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FOURTH-ORDER PARTIALLY SYMMETRIC TENSORS: THEORY AND ALGORITHM*

Mengzhen Li, Haibin Chen† and Guanglu Zhou

Abstract: With the coming of the big data era, high-order tensors have received much attention of researches in recent years, and this makes it to be a useful tool in data analysis. As a special kind of structured tensor, the fourth-order partially symmetric tensor receives a special concern due to its wide applications in nonlinear elasticity materials. In this review, we will give a survey on recent advances of fourth-order partially symmetric tensors such as M-eigenvalue inclusion intervals, M-positive definiteness, strong ellipticity condition and algorithms for computing the largest M-eigenvalue. Some potential research directions in the future are also listed in the paper.

Key words: *furth-order partially symmetric tensor, strong ellipticity condition, M-positive definite, M-eigenvalue, power method, block improvement method*

Mathematics Subject Classification: *65H17, 15A18, 90C30*

1 Introduction

A tensor is a multidimensional array and a physical quantity which is independent from co-ordinate system changes. Mathematically, a zero order tensor is a scalar, a first order tensor is a vector and a second-order tensor is a matrix. Tensors of order three or higher are called higher-order tensors. Fourth-order partially symmetric tensor is a special kind of structured tensors, and it has attracted much attention of researchers' from optimization field and numerical algebra. Fourth-order tensors as a mathematical object have found in the last 20 years a wide use in computational mechanics and especially in the finite element method, they have wide applications in physics and mechanics, such as the piezo-optical tensor, the elasto-optical tensor and the flexo-electric tensor. The most well-known fourth-order tensor is the elasticity tensor which is a typical kind of fourth-order partially symmetric tensor [30, 46, 70]. In this survey article, we mainly analyze and summarize the existing conclusions of fourth-order partially symmetric tensors from several aspects such as M-eigenvalue inclusion intervals, M-positive definiteness, strong ellipticity condition and algorithms for computing the largest M-eigenvalue.

Tensor eigenvalue problems play an important role in numerical multi-linear algebra [5, 17, 29, 36, 53, 63, 67] and nonlinear elastic material analysis, they have a wide range in medical

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†Corresponding author.

resonance [3], imaging spectral hyper-graph theory [37], automatical control [20–22, 43]. Particularly, the eigenvalue problem of the fourth-order elastic modulus tensor was dealt with by Love for the isotropic tensor [40] and for the anisotropic tensor [1,44,47,48,56,59,71]. Some effective algorithms for finding eigenvalues and the corresponding eigenvectors have been implemented [65, 66]. However, it is NP-hard to compute all M-eigenvalues exactly for a given tensor. Thus, some researchers turned to investigate eigenvalue inclusion sets i.e. intervals including all possible M-eigenvalues [7–10, 16, 27, 38, 62, 69].

On the other hand, to identify whether the strong ellipticity holds or not for a given elasticity material is an important problem in mechanics [25]. Fourth-order partially symmetric tensors are closely related to the strong ellipticity condition in nonlinear mechanics which guarantees the existence of solutions of basic boundary-value problems of elastostatics and thus ensures an elastic material to satisfy some mechanical properties. Knowles and Sternberg [31, 32] proposed necessary and sufficient conditions for strong ellipticity of the equations governing finite plane equilibrium deformations of a compressible hyper-elastic solid. Their works were further extended by Simpson and Spector [57] to the special case using the representation theorem for copositive matrices. Rosakis [55], Wang and Aron [64] also established some reformulations. Furthermore, Walton and Wilber [61] provided sufficient conditions for strong ellipticity of a general class of anisotropic hyper-elastic materials, which require the first partial derivatives of the reduced-stored energy function to satisfy several simple inequalities and the second partial derivatives to satisfy a convexity condition. Zubov and Rudev [71] gave sufficient and necessary conditions for the strong ellipticity of certain classes of anisotropic linearly elastic materials. Gourgiotis and Bigoni [24] investigated the strong ellipticity of materials with extreme mechanical anisotropy. As an application of tensor eigenvalues, Qi *et al.* [52] gave a necessary and sufficient condition of the strong ellipticity by introducing the definition of M-eigenvalue for fourth-order partially symmetric tensors and it is shown that the strong ellipticity holds if and only if all the M-eigenvalues of the ellipticity tensor is positive. Therefore, it is important to compute the extreme M-eigenvalues of a given tensor from practical problems.

In the optimization point of view, a fourth-order partially symmetric tensor corresponds to a unique bi-quadratic homogeneous polynomial. Then, the problem of computing extreme M-eigenvalues for a given tensor is equivalent to an bi-quadratic optimization problem with equality constraints. The state-of-the-art solution methods such as the block coordinate ascent method and the sequential quadratic programming method (SQP) can be used to solve the problem [2]. However, due to the specificity of the problem, several other targeted solution methods are developed. The first one is the sum of squares approach, which is based on the decomposition of a nonnegative multivariate polynomial into sum of squares [35, 49, 60]. This method has a strong theoretical appeal as it can theoretically achieve the global optimizer. Its basic idea is to relax the concerned problem to a hierarchy of semi-definite programming problems(SDP). However, the scale of the SDP grows exponentially with that of the problem, and thus the method can only solve small-scaled problems. The second one is the semi-definite programming relaxation approach [28, 39, 41], where the concerned problem is approximated by a specially constructed SDP problem. The computing cost of this method also increases quickly with the scale of the problem. The third one is the power method, which is originated from the computing of the dominant eigenvalue of a square matrix [23] and later, it is extended to compute the best rank-1 approximation to a higher order tensor [15, 33, 51]. It is worthy noted that the advantage of the power method is its less cost and less memory in computing. Its convergence can be guaranteed under convexity assumption for the symmetric tensor case [33, 54], and this restrict can be removed via a shifted technique [34]. Furthermore, the linear convergence rate of Power method is established in [34]. To the best of our knowledge, the power method has became a powerful and popular solution method in tensor computation and multi-linear algebra [42]. Recently, Wang *et al.* applied the power method to give a block improvement method (BIM) for computing the largest M-eigenvalue of fourth-order partially symmetric tensors [66], and several numerical examples show the efficiency of BIM.

2 Preliminaries

In this section, we recall some symbols and basic facts about tensors. First of all, we briefly mention the notation that will be used in the sequel. Let $\mathbb{R}^n$ be the n dimensional real Euclidean space and the set of all positive integers is denoted by $\mathbb{N}$. Suppose $m, n \in \mathbb{N}$ are two natural number. Denote $[n]=\{1, 2, \ldots, n\}$. Vectors are denoted by bold lowercase letters such as $\mathbf{x}$, $\mathbf{y}$, and tensors are written as calligraphic capitals such as $\mathcal{A}$, $\mathcal{B}$. All one tensor and all one vector are denoted by $\mathcal{E}$ and $\mathbf{e}$ respectively.

A general fourth-order tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n_1\times n_2\times n_3\times n_4}$ has $n_1 \times n_2 \times n_3 \times n_4$ entries such that

$$a_{ijkl} \in \mathbb{R}, \quad \forall i \in [n_1], \quad \forall j \in [n_2], \quad \forall k \in [n_3], \quad \forall l \in [n_4].$$

Particularly, when $n_1 = n_3 = m$ and $n_2 = n_4 = n$, the authors of [7, 26, 38, 62] studied the partially symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ with the following structure:

$$a_{ijkl} = a_{kjil} = a_{ilkj} = a_{klij}, \quad \forall i, k \in [m], \quad \forall j, l \in [n]. \tag{2.1}$$

It should be noted that any fourth-order partially symmetric tensor as in (2.1) corresponds to a unique bi-quadratic homogeneous polynomial:

$$f(\mathbf{x}, \mathbf{y}) = \mathcal{A}\mathbf{xyxy} = \sum_{i,k\in[m]} \sum_{j,l\in[n]} a_{ijkl} x_i y_j x_k y_l. \tag{2.2}$$

In addition, there is another formulation for fourth-order partially symmetric tensor $\bar{\mathcal{A}} = (\bar{a}_{ijkl}) \in \mathbb{R}^{m\times m\times n\times n}$ such as

$$\bar{a}_{ijkl} = \bar{a}_{jikl} = \bar{a}_{ijlk} = \bar{a}_{jilk}, \quad \forall i, j \in [m], \quad \forall k, l \in [n], \tag{2.3}$$

and the corresponding bi-quadratic polynomial is

$$\bar{f}(\mathbf{x}, \mathbf{y}) = \bar{\mathcal{A}}\mathbf{xxyy} = \sum_{i,j\in[m]} \sum_{k,l\in[n]} \bar{a}_{ijkl} x_i x_j y_k y_l. \tag{2.4}$$

Actually, (2.1) and (2.3) are equivalent in the sense that (2.2) and (2.4) are equal. For example, let $\mathcal{B} = (b_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be a tensor defined in (2.1). Define $\bar{\mathcal{B}} = (\bar{b}_{ijkl}) \in \mathbb{R}^{m\times m\times n\times n}$ with entries satisfying

$$\bar{b}_{ijkl} = \frac{b_{ikjl} + b_{jkil} + b_{iljk} + b_{jlik}}{4}, \quad i, j \in [m], \quad k, l \in [n].$$

Then, it is not difficult to verify that $\bar{\mathcal{B}}$ is a partially symmetric tensor as in (2.3) and

$\mathcal{B}\mathbf{xyxy} = \bar{\mathcal{B}}\mathbf{xxyy}$. On the other hand, if $\mathcal{B} = (b_{ijkl}) \in \mathbb{R}^{m\times m\times n\times n}$ be a tensor defined in (2.3), then define $\bar{\mathcal{B}} = (\bar{b}_{ijkl}) \in \mathbb{R}^{m\times m\times n\times n}$ with entries such that

$$\bar{b}_{ijkl} = \frac{b_{ikjl} + b_{kijl} + b_{iklj} + b_{kilj}}{4}, \quad i,k \in [m], \quad j,l \in [n].$$

Thus, $\bar{\mathcal{B}}$ is a partially symmetric tensor as in (2.1) and it still holds that$\mathcal{B}\mathbf{xxyy} = \bar{\mathcal{B}}\mathbf{xyxy}$.

For the sake of readers convenience, in the following analysis, we always consider the fourth order tensors with symmetry (2.1) and current conclusions related with (2.3) are rewritten as the form of (2.1).

Next, we recall the definition of M-eigenvalue and M-eigenvectors for partially symmetric tensors which is first defined in [52].

Definition 2.1. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be a partially symmetric tensor as in (2.1). If there are $\lambda \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$, $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that

$$\mathcal{A}\cdot\mathbf{yxy} = \lambda\mathbf{x}, \quad \mathcal{A}\mathbf{xyx}\cdot = \lambda\mathbf{y}, \quad \mathbf{x}^\top\mathbf{x} = 1, \quad \mathbf{y}^\top\mathbf{y} = 1, \tag{2.5}$$

where $\mathcal{A}\cdot\mathbf{yxy}$ and $\mathcal{A}\mathbf{xyx}\cdot$ are vectors with i-th and l-th components such that

$$(\mathcal{A}\cdot\mathbf{yxy})_i = \sum_{k=1}^{m}\sum_{j,l=1}^{n} a_{ijkl}y_jx_ky_l, \qquad (\mathcal{A}\mathbf{xyx}\cdot)_l = \sum_{i,k=1}^{m}\sum_{j=1}^{n} a_{ijkl}x_iy_jx_k,$$

then λ is called an M-eigenvalue of $\mathcal{A}$ and $\mathbf{x}$, $\mathbf{y}$ are called left and right M-eigenvectors of $\mathcal{A}$, respectively, which are associated with the M-eigenvalue λ. The set of all M-eigenvalues of $\mathcal{A}$ is denoted by $\sigma(\mathcal{A})$.

For M-eigenvalues, the extremal cases are always meaningful. Denote the M-spectral radius and the lowest M-eigenvalue of $\mathcal{A}$ as follows:

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}, \quad \tau_M(\mathcal{A}) = \min\{\lambda : \lambda \in \sigma(\mathcal{A})\}.$$

It has been proved as below that each partially symmetric tensor always have M-eigenvalue.

Theorem 2.2 ([52])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be a partially symmetric tensor as in (2.1). Then, its M-eigenvalues always exist. If $\mathbf{x}$ and $\mathbf{y}$ are left and right M-eigenvectors of $\mathcal{A}$, associated with an M-eigenvalue λ, then $\lambda = \mathcal{A}\mathbf{xyxy}$.*

Furthermore, the authors [52] also showed the definition of M-characteristic polynomial and gave an application for the spectral radius. To make full use of the resultant theory of homogeneous systems, by (2.5), one can obtain that

$$\begin{cases} \mathcal{A}\cdot\mathbf{yxy} = \lambda(\mathbf{y}^\top\mathbf{y})\mathbf{x}, \\ \mathcal{A}\mathbf{xyx}\cdot = \lambda(\mathbf{x}^\top\mathbf{x})\mathbf{y}. \end{cases} \tag{2.6}$$

According to the algebraic geometry theory [13], the resultant of (2.6) is a one dimensional polynomial ϕ with variable λ, and $\phi(\lambda)$ is called the M-characteristic polynomial of $\mathcal{A}$. Then, Qi *et al.* have the following results.

Theorem 2.3 ([52]). *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be a partially symmetric tensor as in* (2.1). *Then the following results hold.*
(1) An M-eigenvalue of $\mathcal{A}$ is always a real root of the M-characteristic polynomial $\phi(\lambda)$.
(2) If λ is the spectral radius of $\mathcal{A}$, assume that $\mathbf{x}$ and $\mathbf{y}$ are corresponding left and right M-eigenvectors, then $\lambda\mathbf{xyxy}$ is the best rank-one approximation of $\mathcal{A}$.

Next, we will show that two orthogonal similar tensors have same M-eigenvalues. To continue, we first present the definition of orthogonal similar tensors, which was first defined in [52]. Suppose that $\mathcal{A} = (a_{ijkl}), \mathcal{B} = (b_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ are two partially symmetric tensors. If there are orthogonal matrices $P = (p_{ii'}) \in \mathbb{R}^{m\times m}$ and $Q = (q_{jj'}) \in \mathbb{R}^{n\times n}$ satisfying that

$$b_{ijkl} = \sum_{i',k'=1}^{m} \sum_{j',l'=1}^{n} p_{ii'} q_{jj'} p_{kk'} q_{ll'} a_{i'j'k'l'}.$$

Then $\mathcal{A}$ and $\mathcal{B}$ are orthogonally similar [52].

