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SHERMAN-MORRISON-WOODBURY IDENTITY FOR TENSORS*

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Abstract: In linear algebra, the Sherman-Morrison-Woodbury identity says that the inverse of a rank-k correction of some matrix can be computed by doing a rank-k correction to the inverse of the original matrix. This identity is crucial to accelerate the matrix inverse computation when the matrix involves correction. Many scientific and engineering applications have to deal with this matrix inverse problem after updating the matrix, e.g., sensitivity analysis of linear systems, covariance matrix update in Kalman filter, etc. However, there is no similar identity in tensors. In this work, we will derive the Sherman-Morrison-Woodbury identity for invertible tensors first. Since not all tensors are invertible, we further generalize the Sherman-Morrison-Woodbury identity for tensors with Moore-Penrose generalized inverse by utilizing orthogonal projection of the correction tensor part into the original tensor and its Hermitian tensor. According to this new established the Sherman-Morrison-Woodbury identity for tensors, we can perform sensitivity analysis for multilinear systems by deriving the normalized upper bound for the solution of a multilinear system. Several numerical examples are also presented to demonstrate how the normalized error upper bounds are affected by perturbation degree of tensor coefficients.

Key words: *multilinear algebra, tensor inverse, Moore-Penrose inverse, sensitivity analysis, multilinear system*

Mathematics Subject Classification: *65R10, 33A65, 35K05, 62G20, 65P05*

1 Introduction

Tensors are higher-order generalizations of matrices and vectors, which have been studied abroadly due to the practical applications in many scientific and engineering fields [29, 51, 62], including psychometrics [60], digital image restorations [49], quantum entanglement [24, 50], signal processing [33, 45, 66, 31], high-order statistics [7, 20], automatic control [46], spectral hypergraph theory [23, 12], higher order Markov chains [40, 41, 30], magnetic resonance imaging [49, 48], algebraic geometry [9, 36], Finsler geometry [1], image authenticity verification [65], and so on. More applications about tensors can be found at [29, 51]

In tensor data analysis, the data sets are represented by tensors, while the associated multilinear algebra problems can be formulated for various data-processing tasks, such as web-link analysis [27, 28], document analysis [8, 32], information retrieval [44, 37], model

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learning [15, 4], data-model reduction [56, 10, 47], model prediction [16], movie recommendation [61], and videos analysis [35, 59], numerical PDE [58]. Most of above-stated tensor-formulated methodologies depend on the solution to the following *tensor equation* (a.k.a. *multilinear system of equations* [27, 43, 13, 14]):

$$\mathcal{A} \star_N \mathcal{X} \quad = \quad \mathcal{B}, \tag{1.1}$$

where $\mathcal{A}, \mathcal{B}$ are tensors of appropriate size and $\star_N$ denotes the *Einstein product* with order N [57]. Basically, there are two main approaches to solve the unknown tensor $\mathcal{X}$. The first approach is to solve the Eq. (1.1) iteratively. Three primary iterative algorithms are *Jacobi method*, *Gauss-Seidel method*, and *Successive Over-Relaxation* (SOR) *method* [54]. Nonetheless, in order to make these iterative algorithms converge, one has to provide some constraints during the tensor update at each iteration. For example, the updated tensor is required to be positive-definite and/or diagonally dominant [38, 11, 52, 51]. When the tensor $\mathcal{A}$ becomes a special type of tensor, namely $\mathcal{M}$-tensors, the Eq. (1.1) becomes a $\mathcal{M}$-equation. Ding and Wei [19] prove that a nonsingular $\mathcal{M}$-equation with a positive $\mathcal{B}$ always has a unique positive solution, present some generalized iterative algorithms to solve the $\mathcal{M}$-equations, which are in turn applied in the solution to nonlinear differential equations. We have to note that if we have $\mathcal{A} \in \mathbb{C}^{\overbrace{n \times \cdots \times n}^{m \text{ terms}}}$, $\mathcal{X} \in \mathbb{C}^{\overbrace{n \times \cdots \times n}^{m-1 \text{ terms}}}$, and $\mathcal{B} \in \mathbb{C}^n$, then Eq. (1.1) can be formulated as $\mathcal{A} \star_{m-1} \mathbf{x}^{m-1} = \mathbf{b}$ discussed in [19]. In [64], the authors proposed the rank-1 approximation to the coefficient tensor $\mathcal{A}$ which is combinsed with iterative tensor method to solve symmetric $\mathcal{M}$-equations. This method is shown by some numerical examples more effective than the traditional Newton method.

Sometimes, it is difficult to set a proper value of the underlying parameter (such as step size) in the solution-update equation to accelerate the convergence speed, while people often apply heuristics to determine such a parameter case by case. The other approach is to solve the unknown tensor $\mathcal{X}$ at the Eq. (1.1) through the tensor inversion. Brazell et al. [2] proposed the concept of the inverse of an even-order square tensor by adopting Einstein product, which provides a new direction to study tensors and tensor equations that model many phenomena in engineering and science [34]. In [3], the authors give some basic properties for the left (right) inverse, rank and product of tensors. The existence of order 2 left (right) inverses of tensors is also characterized and several tensor properties, e.g., some equalities and inequalities on the tensor rank, independence between the rank of a uniform hypergraph and the ordering of its vertices, rank characteristics of the Laplacian tensor, are established through inverses of tensors. Since the key step in solving the Eq. (1.1) is to characterize the inverse of the tensor $\mathcal{A}$, Sun et al. in [57] define different types of inverse, namely, i-inverse ($i = 1, 2, 5$) and group inverse of tensors based on a general product of tensors. They explore properties of the generalized inverses of tensors on solving tensor equations and computing formulas of block tensors. The representations for the 1-inverse and group inverse of some block tensors are also established. They then use the 1-inverse of tensors to give the solutions of a multilinear system represented by tensors. The authors in [57] also proved that, for a tensor equation with invertible tensor $\mathcal{A}$, the solution is unique and can be expressed by the inverse of the tensor $\mathcal{A}$. For more related works about tensors/matrices inversion can be found at [63].

However, the coefficient tensor $\mathcal{A}$ in the Eq. (1.1) is not always invertible, for example, when the tensor $\mathcal{A}$ is not square. Sun et al. [55] extend the tensor inverse proposed by Brazell et al. [2] to the Moore-Penrose inverse via Einstein product, and a concrete representation for the Moore-Penrose inverse can be obtained by utilzing the singular value decomposition (SVD) of the tensor. An important application of the Moore-Penrose inverse is the tensor nearness problem associated with tensor equation with Einstein product, which can be expressed as follows [39]. Let $\mathcal{X}_0$ be a given tensor, find the tensor $\hat{\mathcal{X}} \in \Omega$ such that

$$\left\|\hat{\mathcal{X}} - \mathcal{X}_0\right\| = \min_{\mathcal{X} \in \Omega} \|\mathcal{X} - \mathcal{X}_0\|, \tag{1.2}$$

where $\|\cdot\|$ is the Frobenius norm, and Ω is the solution set of tensor equation shown by Eq. (1.1).

