelevich [19]; In [19],

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there is the need to calculate two projections onto C, and convergence is proved under the assumption of Lipschitz continuity and monotonicity. It is well known that, if C is a general closed convex set, this might be computationally expensive and hence it will affect the efficiency of the proposed algorithms. To overcome the difficulty, Censor et al. [7] proposed a subgradient extragradient algorithm for solving single-valued variational inequality, in which the second projection is onto C instead of the half-space; see also [8,20]. We note that the above algorithms need at least two projections per iteration. Further, one-projection methods for solving single-valued variational inequality problems were proposed; see for example [23,24,29].

Projection-type methods for solving multi-valued variational inequality have been proposed. Li and He [22] proposed a projection algorithm for solving multi-valued variational inequality in which the hyperplane strictly separates the current iteration from the solution set; see also [13]. Xia and Huang [31] studied a projection-proximal point algorithm for solving multi-valued variational inequalities in Hilbert spaces and obtained the weak convergence result under the assumption of pseudomonotonicity. Further, Fang and Chen [12] extended the subgradient extragradient algorithm in [7] to solve multi-valued variational inequality (1.1). Recently, Burachik and Milln [6] suggested a projection-type algorithm for solving (1.1), in which the next iteration is a projection of the initial point onto the intersection of some suitable convex subsets. He et al. [16] proposed two projection-type algorithms for solving the multivalued variational inequality and studied the convergence of the proposed algorithms. Inspired by Fang et al. [12,15], Dong et al. [11] presented a projection and contraction method for solving multi-valued variational inequality (1.1) and proved the strong convergence of the proposed algorithm.

The inertial-type methods originate from an implicit discretization method of the heavy-ball with friction(HBF) system, the main feature of which is that each new iteration point depends on the previous two iterations [1]. Subsequently, this inertial technique was extended to solve the inclusion problem of maximal monotone operators [2]. Since then, there has been increasing interest in studying inertial-type algorithms; see, for example, inertial forward-backward splitting methods [3,25], inertial Douglas-Rachford splitting method [5], inertial ADMM [9], inertial-type methods for variational inequalities [10,35].

Motivated by the recent work mentioned above, in this paper, we present an inertial Tseng's extragradient method for solving multi-valued variational inequalities, in which only one projection is needed at each iteration; see Step 3 in Algorithm 3.3. In our method, the projection onto the hypeplane in [12] is replaced by the Tseng's term; see Step 4 in Algorithm 3.3. In addition, the mapping A is assumed to be pseudomonotone with nonempty compact convex values. Under those assumptions above, we prove that the iterative sequence generated by our method converges strongly to a solution of the multi-valued variational inequality (1.1). We also present numerical results of the proposed method.

This paper is organized as follows. In Section 2, we present definitions and auxiliary material. In Section 3, we describe our algorithm and investigate the global convergence of our method. Numerical experiments are reported in Section 4. Finally, some concluding remarks are stated in Section 5.

2 Preliminaries

In this section, we introduce some basic concepts which will be used in this paper.

The multi-valued mapping $A : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is said to be upper semicontinuous at $x \in C$ if for every open set V containing $A(x)$, there is an open set U containing x such that $A(y) \subset V$ for all $y \in C \bigcap U$. A is said to be lower semicontinuous at $x \in C$ if given any sequence $\{x_n\}$ converging to x and any $y \in A(x)$, there exists a sequence $\{y_n\}$ satisfying $y_n \in A(x_n)$ that converges to y. A is said to be continuous at $x \in C$ if it is both upper semicontinuous and lower semicontinuous at x.

Let the set C be given by

$$C := \{x \in \mathbb{R}^n \mid g(x) \leq 0\},$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a convex function. We denote the subdifferential of g at a point x by

$$\partial g(x) := \{w \in \mathbb{R}^n | g(y) \geq g(x) + \langle w, y - x\rangle, \forall y \in \mathbb{R}^n\}.$$

The multi-valued mapping A is called monotone on C, if for any $x, y \in C$,

$$\langle u - \nu, x - y\rangle \geq 0, \quad \forall u \in A(x), \quad \forall \nu \in A(y).$$

The multi-valued mapping A is called pseudomonotone on C, if for any $x, y \in C$,

$$\langle \nu, x - y\rangle \geq 0, \quad \exists \nu \in A(y) \implies \langle u, x - y\rangle \geq 0, \quad \forall u \in A(x). \tag{2.1}$$

Denote by S the solution set of the multi-valued variational inequality (1.1). Throughout this paper, we assume that the solution set S is nonempty satisfying the following property:

$$\langle w, y - x\rangle \geq 0, \quad \forall y \in C \quad \forall w \in A(y) \quad \forall x \in S. \tag{2.2}$$

The property (2.2) holds if A is pseudomonotone on C.