Theorem 2.4 ([52]). *If fourth-order partially symmetric tensors $\mathcal{A}$ and $\mathcal{B}$ are orthogonally similar, then they have the same M-eigenvalues. In particular, if they are orthogonally similar via orthogonal matrices P and Q as above, and λ is an M-eigenvalue of $\mathcal{A}$ with left and right M-eigenvectors $\mathbf{x}$ and $\mathbf{y}$, then λ is also an M-eigenvalue of $\mathcal{B}$ with left and right M-eigenvectors $P\mathbf{x}$ and $Q\mathbf{y}$.*

To end this section, we present the notion of M-identity tensor and some useful equations that will be used in the sequel.

The partially symmetric tensor $\mathcal{I}_M \in \mathbb{R}^{m\times n\times m\times n}$ is called an M-identity tensor if its entries are

$$(\mathcal{I}_M)_{ijkl} = \begin{cases} 1, & \text{if } i = k, j = l, \\ 0, & \text{otherwise,} \end{cases}$$

where $i, k \in [m], j, l \in [n]$.

$\boxed{3}$ M-Eigenvalue Inclusion Intervals for Fourth-Order Partially Symmetric Tensors

M-eigenvalues of a fourth-order partially symmetric tensor are important in the nonlinear elastic material analysis and the entanglement problem in quantum physics. Generally speaking, it is not easy to get all the M-eigenvalues exactly, especially for high dimensional fourth-order partially symmetric tensors. Hence, researchers want to find a interval as tight as possible including all M-eigenvalues.

In this section, we recall some current inclusion intervals about M-eigenvalues of fourth-order partially symmetric tensors, and bounds for the M-spectral radius. To move on, we now list several useful equations based on the elements of tensor $\mathcal{A} = (a_{ijkl})$:

$$R_i(\mathcal{A}) = \sum_{k=1}^{m}\sum_{j,l=1}^{n} |a_{ijkl}|, \qquad R_i^k(\mathcal{A}) = \sum_{j,l\in[n]} |a_{ijkl}|,$$

$$C_l(\mathcal{A}) = \sum_{i,k=1}^{m}\sum_{j=1}^{n} |a_{ijkl}|, \qquad C_j^l(\mathcal{A}) = \sum_{i,k\in[m]} |a_{ijkl}|,$$

$$d_i(\mathcal{A}) = \sum_{j,k,l\in[n],j\neq l} |a_{ijkl}|, \qquad \gamma_i = \sum_{k\in[n],k\neq i} \max_{l\in[n]}\{|a_{ilkl}|\},$$

$$d_i^i = \sum_{j,l\in[n],j\neq l} |a_{ijil}|, \qquad D_i(\mathcal{A}) = d_i(\mathcal{A}) + \gamma_i,$$

$$g_l(\mathcal{A}) = \sum_{i,j,k\in[n],i\neq k} |a_{ijkl}|, \qquad \delta_l = \sum_{j\in[n],j\neq l} \max_{i\in[n]}\{|a_{ijil}|\},$$

$$g_l^l = \sum_{i,k\in[n],i\neq k} |a_{ilkl}|, \qquad G_l(\mathcal{A}) = g_l(\mathcal{A}) + \delta_l,$$

$$D_i^i(\mathcal{A}) = \sum_{j,l\in[n],j\neq l} |a_{ijil}|, \qquad G_l^l(\mathcal{A}) = \sum_{i,k\in[n],i\neq k} |a_{ilkl}|.$$

3.1 M-eigenvalue inclusion intervals

For the sake of readers, we do not change the symbols or structures in the following analysis, and we try to make it consistent with the original literatures.

Theorem 3.1 ([7,38]). *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be a partially symmetric tensor as in (2.1). Let λ be an M-eigenvalue of $\mathcal{A}$, then*

(1) $\lambda \in \Gamma(\mathcal{A}) = \left\{ z \in \mathbb{R} : |z| \leq \min\{\max_{1\leq i\leq m}\{R_i(\mathcal{A})\}, \max_{1\leq l\leq n}\{C_l(\mathcal{A})\}\right\}.$

(2) $\lambda \in \Upsilon(\mathcal{A}) = \bigcup_{i\in[m]} \Upsilon_i(\mathcal{A}),\ \Upsilon_i(\mathcal{A}) = \{z \in \mathbb{R} : |z| \leq R_i(\mathcal{A})\}.$

The inclusion intervals (1) and (2) in Theorem 3.1 were given independently by Li *et al.* [38] and Che *et al.* [7] respectively. Actually, it is not difficult to prove that

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) \subseteq \Upsilon(\mathcal{A}).$$

To verify the result above, we give a simple example below.

Example 3.2. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{2\times 2\times 2\times 2}$ be a partially symmetric tensor with entries such that

$$a_{1111} = 2,\ a_{1211} = 3,\ a_{2111} = 6,\ a_{1121} = 6,\ a_{1112} = 3,$$
$$a_{1212} = 2,\ a_{2212} = 3,\ a_{1222} = 3,\ a_{2222} = 5,$$

and $a_{ijkl} = 0$ otherwise.

By a direct computing, we obtain that

$$R_1(\mathcal{A}) = 19,\ R_2(\mathcal{A}) = 14,\ C_1(\mathcal{A}) = 17,\ C_2(\mathcal{A}) = 16,$$

which implies that

$$\Gamma(\mathcal{A}) = \{z \in R : |z| \leq 17\}, \qquad \Upsilon(\mathcal{A}) = \{z \in R : |z| \leq 19\}.$$

Therefore, it is obvious that

$$\Gamma(\mathcal{A}) \subset \Upsilon(\mathcal{A}).$$

Furthermore, another two inclusion intervals were presented in the following theorem.

Theorem 3.3 ([7,38])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be a partially symmetric tensor as in (2.1). If λ is an M-eigenvalue of $\mathcal{A}$, then*
(1) $\lambda \in \Theta(\mathcal{A}) = U(\mathcal{A}) \bigcap V(\mathcal{A})$,
(2) $\lambda \in \Lambda(\mathcal{A}) = \bigcup\limits_{i\in[m]} (\bigcap\limits_{k\in[m],k\neq i} \Lambda_{i,k}(\mathcal{A}))$,
where

$$U(\mathcal{A}) = \bigcup_{s\neq p,\ s,p=1}^{m} \left\{ z \in \mathbb{R} : (|z| - \sum_{j,l=1}^{n} |a_{pjpl}|)|z| \leq (R_p(\mathcal{A}) - \sum_{j,l=1}^{n} |a_{pjpl}|)R_s(\mathcal{A}) \right\},$$

$$V(\mathcal{A}) = \bigcup_{t\neq q,\ t,q=1}^{n} \left\{ z \in \mathbb{R} : (|z| - \sum_{i,k=1}^{m} |a_{iqkq}|)|z| \leq (C_q(\mathcal{A}) - \sum_{i,k=1}^{m} |a_{iqkq}|)C_t(\mathcal{A}) \right\},$$

$$\Lambda_{i,k}(\mathcal{A}) = \left\{ z \in \mathbb{R} : (|z| - (R_i(\mathcal{A}) - R_i^k(\mathcal{A})))|z| \leq R_i^k(\mathcal{A})R_k(\mathcal{A}) \right\}.$$

In Theorem 3.3, it seems that the relationship between intervals $\Theta(\mathcal{A})$ and $\Lambda(\mathcal{A})$ is not clear. Now, we provide two examples to show that $\Theta(\mathcal{A}) \subseteq \Lambda(\mathcal{A})$ in some cases, while $\Lambda(\mathcal{A}) \subseteq \Theta(\mathcal{A})$ in some other cases.

Example 3.4. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{2\times 2\times 2\times 2}$ be a partially symmetric tensor with entries such as

$$a_{1111} = 2,\ a_{1211} = 0\ a_{2111} = 1,\ a_{1121} = 1,\ a_{1112} = 0,$$

$$a_{1212} = 1,\ a_{2212} = 1,\ a_{1222} = 1,\ a_{2222} = 1,$$

and other $a_{ijkl} = 0$. By direct computing, we know that

$$\Theta(\mathcal{A}) = \{z \in R : |z| \leq 4\}, \qquad \Lambda(\mathcal{A}) = \{z \in R : |z| \leq \frac{3+\sqrt{33}}{2}\},$$

thus $\Theta(\mathcal{A}) \subset \Lambda(\mathcal{A})$.

Example 3.5 ([38])**.** Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{2\times 2\times 2\times 2}$ be a partially symmetric tensor with entries such as

$$a_{1111} = 2,\ a_{1211} = 3\ a_{2111} = 6,\ a_{1121} = 6,\ a_{1112} = 3,$$

$$a_{1212} = 2,\ a_{2212} = 10,\ a_{1222} = 10,\ a_{2222} = 5,$$

and other $a_{ijkl} = 0$. By direct computing, we obtain that

$$\Theta(\mathcal{A}) = \{z \in \mathbb{R} : |z| \leq 24\},\ \Lambda(\mathcal{A}) = \{z \in \mathbb{R} : |z| \leq 23\}.$$

It is obvious that $\Lambda(\mathcal{A}) \subset \Theta(\mathcal{A})$.

The following conclusions will show that the intervals of Theorem 3.3 are tighter than the intervals in Theorem 3.1.

Theorem 3.6 ([7,38])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be a partially symmetric tensor as in (2.1). Then it holds that*

$$\sigma(\mathcal{A}) \subseteq \Theta(\mathcal{A}) \subseteq \Gamma(\mathcal{A}), \quad \sigma(\mathcal{A}) \subseteq \Lambda(\mathcal{A}) \subseteq \Upsilon(\mathcal{A}).$$

Now we use an example to verify that $\Theta(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$ holds.

Example 3.7. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{2\times 2\times 2\times 2}$ be a partially symmetric tensor, whose entries satisfy example 3.5.

By direct computing, we obtain that

$$\Gamma(\mathcal{A}) = \{z \in \mathbb{R} : |z| \leq 26\},\ \Theta(\mathcal{A}) = \{z \in \mathbb{R} : |z| \leq 24\}.$$

It is obvious that $\Theta(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$.

By the intervals (2) in Theorem 3.1 and Theorem 3.3, Wang *et al.* introduced several new inclusion intervals with a suitable parameter [62].

Theorem 3.8 ([62])**.** *Let $\mathcal{A} = (a_{ijkl})$ be a partially symmetric tensor as in Theorem 3.1 and $\mathcal{I}_M$ be an M-identity tensor. For any $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)^T \in \mathbb{R}^m$, then*

$$\sigma(\mathcal{A}) \subseteq \mathscr{G}(\mathcal{A},\alpha) = \bigcup_{i\in[m]} \mathscr{G}_i(\mathcal{A},\alpha), \quad \sigma(\mathcal{A}) \subseteq \mathscr{K}(\mathcal{A},\alpha) = \bigcup_{i\in[m]} (\bigcap_{v\neq i,\ v\in[m]} \mathscr{K}_{i,v}(\mathcal{A},\alpha)),$$

where

$$\mathscr{G}_i(\mathcal{A},\alpha) = \{z \in \mathbb{R} : |z-\alpha_i| \leq R_i(\mathcal{A},\alpha_i)\},\ R_i(\mathcal{A},\alpha_i) = \sum_{k\in[m]\ j,l\in[n]} |a_{ijkl} - \alpha_i(\mathcal{I}_M)_{ijkl}|,$$

$$R_i^v(\mathcal{A},\alpha_i) = \sum_{j,l\in[n]} |a_{ijvl} - \alpha_i(\mathcal{I}_M)_{ijvl}|,$$

and

$$\mathscr{K}_{i,v}(\mathcal{A},\alpha) = \{z \in \mathbb{R} : [|z-\alpha_i| - (R_i(\mathcal{A},\alpha_i) - R_i^v(\mathcal{A},\alpha_i))] \cdot |z-\alpha_v| \leq R_i^v(\mathcal{A},\alpha_i) R_v(\mathcal{A},\alpha_v)\}.$$

Furthermore, by the arbitrary of α, it follows that

$$\sigma(\mathcal{A}) \subseteq \bigcap_{\alpha\in\mathbb{R}^m} \mathscr{G}(\mathcal{A},\alpha), \quad \sigma(\mathcal{A}) \subseteq \bigcap_{\alpha\in\mathbb{R}^m} \mathscr{K}(\mathcal{A},\alpha).$$

Remark 3.9. Noted that, if $\alpha = 0$ in Theorem 3.8, the intervals $\mathscr{G}(\mathcal{A},\alpha)$ and $\mathscr{K}(\mathcal{A},\alpha)$ reduces to $\Upsilon(\mathcal{A})$ in Theorem 3.1 and $\Lambda(\mathcal{A})$ in Theorem 3.3 respectively. Moreover, it is proved in [62] that

$$\sigma(\mathcal{A}) \subseteq \mathscr{K}(\mathcal{A},\alpha) \subseteq \mathscr{G}(\mathcal{A},\alpha).$$

In what follows, for given M-eigenvalue $\lambda \in \sigma(\mathcal{A})$ with left M-eigenvector $\mathbf{x}$, let x_s denote the component of $\mathbf{x}$ with the second largest modulus. Then Che *et al.* [7] obtained the following technical results for $\sigma(\mathcal{A})$.

Theorem 3.10 ([7])**.** *Suppose $\mathcal{A} = (a_{ijkl})$ is a partially symmetric tensor as in (2.1) with $i,k \in [m], j,l \in [n]$. Then, it holds that*

(1) $\sigma(\mathcal{A}) \subseteq \mathscr{M}(\mathcal{A}) = \bigcup_{i,k\in[m],k\neq i} (\mathscr{M}_{i,k}(\mathcal{A}) \bigcup \mathscr{H}_{i,k}(\mathcal{A}))$,

(2) $\sigma(\mathcal{A}) \subseteq \mathscr{N}(\mathcal{A}) = \bigcup_{i,k\in[m],k\neq i} \mathscr{N}_{i,k}(\mathcal{A})$,

where

$$\mathscr{M}_{i,k}(\mathcal{A}) = \left\{z \in \mathbb{R} : (|z| - (R_i(\mathcal{A}) - R_i^k(\mathcal{A})))(|z| - R_k^k(\mathcal{A})) \leq R_i^k(\mathcal{A})(R_k(A) - R_k^k(\mathcal{A}))\right\},$$

$$\mathscr{H}_{i,k}(\mathcal{A}) = \left\{z \in \mathbb{R} : |z| - (R_i(\mathcal{A}) - R_i^k(\mathcal{A})) \leq 0,\ |z| - R_k^k(\mathcal{A}) < 0\right\},$$

$$\mathscr{N}_{i,k}(\mathcal{A}) = \left\{z \in \mathbb{R} : (|\lambda| - R_i^i(\mathcal{A}))|\lambda| \leq (R_i(\mathcal{A}) - R_i^i(\mathcal{A}))R_k(\mathcal{A})\right\}.$$

In [62], Wang *et al.* proved that the intervals $\mathscr{G}(\mathcal{A},\alpha)$ and $\mathscr{K}(\mathcal{A},\alpha)$ are tighter than the intervals showed in Theorems 3.1, 3.3, 3.10. The following example exhibits the superiority of the intervals $\mathscr{G}(\mathcal{A},\alpha)$ and $\mathscr{K}(\mathcal{A},\alpha)$.