The tensor nearness problem is a generalization of the matrix nearness problem that are studied in many areas of applied matrix computations [17, 21]. The tensor $\mathcal{X}_0$ in Eq. (1.2) may be obtained by experimental or statistical distribution information, but it may not satisfy the desired form and the minimum error requirement, while the optimal estimation $\hat{\mathcal{X}}$ is the tensor that not only satisfies these restrictions but also best approximates $\mathcal{X}_0$. Under certain conditions, it will be proved that the solution to the tensor nearness problem (1.2) is unique, and can be represented by means of the Moore-Penrose inverses [5]. Another situation to apply the Moore-Penrose inverse is that the the given tensor equation in Eq. (1.1) has a non-cubic coefficient tensor $\mathcal{A}$. The associated least-squares problem for the solution in Eq. (1.1) can be obtained by using the Moore-Penrose inverse of tensor $\mathcal{A}$ [2]. Some work involving the contribution and applications of the Moore-Penrose inverse to build necessary and sufficient conditions for the existence of the solution to Eq. (1.1) can be found at [26, 25, 5].

In matrix theory, the Sherman-Morrison-Woodbury (SMW) identity says that the inverse of a rank-k correction of a matrix $\mathbf{A}$ can be obtained by computing a rank-k correction to the inverse of $\mathbf{A}$. For rank-1 example, given $\mathbf{A} \in \mathbb{R}^{n \times n}$ be symmetric, $\mathbf{u} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ be a scaler, then the SMW identity becomes

$$(\mathbf{A} + \mathbf{u}\mathbf{u}^T) = \mathbf{A}^{-1} - \alpha \mathbf{x}\mathbf{x}^T, \tag{1.3}$$

where $\mathbf{x}$ is the solution to the equation $\mathbf{A}\mathbf{x} = \mathbf{u}$ and $\alpha \stackrel{\text{def}}{=} \frac{1}{1+\mathbf{u}^T\mathbf{x}}$. This is exactly the expression for the inverse of the rank-1 perturbation to an invertible symmetric matrix $\mathbf{A}$, which can be extended iteratively to a general rank-k perturbation shown by the following SMW as:

$$(\mathbf{A} + \mathbf{U}\mathbf{B}\mathbf{V})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{B}^{-1} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}\mathbf{A}^{-1}, \tag{1.4}$$

where $\mathbf{A}$ is a $n \times n$ matrix, $\mathbf{U}$ is a $n \times k$ matrix, $\mathbf{B}$ is a $k \times k$ matrix, and $\mathbf{V}$ is a $k \times n$ matrix. This identity is useful in numerical computations when $\mathbf{A}^{-1}$ has already been computed but the goal is to compute $(\mathbf{A} + \mathbf{U}\mathbf{B}\mathbf{V})^{-1}$. With the inverse of $\mathbf{A}$ available, it is only necessary to find the inverse of $\mathbf{B}^{-1} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U}$ in order to obtain the result using the right-hand side of the identity. If the matrix $\mathbf{B}$ has a much smaller dimension than $\mathbf{A}$, this is much easier than inverting $\mathbf{A}+\mathbf{U}\mathbf{B}\mathbf{V}$ directly. A common application is finding the inverse of a low-rank update $\mathbf{A} + \mathbf{U}\mathbf{B}\mathbf{V}$ of $\mathbf{A}$ when $\mathbf{U}$ only has a few columns and $\mathbf{V}$ also has only a few rows, or finding an approximation of the inverse of the matrix $\mathbf{A} + \mathbf{C}$ where the matrix $\mathbf{C}$ can be approximated by a low-rank matrix $\mathbf{U}\mathbf{B}\mathbf{V}$ via the singular value decomposition (SVD). The Sherman-Morrison-Woodbury identity involving singular matrices is discussed at [22, 53].

Analogously, we expect to have the Sherman-Morrison-Woodbury identity for tensors to facilitate the tensor inversion computation with those benefits in the matrix inversion computation when the correction of the original tensors is required. The Sherman-Morrison-Woodbury identity for tensors can be applied at various engineering and scientific areas, e.g., the tensor Kalman filter and recursive least squares methods [6]. This identity can significantly speed up the real time calculations of the tensor filter update because each new observation, which can be described with much lower dimension, can be treated as perturbation of the original covariance tensor. Similar to sensitivity analysis for linear systems [18], if we wish to consider how the solution is affected by the perturbed of coefficients in the tensor $\mathcal{A}$ in Eq. (1.1), we need to understand the relationship between the original tensor inverse and the perturbed tensor inverse. Although the work in [42] tries to study perturbation theory for Moore-Penrose inverse of tensor under Einstein product, our perturbation analysis takes different approach by utilizing the Sherman-Morrison-Woodbury identity for tensors. The benefits of this Sherman-Morrison-Woodbury identity based method is that we can have more relax requirements for the perturbed tensors during perturbation analysis. The Sherman-Morrison-Woodbury identity helps us to quantify the difference between the original solution and the perturbed solution of Eq. (1.1). The application of perturbation analysis for linear systems involving tensor equations in data sceince can be found at [14]. The contribution of this work can be summarized as follows.

1. We establish Sherman-Morrison-Woodbury identity for invertible tensors.
2. Because not every tensors are invertible, we generalize the Sherman-Morrison-Woodbury identity for tensors with Moore-Penrose inverse.
3. The sensitivity analysis is provided to the solution of a multilinear system when coefficient tensors are perturbed.

The paper is organized as follows. Preliminaries of tensors are given in Section 2. In Section 3, we will derive the Sherman-Morrison-Woodbury identity for invertible tensors. In Section 4, the Sherman-Morrison-Woodbury identity is generalized for Moore-Penrose tensor inverse, and two illustrative examples about applying this identity are also presented. We apply Sherman-Morrison-Woodbury identity to analyze the sensitivity of perturbed multiplinear systems in Section 5. Finally, the conclusions are given in Section 6.