The projection of a point $x \in \mathbb{R}^n$ onto a closed set C is defined as

$$P_C(x) = \text{argmin}_{y \in C} \parallel y - x \parallel .$$

Lemma 2.1 ([32]). *Let C be a closed convex subset of $\mathbb{R}^n$. For any $x, y \in \mathbb{R}^n$ and $z \in C$, the following statements hold,*

(i) $\langle x - P_C(x), z - P_C(x)\rangle \leq 0$;

(ii) $\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|P_C(x) - x + y - P_C(y)\|^2$.

Proposition 2.2 ([22]). *$x \in C$, and $w \in A(x)$ solve the problem* (1.1) *if and only if*

$$r_\mu(x, w) := x - P_C(x - \mu w) = 0.$$

Proposition 2.3 ([12]). *For any $x \in \mathbb{R}^n$, $w \in A(x)$ and $\mu > 0$,*

$$\min\{1, \mu\}\|r_1(x, w)\| \leq\| r_\mu(x, w) \|\leq \max\{1, \mu\}\|r_1(x, w)\|.$$

Lemma 2.4 ([35]). *For all $x, y \in \mathbb{H}$ and $\lambda \in [0, 1]$,*

$$\| \lambda x + (1 - \lambda)y \|^2 = \lambda \| x \|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2,$$

where $\mathbb{H}$ is a real Hilbert space.

Lemma 2.5 ([2]). *Let $\{\varphi_n\}$, $\{\theta_n\}$, and $\{\alpha_n\}$ be sequences in $[0, +\infty)$, such that*

$$\varphi_{n+1} \leq \varphi_n + \alpha_n(\varphi_n - \varphi_{n-1}) + \theta_n \quad \forall n \geq 1, \quad \sum_{n=1}^{+\infty} \theta_n < +\infty$$

and there exists a real number α with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \in N$. Then, the following hold:

(i) $\Sigma_{n=1}^{+\infty}[\varphi_n - \varphi_{n-1}]_+ < +\infty$, *where* $[t]_+ = \max\{t, 0\}$.

(ii) *There exists* $\varphi^* \in [0, +\infty)$, *such that* $\lim_{n\to+\infty} \varphi_n = \varphi^*$.

Lemma 2.6 ([26]). *Let C be a nonempty set of $\mathbb{H}$ and x_n be a sequence in $\mathbb{H}$ such that the following two conditions hold:*

(i) *For every $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists.*

(ii) *Every sequential weak cluster point of x_n is in C. Then, x_n converges weakly to a point in C.*

3 Main Results

In this section, we introduce the inertial Tseng's extragradient algorithm for solving the multi-valued variational inequality problems. In order to find a point of the set C, we have the following procedure.

Procedure A [18]

Data A Point $x \in \mathbb{R}^n$.

Output A point $R(x)$.

step 0. If $x \in C$, set $R(x) = x$. Otherwise, set $y_0 = x$, $n = 0$.

Step 1. Choose $w_n \in \partial g(y_n)$, set $y_{n+1} - 2g(y_n)\frac{w_n}{\|w_n\|^2}$.

Step 2. If $y_{n+1} \in C$, set $R(x) = y_{n+1}$ and stop. Otherwise, set $n = n + 1$ go to Step 1.

We get the following results from Procedure A.

Proposition 3.1 ([21]). *The number of iterations in Procedure A is finite.*

Proposition 3.2 ([18]). *Let $x \in \mathbb{R}^n$, we have*

$$\|R(x) - y\| \le \|x - y\|, \quad \forall y \in C, \quad R(x) \in C.$$

Algorithm 3.3. Choose $\tilde{x}_0 \in \mathbb{R}^n$, $\tilde{x}_1 \in \mathbb{R}^n$ and two parameters $\mu, \gamma \in (0, 1)$. Set $n = 1$

Step 1. Apply Procedure A with $x = \tilde{x}_0$ and set $x_0 = R(\tilde{x}_0)$.

Step 2. Apply Procedure A with $x = \tilde{x}_n$ and set $x_n = R(\tilde{x}_n)$.

Step 3. Let $w_n = x_n + \alpha_n(x_n - x_{n-1})$, choose $u_n \in A(w_n)$, and compute

$$y_n = P_C(w_n - \lambda_n u_n),$$

where $\lambda_n = \gamma^{m_n}$ and m_n is the smallest nonnegative integer m such that

$$\nu_n \in A(P_C(w_n - \gamma^m u_n)). \tag{3.1}$$

$$\gamma^m \|u_n - \nu_n\| \le \mu \|r_{\gamma^m}(w_n, u_n)\|. \tag{3.2}$$

If $r_{\lambda_n}(w_n, u_n) = 0$, then stop.