Example 3.11 ([62])**.** Let $\mathcal{A}=(a_{ijkl})\in\mathbb{R}^{2\times2\times2\times2}$ be a partially symmetric tensor, whose entries are

$$a_{1111}=20,\ a_{1122}=a_{1221}=1,\ a_{1212}=8,\ a_{2222}=10,\ a_{2112}=a_{2211}=1,\ a_{2121}=7$$

and other $a_{ijkl}=0$. Set $\alpha=(14,8.5)^T$. For this tensor, the bounds via different estimations given in this literature are shown in Table 1.

Table 1: Number results of Example 3.11

References	Inclusion interval
Theorem 3.1(1)	$\Gamma(\mathcal{A})=[-29,29]$
Theorem 3.1(2)	$\Upsilon(\mathcal{A})=[-30,30]$
Theorem 3.3(1)	$\Theta(\mathcal{A})=[-28.4081,28.4081]$
Theorem 3.3(2)	$\Lambda(\mathcal{A})=[-29.2971,29.2971]$
Theorem 3.8(1)	$\mathscr{M}(\mathcal{A})=[-28.3523,28.3523]$
Theorem 3.8(2)	$\mathscr{N}(\mathcal{A})=[-29.2971,29.2971]$
Theorem 3.6(1)	$\mathscr{G}(\mathcal{A},(14,8.5))=[0,28]$
Theorem 3.6(2)	$\mathscr{K}(\mathcal{A},(14,8.5))=[0.7154,26.5539]$

By Theorems 3.1 and 3.10, Che *et al.* have the following results.

Theorem 3.12 ([7])**.** *Suppose $\mathcal{A}=(a_{ijkl})$ is a partially symmetric tensor as in Theorem 3.10. Then it holds that*

$$\sigma(\mathcal{A})\subseteq\mathscr{M}(\mathcal{A})\subseteq\Upsilon(\mathcal{A}),\quad \sigma(\mathcal{A})\subseteq\mathscr{N}(\mathcal{A})\subseteq\Upsilon(\mathcal{A}).$$

In the following, an example is given to illustrate the relationship between $\Upsilon(\mathcal{A})$, $\Lambda(\mathcal{A})$ $\mathscr{M}(\mathcal{A})$, $\mathscr{N}(\mathcal{A})$. Figure 1 presents the comparisons of these intervals.

Example 3.13 ([7])**.** Let $\mathcal{A}=(a_{ijkl})\in\mathbb{R}^{2\times2\times2\times2}$ be a partially symmetric tensor, whose entries are

$$a_{1111}=-1,\ a_{1112}=2,\ a_{1131}=3,\ a_{1121}=-1,\ a_{1211}=2,\ a_{1221}=1,\ a_{1122}=1,\ a_{2111}=-1,$$
$$a_{2211}=1,\ a_{2112}=1,\ a_{2131}=-2,\ a_{2222}=2,\ a_{3111}=3,\ a_{3232}=-1,\ a_{3131}=-2,$$

and $a_{ijkl}=0$ otherwise. By computation, we obtain that

$$\Upsilon(\mathcal{A})=\bigcup_{i\in[m]}\Upsilon_i(\mathcal{A})=\{\lambda\in\mathbb{R}:|\lambda|\le 11\},$$
$$\Lambda(\mathcal{A})=\bigcup_{i\in[m]}\Big(\bigcap_{k\in[m],k\neq i}\Lambda_{i,k}(\mathcal{A})\Big)=\{\lambda\in\mathbb{R}:|\lambda|\le 4+\sqrt{34}\},$$
$$\mathscr{M}(\mathcal{A})=\bigcup_{i,k\in[m],k\neq i}\Big(\mathscr{M}_{i,k}(\mathcal{A})\bigcup\mathscr{H}_{i,k}(\mathcal{A})\Big)=\{\lambda\in\mathbb{R}:|\lambda|\le 5+2\sqrt{6}\},$$
$$\mathscr{N}(\mathcal{A})=\bigcup_{i,k\in[m],k\neq i}(\mathscr{N}_{i,k})=\{\lambda\in\mathbb{R}:|\lambda|\le \tfrac{5+\sqrt{193}}{2}\}.$$

The M-eigenvalue inclusion sets $\Upsilon(\mathcal{A})$, $\Lambda(\mathcal{A})$, $\mathscr{M}(\mathcal{A})$ and $\mathscr{N}(\mathcal{A})$ of example 3.13 are drawn in Figure 1, where $\Upsilon(\mathcal{A})$, $\Lambda(\mathcal{A})$, $\mathscr{M}(\mathcal{A})$ and $\mathscr{N}(\mathcal{A})$ are represented by red, blue, green and black boundary, respectively, and the exact eigenvalues is plotted by $*$. From Figure 1, the example shows that the M-eigenvalue inclusion sets $\Upsilon(\mathcal{A})$, $\Lambda(\mathcal{A})$, $\mathscr{M}(\mathcal{A})$ and $\mathscr{N}(\mathcal{A})$ are different.

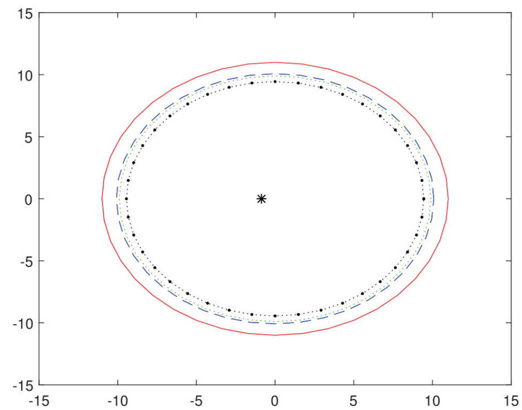

Figure 1: The comparisons of $\Upsilon(\mathcal{A})$, $\Lambda(\mathcal{A})$, $\mathscr{M}(\mathcal{A})$, $\mathscr{N}(\mathcal{A})$.

Theorem 3.14 ([27])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ as in* (2.1)*. Let λ be an M-eigenvalue of $\mathcal{A}$. Then He et al. have the following results.*
(1) *If $a_{i1i1} = \cdots = a_{inin} = \alpha_i$, $i \in [n]$, it holds that*

$$\lambda \in \mathscr{U}_1(\mathcal{A}) = \bigcup_{i\in[n]} \{z \in \mathbb{R} : z \in [\alpha_i - D_i(\mathcal{A}), \alpha_i + D_i(\mathcal{A})]\}.$$

(2) *If $a_{1l1l} = \cdots = a_{nlnl} = \beta_l$, $l \in [n]$, it holds that*

$$\lambda \in \mathscr{U}_2(\mathcal{A}) = \bigcup_{l\in[n]} \{z \in \mathbb{R} : z \in [\beta_l - G_l(\mathcal{A}), \beta_l + G_l(\mathcal{A})]\}.$$

(3) *If $a_{i1i1} = \cdots = a_{inin} = \alpha_i = a_{1i1i} = \cdots = a_{nini}$, $i \in [n]$. If λ is an M-eigenvalue of $\mathcal{A}$, then it holds that*

$$\lambda \in \mathscr{U}(\mathcal{A}) = \bigcup_{i\in[n]} \{z \in \mathbb{R} : z \in [\alpha_i - \min\{D_i(\mathcal{A}), G_i(\mathcal{A})\}, \alpha_i + \min\{D_i(\mathcal{A}), G_i(\mathcal{A})\}]\}.$$

Theorem 3.15 ([10])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ as in* (2.1)*. Let λ be an M-eigenvalue of $\mathcal{A}$. Then Che et al. have the following results.*
(1) *If $a_{i1i1} = \cdots = a_{inin} = \alpha_i$, $i \in [n]$, it holds that*

$$\begin{aligned}&\lambda \in \Omega(\mathcal{A})\\ =& \bigcup_{p,s\in[n],p\neq s} \left\{z \in \mathbb{R} : (|z-\alpha_p| - D_p^p(\mathcal{A}) - \gamma_p)(|z-\alpha_s|) \leq (D_p(\mathcal{A}) - D_p^p(\mathcal{A}) - \gamma_p)D_s(\mathcal{A})\right\}.\end{aligned}$$

(2) *If $a_{1l1l} = \cdots = a_{nlnl} = \beta_l$, $l \in [n]$, it holds that*

$$\begin{aligned}&\lambda \in \Phi(\mathcal{A})\\ =& \bigcup_{q,t\in[n],q\neq t} \left\{z \in \mathbb{R} : (|z-\beta_q| - G_q^q(\mathcal{A}) - \delta_q)(|z-\beta_t|) \leq (G_q(\mathcal{A}) - G_q^q(\mathcal{A}) - \delta_q)G_t(\mathcal{A})\right\}.\end{aligned}$$

(3) *If $a_{i1i1} = \cdots = a_{inin} = \alpha_i = a_{1i1i} = \cdots = a_{nini}$, $i \in [n]$, it holds that*

$$\lambda \in (\Omega(\mathcal{A}) \cap \Phi(\mathcal{A}))$$

By Theorems 3.14 and 3.15, Che *et al.* have the following results.

Theorem 3.16 ([10])**.** *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ *as in* (2.1)*. If* $a_{i1i1} = \cdots = a_{inin} = \alpha_i = a_{1i1i} = \cdots = a_{nini}$, $i \in [n]$*. Then, it follows that*

$$\Omega(\mathcal{A}) \cap \Phi(\mathcal{A}) \subseteq \mathscr{U}_1(\mathcal{A}) \cap \mathscr{U}_2(\mathcal{A}).$$

In what follows, an example is given to reveal $\Omega(\mathcal{A}) \cap \Phi(\mathcal{A}) \subseteq \mathscr{U}_1(\mathcal{A}) \cap \mathscr{U}_2(\mathcal{A})$.

Example 3.17 ([10])**.** Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ as in (2.1), whose entries are

$$a_{1111} = a_{1212} = a_{2121} = a_{2222} = 4,\ a_{1122} = 1,\ a_{2112} = 1,$$
$$a_{1211} = 2,\ a_{2122} = 3,\ a_{1222} = 5,\ a_{2111} = 4.$$

By computation, we have

$$\Omega(\mathcal{A}) \cap \Phi(\mathcal{A}) = \left\{\lambda \in \mathbb{R} : |\lambda - 4| \le 8 + 4\sqrt{6}\right\} \subseteq \mathscr{U}_1(\mathcal{A}) \cap \mathscr{U}_2(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda - 4| \le 18\}.$$

$\boxed{3.2}$ Bound estimations on the M-Spectral radius

In several cases, the researchers pay more attention to the M-spectral radius than all M-eigenvalues of nonnegative tensors. Therefore, it is interesting to find tight estimations or bounds for the M-spectral radius. In this section, we conclude some theorems about bound estimations on the M-spectral radius of nonnegative partially symmetric tensors [7,16,38,62].

Theorem 3.18 ([16])**.** *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ *be a partially symmetric nonnegative tensor as in (2.1). Then, the M-spectral radius of* $\mathcal{A}$ *is exactly its greatest M-eigenvalue. Furthermore, there is a pair of nonnegative M-eigenvectors corresponding to the M-spectral radius.*

Theorem 3.18 tells us that the M-spectral radius $\rho(\mathcal{A})$ of a nonnegative fourth-order partially symmetric tensor $\mathcal{A}$ is exactly an M-eigenvalue of $\mathcal{A}$. This result is very similar to the Perron-Frobenius theorem for nonnegative matrixs and nonnegative tensors [18, 45, 67, 68].

Several sharp upper bounds on the largest M-eigenvalue for nonnegative fourth-order partially symmetric tensors were presented by Che *et al.* [7], which improved the corresponding results in [6, 58].

Theorem 3.19 ([38])**.** *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ *be a fourth-order partially symmetric tensor as in (2.1). Then*

(1) $\rho(\mathcal{A}) \le t_1$*, where* $t_1 = \min\left\{\max\limits_{1\le i\le m} R_i(\mathcal{A}), \max\limits_{1\le l\le n} C_l(\mathcal{A})\right\}$,

(2) $\rho(\mathcal{A}) \le t_2$*, where* $t_2 = \min\{P_1(\mathcal{A}), P_2(\mathcal{A})\}$,

$$P_1(\mathcal{A}) = \max_{s\ne p, 1\le s,p\le m} \frac{1}{2}\left\{\sum_{j,l=1}^{n} |a_{pjpl}| + \sqrt{(\sum_{j,l=1}^{n} |a_{pjpl}|)^2 + 4(R_p(\mathcal{A}) - \sum_{j,l=1}^{n} |a_{pjpl}|)R_s(\mathcal{A})}\right\},$$

$$P_2(\mathcal{A}) = \max_{t\ne q, 1\le t,q\le n} \frac{1}{2}\left\{\sum_{i,k=1}^{m} |a_{iqkq}| + \sqrt{(\sum_{i,k=1}^{m} |a_{iqik}|)^2 + 4(C_q(\mathcal{A}) - \sum_{i,k=1}^{m} |a_{iqkq}|)C_t(\mathcal{A})}\right\}.$$

Theorem 3.20 ([7]). *Suppose $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ is a nonnegative fourth-order partially symmetric tensor as in (2.1). Then, Che et al. have the following results:*
(1) $\rho(\mathcal{A}) \le \max\limits_{i\in[m]} \min\limits_{k\in[m],\ k\neq i} \frac{1}{2}\left\{R_i(\mathcal{A}) - R_i^k(\mathcal{A}) + \sqrt{(R_i(\mathcal{A}) - R_i^k(\mathcal{A}))^2 + 4R_i^k(\mathcal{A})R_k(\mathcal{A})}\right\}$,
(2) $\rho(\mathcal{A}) \le \max\limits_{i,k\in[m]\ k\neq i} \left\{\frac{1}{2}(R_i(\mathcal{A}) - R_i^k(\mathcal{A}) + R_k^k(\mathcal{A}) + \delta_i^k), R_i(\mathcal{A}) - R_i^k(\mathcal{A}), R_k^k(\mathcal{A})\right\}$, *where*

$$\delta_i^k = \sqrt{(R_i(\mathcal{A}) - R_i^k(\mathcal{A}) + R_k^k(\mathcal{A}))^2 - 4(R_k^k(\mathcal{A})R_i(\mathcal{A}) - R_i^k(\mathcal{A})R_k(\mathcal{A}))},$$

(3) $\rho(\mathcal{A}) \le \max\limits_{i,k\in[m]\ k\neq i} \left\{\frac{1}{2}(R_i^i(\mathcal{A}) + \sqrt{(R_i^i(\mathcal{A}))^2 + 4(R_i(\mathcal{A}) - R_i^i(\mathcal{A}))R_k(\mathcal{A}))}\right\}$.