2 Preliminaries of Tensors

In this work, we denote scalars by lower-case letters (e.g., d, e, f), vectors by boldface lower-case letters (e.g., $\mathbf{d}, \mathbf{e}, \mathbf{f}$), matrices by boldface capital letters (e.g., $\mathbf{D}, \mathbf{E}, \mathbf{F}$), and tensors by calligraphic letters (e.g., $\mathcal{D}, \mathcal{E}, \mathcal{F}$), respectively. Tensors are multiarray of values which are higher-dimensional generlization of vectors and matrices. Given a positive integer N, let $[N] = 1, \cdots, N$. An order N tensor $\mathcal{A} = (a_{i_1,\cdots,i_N})$, where $1 \leq i_j \leq I_j$ for $j \in [N]$, is a multidimensional array with $I_1 \times I_2 \times \cdots \times I_N$ entries. Let $\mathbb{C}^{I_1\times\cdots\times I_N}$ and $\mathbb{R}^{I_1\times\cdots\times I_N}$ be the sets of the order N dimension $I_1 \times \cdots \times I_N$ tensors over the complex field $\mathbb{C}$ and the real field $\mathbb{R}$, respectively. For example, $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ is a multiway array with N-th order and $I_1, I_2, \cdots, I_N$ dimension in the first, second, …, Nth direction, respectively. Each entry of $\mathcal{A}$ is represented by $a_{i_1,\cdots,i_N}$. For $N = 4$, $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times I_3\times I_4}$ is a fourth order tensor with entries as a_{i_1,i_2,i_3,i_4}.

For tensors $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$ and $\mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$, the *tensor addition* is defined as

$$(\mathcal{A}+\mathcal{B})_{i_1,\cdots,i_M,j_1,\cdots,j_N} = a_{i_1,\cdots,i_M,j_1,\cdots,j_N} + b_{i_1,\cdots,i_M,j_1,\cdots,j_N}. \tag{2.1}$$

If $M = N$ for the tensor $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$, the tensor $\mathcal{A}$ is named as a *square tensor*.

For tensors $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$ and $\mathcal{B} \in \mathbb{C}^{J_1\times\cdots\times J_N\times K_1\times\cdots\times K_L}$, the *Einstein product* with order N $\mathcal{A} \star_N \mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_M\times K_1\times\cdots\times K_L}$ is defined as

$$(\mathcal{A} \star_N \mathcal{B})_{i_1,\cdots,i_M,k_1,\cdots,k_L} = \sum_{j_1,\cdots,j_N} a_{i_1,\cdots,i_M,j_1,\cdots,j_N} b_{j_1,\cdots,j_N,k_1,\cdots,k_L}. \tag{2.2}$$

This tensor product reduces to the standard matrix multiplication when we have $L = M = N = 1$, which is just the standard matrix-matrix multiplication. We need following definitions about tensors.

The identity tensor is defined as following:

Definition 2.1. An identity tensor $\mathcal{I} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$ is defined as

$$(\mathcal{I})_{i_1\times\cdots\times i_N\times j_1\times\cdots\times j_N} = \prod_{k=1}^{N} \delta_{i_k,j_k}, \tag{2.3}$$

where $\delta_{i_k,j_k} = 1$ if $i_k = j_k$, otherwise $\delta_{i_k,j_k} = 0$.

In order to define a *Hermitian* tensor, we need following conjugate transpose operation of a tensor.

Definition 2.2. Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$ be a given tensor, then its conjugate transpose, denoted as $\mathcal{A}^H$, is defined as

$$(\mathcal{A}^H)_{j_1,\cdots,j_N,i_1,\cdots,i_M} = \overline{a}_{i_1,\cdots,i_M,j_1,\cdots,j_N}, \tag{2.4}$$

where the over line indicates the complex conjugate of the complex number $a_{i_1,\cdots,i_M,j_1,\cdots,j_N}$. If a tenser with the property $\mathcal{A}^H = \mathcal{A}$, this tensor is named as *Hermitian* tensor.

The inverse of a tensor here i sdefined as:

Definition 2.3. For a square tensor $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$, if there exists $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ such that

$$\mathcal{A} \star_M \mathcal{X} = \mathcal{X} \star_M \mathcal{A} = \mathcal{I}, \tag{2.5}$$

then such $\mathcal{X}$ is called as the inverse of the tensor $\mathcal{A}$, represented by $\mathcal{A}^{-1}$.

Definition 2.4. Given a tensor $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$. The tensor $\mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_N\times I_1\times\cdots\times I_M}$, satisfying the following tensor equations:

$$\begin{array}{ll} (1)\mathcal{A} \star_N \mathcal{X} \star_M \mathcal{A} = \mathcal{A}, & (2)\mathcal{X} \star_M \mathcal{A} \star_N \mathcal{X} = \mathcal{X}, \\ (3)(\mathcal{A} \star_N \mathcal{X})^H = \mathcal{A} \star_M \mathcal{X}, & (4)(\mathcal{X} \star_M \mathcal{A})^H = \mathcal{X} \star_M \mathcal{A}, \end{array} \tag{2.6}$$

is called the Moore-Penrose inverse of the tensor $\mathcal{A}$, denoted as $\mathcal{A}^\dagger$.

The partial trace of a tensor is defined as the summation of all the diagonal entries as

$$\mathrm{Tr}(\mathcal{A}) = \sum_{1\leq i_j\leq I_j, j\in[N]} \mathcal{A}_{i_1,\cdots,i_M,i_1,\cdots,i_M}. \tag{2.7}$$

Then, we can define the inner product of two tensors $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$ as

$$\langle \mathcal{A}, \mathcal{B} \rangle = \mathrm{Tr}(\mathcal{A}^H \star_M \mathcal{B}). \tag{2.8}$$

From the definition of tensor inner product, the Frobenius norm of a tensor $\mathcal{A}$ can be defined as

$$\|\mathcal{A}\| = \sqrt{\langle \mathcal{A}, \mathcal{A} \rangle}. \tag{2.9}$$

An unfolded tensor is a matrix obtained by reorganizing the entries of a tensor into a two-dimensional array. For the tensor space $\mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$ and the matrix space $\mathbb{C}^{(I_1\cdots I_M)\times(J_1\cdots J_N)}$, we define a map φ as follows:

$$\begin{aligned}\varphi : \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N} &\to \mathbb{C}^{(I_1\cdots I_M)\times(J_1\cdots J_N)}\\ \mathcal{A} = (a_{i_1,\cdots,i_M,j_1,\cdots,j_N}) &\to (\mathbf{A}_{\phi(\mathbf{i},\mathbb{I}),\phi(\mathbf{j},\mathbb{J})}),\end{aligned} \tag{2.10}$$

where ϕ is an index mapping function from tensor indices to matrix indices with arguments of row subscripts $\mathbf{i} = \{i_1,\cdots,i_M\}$ and row dimensions of $\mathcal{A}$, denoted as $\mathbb{I} = \{I_1,\cdots,I_M\}$. The relation $\phi(\mathbf{i},\mathbb{I})$ can be expressed as

$$\phi(\mathbf{i},\mathbb{I}) = i_1 + \sum_{m=2}^{M}(i_m - 1)\prod_{u=1}^{m-1} I_u. \tag{2.11}$$