Step 4. Compute

$$\tilde{x}_{n+1} = y_n - \lambda_n(\nu_n - u_n). \tag{3.3}$$

Set $n := n + 1$ and return to Step 2.

We first show that Algorithm 3.3 is well defined.

Lemma 3.4. *Suppose that the assumption* (2.2) *holds, then for any* $\gamma \in (0,1)$ *and* $x_n \in C$, *the linesearch procedure in Algrithm 3.3 is well defined.*

Proof. If $r_1(w_n, u_n) = 0$, then by Proposition 2.3 we have $r_{\gamma^m}(w_n, u_n) = 0$, i.e., $w_n = P_C(w_n - \gamma^m u_n)$ and hence we can take $\nu_n = u_n$ which satisfies (3.1) and (3.2).

Assume now that $\|r_1(w_n, u_n)\| > 0$. Suppose that for all m and $\nu \in A(y_m) = A(P_C(w_n - \gamma^m u_n))$, we have

$$\gamma^m \|u_n - \nu\| > \mu \|r_{\gamma^m}(w_n, u_n)\|, \tag{3.4}$$

i.e.,

$$\|u_n - \nu\| > \frac{\mu}{\gamma^m} \|r_{\gamma^m}(w_n, u_n)\| \geq \frac{\mu}{\gamma^m} \min\{1, \gamma^m\} \|r_1(w_n, u_n)\| = \mu \|r_1(w_n, u_n)\|, \tag{3.5}$$

where the second inequality follows from Proposition 2.3 and the equality follows from $\gamma \in (0,1)$ and $m \geq 0$.

We now consider the two cases, $w_n \in C$ and $w_n \notin C$.

(i) If $w_n \in C$. Since $P_C(\cdot)$ is continuous, $y_m = P_C(w_n - \gamma^m u_n) \to w_n (m \to \infty)$. Since A is lower semicontinuous, $u_n \in A(w_n)$ and $y_m \to w_n (m \to \infty)$, there is $\nu_m \in A(y_m)$ such that $\nu_m \to u_n (m \to \infty)$. Therefore, from (3.5) we have

$$\|u_n - \nu_m\| > \mu \|r_1(w_n, u_n)\|, \quad \forall m. \tag{3.6}$$

Letting $m \to \infty$ in (3.6), we have

$$0 = \|u_n - u_n\| \geq \mu \|r_1(w_n, u_n)\| > 0.$$

This is a contradiction.

(ii) If $w_n \notin C$, then $\|r_{\gamma^m}(w_n, u_n)\| \to \|w_n - P_C(w_n)\| \neq 0 (m \to \infty)$. Letting $m \to \infty$ in (3.4), we have

$$0 = \gamma^m \|u_n - \nu\| \geq \mu \|w_n - P_C(w_n)\| > 0,$$

as A is continuous. This is a contradiction. Thus, Algorithm 3.3 is well defined and implementable.

□

Next we show that the stopping criterion in Step 3 is valid.

Lemma 3.5. *If* $r_{\lambda_n}(w_n, u_n) = 0$ *in Algorithm 3.1, then* $w_n \in S$.

Proof. If $r_{\lambda_n}(w_n, u_n) = 0$, then $w_n = P_C(w_n - \lambda_n u_n)$. Since $\lambda_n > 0$, it follows from Proposition 2.2 that $w_n \in S$. □

The following two lemmas play an important role in proving the convergence of Algorithm 3.3.

Lemma 3.6. *Let* $\{x_n\}$ *be a sequence generated by Algorithm 3.1. Then for every* $x^* \in S$

$$\| x_{n+1} - x^* \|^2 \leq \| w_n - x^* \|^2 - (1 - \mu^2) \| r_{\lambda_n}(w_n, u_n) \|^2 \tag{3.7}$$

Proof. From (3.3) we have

$$\begin{aligned}\|\tilde{x}_{n+1}-x^*\|^2 &= \|y_n-\lambda_n(\nu_n-u_n)-x^*\|^2\\ &= \|y_n-x^*\|^2+\lambda_n^2\|\nu_n-u_n\|^2-2\lambda_n\langle y_n-x^*,\nu_n-u_n\rangle\\ &= \|w_n-x^*\|^2+\|w_n-y_n\|^2\\ &\quad +2\langle y_n-w_n,w_n-x^*\rangle+\lambda_n^2\|\nu_n-u_n\|^2-2\lambda_n\langle y_n-x^*,\nu_n-u_n\rangle\\ &= \|w_n-x^*\|^2+\|w_n-y_n\|^2-2\langle y_n-w_n,y_n-w_n\rangle+2\langle y_n-w_n,y_n-x^*\rangle\\ &\quad +\lambda_n^2\|\nu_n-u_n\|^2-2\lambda_n\langle y_n-x^*,\nu_n-u_n\rangle\\ &= \|w_n-x^*\|^2-\|w_n-y_n\|^2+2\langle y_n-w_n,y_n-x^*\rangle\\ &\quad +\lambda_n^2\|\nu_n-u_n\|^2-2\lambda_n\langle y_n-x^*,\nu_n-u_n\rangle.\end{aligned}\tag{3.8}$$