Based on Theorem 3.8, Wang *et al.* [62] presented another bound estimations on M-spectral radius of fourth-order partially symmetric nonnegative tensors, which improved the corresponding results in [7,38]. What's more, a lower bound estimation on M-spectral radius of fourth-order partially symmetric nonnegative tensors is also presented in [62].

Theorem 3.21 ([62]). *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be a partially symmetric nonnegative tensor and $\mathcal{I}_M$ be an M-identity tensor. For vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_m)^\top \in \mathbb{R}^m$ with $\alpha_i \le \max\limits_{i\in[m],j\in[n]}\{a_{ijij}\}$, then*
(1) $\rho(\mathcal{A}) \le \max\limits_{i\in[m]}\{\alpha_i + R_i(\mathcal{A}, \alpha_i)\}$,
(2) $\rho(\mathcal{A}) \le \max\limits_{i\in[m]} \min\limits_{v\neq i,\ v\in[m]} \frac{1}{2}(\alpha_i + \alpha_v + [(R_i(\mathcal{A}, \alpha_i) - R_i^v(\mathcal{A}, \alpha_i))] + \Delta_{i,v}^{\frac{1}{2}}(\mathcal{A}))$, *where* $\Delta_{i,v}(\mathcal{A}) = (\alpha_i - \alpha_v + [(R_i(\mathcal{A}, \alpha_i) - R_i^v(\mathcal{A}, \alpha_i))])^2 + 4(R_i^v(\mathcal{A}, \alpha_i)R_v(\mathcal{A}, \alpha_v))$.

Lemma 3.22 ([62]). *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be a nonnegative fourth-order partially symmetric tensor as in (2.1), then*

$$\rho(\mathcal{A}) \ge \max\left\{\max_{i\in[m],j\in[n]} a_{ijij}, \frac{\sum_{i\in[m]} R_i(\mathcal{A})}{mn}\right\}.$$

In the following, we use Example 3.5 to show the superiority of the results in Theorem 3.21. In fact, $\sigma(\mathcal{A}) = \{-7.6841, 13.8616, -4.2541, 6.6751\}$. From Lemma 3.22, we compute $11.75 \le \rho(\mathcal{A})$. Set $\boldsymbol{\alpha} = (2,5)^\top$. For this tensor, the bounds via different estimations given in the literature are shown in Table 2.

Table 2: Number results of Example 3.5

References	Inclusion interval
Theorem 3.18(1)	$\rho(\mathcal{A}) \le 24$
Theorem 3.18(2)	$\rho(\mathcal{A}) \le 24$
Theorem 3.18(3)	$\rho(\mathcal{A}) \le 24$
Theorem 3.17(1)	$\rho(\mathcal{A}) \le 26$
Theorem 3.17(2)	$\rho(\mathcal{A}) \le 24$
Theorem 3.19(1) and Lemma 3.20	$11.75 \le \rho(\mathcal{A}) \le 24$
Theorem 3.19(2) and Lemma 3.20	$11.75 \le \rho(\mathcal{A}) \le 23.6941$

3.3 The M-eigenvalue inclusion sets for fourth-order Cauchy tensors

Cauchy tensor is a kind of important structured tensors, which is a natural extension from Cauchy matrices [11]. In the fourth-order case, Che *et al.* [8] established several M-eigenvalue inclusion theorems for fourth-order partially symmetric Cauchy tensors. To continue, we first list the corresponding notions below.

Definition 3.23 ([11]). Let $\mathbf{c} = (c_1, c_2, \cdots, c_n) \in \mathbb{R}^n$ be a given vector. Suppose that a real tensor $\mathcal{C} = (c_{i_1 i_2 \cdots i_m})$ is defined by

$$c_{i_1 i_2 \cdots i_m} = \frac{1}{c_{i_1} + c_{i_2} + \cdots + c_{i_m}}, \qquad i_j \in [n], j \in [m].$$

Then $\mathcal{C}$ is called an order m dimension n symmetric Cauchy tensor and the vector $\mathbf{c} \in \mathbb{R}^n$ is called the generating vector of $\mathcal{C}$.

Following the ideas of Cauchy matrix [50] and Cauchy tensor [11], Che *et al.* [8] introduced the definition of fourth-order Cauchy tensors.

Definition 3.24 ([8]). Suppose that $\mathbf{a} = (a_1, a_2, \cdots, a_m) \in \mathbb{R}^m$ and $\mathbf{b} = (b_1, b_2, \cdots, b_n) \in \mathbb{R}^n$ are two given vectors. The fourth-order partially symmetric Cauchy tensor $\mathcal{C} = (c_{ijkl})$ is defined as follows:

$$c_{ijkl} = \frac{1}{a_i + b_j + a_k + b_l}, \qquad i, k \in [m], j, l \in [n],$$

where vectors $\mathbf{a} \in \mathbb{R}^m, \mathbf{b} \in \mathbb{R}^n$ are called generating vectors of $\mathcal{C}$.

Obviously, the fourth-order Cauchy tensor has the following partially symmetric property:

$$c_{ijkl} = c_{kjil} = c_{ilkj} = c_{klij} = \frac{1}{a_i + b_j + a_k + b_l}, \qquad i, k \in [m], j, l \in [n].$$

Furthermore, if $\mathbf{a} = \mathbf{b}$ and $m = n$, then the fourth-order partially symmetric Cauchy tensor reduces to the fourth-order symmetric Cauchy tensor. In [8], the authors gave the following two inclusion sets for M-eigenvalues.

Theorem 3.25 ([8]). *Suppose that the tensor $\mathcal{C}$ is a fourth-order Cauchy tensor with generating vectors $\mathbf{a} \in \mathbb{R}^m, \mathbf{b} \in \mathbb{R}^n$. Then*

$$\sigma(\mathcal{C}) \subseteq \mathscr{C}(\mathcal{C}) = \bigcup_{i \in [m]} \mathscr{C}_i(\mathcal{C}),$$

where $\mathscr{C}_i(\mathcal{C}) = \left\{ z \in \mathbb{R} : |z| \leq \sum_{k \in [m], j, l \in [n]} \frac{1}{|a_i + b_j + a_k + b_l|} \right\}$.

Theorem 3.26 ([8]). *Suppose that the tensor $\mathcal{C}$ is a fourth-order Cauchy tensor with generating vectors $\mathbf{a} \in \mathbb{R}^m, \mathbf{b} \in \mathbb{R}^n$. If there exists an index $i \in [m]$ such that $c_{i1i1} = c_{i2i2} = \cdots = c_{inin} = d$, then*

$$\sigma(\mathcal{C}) \subseteq \mathscr{J}(\mathcal{C}) = \bigcup_{i \in [m]} \mathscr{J}_i(\mathcal{C}),$$

where $\mathscr{J}_i(\mathcal{C}) = \left\{ z \in \mathbb{R} : |z - d| \leq \sum_{j, l \in [n],\ j \neq l} \frac{1}{|a_i + b_j + a_k + b_l|} + \sum_{k \in [m],\ j, l \in [n],\ k \neq i} \frac{1}{|a_i + b_j + a_k + b_l|} \right\}$.

$\boxed{3.4}$ Inclusion sets for elasticity M-tensors

In this section, we study bounds for the minimum M-eigenvalue of another kind of fourth-order partially symmetric tensors i.e. elasticity M-tensors. The elasticity tensor has partially symmetric structure as in (2.1). To move on, we first give some useful descriptions.

Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ be a partially symmetric tensor as in (2.1). The entries a_{ilil}, $i, l = 1, 2, \cdots, n$, are called diagonal entries and other entries are called off-diagonal entries of $\mathcal{A}$. A fourth-order partially symmetric tensor is called an elasticity Z-tensor, if all its off-diagonal entries are non-positive [16]. Noted that the elasticity Z-tensor can always be written as $\mathcal{A} = s\mathcal{I}_M - \mathcal{B}$, where $\mathcal{B}$ is a nonnegative partially symmetric tensor and $\mathcal{I}_M$ is the identity tensor as defined in Section 2.

Definition 3.27 ([9, 16, 69])**.** A fourth-order partially symmetric tensor $\mathcal{A}$ is called an elasticity M-tensor if there exists a nonnegative partially symmetric tensor $\mathcal{B}$ and a real number $s \geq \rho(\mathcal{B})$ such that $\mathcal{A} = s\mathcal{I}_M - \mathcal{B}$, where $\rho(\mathcal{B})$ is the M-spectral radius of $\mathcal{B}$. Furthermore, if $s > \rho(\mathcal{B})$, then $\mathcal{A}$ is called a nonsingular elasticity M-tensor.

By Definitions 3.27, Ding *et al.* have the following conclusions.

Theorem 3.28 ([16])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ be an elasticity Z-tensor. Then $\mathcal{A}$ is a nonsingular elasticity M-tensor if and only if $\alpha > \rho(\alpha\mathcal{I}_M - \mathcal{A})$, where $\alpha = \max\{a_{ilil} : i, l = 1, 2, \cdots, n\}$.*

Theorem 3.29 ([16])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ be an elasticity Z-tensor. The following conditions are equivalent:*
(1) *$\mathcal{A}$ is a nonsingular elasticity M-tensor;*
(2) *For each $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$, there exists $\mathbf{y} > \mathbf{0}$ such that $\mathcal{A}\mathbf{x}\mathbf{y}\mathbf{x}\cdot > \mathbf{0}$;*
(3) *For each $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$, there exists $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$, such that $\mathcal{A}\mathbf{x}\mathbf{y}\mathbf{x}\cdot > \mathbf{0}$;*
(4) *For each $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$, there exists $\mathbf{x} > \mathbf{0}$ such that $\mathcal{A}\cdot\mathbf{y}\mathbf{x}\mathbf{y} > \mathbf{0}$;*
(5) *For each $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$, there exists $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$, such that $\mathcal{A}\cdot\mathbf{y}\mathbf{x}\mathbf{y} > \mathbf{0}$.*

Motivated by Corollary 3.16, Che *et al.* obtained the following conclusions [9]. To continue, the following definitions of reducible tensors and irreducible tensors are needed.

Definition 3.30 ([19])**.** A tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ with order m and dimension n is called reducible if there exists a nonempty proper index subset $J \subseteq \{1, 2, \cdots, n\} = [n]$ such that

$$a_{i_1i_2\cdots i_m} = 0, \quad \forall i_1 \in J, \forall i_2\cdots i_m \notin J.$$

If $\mathcal{A}$ is not reducible, then $\mathcal{A}$ is called irreducible.

Theorem 3.31 ([9])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ be an irreducible elasticity M-tensor. Let $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue of $\mathcal{A}$. Then it holds that*
(1)

$$\begin{aligned}\tau_M(\mathcal{A}) &\geq \max\left\{\min_{i,k\in[n],i\neq k}\{\eta_1(\mathcal{A})\}, \min_{j,l\in[n],j\neq l}\{\eta_2(\mathcal{A})\}\right\}\\ &\geq \max\left\{\min_{i\in[n]}\{\alpha_i - D_i(\mathcal{A})\}, \min_{l\in[n]}\{\beta_l - G_l(\mathcal{A})\}\right\}\end{aligned}$$

(2)

$$\begin{aligned}\tau_M(\mathcal{A}) \geq \max\Bigg\{&\min_{i,k\in[n],i\neq k}\{\varphi_1(\mathcal{A}), \alpha_i - d_i^i(\mathcal{A}), \alpha_k - d_k^k(\mathcal{A})\},\\ &\min_{j,l\in[n],j\neq l}\{\varphi_2(\mathcal{A}), \beta_j - g_j^j(\mathcal{A}), \beta_l - g_l^l(\mathcal{A})\}\Bigg\}\\ \geq \max&\left\{\min_{i\in[n]}\{\alpha_i - D_i(\mathcal{A})\}, \min_{l\in[n]}\{\beta_l - G_l(\mathcal{A})\}\right\},\end{aligned}$$

where

$$
\begin{aligned}
&\eta_1(\mathcal{A}) = \frac{\alpha_i - d_i^i(\mathcal{A}) + \alpha_k - \theta_{i,k}^{\frac{1}{2}}}{2}, \quad \eta_2(\mathcal{A}) = \frac{\beta_j - g_j^j(\mathcal{A}) + \beta_l - \omega_{j,l}^{\frac{1}{2}}}{2},\\
&\theta_{i,k} = (\alpha_i - d_i^i(\mathcal{A}) - \alpha_k)^2 + 4(d_i(\mathcal{A}) - d_i^i(\mathcal{A}) + \gamma_i)D_k(\mathcal{A}),\\
&\omega_{j,l} = (\beta_j - g_j^j(\mathcal{A}) - \beta_l)^2 + 4(g_j(\mathcal{A}) - g_j^j(\mathcal{A}) + \delta_j)G_l(\mathcal{A}),\\
&\varphi_1(\mathcal{A}) = \frac{(\alpha_i - d_i^i(\mathcal{A})) + (\alpha_k - d_k^k(\mathcal{A})) - \mathfrak{V}_{i,j}^{\frac{1}{2}}}{2}, \; \varphi_2(\mathcal{A}) = \frac{(\beta_j - g_j^j(\mathcal{A})) + (\beta_l - d_l^l(\mathcal{A})) - \mathfrak{Y}_{j,l}^{\frac{1}{2}}}{2},\\
&\mathfrak{V}_{i,k} = (\alpha_i - d_i^i(\mathcal{A}) - (\alpha_k - d_k^k(\mathcal{A})))^2 + 4(D_i(\mathcal{A}) - d_i^i(\mathcal{A}))(D_k(\mathcal{A}) - d_k^k(\mathcal{A})),\\
&\mathfrak{Y}_{j,l} = (\beta_j - g_j^j(\mathcal{A}) - (\beta_l - g_l^l(\mathcal{A})))^2 + 4(G_j(\mathcal{A}) - g_j^j(\mathcal{A}))(G_l(\mathcal{A}) - g_l^l(\mathcal{A})).
\end{aligned}
$$

By Theorem 3.31, the conclusion below summarizes bounds for the minimum M-eigenvalue without irreducible conditions.