Similarly, $\phi(\mathbf{j},\mathbb{J})$ is an index mapping relation for column dimensions of $\mathcal{A}$ which can be expressed as

$$\phi(\mathbf{j},\mathbb{J}) = j_1 + \sum_{n=2}^{N}(j_n - 1)\prod_{v=1}^{n-1} J_v, \tag{2.12}$$

where $\mathbf{j} = \{j_1,\cdots,j_N\}$ and column dimensions of $\mathcal{A}$, denoted as $\mathbb{J} = \{J_1,\cdots,J_N\}$. We will use this unfolding mapping φ to build the condition of the existence of an inverse of a tensor.

Following definition, which is based on the tensor unfolding map introduced by the Eq. (2.10), is required to determine when a given square tensor is invertible.

Definition 2.5. For a tensor $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$, and the map φ defined by the Eq. (2.10), the unfolding rank of a tensor $\mathcal{A}$ is defined as the rank of the mapped matrix $\varphi(\mathcal{A})$. If we have $\varphi(\mathcal{A}) = I_1\cdots I_M$ (the multiplication of all integers $I_1,\cdots,I_M$ together), we say that $\mathcal{A}$ is full row rank. On ther other hand, if we have $\varphi(\mathcal{A}) = J_1\cdots J_N$ (the multiplication of all integers $J_1,\cdots,J_N$ together), we say that $\mathcal{A}$ is full column rank.

Now we are able to present the sufficient and the necessary conditions for the existence of a given tensor. Following Lemma 2.6 can be deduced from Section 2.2 in [39] since, for an invertible square tensor, the image under the map φ must be a nonsingular matrix.

Lemma 2.6. *A tensor $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ is invertible if and only if the matrix $\mathbf{A} = \varphi(\mathcal{A})$ is a full rank, i.e., rank$(\mathbf{A}) = I_1\cdots I_M$.*

The concepts such as "identity tensor", "Hermitian tensor", and "tensor (pseudo) inverse" under the Einstein product can also be found at [39] and references therein.

Remark: A tensor that all its entries are zero is called zero tensor, denoted as $\mathcal{O}$.

3 Identity for Invertible Tensors

The purpose of this section is to prove Sherman-Morrison-Woodbury identity for invertible tensors.

Theorem 3.1 (Sherman-Morrison-Woodbury identity for invertible tensors.)**.** *Given invertible tensors $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ and $\mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_K\times I_1\times\cdots\times I_K}$, and tensors $\mathcal{U} \in$*

$\mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_K}$ *and* $\mathcal{V}\in\mathbb{C}^{I_1\times\cdots\times I_K\times I_1\times\cdots\times I_M}$*, if the tensor* $(\mathcal{B}^{-1}+\mathcal{V}\star_M\mathcal{A}^{-1}\star_M\mathcal{U})$ *is invertible, we have following identiy:*

$$(\mathcal{A}+\mathcal{U}\star_K\mathcal{B}\star_K\mathcal{V})^{-1}=\mathcal{A}^{-1}- \\ \mathcal{A}^{-1}\star_M\mathcal{U}\star_K(\mathcal{B}^{-1}+\mathcal{V}\star_M\mathcal{A}^{-1}\star_M\mathcal{U})^{-1}\star_K\mathcal{V}\star_M\mathcal{A}^{-1}. \tag{3.1}$$

Proof. The identiy can be proven by checking that $(\mathcal{A}+\mathcal{U}\star_K\mathcal{B}\star_K\mathcal{V})$ multiplies its alleged inverse on the right side of the Sherman-Morrison-Woodbury identity gives the identity matrix (To save space, we omit Einstein product symbol, $\star$, between two tensors):

$$\begin{aligned}&(\mathcal{A}+\mathcal{U}\mathcal{B}\mathcal{V})[\mathcal{A}^{-1}-\mathcal{A}^{-1}\mathcal{U}(\mathcal{B}^{-1}+\mathcal{V}\mathcal{A}^{-1}\mathcal{U})^{-1}\mathcal{V}\mathcal{A}^{-1}]\\
&=\mathcal{I}+\mathcal{U}\mathcal{B}\mathcal{V}\mathcal{A}^{-1}-\mathcal{U}(\mathcal{B}^{-1}+\mathcal{V}\mathcal{A}^{-1}\mathcal{U})^{-1}\mathcal{V}\mathcal{A}^{-1}-\mathcal{U}\mathcal{B}\mathcal{V}\mathcal{A}^{-1}\mathcal{U}(\mathcal{B}^{-1}+\mathcal{V}\mathcal{A}^{-1}\mathcal{U})^{-1}\mathcal{V}\mathcal{A}^{-1}\\
&=(\mathcal{I}+\mathcal{U}\mathcal{B}\mathcal{V}\mathcal{A}^{-1})-\left[\mathcal{U}(\mathcal{B}^{-1}+\mathcal{V}\mathcal{A}^{-1}\mathcal{U})^{-1}\mathcal{V}\mathcal{A}^{-1}+\mathcal{U}\mathcal{B}\mathcal{V}\mathcal{A}^{-1}\mathcal{U}(\mathcal{B}^{-1}+\mathcal{V}\mathcal{A}^{-1}\mathcal{U})^{-1}\mathcal{V}\mathcal{A}^{-1}\right]\\
&=\mathcal{I}+\mathcal{U}\mathcal{B}\mathcal{V}\mathcal{A}^{-1}-(\mathcal{U}+\mathcal{U}\mathcal{B}\mathcal{V}\mathcal{A}^{-1}\mathcal{U})(\mathcal{B}^{-1}+\mathcal{V}\mathcal{A}^{-1}\mathcal{U})^{-1}\mathcal{V}\mathcal{A}^{-1}\\
&=\mathcal{I}+\mathcal{U}\mathcal{B}\mathcal{V}\mathcal{A}^{-1}-(\mathcal{U}\mathcal{B}(\mathcal{B}^{-1}+\mathcal{V}\mathcal{A}^{-1}\mathcal{U})(\mathcal{B}^{-1}+\mathcal{V}\mathcal{A}^{-1}\mathcal{U})^{-1}\mathcal{V}\mathcal{A}^{-1}\\
&\qquad=\mathcal{I}+\mathcal{U}\mathcal{B}\mathcal{V}\mathcal{A}^{-1}-\mathcal{U}\mathcal{B}\mathcal{V}\mathcal{A}^{-1}=\mathcal{I}\end{aligned} \tag{3.2}$$