Since $y_n=P_C(w_n-\lambda_n u_n)$,

$$\langle y_n-w_n+\lambda_n u_n,y_n-x^*\rangle\le 0,$$

or equivalently

$$\langle y_n-w_n,y_n-x^*\rangle\le-\lambda_n\langle u_n,y_n-x^*\rangle.\tag{3.9}$$

From (3.8) and (3.9), we get

$$\begin{aligned}\|\tilde{x}_{n+1}-x^*\|^2 &\le \|w_n-x^*\|^2-\|w_n-y_n\|^2-2\lambda_n\langle u_n,y_n-x^*\rangle\\ &\quad +\lambda_n^2\|\nu_n-u_n\|^2-2\lambda_n\langle y_n-x^*,\nu_n-u_n\rangle\\ &= \|w_n-x^*\|^2-\|w_n-y_n\|^2+\lambda_n^2\|\nu_n-u_n\|^2-2\lambda_n\langle y_n-x^*,\nu_n\rangle\\ &\le \|w_n-x^*\|^2-\|r_{\lambda_n}(w_n,u_n)\|^2\\ &\quad +\mu^2\|r_{\lambda_n}(w_n,u_n)\|^2-2\lambda_n\langle y_n-x^*,\nu_n\rangle\\ &\le \|w_n-x^*\|^2-(1-\mu^2)\|r_{\lambda_n}(w_n,u_n)\|^2-2\lambda_n\langle y_n-x^*,\nu_n\rangle.\end{aligned}\tag{3.10}$$

Since $\nu_n\in A(y_n)$ and $x^*\in S$, it follows from (2.2) that

$$\langle\nu_n,y_n-x^*\rangle\ge 0.\tag{3.11}$$

Combining (3.10) and (3.11), we have

$$\|\tilde{x}_{n+1}-x^*\|^2\le\|w_n-x^*\|^2-(1-\mu^2)\|r_{\lambda_n}(w_n,u_n)\|^2,$$

and hence from Proposition 3.2 we get

$$\|x_{n+1}-x^*\|^2\le\|\tilde{x}_{n+1}-x^*\|^2\le\|w_n-x^*\|^2-(1-\mu^2)\|r_{\lambda_n}(w_n,u_n)\|^2.$$

This completes the proof. □

Lemma 3.7. *Assume that the sequence* $\{\alpha_n\}$ *satisfies* $0\le\alpha_n\le\alpha_{n+1}\le\alpha$ *and*

$$\alpha<1-\frac{4}{\sqrt{8\tau+1}+3},\tag{3.12}$$

where $\tau=\frac{2}{\mu+1}-1$, *and that* $x^*\in S$. *Then,*

(1) $\lim_{n\to\infty} \|x_n - x^*\|$ *exists;*

(2) $\lim_{n\to\infty} \|w_n - x_n\| = 0$.

Proof. By the definition of $\tilde{x}_{n+1}$, we have

$$\begin{aligned}\|\tilde{x}_{n+1} - y_n\| &= \|y_n - \lambda_n(\nu_n - u_n) - y_n\| \\ &\le \lambda_n\|\nu_n - u_n\| \\ &\le \mu\|r_{\lambda_n}(w_n, u_n)\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\tilde{x}_{n+1} - w_n\| &\le \|\tilde{x}_{n+1} - y_n\| + \|y_n - w_n\| \\ &\le (1+\mu)\|r_{\lambda_n}(w_n, u_n)\|,\end{aligned}$$

which implies

$$\|r_{\lambda_n}(w_n, u_n)\| \ge \frac{1}{1+\mu}\|\tilde{x}_{n+1} - w_n\|. \tag{3.13}$$

Let $x^* \in S$. By Lemma 3.6, we have

$$\|x_{n+1} - x^*\|^2 \le \|w_n - x^*\|^2 - (1-\mu^2)\|r_{\lambda_n}(w_n, u_n)\|^2. \tag{3.14}$$

From (3.13) and (3.14), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \|w_n - x^*\|^2 - \frac{1-\mu^2}{(1+\mu)^2}\|\tilde{x}_{n+1} - w_n\|^2 \\ &= \|w_n - x^*\|^2 - (\frac{2}{\mu+1} - 1)\|\tilde{x}_{n+1} - w_n\|^2 \\ &= \|w_n - x^*\|^2 - \tau\|\tilde{x}_{n+1} - w_n\|^2 \\ &\le \|w_n - x^*\|^2 - \tau\|x_{n+1} - w_n\|^2.\end{aligned} \tag{3.15}$$