Theorem 3.32 ([69]). *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ be an elasticity M-tensor. Then*
(1) $\tau_M(\mathcal{A}) \le \min\left\{\min_{i,l\in[n]} a_{ilil}, \frac{\sum_{i\in[n]} S_i(\mathcal{A})}{n^2}\right\}$,
(2) $\tau_M(\mathcal{A}) \ge \max\left\{\min_{i\in[n]}\{\alpha_i - B_i(\mathcal{A})\}, \min_{l\in[n]}\{\beta_l - M_l(\mathcal{A})\}\right\}$,
(3) $\tau_M(\mathcal{A}) \ge \max\{\zeta_1(\mathcal{A}), \zeta_2(\mathcal{A})\}$,
where

$$
\begin{aligned}
&S_i(\mathcal{A}) = \sum_{j,k,l\in[n]} a_{ijkl}, \; r_i(\mathcal{A}) = \sum_{j,k,l=1, j\neq l}^{n} a_{ijkl}, \; c_l(\mathcal{A}) = \sum_{i,j,k=1, i\neq k}^{n} a_{ijkl},\\
&B_i(\mathcal{A}) = \kappa_i(\mathcal{A}) - \frac{1}{2} r_i(\mathcal{A}), \; \alpha_i = \min_{l\in[n]}\{a_{ilil}\}, \; \kappa_i(\mathcal{A}) = \max_{l\in[n]}(\alpha_i - a_{ilil} - \sum_{k=1,k\neq i}^{n} a_{ilkl}),\\
&M_l(\mathcal{A}) = m_l(\mathcal{A}) - \frac{1}{2} c_l(\mathcal{A}), \; \beta_l = \min_{i\in[n]}\{a_{ilil}\}, \; m_l(\mathcal{A}) = \max_{i\in[n]}(\beta_l - a_{ilil} - \sum_{j=1,j\neq l}^{n} a_{ijil}),\\
&\zeta_1(\mathcal{A}) = \min_{i,v\in[n], v\neq i}\left\{\frac{1}{2}\{\alpha_i + \frac{1}{2} r_i(\mathcal{A}) + \alpha_v - \xi_v^i(\mathcal{A}) - \chi_{i,v}^{\frac{1}{2}}(\mathcal{A})\}, \alpha_i + \frac{1}{2} r_i(\mathcal{A}), \alpha_v - \xi_v^i(\mathcal{A})\right\},\\
&\zeta_2(\mathcal{A}) = \min_{u,l\in[n], u\neq l}\left\{\frac{1}{2}\{\beta_l + \frac{1}{2} c_l(\mathcal{A}) + \beta_u - m_u^l(\mathcal{A}) - \vartheta_{l,u}^{\frac{1}{2}}(\mathcal{A})\}, \beta_l + \frac{1}{2} c_l(\mathcal{A}), \beta_u - m_u^l(\mathcal{A})\right\},\\
&\chi_{i,v}(\mathcal{A}) = (\alpha_i + \frac{1}{2} r_i(\mathcal{A}) - \alpha_v + \xi_v^i(\mathcal{A}))^2 + 4\xi_i(\mathcal{A})(\varepsilon_v^i(\mathcal{A}) - \frac{1}{2} r_v(\mathcal{A})),\\
&\iota_{i,v}(\mathcal{A}) = (\beta_l + \frac{1}{2} c_l(\mathcal{A}) - \beta_u + m_u^l(\mathcal{A}))^2 + 4m_l(\mathcal{A})(\epsilon_u^l(\mathcal{A}) - \frac{1}{2} c_u(\mathcal{A})),\\
&\varepsilon_v^i(\mathcal{A}) = \max_{l\in[n]}(-a_{vlil}), \; \xi_v^i(\mathcal{A}) = \max_{l\in[n]}(\alpha_v - a_{vlvl} - \sum_{k=1,\; k\neq v,i}^{n} a_{vlkl}),\\
&\epsilon_u^l(\mathcal{A}) = \max_{i\in[n]}(-a_{iliu}), \; m_u^l(\mathcal{A}) = \max_{i\in[n]}(\beta_u - a_{iuiu} - \sum_{j=1,\; j\neq u,l}^{n} a_{ijil}).
\end{aligned}
$$

4 Applications of Fourth-Order Partially Symmetric Tensors

The strong ellipticity condition plays an important role in nonlinear elasticity materials. Elasticity tensors are fourth-order partially symmetric tensors. In this section, we conclude some necessary or sufficient conditions for the strong ellipticity condition, M-positive definiteness and rank-one positive definiteness of fourth-order partially symmetric tensors.

4.1 M-positive definiteness and M-positive semi-definiteness

In applications, suppose that the potential elasticity tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ is a partially symmetric tensor as in (2.1). Noted that the strong ellipticity condition is stated by

$$f(\mathbf{x},\mathbf{y}) = \mathcal{A}\mathbf{xyxy} = \sum_{i,k\in[m]}\sum_{j,l\in[n]} a_{ijkl}x_iy_jx_ky_l > 0, \tag{4.1}$$

for any nonzero $\mathbf{x}\in\mathbb{R}^m, \mathbf{y}\in\mathbb{R}^n$ [16]. If the above inequality holds with equality i.e.

$$\sum_{i,k\in[m]}\sum_{j,l\in[n]} a_{ijkl}x_iy_jx_ky_l \geq 0,$$

for all $\mathbf{x}\in\mathbb{R}^m, \mathbf{y}\in\mathbb{R}^n$, then the tensor $\mathcal{A}$ satisfies the ordinary ellipticity condition. A tensor $\mathcal{A}$ satisfying the strong ellipticity condition is M-positive definite(M-PD) and a tensor $\mathcal{A}$ is said to be M-positive semi-definite(M-PSD) if the ordinary ellipticity condition holds.

Theorem 4.1 ([16])**.** *The given tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ is M-positive definite if and only if all of its M-eigenvalues are positive, and $\mathcal{A}$ is M-positive semi-definite if and only if all of its M-eigenvalues are nonnegative.*

To verify the M-positive definiteness or M-positive semi-definiteness of partially symmetric tensors $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$, the following symbols are needed.

$$\mathbf{A}_x = \begin{bmatrix} \mathbf{A}_x^{(1,1)} & \mathbf{A}_x^{(1,2)} & \cdots & \mathbf{A}_x^{(1,n)} \\ \mathbf{A}_x^{(2,1)} & \mathbf{A}_x^{(2,2)} & \cdots & \mathbf{A}_x^{(2,n)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_x^{(n,1)} & \mathbf{A}_x^{(n,2)} & \cdots & \mathbf{A}_x^{(n,n)} \end{bmatrix}, \quad \mathbf{A}_y = \begin{bmatrix} \mathbf{A}_y^{(1,1)} & \mathbf{A}_y^{(1,2)} & \cdots & \mathbf{A}_y^{(1,n)} \\ \mathbf{A}_y^{(2,1)} & \mathbf{A}_y^{(2,2)} & \cdots & \mathbf{A}_y^{(2,n)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_y^{(n,1)} & \mathbf{A}_y^{(n,2)} & \cdots & \mathbf{A}_y^{(n,n)} \end{bmatrix},$$

where $\mathbf{A}_x^{(j,l)} := \mathcal{A}(:,j,:,l), j,l = 1,\cdots,n$ and $\mathbf{A}_y^{(i,k)} := \mathcal{A}(i,:,k,:), i,k = 1,\cdots,n$.

In [16], Ding *et al.* defined two matrices $\mathcal{A}\mathbf{x^2} \in \mathbb{R}^{n\times n}$ and $\mathcal{A}\mathbf{y^2} \in \mathbb{R}^{n\times n}$ such that

$$(\mathcal{A}\mathbf{x^2})_{jl} := \sum_{i,k=1}^{n} a_{ijkl}x_ix_k, \quad j,l = 1,2,\cdots,n,$$

$$(\mathcal{A}\mathbf{y^2})_{ik} := \sum_{j,l=1}^{n} a_{ijkl}y_jy_l, \quad i,k = 1,2,\cdots,n.$$

Note that

$$\mathcal{A}\mathbf{x}^2 = \begin{bmatrix} \mathbf{x}^\top\mathbf{A}_x^{(1,1)}\mathbf{x} & \mathbf{x}^\top\mathbf{A}_x^{(1,2)}\mathbf{x} & \cdots & \mathbf{x}^\top\mathbf{A}_x^{(1,n)}\mathbf{x} \\ \mathbf{x}^\top\mathbf{A}_x^{(2,1)}\mathbf{x} & \mathbf{x}^T\mathbf{A}_x^{(2,2)}\mathbf{x} & \cdots & \mathbf{x}^\top\mathbf{A}_x^{(2,n)}\mathbf{x} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}^\top\mathbf{A}_x^{(n,1)}\mathbf{x} & \mathbf{x}^\top\mathbf{A}_x^{(n,2)}\mathbf{x} & \cdots & \mathbf{x}^\top\mathbf{A}_x^{(n,n)}\mathbf{x} \end{bmatrix},$$

and

$$\mathcal{A}\mathbf{y}^2 = \begin{bmatrix} \mathbf{y}^\top \mathbf{A}_y^{(1,1)}\mathbf{y} & \mathbf{y}^\top \mathbf{A}_y^{(1,2)}\mathbf{y} & \cdots & \mathbf{y}^\top \mathbf{A}_y^{(1,n)}\mathbf{y} \\ \mathbf{y}^\top \mathbf{A}_y^{(2,1)}\mathbf{y} & \mathbf{y}^\top \mathbf{A}_y^{(2,2)}\mathbf{y} & \cdots & \mathbf{y}^\top \mathbf{A}_y^{(2,n)}\mathbf{y} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}^\top \mathbf{A}_y^{(n,1)}\mathbf{y} & \mathbf{y}^\top \mathbf{A}_y^{(n,2)}\mathbf{y} & \cdots & \mathbf{y}^\top \mathbf{A}_y^{(n,n)}\mathbf{y} \end{bmatrix}.$$

Furthermore, it is straightforward to verify that

$$\mathcal{A}\mathbf{xyxy} = \mathbf{y}^\top(\mathcal{A}\mathbf{x}^2)\mathbf{y} = \mathbf{x}^\top(\mathcal{A}\mathbf{y}^2)\mathbf{x},\ \mathcal{A}\mathbf{x}^2\mathbf{y} = (\mathcal{A}\mathbf{x}^2)\mathbf{y},\ \mathcal{A}\mathbf{x}\mathbf{y}^2 = (\mathcal{A}\mathbf{y}^2)\mathbf{x}. \tag{4.2}$$

It should be noted that the symmetries of $\mathcal{A}$ implies that both $\mathcal{A}\mathbf{x}^2$ and$\mathcal{A}\mathbf{y}^2$ are symmetric matrices [16].

Note that $\mathbf{A}_x$ and $\mathbf{A}_y$ are permutation similar to each other i.e. there is a permutation matrix P such that $\mathbf{A}_x = P^\top \mathbf{A}_y P$. Then $\mathcal{A}$ is M-PD or M-PSD if $\mathbf{A}_x$ (or equivalently $\mathbf{A}_y$) is PD or PSD, respectively. This can be proved by noticing that

$$\mathcal{A}\mathbf{xyxy} = (\mathbf{y}\otimes\mathbf{x})^\top \mathbf{A}_x(\mathbf{y}\otimes\mathbf{x}) = (\mathbf{x}\otimes\mathbf{y})^\top \mathbf{A}_y(\mathbf{x}\otimes\mathbf{y}),$$

where $\otimes$ denotes the Kronecker product. Thus $\mathcal{A}$ is S-positive (semi)definite if $\mathbf{A}_x$ or $\mathbf{A}_y$ is positive (semi)definite, and call the eigenvalues of $\mathbf{A}_x$ or $\mathbf{A}_y$ the S-eigenvalues of $\mathcal{A}$. The S-positive definiteness is a sufficient condition for the M-positive definiteness, but the converse is not true.

The symmetries in $\mathcal{A}$ imply that both $\mathcal{A}\mathbf{x}^2$ and $\mathcal{A}\mathbf{y}^2$ are symmetric matrix. According to Eq.(4.2), one can prove the following necessary and sufficient condition for the M-positive (semi)definiteness.

Proposition 4.2 ([16]). *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$. Then $\mathcal{A}$ is M-PD or M-PSD if and only if the matrix $\mathcal{A}\mathbf{x}^2$ ($\mathcal{A}\mathbf{y}^2$) is PD or PSD for each nonzero $\mathbf{x}\in\mathbb{R}^n$ ($\mathbf{y}\in\mathbb{R}^n$), respectively.*

Given a fourth-order tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$, denote

$$\mathbb{T}_{\mathcal{A}} := \{\mathfrak{J} = (t_{ijkl}) : t_{ijkl} = t_{klij},\ t_{ijkl} + t_{klij} = 2a_{ijkl}\}.$$

Denote the set of all fourth-order S-PSD tensors as

$$\mathbb{S} := \{\mathfrak{J} : t_{ijkl} = t_{klij},\ \mathfrak{J} \textit{ is S-PSD}\}.$$

Note that both $\mathbb{T}_{\mathcal{A}}$ and $\mathbb{S}$ are closed convex sets, where $\mathbb{T}_{\mathcal{A}}$ is a linear subspace of the whole space of all the fourth-order n dimensional tensor with $t_{ijkl} = t_{klij}$ and $\mathbb{S}$ is isomorphic with the nine-by-nine symmetric PSD matrix cone. Furthermore, Ding *et al.* [16] proved the following sufficient condition for a tensor to be M-PD or M-PSD.

Theorem 4.3 ([16]). *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$. If $\mathbb{T}_{\mathcal{A}} \cap \mathbb{S} \neq \emptyset$, then $\mathcal{A}$ is M-PSD; If $\mathbb{T}_{\mathcal{A}} \cap (\mathbb{S}\backslash\partial\mathbb{S}) \neq \emptyset$, then $\mathcal{A}$ is M-PD.*

Next, we recall an alternating projection method, which was provided by Ding *et al.* in [16]. The projection onto convex sets(POCS) is often employed to check whether the intersection of two closed convex sets is empty or not. Assume $\mathcal{P}_1$ and $\mathcal{P}_2$ are the projection operators onto $\mathbb{T}_{\mathcal{A}}$ and $\mathbb{S}$, respectively such that

$$\mathcal{B}^{(t+1)} = \mathcal{P}_2(\mathcal{A}^{(t)}),\ \ \mathcal{A}^{(t+1)} = \mathcal{P}_1(\mathcal{B}^{(t+1)}),\ t = 0,1,2,\cdots.$$

Based the projection technique above, Ding *et al.* [16] provided the following iterative scheme:

Step 0. Give the eigenvalue decomposition of matrix $\mathbf{A}^{(t)}$ such that

$$\mathbf{A}^{(t)} = \mathbf{V}^{(t)}\mathbf{D}^{(t)}(\mathbf{V}^{(t)})^\top;$$

Step 1. let $\mathbf{B}^{(t+1)} = \mathbf{V}^{(t)}\mathbf{D}_+^{(t)}(\mathbf{V}^{(t)})^\top$, where $\mathbf{D}_+^{(t)} = \mathrm{diag}(\max\{d_{ii}^{(t)}, 0\})$;
Step 2. set

$$a_{ijil}^{(t+1)} = a_{ijil} \quad for\ i,j,l = 1,2,3, \quad a_{ijkj}^{(t+1)} = a_{ijkj} \quad for\ i,j,k = 1,2,3;$$

$$a_{ijkl}^{(t+1)} = a_{ijkl} + \frac{1}{2}(b_{ijkl}^{(t+1)} - b_{kjil}^{(t+1)}) \quad for\ i \neq k, j \neq l;$$

Step 3. take $t = t+1$ and return to **Step 0**.
Here, $\mathcal{A}^{(0)} = \mathcal{A}$, $\mathbf{A}^{(t)}$ and $\mathbf{B}^{(t)}$ are the unfolding matrices of $\mathcal{A}^{(t)}$ and $\mathcal{B}^{(t)}$ respectively.