Similar steps can be applied to prove this identity by multiplying the alleged inverse from the left side of $(\mathcal{A}+\mathcal{U}\mathcal{B}\mathcal{V})$:

$$\begin{aligned}&[\mathcal{A}^{-1}-\mathcal{A}^{-1}\mathcal{U}(\mathcal{B}^{-1}+\mathcal{V}\mathcal{A}^{-1}\mathcal{U})^{-1}\mathcal{V}\mathcal{A}^{-1}](\mathcal{A}+\mathcal{U}\mathcal{B}\mathcal{V})\\
&\quad=\mathcal{I}+\mathcal{A}^{-1}\mathcal{U}(\mathcal{B}^{-1}+\mathcal{V}\mathcal{A}^{-1}\mathcal{U})^{-1})^{-1}\mathcal{V}+\mathcal{A}^{-1}\mathcal{U}\mathcal{B}\mathcal{V}-\\
&\qquad\mathcal{A}^{-1}\mathcal{U}(\mathcal{B}^{-1}+\mathcal{V}\mathcal{A}^{-1}\mathcal{U})^{-1})^{-1}\mathcal{V}\mathcal{A}^{-1}\mathcal{U}\mathcal{B}\mathcal{V}\\
&\quad=(\mathcal{I}+\mathcal{A}^{-1}\mathcal{U}\mathcal{B}\mathcal{V})-\mathcal{A}^{-1}\mathcal{U}(\mathcal{B}^{-1}+\mathcal{V}\mathcal{A}^{-1}\mathcal{U})^{-1}(\mathcal{V}+\mathcal{V}\mathcal{A}^{-1}\mathcal{U}\mathcal{B}\mathcal{V})\\
&\quad=(\mathcal{I}+\mathcal{A}^{-1}\mathcal{U}\mathcal{B}\mathcal{V})-\mathcal{A}^{-1}\mathcal{U}(\mathcal{B}^{-1}+\mathcal{V}\mathcal{A}^{-1}\mathcal{U})^{-1}(\mathcal{B}^{-1}+\mathcal{V}\mathcal{A}^{-1}\mathcal{U})\mathcal{B}\mathcal{V}\\
&\quad=(\mathcal{I}+\mathcal{A}^{-1}\mathcal{U}\mathcal{B}\mathcal{V})-\mathcal{A}^{-1}\mathcal{U}\mathcal{B}\mathcal{V}=\mathcal{I}\end{aligned} \tag{3.3}$$

Therefore, the identiy is established. □

4 Identity for Tensors with Moore-Penrose Inverse

In this section, we will extend our tensor inverse result from previous section to the Sherman-Morrison- Woodbury identity for Moore-Penrose inverse in section 4.1. Two illustrative examples for the Sherman-Morrison-Woodbury identity for Moore-Penrose inverse will be provided in section 4.2.

4.1 Identity for Moore-Penrose Inverse Tensors

The goal of this subsection is to establish our main result: the Sherman-Morrison-Woodbury identity for Moore-Penrose inverse. We begin with the definitions about row space and column space of a given tensor. Let us define two symbols $\mathbb{I}_M \stackrel{\text{def}}{=} \underbrace{1\times\cdots\times 1}_{M}$ and $\mathbb{I}_N \stackrel{\text{def}}{=} \underbrace{1\times\cdots\times 1}_{N}$.

We define *row-tensors* of a tensor $\mathcal{A}=(a_{i_1,\cdots,i_M,j_1,\cdots,j_N})\in\mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$ as subtensors $\mathbf{a}^{\mathrm{R}}_{i_1,\cdots,i_M}$ where $1\le i_k\le I_k$ for $k\in[M]$. The entries in the row-tensor $\mathbf{a}^{\mathrm{R}}_{i_1,\cdots,i_M}$ are entries $a_{i_1,\cdots,i_M,j_1\cdots,j_N}$ where $1\le j_k\le J_k$ for $k\in[N]$ but fix the indices of $i_1,\cdots,i_M$. Similarly, *column-tensors* of a tensor $\mathcal{A}$ are subtensors $\mathbf{a}^{\mathrm{C}}_{j_1,\cdots,j_N}$ where $1\le j_k\le J_k$ for $k\in$

$[N]$. The entries in the column-tensor $\mathbf{a}^{\mathrm{C}}_{j_1,\cdots,j_N}$ are entries $a_{i_1,\cdots,i_M,j_1\cdots,j_N}$ where $1 \le i_k \le I_k$ for $k \in [M]$ but fix the indices of $j_1, \cdots, j_N$.

Let the tensor $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$. The right null space is defined as

$$\mathfrak{N}_R(\mathcal{A}) \stackrel{\text{def}}{=} \left\{\mathbf{z} \in \mathbb{C}^{J_1\times\cdots\times J_N\times \mathbb{I}_M} : \mathcal{A} \star_N \mathbf{z} = \mathcal{O}\right\} \tag{4.1}$$

Then the row space of $\mathcal{A}$ is defined as

$$\mathfrak{R}(\mathcal{A}) \stackrel{\text{def}}{=} \Big\{\mathbf{y} \in \mathbb{C}^{J_1\times\cdots\times J_N\times \mathbb{I}_M} : \mathbf{y} = \sum_{i_1,\cdots,i_M} \mathbf{a}^{\mathrm{R}}_{i_1,\cdots,i_M} x_{i_1,\cdots,i_M}, \text{ where } x_{i_1,\cdots,i_M} \in \mathbb{C} \text{ and } \mathbf{a}^{\mathrm{R}}_{i_1,\cdots,i_M} \in \mathbb{C}^{J_1\times\cdots\times J_N\times \mathbb{I}_M}.\Big\} \tag{4.2}$$