By the definition of w_n, we have

$$\begin{aligned}\|w_n - x^*\|^2 &= \|x_n + \alpha_n(x_n - x_{n-1}) - x^*\|^2 \\ &= \|(1+\alpha_n)(x_n - x^*) - \alpha_n(x_{n-1} - x^*)\|^2 \\ &= (1+\alpha_n)\|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2 + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2.\end{aligned} \tag{3.16}$$

Thus, it follows from (3.15) and (3.16) that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1+\alpha_n)\|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2 + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 \\ &\le (1+\alpha_n)\|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2 + 2\alpha\|x_n - x_{n-1}\|^2.\end{aligned} \tag{3.17}$$

Also,

$$\begin{aligned}\|x_{n+1} - w_n\|^2 &= \|x_{n+1} - x_n - \alpha_n(x_n - x_{n-1})\|^2 \\ &= \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \\ &\ge \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\|x_{n+1} - x_n\|\|x_n - x_{n-1}\| \\ &\ge (1-\alpha_n)\|x_{n+1} - x_n\|^2 + (\alpha_n^2 - \alpha_n)\|x_n - x_{n-1}\|^2.\end{aligned} \tag{3.18}$$

Combining (3.15), (3.16) and (3.18), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1+\alpha_n)\|x_n-x^*\|^2-\alpha_n\|x_{n-1}-x^*\|^2+\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2\\ &\quad -\tau(1-\alpha_n)\|x_{n+1}-x_n\|^2-\tau(\alpha_n^2-\alpha_n)\|x_n-x_{n-1}\|^2\\ &= (1+\alpha_n)\|x_n-x^*\|^2-\alpha_n\|x_{n-1}-x^*\|^2-\tau(1-\alpha_n)\|x_{n+1}-x_n\|^2\\ &\quad +[\alpha_n(1+\alpha_n)-\tau(\alpha_n^2-\alpha_n)]\|x_n-x_{n-1}\|^2\\ &= (1+\alpha_n)\|x_n-x^*\|^2-\alpha_n\|x_{n-1}-x^*\|^2\\ &\quad -\sigma_n\|x_{n+1}-x_n\|^2+\delta_n\|x_n-x_{n-1}\|^2,\end{aligned} \tag{3.19}$$

where $\sigma_n=\tau(1-\alpha_n)>0$ and $\delta_n=\alpha_n(1+\alpha_n)-\tau(\alpha_n^2-\alpha_n)\ge 0$.
Set

$$\Phi_n=\|x_n-x^*\|^2-\alpha_n\|x_{n-1}-x^*\|^2+\delta_n\|x_n-x_{n-1}\|^2,$$

and hence

$$\Phi_{n+1}=\|x_{n+1}-x^*\|^2-\alpha_{n+1}\|x_n-x^*\|^2+\delta_{n+1}\|x_{n+1}-x_n\|^2.$$

Therefore, from (3.19) we have

$$\begin{aligned}\Phi_{n+1}-\Phi_n &= \|x_{n+1}-x^*\|^2-(1+\alpha_{n+1})\|x_n-x^*\|^2+\alpha_n\|x_{n-1}-x^*\|^2\\ &\quad +\delta_{n+1}\|x_{n+1}-x_n\|^2-\delta_n\|x_n-x_{n-1}\|^2\\ &\le \|x_{n+1}-x^*\|^2-(1+\alpha_n)\|x_n-x^*\|^2+\alpha_n\|x_{n-1}-x^*\|^2\\ &\quad +\delta_{n+1}\|x_{n+1}-x_n\|^2-\delta_n\|x_n-x_{n-1}\|^2\\ &\le -(\sigma_n-\delta_{n+1})\|x_{n+1}-x_n\|^2.\end{aligned} \tag{3.20}$$

Since $0\le\alpha_n\le\alpha_{n+1}\le\alpha$,

$$\begin{aligned}\sigma_n-\delta_{n+1} &= \tau(1-\alpha_n)-\alpha_{n+1}(1+\alpha_{n+1})+\tau(\alpha_{n+1}^2-\alpha_{n+1})\\ &\ge \tau(1-\alpha_{n+1})-\alpha_{n+1}(1+\alpha_{n+1})+\tau(\alpha_{n+1}^2-\alpha_{n+1})\\ &\ge \tau(1-\alpha)-\alpha(1+\alpha)+\tau(\alpha^2-\alpha)\\ &= \tau-2\tau\alpha-\alpha-\alpha^2+\tau\alpha^2\\ &= -(1-\tau)\alpha^2-(1+2\tau)\alpha+\tau.\end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21), we get