Noteworthy that the convergence of the alternating projection method between two closed convex sets has been known for a long time [12].

Theorem 4.4 ([16])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$. If $\mathbb{T}_\mathcal{A} \cap \mathbb{S} \neq \emptyset$, then the sequences $\{\mathcal{A}^{(t)}\}$ and $\{\mathcal{B}^{(t)}\}$ produced by the above algorithm both converge to a point $\mathcal{A}^* \in \mathbb{T}_\mathcal{A} \cap \mathbb{S}$.*

Because the convergence of POCS requires the involved convex sets to be closed. The above algorithm is only suitable for identifying the M-positive semi-definiteness. If one want to check the M-positive definiteness, then some modifications are needed. Note that $\mathcal{I}_M\mathbf{xyxy} = (\mathbf{x}^\top\mathbf{x})(\mathbf{y}^\top\mathbf{y})$. Hence $\mathcal{I}_M$ is M-PD, which implies that $\mathcal{A}$ is M-PD if and only if $\mathcal{A} - \epsilon\mathcal{I}_M$ is M-PSD for some sufficiently small $\epsilon > 0$. From such observation, Ding *et al.* applyed POCS to $\mathcal{A} - \epsilon\mathcal{I}_M$ with a very small ϵ. If the iteration converges and both $\{\mathcal{A}^{(t)}\}$ and $\{\mathcal{B}^{(t)}\}$ converge to the same tensor, then they concluded that $\mathcal{A}$ is M-PD i.e. the strong ellipticity holds.

Lemma 4.5 ([27])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ be M-positive definite. Then $a_{ilil} > 0$, $i,l \in [n]$.*

Lemma 4.5 provides a necessary condition for the M-positive definiteness, also the strong ellipticity condition. Several more sufficient conditions are listed below.

Theorem 4.6 ([10, 27])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ be a given tensor.*
(1) *With $a_{i1i1} = \cdots = a_{inin} = \alpha_i > 0, i \in [n]$. If for all $i \in [n]$, $\alpha_i > D_i(\mathcal{A})$, then $\mathcal{A}$ is M-positive definite.*
(2) *With $a_{i1i1} = \cdots = a_{inin} = \alpha_i > 0$ for $i \in [n]$. If for all $p \neq s \in [n]$, the following statement hold*

$$(\alpha_p - D_p^p(\mathcal{A}) - \gamma_p)\alpha_s > (D_p(\mathcal{A}) - D_p^p(\mathcal{A}) - \gamma_p)D_s(\mathcal{A}), \tag{4.3}$$

then $\mathcal{A}$ is M-positive definite.
(3) *With $a_{1l1l} = \cdots = a_{nlnl} = \beta_l > 0, l \in [n]$. If for all $l \in [n]$, $\beta_l > G_l(\mathcal{A})$, then $\mathcal{A}$ is M-positive definite.*
(4) *With $a_{1l1l} = \cdots = a_{nlnl} = \beta_l > 0$ for $l \in [n]$. If for all $q \neq t \in [n]$, the following statement hold*

$$(\beta_q - G_q^q(\mathcal{A}) - \delta_q)\beta_t > (G_q(\mathcal{A}) - G_q^q(\mathcal{A}) - \delta_q)G_t(\mathcal{A}), \tag{4.4}$$

then $\mathcal{A}$ is M-positive definite.
(5) *With $a_{i1i1} = \cdots = a_{inin} = \alpha_i = a_{1i1i} = \cdots = a_{nini}, i \in [n]$. If for all $i \in [n], \alpha_i >$ $\min\{D_i(\mathcal{A}), G_i(\mathcal{A})\}$, then $\mathcal{A}$ is M-positive definite.*
(6) *With $a_{i1i1} = \cdots = a_{inin} = \alpha_i = a_{1i1i} = \cdots = a_{nini}$ for $i \in [n]$. If for all $q \neq t \in [n], p \neq s \in [n]$, (4.3) and (4.4) hold, then $\mathcal{A}$ is M-positive definite.*

Theorem 4.6 provides several checkable sufficient conditions for the strong ellipticity condition of three classes of structured positive semi-definite tensors. Later, He *et al.* [27] and Che *et al.* [10] extended these conditions to general positive semi-definite tensors independently.

Theorem 4.7 ([10, 27])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ be a given tensor with $a_{ilil} > 0$, $i, l \in [n]$ and $\min_{i\in[n]}\{a_{i1i1}, \cdots, a_{inin}\} = \alpha_i$, $\min_{l\in[n]}\{a_{1l1l}, \cdots, a_{nlnl}\} = \beta_l$.*
(1) If for all $i, l \in [n]$, $\alpha_i > D_i(\mathcal{A})$ or $\beta_l > G_l(\mathcal{A})$, then $\mathcal{A}$ is M-positive definite.
(2) If for all $q \neq t \in [n], p \neq s \in [n]$, (4.3) or (4.4) holds, then $\mathcal{A}$ is M-positive definite.

4.2 Strong ellipticity, rank-one positive definiteness

The strong ellipticity is related to another kind of positive definiteness of $\mathcal{A}$ [52] named rank-one positive definiteness.

The tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ is called rank-one positive definite if for all $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$,

$$f(\mathbf{x}, \mathbf{x}) = \mathcal{A}\mathbf{x}^4 = \mathcal{A}\mathbf{xxxx} = \sum_{i,j,k,l=1}^{n} a_{ijkl}x_i x_j x_k x_l > 0. \tag{4.5}$$

Clearly, if the strong ellipticity holds, then $\mathcal{A}$ is rank-one positive definite but not vice versa.

It is also easy to see that the rank-one positivity condition (4.5) holds if and only if the optimal value of the following global polynomial optimization problem is positive

$$\begin{array}{ll} \min & f(\mathbf{x}, \mathbf{x}) = \mathcal{A}\mathbf{x}^4 = \sum\limits_{ijkl=1}^{n} a_{ijkl}x_i x_j x_k x_l \\ \text{s.t.} & \mathbf{x}^\top \mathbf{x} = 1. \end{array} \tag{4.6}$$

The KKT-condition for (4.6) is:

$$\begin{cases} \mathcal{A}\mathbf{x}^3 = \lambda\mathbf{x}, \\ \mathbf{x}^\top\mathbf{x} = 1. \end{cases} \tag{4.7}$$

where $\mathcal{A}\mathbf{x}^3 = \mathcal{A}\cdot\mathbf{xxx} = \mathcal{A}\mathbf{xxx}\cdot$. In [51], Qi *et al.* first defined that if $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$ satisfy (4.7), λ is called a Z-eigenvalue of $\mathcal{A}$, and $\mathbf{x}$ is called the Z-eigenvector of $\mathcal{A}$, associated with the Z-eigenvalue λ. Thus, if the smallest Z-eigenvalue of $\mathcal{A}$ is positive, then $\mathcal{A}$ is rank-one positive.

By (4.7) and the definition of M-eigenvalue, it is easy to see that a Z-eigenvalue ia an M-eigenvalue. However, an M-eigenvalue is not necessarily a Z-eigenvalue.

Theorem 4.8 ([52])**.** *For the given partially symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$. The strong ellipticity condition holds if and only if the smallest M-eigenvalue of $\mathcal{A}$ is positive. Furthermore, $\mathcal{A}$ is rank-one positive definite if and only if its smallest Z-eigenvalue is positive.*

In some cases, it is possible that all the M-eigenvalues are Z-eigenvalues. If, furthermore, for all i, j, k, l, $a_{ijkl} = a_{\pi(ijkl)}$, where $\pi(ijkl)$ is any permutation of i, j, k, l. Then $\mathcal{A}$ is a symmetric tensor. In this case, Qi *et al.* [52] denoted

$$\mathcal{A}\mathbf{xy} = \mathcal{A}\cdot\cdot\mathbf{xy} = \left(\sum_{k,l}^{n} a_{ijkl}x_k y_l\right),$$

which is a symmetric matrix. Then, they gave the following theorem.

Theorem 4.9 ([52]). *Suppose $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ with $n = 2$ and $\mathcal{A}$ is symmetric. Then all the M-eigenvalues of $\mathcal{A}$ are Z-eigenvalues if there are no $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, satisfying the following three conditions:*
(1) $\mathbf{x}^\top \mathbf{x} = 1,\ \mathbf{y}^\top \mathbf{y} = 1$;
(2) $\mathbf{x}$ *and* $\mathbf{y}$ *are linearly independent;*
(3) $(\mathcal{A}\mathbf{x}\mathbf{y})^2 = \lambda^2 I$, *where I is the 2×2 unit matrix.*

Theorem 4.9 implies that when $\mathcal{A}$ is symmetric and $n = 2$, it is very possible that all the M-eigenvalues are Z-eigenvalues. Combining this with Theorem 4.8, it forms several checkable conditions for the strong ellipticity of a given tensor $\mathcal{A}$ with $n = 2$. Next we study the case of partially symmetric tensor with $n = 3$.

Suppose that $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{3\times 3\times 3\times 3}$ is a partially symmetric tensor. Let $Q(\mathbf{x}) = (q_{jl})$ denote a symmetric matrix such that

$$q_{jl} = \sum_{i,k=1}^{3} a_{ijkl} x_i x_k.$$

Then the strong ellipticity of $\mathcal{A}$ is equivalent to the positive definiteness of $Q(\mathbf{x})$ for all unit vector $\mathbf{x}$. Furthermore, several new matrices and tensors related with $\mathcal{A}$ are needed.

$$M_1 = \begin{bmatrix} a_{1111} & a_{1121} & a_{1131} \\ a_{1121} & a_{2121} & a_{2131} \\ a_{1131} & a_{2131} & a_{3131} \end{bmatrix},\ M_2 = \begin{bmatrix} a_{1212} & a_{1222} & a_{1232} \\ a_{1222} & a_{2222} & a_{2232} \\ a_{1232} & a_{2232} & a_{3232} \end{bmatrix},\ M_3 = \begin{bmatrix} a_{1313} & a_{1323} & a_{1333} \\ a_{1323} & a_{2323} & a_{2333} \\ a_{1333} & a_{2333} & a_{3333} \end{bmatrix}$$

Let $\mathcal{T}^1, \mathcal{T}^2$ and $\mathcal{T}^3$ be the fourth-order three-dimensional tensors, $\mathcal{W}$ be the sixth-order three-dimensional tensor, such that

$$\mathcal{T}^1\mathbf{x}^4 = \sum_{i,j,k,l=1}^{3} t^1_{ijkl} x_i x_j x_k x_l = q_{11}q_{22} - q_{12}^2,\ \mathcal{T}^2\mathbf{x}^4 = \sum_{i,j,k,l=1}^{3} t^2_{ijkl} x_i x_j x_k x_l = q_{11}q_{33} - q_{13}^2,$$
$$\mathcal{T}^3\mathbf{x}^4 = \sum_{i,j,k,l=1}^{3} t^3_{ijkl} x_i x_j x_k x_l = q_{22}q_{33} - q_{23}^2,\ \mathcal{W}\mathbf{x}^6 = \sum_{i_1,\cdots,i_6=1}^{3} w_{i_1\cdots i_6} x_{i_1} x_{i_2} \cdots x_{i_6} = \det Q(\mathbf{x}).$$

Theorem 4.10 ([26]). *The partially symmetric tensor $\mathcal{A}$ satisfies the strong ellipticity condition if and only if the following conditions hold*
(1) *The matrices M_1, M_2 and M_3 are positive definite;*
(2) *The fourth-order tensors $\mathcal{T}^1, \mathcal{T}^2$ and $\mathcal{T}^3$ are rank-one positive;*
(3) *The sixth-order tensor $\mathcal{W}$ is rank-one positive.*

To end this section, we recall the strong ellipticity condition of a partially symmetric tensor from a practical problem. In [26], Han *et al.* discussed the strong ellipticity condition for the rhombic system, where the partially symmetric tensor $\mathcal{A}$ with entries such that:

$$\begin{aligned} &a_{1123} = a_{1131} = a_{1112} = a_{2223} = a_{2231} = a_{2212} = 0, \\ &a_{3323} = a_{3331} = a_{3312} = a_{2331} = a_{2312} = a_{3112} = 0. \end{aligned} \tag{4.8}$$

For the sake of simplicity, nonzero components of the tensor $\mathcal{A}$ are denoted as follows

$$a_{11} = a_{1111},\quad a_{22} = a_{2222},\quad a_{33} = a_{3333},\quad a_{12} = a_{1122},\quad a_{23} = a_{2233},$$
$$a_{31} = a_{3311},\quad a_{44} = a_{2323},\quad a_{55} = a_{1313},\quad a_{66} = a_{1212}.$$

Then, they gave the following result.

Theorem 4.11 ([26]). *Let $\mathcal{A}$ be an tensor as in* (4.8)*. Then $\mathcal{A}$ satisfies the strong ellipticity condition if and only if the following conditions hold.*
(1) $a_{11} > 0, a_{22} > 0, a_{33} > 0, a_{44} > 0, a_{55} > 0, a_{66} > 0$.
(2) *The matrices P^1, P^2 and P^3 defined in* (4.9) *are copositive.*
(3) *The sixth-order tensor $\mathcal{W}$ is rank-one positive, where*

$$P^1 = \begin{bmatrix} a_{11}a_{66} & \frac{(a_{11}a_{22}+a_{66}^2-4a_{12}^2)}{2} & \frac{(a_{11}a_{44}+a_{55}a_{66})}{2} \\ \frac{(a_{11}a_{22}+a_{66}^2-4a_{12}^2)}{2} & a_{22}a_{66} & \frac{(a_{44}a_{66}+a_{22}a_{55})}{2} \\ \frac{(a_{11}a_{44}+a_{55}a_{66})}{2} & \frac{(a_{44}a_{66}+a_{22}a_{55})}{2} & a_{44}a_{55} \end{bmatrix},$$
$$P^2 = \begin{bmatrix} a_{11}a_{55} & \frac{(a_{11}a_{44}+a_{55}a_{66})}{2} & \frac{(a_{11}a_{33}+a_{55}^2-4a_{31}^2)}{2} \\ \frac{(a_{11}a_{44}+a_{55}a_{66})}{2} & a_{44}a_{66} & \frac{(a_{33}a_{66}+a_{44}a_{55})}{2} \\ \frac{(a_{11}a_{33}+a_{55}^2-4a_{31}^2)}{2} & \frac{(a_{33}a_{66}+a_{44}a_{55})}{2} & a_{33}a_{55} \end{bmatrix}, \tag{4.9}$$
$$P^3 = \begin{bmatrix} a_{55}a_{66} & \frac{(a_{44}a_{66}+a_{22}a_{55})}{2} & \frac{(a_{33}a_{66}+a_{44}a_{55})}{2} \\ \frac{(a_{44}a_{66}+a_{22}a_{55})}{2} & a_{22}a_{44} & \frac{(a_{22}a_{33}+a_{44}^2-4a_{23}^2)}{2} \\ \frac{(a_{33}a_{66}+a_{44}a_{55})}{2} & \frac{(a_{22}a_{33}+a_{44}^2-4a_{23}^2)}{2} & a_{33}a_{44} \end{bmatrix}.$$

4.3 The M-positive definiteness and the strong ellipticity condition of fourth-order Cauchy tensors

In this section, we are interested in necessary and sufficient conditions for the M-positive semi-definiteness and M-positive definiteness of fourth-order Cauchy tensors as defined in Definition 3.23. Moreover, the necessary and sufficient conditions of the strong ellipticity conditions for fourth-order Cauchy tensors are obtained. Furthermore, fourth-order Cauchy tensors are M-positive semi-definite if and only if the homogeneous polynomial for fourth-order Cauchy tensors is monotonically increasing.