Now from the definition of right null space we have $(\mathbf{a}^{\mathrm{R}}_{i_1,\cdots,i_M})^H\mathbf{z} = \mathcal{O}$, where H is the Hermitian operator and $(\mathbf{a}^{\mathrm{R}}_{i_1,\cdots,i_M})^H \in \mathbb{C}^{\mathbb{I}_M\times J_1\times\cdots\times J_N}$. If we take any tensor $\mathbf{y} \in \mathfrak{R}(\mathcal{A})$, then $\mathbf{y} = \sum_{i_1,\cdots,i_M} \mathbf{a}^{\mathrm{R}}_{i_1,\cdots,i_M} x_{i_1,\cdots,i_M}$, where $x_{i_1,\cdots,i_M} \in \mathbb{C}$. Hence,

$$\begin{aligned} \mathbf{y}^H\mathbf{z} &= (\sum_{i_1,\cdots,i_M} \mathbf{a}^{\mathrm{R}}_{i_1,\cdots,i_M} x_{i_1,\cdots,i_M})^H\mathbf{z} \\ &= (\sum_{i_1,\cdots,i_M} x_{i_1,\cdots,i_M} (\mathbf{a}^{\mathrm{R}}_{i_1,\cdots,i_M})^H)\mathbf{z} \\ &= \sum_{i_1,\cdots,i_M} x_{i_1,\cdots,i_M} ((\mathbf{a}^{\mathrm{R}}_{i_1,\cdots,i_M})^H\mathbf{z}) = \mathcal{O} \end{aligned} \tag{4.3}$$

This shows that row space is orthogonal to the right null space.

Following this right null space approach, we also can define the left null space as

$$\mathfrak{N}_L(\mathcal{A}) \stackrel{\text{def}}{=} \left\{\mathbf{z} \in \mathbb{C}^{I_1\times\cdots\times I_M\times \mathbb{I}_N} : \mathbf{z}^H \star_M \mathcal{A} = \mathcal{O}\right\} \tag{4.4}$$

Then the column space of $\mathcal{A}$ is defined as

$$\mathfrak{C}(\mathcal{A}) \stackrel{\text{def}}{=} \Big\{\mathbf{y} \in \mathbb{C}^{I_1\times\cdots\times I_M\times \mathbb{I}_N} : \mathbf{y} = \sum_{j_1,\cdots,j_N} \mathbf{a}^{\mathrm{C}}_{j_1,\cdots,j_N} x_{j_1,\cdots,j_N}, \text{ where } x_{j_1,\cdots,j_N} \in \mathbb{C} \text{ and } \mathbf{a}^{\mathrm{C}}_{j_1,\cdots,j_N} \in \mathbb{C}^{I_1\times\cdots\times I_M\times \mathbb{I}_N}.\Big\} \tag{4.5}$$

From the definition of left null space we have $\mathbf{z}^H\mathbf{a}^{\mathrm{C}}_{j_1,\cdots,j_N} = \mathcal{O}$, where H is the Hermitian operator and $\mathbf{z}^H \in \mathbb{C}^{\mathbb{I}_N\times I_1\times\cdots\times I_M}$. By taking any tensor $\mathbf{y} \in \mathfrak{C}(\mathcal{A})$, then $\mathbf{y} = \sum_{j_1,\cdots,j_N} \mathbf{a}^{\mathrm{C}}_{j_1,\cdots,j_N} x_{j_1,\cdots,j_N}$, where $x_{j_1,\cdots,j_N} \in \mathbb{C}$. We have,

$$\begin{aligned} \mathbf{z}^H\mathbf{y} &= \mathbf{z}^H(\sum_{j_1,\cdots,j_N} \mathbf{a}^{\mathrm{C}}_{j_1,\cdots,j_N} x_{j_1,\cdots,j_N}) \\ &= (\sum_{j_1,\cdots,j_N} \mathbf{z}^H\mathbf{a}^{\mathrm{C}}_{j_1,\cdots,j_N} x_{j_1,\cdots,j_N}) \\ &= \sum_{j_1,\cdots,j_N} x_{i_1,\cdots,i_M} (\mathbf{z}^T\mathbf{a}^{\mathrm{C}}_{j_1,\cdots,j_N}) = \mathcal{O} \end{aligned} \tag{4.6}$$

This shows that column space is orthogonal to the left null space.

Given following tensor relation:

$$\mathcal{S} \;=\; \mathcal{A} + \mathcal{U}\mathcal{B}\mathcal{V}, \tag{4.7}$$

the goal is to expresse the Moore-Penrose inverse of $\mathcal{S}$ in terms of tensors related to $\mathcal{A}, \mathcal{U}, \mathcal{B}, \mathcal{V}$. From the definition of column space, we can decompose the tensor $\mathcal{U}$ into $\mathcal{X}_1 + \mathcal{Y}_1$, wherer the column-tensors of $\mathcal{X}_1$ are contained in the clumn space of $\mathcal{A}$, denoted as $\mathfrak{C}(\mathcal{A})$, and the column-tensors of $\mathcal{Y}_1$ are contained in the left null space of $\mathcal{A}$. Correspondingly, we also can decompose the tensor $\mathcal{V}^H$ into $\mathcal{X}_2 + \mathcal{Y}_2$, wherer the column-tensors of $\mathcal{X}_2$ are contained in the clumn space of $\mathcal{A}^H$, denoted as $\mathfrak{C}(\mathcal{A}^H)$, and the column-tensors of $\mathcal{Y}_2$ are contained in the left null space of $\mathcal{A}^H$. Define tensors $\mathcal{E}_i$ as $\mathcal{E}_i \stackrel{\text{def}}{=} \mathcal{Y}_i(\mathcal{Y}_i^H\mathcal{Y}_i)^\dagger$ for $i = 1, 2$. We are ready to present the following theorem about the identity for tensors with Moore-Penrose inverse.

Theorem 4.1 (Sherman-Morrison-Woodbury identity for Moore-Penrose inverse). *Given tensors* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_N}$, $\mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_K\times I_1\times\cdots\times I_K}$, $\mathcal{U} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_K}$ *and* $\mathcal{V} \in \mathbb{C}^{I_1\times\cdots\times I_K\times I_1\times\cdots\times I_N}$, *if following conditions are satisfied:*

1. $\mathcal{U} = \mathcal{X}_1 + \mathcal{Y}_1$, *where* $\mathcal{X}_1 \in \mathfrak{C}(\mathcal{A})$ *and* $\mathcal{Y}_1$ *is orthgonal to* $\mathfrak{C}(\mathcal{A})$;
2. $\mathcal{V}^H = \mathcal{X}_2 + \mathcal{Y}_2$, *where* $\mathcal{X}_2 \in \mathfrak{C}(\mathcal{A}^H)$ *and* $\mathcal{Y}_2$ *is orthgonal to* $\mathfrak{C}(\mathcal{A}^H)$;
3. *(1)* $\mathcal{E}_2\mathcal{B}^\dagger\mathcal{E}_1^H\mathcal{Y}_1\mathcal{B} = \mathcal{E}_2$, *(2)* $\mathcal{X}_1\mathcal{E}_1^H\mathcal{Y}_1\mathcal{B} = \mathcal{X}_1\mathcal{B}$, *(3)* $\mathcal{Y}_1\mathcal{E}_1^H\mathcal{Y}_1 = \mathcal{Y}_1$;
4. *(1)* $\mathcal{B}\mathcal{Y}_2^H\mathcal{E}_2\mathcal{B}^\dagger\mathcal{E}_1^H = \mathcal{E}_1^H$, *(2)* $\mathcal{B}\mathcal{Y}_2^H\mathcal{E}_2\mathcal{X}_2^H = \mathcal{B}\mathcal{X}_2^H$, *(3)* $\mathcal{E}_2\mathcal{Y}_2^H\mathcal{E}_2 = \mathcal{E}_2$.