$$\Phi_{n+1}-\Phi_n\le -\xi\|x_{n+1}-x_n\|^2, \tag{3.22}$$

where $\xi=-(1-\tau)\alpha^2-(1+2\tau)\alpha+\tau$. From (3.12) we know that $\xi>0$. Therefore,

$$\Phi_{n+1}-\Phi_n\le 0. \tag{3.23}$$

Thus, the sequence $\{\Phi_n\}$ is nonincreasing. Since

$$\begin{aligned}\Phi_n &= \|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2 + \delta_n\|x_n - x_{n-1}\|^2\\ &\geq \|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2,\\ \|x_n - x^*\|^2 &\leq \alpha_n\|x_{n-1} - x^*\|^2 + \Phi_n \leq \alpha\|x_{n-1} - x^*\|^2 + \Phi_1\\ &\leq \cdots \leq \alpha^n\|x_0 - x^*\|^2 + \Phi_1(\alpha^{n-1} + \cdots + 1)\\ &\leq \alpha^n\|x_0 - x^*\|^2 + \frac{\Phi_1}{1-\alpha}. \end{aligned} \tag{3.24}$$

Similarly, we have

$$\begin{aligned}\Phi_{n+1} &= \|x_{n+1} - x^*\|^2 - \alpha_{n+1}\|x_n - x^*\|^2 + \delta_{n+1}\|x_{n+1} - x_n\|^2\\ &\geq -\alpha_{n+1}\|x_n - x^*\|^2.\end{aligned} \tag{3.25}$$

Thus, it follows from (3.24) and (3.25) that

$$-\Phi_{n+1} \leq \alpha_{n+1}\|x_n - x^*\|^2 \leq \alpha\|x_n - x^*\|^2 \leq \alpha^{n+1}\|x_0 - x^*\|^2 + \frac{\alpha\Phi_1}{1-\alpha},$$

and hence from (3.22) we get

$$\xi\sum_{n=1}^{k}\|x_{n+1} - x_n\|^2 \leq \Phi_1 - \Phi_{k+1} \leq \alpha^{k+1}\|x_0 - x^*\|^2 + \frac{\Phi_1}{1-\alpha} \leq \|x_0 - x^*\|^2 + \frac{\Phi_1}{1-\alpha},$$

which implies that $\Sigma_{n=1}^{\infty}\|x_{n+1} - x_n\|^2 < +\infty$. Therefore, $\|x_{n+1} - x_n\| \to 0 (n \to \infty)$. Since $\{\alpha_n\}$ is bounded, from (3.18) we have $\|x_{n+1} - w_n\| \to 0 (n \to \infty)$. Since

$$\begin{aligned}0 \leq \|w_n - x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\|,\\ \|w_n - x_n\| &\to 0 \quad as \quad n \to \infty.\end{aligned}$$

In addition, using Lemma 2.5, from (3.17) we have

$$\lim_{n\to\infty}\|x_n - x^*\| = \rho,$$

for some $\rho \geq 0$. Applying the boundedness of $\{\alpha_n\}$, from (3.16) we also have

$$\lim_{n\to\infty}\|w_n - x^*\| = \rho.$$

□

Theorem 3.8. *If $A : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is continuous with nonempty compact convex values on C and the suppose $S \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 3.3 converges to a solution $\bar{x}$ of* (1.1).

Proof. Let $x^* \in S$. Since $\mu \in (0,1)$, $(1-\mu) \in (0,1)$. It follows from Lemma 3.6 that

$$0 \leq (1-\mu^2)\|r_{\lambda_n}(w_n, u_n)\|^2 \leq \|w_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \to 0 \quad as \quad n \to \infty,$$

which implies that

$$\lim_{n\to\infty} \|r_{\lambda_n}(w_n, u_n)\|^2 = 0. \tag{3.26}$$

By the boundedness of $\{x_n\}$, there exists a convergent subsequence $\{x_{n_j}\}$ converging to $\bar{x}$. By Lemma 3.7 (2), there also exists a convergent subsequence $\{w_{n_j}\}$ converging to $\bar{x}$.

If $\bar{x}$ is a solution of the problem (1.1), i.e., $\bar{x} \in S$. In view of Lemma 3.7, we know that $\lim_{n\to\infty} \|x_n - \bar{x}\|$ exists. Hence, by Lemma 2.6 we have that the sequence $\{x_n\}$ converges to $\bar{x}$.