Theorem 4.12 ([8]). *Let vectors $\mathbf{a} \in \mathbb{R}^m,\ \mathbf{b} \in \mathbb{R}^n$ be generating vectors of the fourth-order Cauchy tensor $\mathcal{C}$. Then the following conclusions hold.*
(1) *The tensor $\mathcal{C}$ is M-positive semi-definite if and only if $a_i + b_j > 0$ for all $i \in [m],\ j \in [n]$.*
(2) *The tensor $\mathcal{C}$ is M-positive definite and the strong ellipticity condition of this tensor holds if and only if $a_i + b_j > 0$ for all $i \in [m], j \in [n]$, and the elements of generating vectors $\mathbf{a}, \mathbf{b}$ are mutually distinct, respectively.*
(3) *For all $i \in [m], j \in [n]$ such that $a_i + b_j > 0$, the tensor $\mathcal{C}$ is M-positive definite if and only if its M-eigenvalues are positive.*

From Theorem 4.12, the following corollary holds directly.

Corollary 4.13. *Assume that tensor $\mathcal{C}$ is defined as in Theorem* 4.12*. Then it follows that*
(1) *it is M-negative semi-definite if and only if $a_i + b_j < 0$ for all $i \in [m], j \in [n]$;*
(2) *it is not M-positive semi-definite if and only if there exist at least $i \in [m], j \in [n], a_i+b_j < 0$ holds.*

In [8], the following conclusions presented by Che *et al.* reveal the relationship between M-positive semi-definiteness of a fourth-order Cauchy tensor and the monotonicity of a homogeneous polynomial with respect to the proposed Cauchy tensor. To continue, the definition of monotonicity for the corresponding function $f(\mathbf{x}, \mathbf{y}) = \mathcal{C}\mathbf{xyxy}$ is needed.

For any $\mathbf{x},\ \bar{\mathbf{x}} \in \mathbb{R}^m$ and $\mathbf{y},\ \bar{\mathbf{y}} \in \mathbb{R}^n$, if $f(\mathbf{x}, \mathbf{y}) \geq f(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ when $\mathbf{x} \geq \bar{\mathbf{x}}$ and $\mathbf{y} \geq \bar{\mathbf{y}}$, ($\mathbf{x} \leq \bar{\mathbf{x}}$ and $\mathbf{y} \leq \bar{\mathbf{y}}$), then $f(\mathbf{x}, \mathbf{y})$ is called monotonically increasing (monotonically decreasing respectively). If $f(\mathbf{x}, \mathbf{y}) > f(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ when $\mathbf{x} \geq \bar{\mathbf{x}}$, $\mathbf{x} \neq \bar{\mathbf{x}}$ and $\mathbf{y} \geq \bar{\mathbf{y}}$, $\mathbf{y} \neq \bar{\mathbf{y}}$ ($\mathbf{x} \leq \bar{\mathbf{x}}$, $\mathbf{x} \neq \bar{\mathbf{x}}$

and $\mathbf{y} \leq \bar{\mathbf{y}}$, $\mathbf{y} \neq \bar{\mathbf{y}}$), then $f(\mathbf{x}, \mathbf{y})$ is called strictly monotone increasing (strictly monotone decreasing respectively).

Theorem 4.14 ([8])**.** *Let $\mathcal{C}$ be a fourth-order Cauchy tensor with generating vectors $\mathbf{a} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^n$. Then the tensor $\mathcal{C}$ is M-positive semi-definite if and only if the homogeneous polynomial $f(x, y)$ in* (2.2) *is monotonically increasing in $\mathbb{R}^m_+ \times \mathbb{R}^n_+$; if the tensor $\mathcal{C}$ is M-positive definite, then $f(x, y)$ is strictly monotone increasing in $\mathbb{R}^m_+ \times \mathbb{R}^n_+$.*

To end this section, we recall the necessary and sufficient conditions for the strong ellipticity condition of fourth-order Cauchy tensors.

Theorem 4.15 ([8])**.** *Let vectors $\mathbf{a} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^n$ be generating vectors of the fourth-order Cauchy tensor $\mathcal{C}$. The strong ellipticity condition holds if and only if the smallest M-eigenvalue of $\mathcal{C}$ is positive.*

4.4 The positive definiteness and the strong ellipticity condition of elasticity M-tensors

In this section, based on the results of Theorem 3.31, we study several sufficient conditions for the strong ellipticity and positive definiteness of elasticity M-tensors.

Theorem 4.16 ([9])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ be an irreducible elasticity M-tensor. If*

$$\max\left\{\min_{i,j\in[n],i\neq j}\{\eta_1(\mathcal{A})\}, \min_{k,l\in[n],k\neq l}\{\eta_2(\mathcal{A})\}\right\} > 0,$$

or

$$\max\left\{\min_{i,j\in[n],i\neq j}\{\varphi_1(\mathcal{A}), \alpha_i - d^i_i(\mathcal{A}), \alpha_j - d^j_j(\mathcal{A})\},\right.$$
$$\left.\min_{k,l\in[n],k\neq l}\{\varphi_2(\mathcal{A}), \beta_k - g^k_k(\mathcal{A}), \beta_l - g^l_l(\mathcal{A})\}\right\} > 0,$$

then $\mathcal{A}$ is positive definite, and the strong ellipticity condition holds.

Theorem 4.17 ([16])**.** *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{n\times n\times n\times n}$ be an elasticity Z-tensor. Then $\mathcal{A}$ is a nonsingular elasticity M-tensor if and only if $\mathcal{A}$ is M-positive definite; and $\mathcal{A}$ is an elasticity M-tensor if and only if $\mathcal{A}$ is M-positive semi-definite.*

5 Algorithm for Computing the Largest M-Eigenvalue of a Fourth-Order Partially Symmetric Tensor

Although it is NP-hard to compute all M-eigenvalues of a fourth-order partially symmetric tensor, it is possible to compute or obtain an approximate value for the largest M-eigenvalue in some cases. In this section, we recall two algorithms to compute the largest M-eigenvalue.

By the definition of M-eigenvalue in (2.5), the problem above can be transformed equivalently as follows:

$$\begin{cases} \max\ f(\mathbf{x}, \mathbf{y}) = \mathcal{A}\mathbf{xyxy} = \sum\limits_{i,k\in[m]}\sum\limits_{j,l\in[n]} a_{ijkl}x_iy_jx_ky_l \\ \text{s.t. } \mathbf{x}^\top\mathbf{x} = 1,\ \mathbf{y}^\top\mathbf{y} = 1,\ \mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^n. \end{cases} \tag{5.1}$$

Noted that this problem has been shown to be NP-hard [39] since neither equation of system (2.5) is linear. In [66], Wang *et al.* proposed a practical method to compute the largest M-eigenvalue of tensor $\mathcal{A}$ based on the power method which is well known to compute the largest eigenvalue in magnitude of a matrix [23]. Compared with the alternating eigenvalue maximization method for solving (5.1) proposed by Dahl *et al.* [14], the computation cost of this method is less.

The power method was successfully extended to compute the best rank-1 approximations of higher-order tensors [15, 33] i.e. the largest Z-eigenvalue in magnitude of higher-order tensors [51]. Motivated by this, Wang *et al.* [66] proposed this modified power method.

The theoretical analysis of the method was given by Wang *et al.* [66]. For the objective function $f(x, y)$, from (2.5), we know that it is a bi-quadratic function with respect to $\mathbf{x}, \mathbf{y}$, respectively. That is, the function can be written as

$$f(\mathbf{x}, \mathbf{y}) = \mathcal{A}\mathbf{xyxy} = \mathbf{x}^\top B(\mathbf{y})\mathbf{x} = \mathbf{y}^\top C(\mathbf{x})\mathbf{y}$$

where B($\mathbf{y}$) and C($\mathbf{x}$) are, respectively, symmetric matrices in $\mathbb{R}^{m\times m}$ and $\mathbb{R}^{n\times n}$ with entries

$$B_{ik}(\mathbf{y}) = \sum_{j,l=1}^{n} \mathcal{A}_{ijkl} y_j y_l, \qquad C_{jl}(\mathbf{x}) = \sum_{i,k=1}^{m} \mathcal{A}_{ijkl} x_i x_k$$

Now, we present the modified power method in Algorithm 5.1.

Algorithm 5.1: A modified power method

Initial step:
Input $\mathcal{A}$ and unfold it to obtain matrix A.
Substep 1:
Take $\tau = \sum_{1\le i\le j\le mn} |A_{ij}|$, set $\bar{\mathcal{A}} = \tau \mathcal{I}_M + \mathcal{A}$ and unfold $\bar{\mathcal{A}}$ to matrix $\bar{A}$.
Substep 2:
Compute the eigenvector $\mathbf{w}$ of matrix $\bar{A}$ associated with the largest eigenvalue and fold it into the matrix W.
Substep 3:
Compute the singular vectors $\mathbf{u_1}$ and $\mathbf{v_1}$ corresponding to the largest singular value of the matrix W.
Substep 4:
Take $\mathbf{x}_0 = \mathbf{u}_1, \mathbf{y}_0 = \mathbf{v}_1$, and let $k = 0$.
Iterative step:
Execute the following procedures alternatively until certain convergence criterion is satisfied and output $\mathbf{x}^*, \mathbf{y}^*$:

$$\begin{aligned}&\bar{\mathbf{x}}_{k+1} = \bar{\mathcal{A}} \cdot \mathbf{y}_k \mathbf{x}_k \mathbf{y}_k, \quad \mathbf{x}_{k+1} = \tfrac{\bar{\mathbf{x}}_{k+1}}{\|\bar{\mathbf{x}}_{k+1}\|}\\ &\bar{\mathbf{y}}_{k+1} = \bar{\mathcal{A}} \cdot \mathbf{x}_{k+1} \mathbf{y}_k \mathbf{x}_{k+1}, \quad \mathbf{y}_{k+1} = \tfrac{\bar{\mathbf{y}}_{k+1}}{\|\bar{\mathbf{y}}_{k+1}\|}\\ &k = k + 1.\end{aligned}$$

Final step:
Output the largest M-eigenvalue of tensor $\mathcal{A} : \lambda = f(\mathbf{x}^*, \mathbf{y}^*) - \tau$, and the associated M-eigenvectors: $\mathbf{x}^*, \mathbf{y}^*$.

Certainly, the algorithm contains two parts: the initial step and the iterative step. In fact, the initial step i.e. computing the largest eigenvalue and the corresponding eigenvector of a matrix, is also an iterative scheme [23]. For Algorithm 5.1, the computation complexity at each iterative step is of order $O(m^2n + mn^2)$. Thus, if the largest eigenvalue of tensor $\mathcal{A}$ can be generated within few steps, this algorithm can be said to be practical. To check the efficiency of the algorithm, Wang *et al.* [66] provided several numerical experiments on two fourth-order three-dimensional partially symmetric tensors, and the global optimal values were obtained with the help of the uniform grid method in high-order accuracy.

To give the numerical experiments, we first introduce the following results.

Theorem 5.1 ([66]). *Suppose that for any $\mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^n$, the matrices $B(\mathbf{y})$ and $C(\mathbf{x})$ are both positive definite. Then the generated sequence $\{f(\mathbf{x}_k, \mathbf{y}_k)\}$ by Algorithm 5.1 is nondecreasing.*

Example 5.2. Consider the tensor whose entries are uniformly generated in $(-1, 1)$:

$$A(:,:,1,1) = \begin{bmatrix} -0.9727 & 0.3169 & -0.3437 \\ -0.6332 & -0.7866 & 0.4257 \\ -0.3350 & -0.9896 & -0.4323 \end{bmatrix}, \quad A(:,:,2,1) = \begin{bmatrix} -0.6332 & -0.7866 & 0.4257 \\ 0.7387 & 0.6873 & -0.3248 \\ -0.7986 & -0.5988 & -0.9485 \end{bmatrix},$$

$$A(:,:,3,1) = \begin{bmatrix} -0.3350 & -0.9896 & -0.4323 \\ -0.7986 & -0.5988 & -0.9485 \\ 0.5853 & 0.5921 & 0.6301 \end{bmatrix}, \quad A(:,:,1,2) = \begin{bmatrix} 0.3169 & 0.6158 & -0.0184 \\ -0.7866 & 0.0160 & 0.0085 \\ -0.98966 & -0.6663 & 0.2559 \end{bmatrix},$$

$$A(:,:,2,2) = \begin{bmatrix} -0.7866 & 0.0160 & 0.0085 \\ 0.6873 & 0.5160 & -0.0216 \\ -0.5988 & 0.0411 & 0.9857 \end{bmatrix}, \quad A(:,:,3,2) = \begin{bmatrix} -0.9896 & -0.6663 & 0.2559 \\ -0.5988 & 0.0411 & 0.9857 \\ 0.5921 & -0.2907 & -0.3881 \end{bmatrix},$$

$$A(:,:,1,3) = \begin{bmatrix} -0.3437 & -0.0184 & 0.5649 \\ 0.4257 & 0.0085 & -0.1439 \\ -0.4323 & 0.2559 & 0.6162 \end{bmatrix}, \quad A(:,:,2,3) = \begin{bmatrix} 0.4257 & 0.0085 & -0.1439 \\ -0.3248 & -0.0216 & -0.0037 \\ -0.9485 & 0.9857 & -0.7734 \end{bmatrix},$$

$$A(:,:,3,3) = \begin{bmatrix} -0.4323 & 0.2559 & 0.6162 \\ -0.9485 & 0.9857 & -0.7734 \\ 0.6301 & -0.3881 & -0.8526 \end{bmatrix}.$$

The variation of the objective function value corresponding to this tensor during the interation can be seen in Figure 2. For this tensor, its largest M-eigenvalue is 2.3227, which is marked in Figure 2 by the horizontal line.