Then the tensor

$$\begin{aligned}\mathcal{S} &= \mathcal{A} + \mathcal{U} \star_K \mathcal{B} \star_K \mathcal{V}\\ &= \mathcal{A} + (\mathcal{X}_1 + \mathcal{Y}_1) \star_K \mathcal{B} \star_K (\mathcal{X}_2 + \mathcal{Y}_2)^H,\end{aligned} \tag{4.7}$$

has the following Moore-Penrose generalized inverse identiy:

$$\begin{aligned}\mathcal{S}^\dagger &= \mathcal{A}^\dagger - \mathcal{E}_2 \star_K \mathcal{X}_2^H \star_N \mathcal{A}^\dagger - \mathcal{A}^\dagger \star_M \mathcal{X}_1 \star_K \mathcal{E}_1^H\\ &\quad + \mathcal{E}_2 \star_K (\mathcal{B}^\dagger + \mathcal{X}_2^H \star_N \mathcal{A}^\dagger \star_M \mathcal{X}_1) \star_K \mathcal{E}_1^H,\end{aligned} \tag{4.7}$$

where $\mathcal{E}_i \stackrel{def}{=} \mathcal{Y}_i(\mathcal{Y}_i^H\mathcal{Y}_i)^\dagger$ *for* $i = 1, 2$.

Proof. From the definition 2.4, the identity is established by direct computation (To save space, we also omit $\star$ product symbol between two tensors in this proof) to verify following four rules:

$$\begin{aligned}&(1)\mathcal{S}\mathcal{S}^\dagger\mathcal{S} = \mathcal{S}, &&(2)\mathcal{S}^\dagger\mathcal{S}\mathcal{S}^\dagger = \mathcal{S}^\dagger,\\ &(3)(\mathcal{S}\mathcal{S}^\dagger)^H = \mathcal{S}\mathcal{S}^\dagger, &&(4)(\mathcal{S}^\dagger\mathcal{S})^H = \mathcal{S}^\dagger\mathcal{S}.\end{aligned} \tag{4.8}$$

Verify : $(\mathcal{S}\mathcal{S}^\dagger)^H = \mathcal{S}\mathcal{S}^\dagger$

By expansion of $\mathcal{S}\mathcal{S}^\dagger$, we have

$$\begin{aligned}\mathcal{S}\mathcal{S}^\dagger &= \mathcal{A}\mathcal{A}^\dagger - \mathcal{A}\mathcal{E}_2\mathcal{X}_2^H\mathcal{A}^\dagger - \mathcal{A}\mathcal{A}^\dagger\mathcal{X}_1\mathcal{E}_1^H + \mathcal{A}\mathcal{E}_2(\mathcal{B}^\dagger + \mathcal{X}_2^H\mathcal{A}^\dagger\mathcal{X}_1)\mathcal{E}_1^H +\\ &\quad (\mathcal{X}_1 + \mathcal{Y}_1)\mathcal{B}(\mathcal{X}_2 + \mathcal{Y}_2)^H\mathcal{A}^\dagger - (\mathcal{X}_1 + \mathcal{Y}_1)\mathcal{B}(\mathcal{X}_2 + \mathcal{Y}_2)^H\mathcal{E}_2\mathcal{X}_2^H\mathcal{A}^\dagger -\\ &\quad (\mathcal{X}_1 + \mathcal{Y}_1)\mathcal{B}(\mathcal{X}_2 + \mathcal{Y}_2)^H\mathcal{A}^\dagger\mathcal{X}_1\mathcal{E}_1^H + (\mathcal{X}_1 + \mathcal{Y}_1)\mathcal{B}(\mathcal{X}_2 + \mathcal{Y}_2)^H\mathcal{E}_2\mathcal{X}_2^H\mathcal{A}^\dagger\mathcal{X}_1\mathcal{E}_1^H +\\ &\quad (\mathcal{X}_1 + \mathcal{Y}_1)\mathcal{B}(\mathcal{X}_2 + \mathcal{Y}_2)^H\mathcal{E}_2\mathcal{B}^\dagger\mathcal{E}_1^H,\end{aligned} \tag{4.9}$$

and, since column-tensors of $\mathcal{Y}_2$ are orthgonal to $\mathfrak{C}(\mathcal{A}^H)$, we also have $\mathcal{A}\mathcal{Y}_2 = \mathcal{O}$, $\mathcal{Y}_2^H\mathcal{A}^\dagger = \mathcal{O}$, and $\mathcal{X}_2^H\mathcal{Y}_2 = \mathcal{O}$. From these relations, the Eq. (4.9) can be simplfies as

$$\begin{aligned}\mathcal{S}\mathcal{S}^\dagger &= \mathcal{A}\mathcal{A}^\dagger - \mathcal{A}\mathcal{A}^\dagger\mathcal{X}_1\mathcal{E}_1^H + \\ &\quad (\mathcal{X}_1+\mathcal{Y}_1)\mathcal{B}\mathcal{X}_2^H\mathcal{A}^\dagger - (\mathcal{X}_1+\mathcal{Y}_1)\mathcal{B}\mathcal{Y}_2^H\mathcal{E}_2\mathcal{X}_2^H\mathcal{A}^\dagger - \\ &\quad (\mathcal{X}_1+\mathcal{Y}_1)\mathcal{B}\mathcal{X}_2^H\mathcal{A}^\dagger\mathcal{X}_1\mathcal{E}_1^H + (\mathcal{X}_1+\mathcal{Y}_1)\mathcal{B}\mathcal{Y}_2^H\mathcal{E}_2\mathcal{X}_2^H\mathcal{A}^\dagger\mathcal{X}_1\mathcal{E}_1^H + \\ &\quad (\mathcal{X}_1+\mathcal{Y}_1)\mathcal{B}\mathcal{Y}_2^H\mathcal{E}_2\mathcal{B}^\dagger\mathcal{E}_1^H, \end{aligned} \tag{4.10}$$

and from the fourth condition at this Theorem 4.1, i.e., $\mathcal{B}\mathcal{Y}_2^H\mathcal{E}_2\mathcal{B}^\dagger\mathcal{E}_1^H = \mathcal{E}_1^H$, and $\mathcal{B}\mathcal{Y}_2^H\mathcal{E}_2\mathcal{X}_2^H = \mathcal{B}\mathcal{X}_2^H$ and $\mathcal{A}\mathcal{A}^\dagger\mathcal{X}_1 = \mathcal{X}_1$, we can further simplify the Eq. (4.10) as :

$$\mathcal{S}\mathcal{S}^\dagger = \mathcal{A}\mathcal{A}^\dagger + \mathcal{Y}_1\mathcal{E}_1^H. \tag{4.11}$$