Suppose now that $\bar{x}$ is not a solution of the problem (1.1), i.e., $\bar{x} \notin S$. We first show that $\{m_n\}$ in Algorithm 3.3 cannot tend to ∞. Since A is continuous with compact values, Proposition 3.11 in [4] implies that $\{A(w_n)|n \in N\}$ is bounded set, and so the sequence $\{u_n\}$ is a bounded set. Therefore, there exists a subsequence $\{u_{n_j}\}$ converging to $\bar{u}$. Since A is upper semicontinuous with compact values, Proposition 3.7 [4] implies that A is closed, and so $\bar{u} \in A(\bar{x})$. By the definition of m_n, we have

$$\gamma^{m_n-1}\|u_n - \nu\| > \mu\|r_{\gamma^{m_n-1}}(w_n, u_n)\|. \qquad \forall \nu \in A(P_C(w_n - \gamma^{m_n-1}u_n)).$$

i.e.,

$$\begin{aligned}\|u_n - \nu\| &> \frac{\mu}{\gamma^{m_n-1}}\|r_{\gamma^{m_n-1}}(w_n, u_n)\| \\ &\geq \frac{\mu}{\gamma^{m_n-1}} \min\{1, \gamma^{m_n-1}\}\|r_1(w_n, u_n)\| \\ &= \mu\|r_1(w_n, u_n)\|, \quad \forall\, \nu \in A(P_C(w_n - \gamma^{m_n-1}u_n))\forall \;\; m_n \geq 1,\end{aligned}$$

where the second inequality follows from Proposition 2.3 and the equality follows from $\gamma \in (0, 1)$.

If $m_{n_j} \to \infty$, then $P_C(w_{n_j} - \gamma^{m_{n_j}-1}u_n) \to \bar{x}$. By the lower semicontinuity of A, we get that there exists $\bar{u}_{n_j} \in A(P_C(w_{n_j} - \gamma^{m_{n_j}-1}u_{n_j}))$ such that $\{\bar{u}_{n_j}\}$ converges to $\bar{u}$. Therefore,

$$\|u_{n_j} - \bar{u}_{n_j}\| > \mu\|r_1(w_{n_j}, u_{n_j})\| \tag{3.27}$$

Letting $j \to \infty$ in (3.27), we obtain the contradiction

$$0 \geq \mu\|r_1(\bar{x}, \bar{u})\| > 0.$$

Therefore, $\{m_n\}$ is bounded, and so is$\{\lambda_n\}$. By Proposition 2.3 ,

$$\|r_{\lambda_n}(w_n, u_n)\| \geq \min\{1, \lambda_n\}\|r_1(w_n, u_n)\| = \lambda_n\|r_1(w_n, u_n)\|. \tag{3.28}$$

It follows from (3.26) and (3.28) that

$$\lim_{n\to\infty} \lambda_n\|r_1(w_n, u_n)\| = 0.$$

Hence,

$$\lim_{n\to\infty} \|r_1(w_n, u_n)\| = 0.$$

Since $r_1(\cdot, \cdot)$ is continuous and the sequences $\{w_n\}$ and $\{u_n\}$ are bounded, there exists an accumulation point $(\bar{x}, \bar{u})$ of $\{(w_n, u_n)\}$ such that $r_1(\bar{x}, \bar{u}) = 0$. Hence $\bar{x}$ is a solution of the multi-valued variational inequality (1.1). Similar to the preceding proof, we obtain that $\{x_n\}$ converges to $\bar{x}$.

□

4 Numerical Experiments

In this section, we present some numerical experiments for the proposed algorithm. The Matlab codes are run on a PC (with Intel(R) Core(TM) i3-4010U CPU @ 1.70GHZ) under MATLAB Version 8.4.0.150421 (R2014b) Service Pack 1. Now, we apply our algorithms to solve the VIP and compare numerical results with other algorithms.

In the following tables, "Iter." denotes the number of iterations and "CPU" denotes the CPU time in seconds. The tolerance ε means when $\|r_\mu(x,w)\| \leq \varepsilon$, the procedure stops.

Example 4.1. Let

$$C := \{x = (x_1, x_2) \in \mathbb{R}^2_+ : 0 \leq x_n \leq 10, n = 1, 2\},$$

and $A : C \to 2^{\mathbb{R}^2}$ be defined by

$$A(x) = \{(x_1^2 + t, x_2^2),\ \ \forall\ x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in [0, 1/5]\}.$$

It is obvious that A satisfies the assumptions in Theorem 3.8. We choose $\mu = 0.98$, $\gamma = 0.91$, $\alpha = 0.03$ for our Algorithm 3.3; $\mu = 0.35$, $\gamma = 0.55$ for Algorithm 2.1 in [12]; $\mu = 0.54$, $\gamma = 0.74$ for Algorithm 3.1 in [11]. See Figure 1 and Table 1.

Table 1 Example 4.1 .