Example 5.3. Consider the tensor whose entries are uniformly generated in $(0, 5)$:

$$A(:,:,1,1) = \begin{bmatrix} 1.9832 & 1.0023 & 4.2525 \\ 2.6721 & 3.2123 & 2.8761 \\ 4.6384 & 2.9484 & 4.0319 \end{bmatrix}, \quad A(:,:,2,1) = \begin{bmatrix} 2.6721 & 3.2123 & 2.8761 \\ 3.0871 & 0.1393 & 4.4704 \\ 1.7450 & 3.0394 & 4.6836 \end{bmatrix},$$

$$A(:,:,3,1) = \begin{bmatrix} 4.6384 & 2.9484 & 4.0319 \\ 1.7450 & 3.0394 & 4.6836 \\ 0.3741 & 1.6947 & 2.7677 \end{bmatrix}, \quad A(:,:,1,2) = \begin{bmatrix} 1.0023 & 4.9748 & 2.3701 \\ 3.2123 & 1.3024 & 3.2064 \\ 2.9484 & 4.9946 & 3.8951 \end{bmatrix},$$

$$A(:,:,2,2) = \begin{bmatrix} 3.2123 & 1.3024 & 3.2064 \\ 0.1393 & 4.9456 & 2.9980 \\ 3.0394 & 4.3263 & 0.5925 \end{bmatrix}, \quad A(:,:,3,2) = \begin{bmatrix} 2.9484 & 4.9946 & 3.8951 \\ 3.0394 & 4.3263 & 0.5925 \\ 1.6947 & 4.2633 & 0.1524 \end{bmatrix},$$

$$A(:,:,1,3) = \begin{bmatrix} 4.2525 & 2.3701 & 2.4709 \\ 2.8761 & 3.2064 & 3.4492 \\ 4.0319 & 3.8951 & 0.6581 \end{bmatrix}, \quad A(:,:,2,3) = \begin{bmatrix} 2.8761 & 3.2064 & 3.4492 \\ 4.4704 & 2.9980 & 0.4337 \\ 4.6836 & 0.5925 & 4.3514 \end{bmatrix},$$

$$A(:,:,3,3) = \begin{bmatrix} 4.0319 & 3.8951 & 0.6581 \\ 4.6836 & 0.5925 & 4.3514 \\ 2.7677 & 0.1524 & 2.2336 \end{bmatrix}.$$

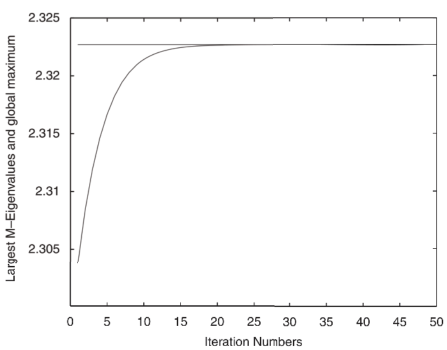

Figure 2: Numerical result of Example 5.2

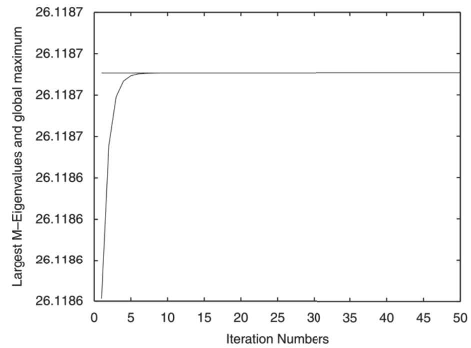

Figure 3: Numerical result of Example 5.3

The variation of the objective function value corresponding to this tensor during the iteration can be seen in Figure 3. For this tensor, its largest M-eigenvalue is 26.1187, which is marked in Figure 3 by the horizontal line.

From Figures 2 and 3, we can see that the largest M-eigenvalue can be highly approximated within few steps especially for the second example. Recently, a modified version named the block improvement method (BIM) was given in [65]. Different to the Algorithm 5.1, the convergence detail of BIM was established for the bi-quadratic polynomial optimization problem over unit spheres. It was proved that the global convergence of BIM hold. Furthermore, its linear convergence rate was given under second-order sufficient conditions.

Algorithm 5.2: Block improvement method

Initial step:

Input the fourth-order partially symmetric tensor $\mathcal{A}$. Take shift parameter $\tau > 0$, initial points $\mathbf{x}_0, \mathbf{y}_0$ such that $\|\mathbf{x}_0\| = 1, \|\mathbf{y}_0\| = 1$, and tolerance $\varepsilon \geq 0$, set $k = 0$. Compute

$\lambda_k = \mathcal{A}\mathbf{x}_k\mathbf{y}_k\mathbf{x}_k\mathbf{y}_k + \tau\mathcal{I}_M\mathbf{x}_k\mathbf{y}_k\mathbf{x}_k\mathbf{y}_k$.

Iterative step:

For $k = 0, 1, 2, \cdots$, do

$\bar{\mathbf{x}}_{k+1} = \mathcal{A} \cdot \mathbf{y}_k\mathbf{x}_k\mathbf{y}_k + \tau\mathcal{I}_M \cdot \mathbf{y}_k\mathbf{x}_k\mathbf{y}_k$, if $\|\bar{\mathbf{x}}_{k+1}\| \leq \epsilon$, then $\mathbf{x}_{k+1} = \mathbf{x}_k$; otherwise, $\mathbf{x}_{k+1} = \frac{\bar{\mathbf{x}}_{k+1}}{\|\bar{\mathbf{x}}_{k+1}\|}$.

$\bar{\mathbf{y}}_{k+1} = \mathcal{A}\mathbf{x}_{k+1} \cdot \mathbf{x}_{k+1}\mathbf{y}_k + \tau\mathcal{I}_M\mathbf{x}_{k+1} \cdot \mathbf{x}_{k+1}\mathbf{y}_k$, if $\|\bar{\mathbf{y}}_{k+1}\| \leq \epsilon$, then $\mathbf{y}_{k+1} = \mathbf{y}_k$; otherwise, $\mathbf{y}_{k+1} = \frac{\bar{\mathbf{y}}_{k+1}}{\|\bar{\mathbf{y}}_{k+1}\|}$.

$\lambda_{k+1} = \mathcal{A}\mathbf{x}_{k+1}\mathbf{y}_{k+1}\mathbf{x}_{k+1}\mathbf{y}_{k+1} + \tau\mathcal{I}_M\mathbf{x}_{k+1}\mathbf{y}_{k+1}\mathbf{x}_{k+1}\mathbf{y}_{k+1}$.

If $|\lambda_{k+1} - \lambda_k| \leq \epsilon$, terminate; otherwise let $k := k + 1$. End if.

Output:

Dominant singular eigenpair of tensor $\mathcal{A} : \lambda = f(\mathbf{x}, \mathbf{y}) - \tau(\mathbf{x}, \mathbf{y})$.

End for.

A few remarks on the algorithm are as follows.

$$\bar{\mathbf{x}}_{k+1} = \mathcal{A} \cdot \mathbf{y}_k\mathbf{x}_k\mathbf{y}_k + \tau\mathcal{I}_M \cdot \mathbf{y}_k\mathbf{x}_k\mathbf{y}_k = \frac{1}{2}\nabla_x g_\tau(\mathbf{x}_k, \mathbf{y}_k),$$

and

$$\bar{\mathbf{y}}_{k+1} = \mathcal{A}\mathbf{x}_{k+1} \cdot \mathbf{x}_{k+1}\mathbf{y}_k + \tau\mathcal{I}_M\mathbf{x}_{k+1} \cdot \mathbf{x}_{k+1}\mathbf{y}_k = \frac{1}{2}\nabla_y g_\tau(\mathbf{x}_{k+1}, \mathbf{y}_k).$$

Because the generated sequence of the BIM lies on the unit sphere, so an alternative choice of the shifted term is $\alpha\mathbf{x}^\top\mathbf{x} + \beta\mathbf{y}^\top\mathbf{y}$, where $\alpha, \beta > 0$ are shift parameters.

The convergence of the algorithm is considered in the following theorem.

Theorem 5.4 ([65])**.** *For Algorithm 5.2 with $\varepsilon = 0$, if $\tau \geq 0$ such that*

$$g_\tau(\mathbf{x}, \mathbf{y}) = \mathcal{A}\mathbf{xyxy} + \tau\mathcal{I}_M\mathbf{xyxy}$$

is convex w.r.t. $\mathbf{x}$ (respectively $\mathbf{y}$) for any fixed $\mathbf{y} \in \mathbb{R}^n$ (respectively $\mathbf{x} \in \mathbb{R}^{\mathbf{m}}$), then
(1) *the generated sequence $\{g_\tau(\mathbf{x_k}, \mathbf{y_k})\}$ is strictly increasing.*

(2) *if the algorithm terminates in finite steps, then the final point is a KKT point of problem* (5.1)*, and if the algorithm generates an infinite sequence $\{(\mathbf{x}_k, \mathbf{y}_k)\}$, then its any accumulation point is a KKT point of the problem.*

(3) *if function $g_\tau(\mathbf{x}, \mathbf{y})$ is strictly convex w.r.t. $\mathbf{x}$ (respectively $\mathbf{y}$) for any fixed $\mathbf{y}$ (respectively $\mathbf{x}$), then*

$$\lim_{k\to\infty} \|(\mathbf{x}_{k+1}, \mathbf{y}_{k+1}) - (\mathbf{x}_k, \mathbf{y}_k)\| = 0.$$

To test the effect of the shifted parameter on the behavior of Algorithm 5.2 and then test the efficiency of the method by giving a numerical comparison with some state-of-the-art solution methods, Wang *et al.* still consider examples 5.2 and 5.3.

For the tensor in Example 5.2, the numerical results of Algorithm 5.2 on four different shift parameters $\tau = \rho(\mathbf{A}), 5\rho(\mathbf{A}), 10\rho(\mathbf{A}), 20\rho(\mathbf{A})$ are shown in Figure 4, and then the numerical results of Example 5.3 are shown in Figure 5.

From Figures 4 and 5, we can see that the numerical efficiency of Algorithm 5.2 is seriously affected by the shift parameter τ: provided that the shifted tensor is positive definite, Algorithm 5.2 is more efficient for smaller shift parameter and it is less efficient for larger shift parameter. This means that, for Algorithm 5.2, as far as the shifted function is convex, the smaller the shifted parameter is, the better behavior the algorithm has. To test

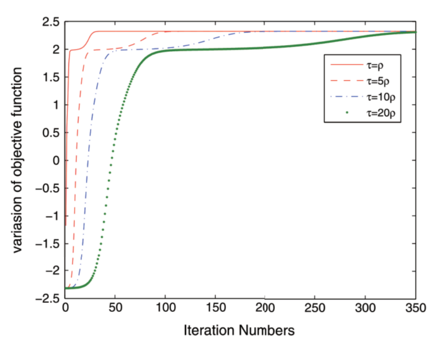

Figure 4: Numerical result of Example 5.2

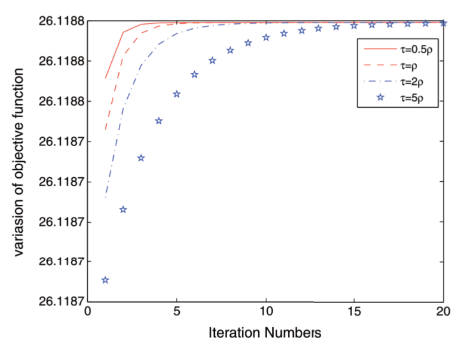

Figure 5: Numerical result of Example 5.3

the efficiency of Algorithm 5.2, the comparison with the GSM method and the SQP method is given in Table 5 [65].

Table 3: Numerical comparisons of BIM, BGM, GSM and SQP

	GSM	GSM	BIM	BIM	BGM	BGM	SQP	SQP
Dimension	vf	rt	vf	rt	vf	rt	vf	rt
	8.92	0.016	9.42	0.0001	9.42	0.015	5.32	0.26
	8.84	0.033	9.36	0.0150	9.37	0.016	7.31	0.18
m=6	7.54	0.016	7.54	0.0160	7.54	0.016	-	0.15
n=8	8.40	0.016	9.21	0.0001	9.21	0.015	2.49	0.61
	9.69	0.016	9.69	0.0001	9.69	0.013	6.37	0.12
	8.35	0.032	8.34	0.0001	8.35	0.015	-	0.33
	20.65	0.81	21.66	0.08	21.69	0.34	-	222.37
	20.91	2.36	21.90	0.03	22.05	1.62	-	37.28
m=30	20.41	1.28	18.55	0.06	20.41	1.45	-	13.64
n=20	21.29	0.58	20.45	0.06	21.19	0.43	-	17.46
	18.15	0.51	20.15	0.05	20.66	0.62	-	20.12
	20.97	0.72	19.60	0.06	19.85	0.72	-	38.48
	34.05	61.86	31.40	1.56	34.63	79.28	-	1559
	34.22	20.44	31.13	1.51	33.63	71.11	-	530
m=50	32.01	65.89	32.56	1.63	33.52	52.92	-	510
n=60	33.68	59.92	31.10	1.09	34.14	61.72	-	164
	33.84	45.84	30.95	1.53	34.07	105.62	-	278
	33.32	96.72	29.47	1.53	33.73	38.37	-	596

6 Conclusions

In this survey, we have provided an overview from several aspects of fourth-order partially symmetric tensors. We mainly focus on M-eigenvalue inclusion intervals, M-positive (semi)definiteness of fourth-order partially symmetric tensors, necessary and sufficient conditions for strong ellipticity condition of tensors from elasticity materials and algorithms to compute the largest M-eigenvalue.

However, there are still some interesting questions need to be considered in the future. As we know that there are many kinds of eigenvalues for high-order tenors such as H-eigenvalue, Z-eigenvalue, C-eigenvalue and D-eigenvalue. Thus, what are the relationships between eigenvalues above and M-eigenvalues here? On the other hand, from the numerical comparison between the BIM method and the GSM method, we know that the BIM method is not much superior, because the BIM method is established on the first-order approximation of the objective function. Now, a question is raised naturally: Can the efficiency of the method be improved if we use the second-order approximation of the objective function? These questions will be discussed in our future research.

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Mengzhen Li
School of Management Science, Qufu Normal University
Address: No. 80, Yantai Road, Donggang District
Rizhao Shandong, 276800, P.R. China
E-mail address: lmzdy9710@163.com

Haibin Chen
School of Management Science, Qufu Normal University
Address: No. 80, Yantai Road, Donggang District
Rizhao Shandong, 276800, P.R. China
E-mail address: chenhaibin508@163.com

Guanglu Zhou
Department of Mathematics and Statistics, Curtin University
Address: Perth, WA, Australia
E-mail address: g.zhou@curtin.edu.au