	Algorithm3.1		Algorithm 2.1 [12]		Algorithm 3.1 [11]	
Tolerance ε	Iter.	CPU	Iter.	CPU	Iter.	CPU
10^{-1}	11	0.0780	13	0.2964	15	0.0780
10^{-2}	13	0.0936	14	0.3076	29	0.1560
10^{-3}	14	0.0936	15	0.3120	42	0.2340
10^{-4}	15	0.0936	-	-	-	-

Example 4.2. Let $n = 4$ The feasible set C is given by

$$C := \left\{ x \in \mathbb{R}^n | \sum_{i=1}^{n} x_i = 1, -10 \leq x_i \leq 10, i = 1, \cdots, n \right\}.$$

and $A : C \to 2^{\mathbb{R}^2}$ be defined by

$$A(x) = \{(t + x_1, x_1, x_1, x_1) : t \in \Big[\frac{1}{10}, \frac{1}{5}]\Big\}$$

Example 4.2 is tested in [34]. It is obvious that A is pseudomonotone and all the assumptions in Theorem 3.8 are satisfied. We choose $\mu = 0.14$, $\gamma = 0.10$, $\alpha = 0.72$ for our Algorithm 3.3; $\mu = 0.52$, $\gamma = 0.49$ for Algorithm 2.1 in [12]; $\mu = 0.37$, $\gamma = 0.34$ for Algorithm 3.1 in [11]; See Figure 2 and Table 2.

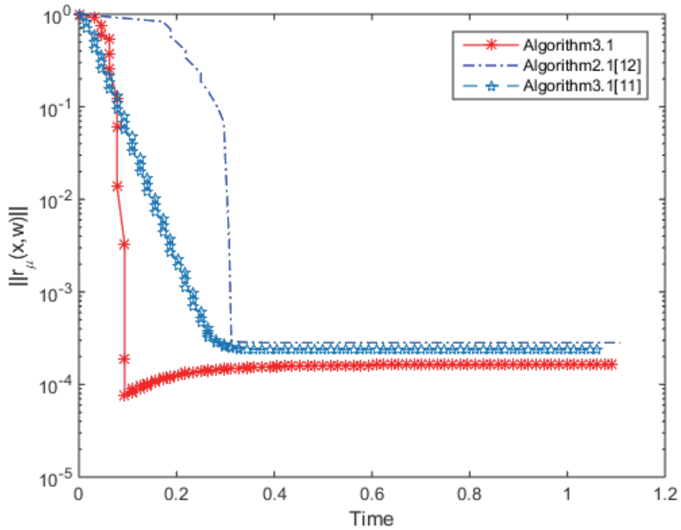

Figure 1: $\|r_\mu(x, w)\|$ and time in Example 4.1

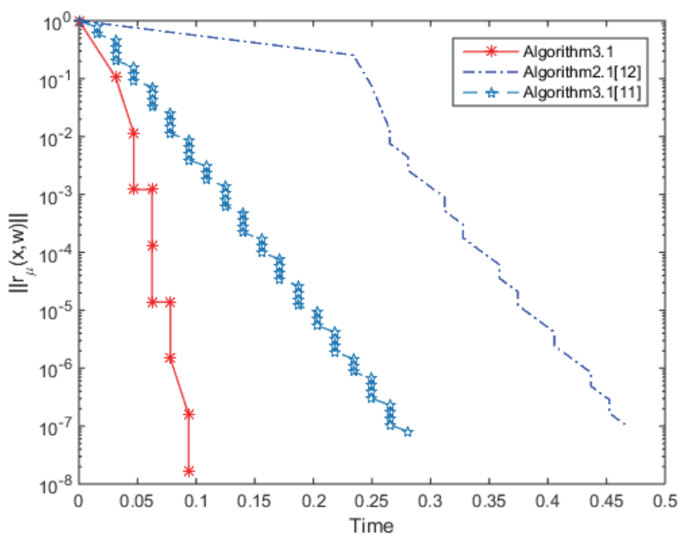

Figure 2: $\|r_\mu(x, w)\|$ and time in Example 4.2

Table 2 Example 4.2.

	Alrithm3.1		Algorithm 2.1 [12]		Algorithm 3.1 [11]	
Tolerance ε	Iter.	CPU	Iter.	CPU	Iter.	CPU
10^{-3}	8	0.0624	9	0.3120	28	0.1248
10^{-5}	12	0.7800	18	0.3900	45	0.2028
10^{-7}	16	0.0936	26	0.4680	63	0.2808

5 Conclusion

In this paper, we proposed an inertial Tseng's extragradient algorithm for solving multi-valued variational inequalities. We proved the convergence of the sequences generated by the proposed algorithm and presented some numerical experiments to illustrate the efficiency of our method. Compared with those algorithms in [6, 16], only one projection is needed at each iteration in our method. Our method is also different from that in [11]. First, we incorporate the inertial effects in our method. Secondly, the next iteration is related to Tseng's technique in our method while in [11] the next iteration is based on contraction method studied in [15].

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