Then,

$$\begin{aligned}(\mathcal{S}\mathcal{S}^\dagger)^H &= (\mathcal{A}\mathcal{A}^\dagger + \mathcal{Y}_1\mathcal{E}_1^H)^H \\ &= (\mathcal{A}\mathcal{A}^\dagger)^H + (\mathcal{Y}_1\mathcal{E}_1^H)^H \\ &= \mathcal{A}\mathcal{A}^\dagger + \mathcal{Y}_1\mathcal{E}_1^H = \mathcal{S}\mathcal{S}^\dagger \end{aligned} \tag{4.12}$$

the third requirement of the definition 2.4 is valid from the Eq. (4.12).

Verify : $(\mathcal{S}^\dagger\mathcal{S})^H = \mathcal{S}^\dagger\mathcal{S}$

By expansion $\mathcal{S}^\dagger\mathcal{S}$ as the Eq. (4.9), we can simplify such expansion with following relations $\mathcal{A}^\dagger\mathcal{Y}_1 = \mathcal{O}$, $\mathcal{Y}_1^H\mathcal{A} = \mathcal{O}$ and $\mathcal{X}_1^H\mathcal{Y}_1 = \mathcal{O}$ due to column-tensors of $\mathcal{Y}_1$ are orthgonal to $\mathfrak{C}(\mathcal{A})$, $\mathcal{X}_2\mathcal{A}^\dagger\mathcal{A} = \mathcal{X}_2$ (from the definition of Moore-Penrose inverse of $\mathcal{A}$), and the third condition at this Theorem 4.1, i.e., $\mathcal{E}_2\mathcal{B}^\dagger\mathcal{E}_1^H\mathcal{Y}_1\mathcal{B} = \mathcal{E}_2$, and $\mathcal{X}_1\mathcal{E}_1^H\mathcal{Y}_1\mathcal{B} = \mathcal{X}_1\mathcal{B}$, then we have

$$\mathcal{S}^\dagger\mathcal{S} = \mathcal{A}^\dagger\mathcal{A} + \mathcal{E}_2\mathcal{Y}_2^H. \tag{4.13}$$

Hence, the fourth requirement of the definition 2.4 is valid from the Eq. (4.13).

Verify : $\mathcal{S}\mathcal{S}^\dagger\mathcal{S} = \mathcal{S}$

Since, we have

$$\begin{aligned}\mathcal{S}\mathcal{S}^\dagger\mathcal{S} &= (\mathcal{A}\mathcal{A}^\dagger + \mathcal{Y}_1\mathcal{E}_1^H)(\mathcal{A} + (\mathcal{X}_1+\mathcal{Y}_1)\mathcal{B}(\mathcal{X}_2+\mathcal{Y}_2)^H) \\ &= \mathcal{A}\mathcal{A}^\dagger\mathcal{A} + \mathcal{Y}_1\mathcal{E}_1^H\mathcal{A} + \mathcal{A}\mathcal{A}^\dagger(\mathcal{X}_1+\mathcal{Y}_1)\mathcal{B}(\mathcal{X}_2+\mathcal{Y}_2)^H + \\ &\quad \mathcal{Y}_1\mathcal{E}_1^H(\mathcal{X}_1+\mathcal{Y}_1)\mathcal{B}(\mathcal{X}_2+\mathcal{Y}_2)^H \\ &= \mathcal{A} + \mathcal{Y}_1\mathcal{E}_1^H\mathcal{A} + \mathcal{A}\mathcal{A}^\dagger\mathcal{X}_1\mathcal{B}(\mathcal{X}_2+\mathcal{Y}_2)^H + \mathcal{A}\mathcal{A}^\dagger\mathcal{Y}_1\mathcal{B}(\mathcal{X}_2+\mathcal{Y}_2)^H + \\ &\quad \mathcal{Y}_1\mathcal{E}_1^H\mathcal{X}_1\mathcal{B}(\mathcal{X}_2+\mathcal{Y}_2)^H + \mathcal{Y}_1\mathcal{E}_1^H\mathcal{Y}_1\mathcal{B}(\mathcal{X}_2+\mathcal{Y}_2)^H \\ &\stackrel{1}{=} \mathcal{A} + \mathcal{X}_1\mathcal{B}(\mathcal{X}_2+\mathcal{Y}_2)^H + \mathcal{Y}_1\mathcal{B}(\mathcal{X}_2+\mathcal{Y}_2)^H \\ &= \mathcal{A} + (\mathcal{X}_1+\mathcal{Y}_1)\mathcal{B}(\mathcal{X}_2+\mathcal{Y}_2)^H = \mathcal{S} \end{aligned} \tag{4.14}$$

where we apply $\mathcal{A}\mathcal{A}^\dagger\mathcal{X}_1 = \mathcal{X}_1$, $\mathcal{A}^\dagger\mathcal{Y}_1 = \mathcal{O}$, $\mathcal{Y}_1^H\mathcal{A} = \mathcal{O}$, $\mathcal{Y}_1^H\mathcal{X}_1 = \mathcal{O}$ and (3) of the third condition at this Theorem 4.1, i.e., $\mathcal{Y}_1\mathcal{E}_1^H\mathcal{Y}_1 = \mathcal{Y}_1$ at the equality $\stackrel{1}{=}$ in simplification.

Verify : $\mathcal{S}^\dagger\mathcal{S}\mathcal{S}^\dagger = \mathcal{S}^\dagger$

Since, we have

$$\begin{aligned}\mathcal{S}^\dagger\mathcal{S}\mathcal{S}^\dagger &= (\mathcal{A}^\dagger\mathcal{A} + \mathcal{E}_2\mathcal{Y}_2^H)(\mathcal{A}^\dagger - \mathcal{E}_2\mathcal{X}_2^H\mathcal{A}^\dagger - \mathcal{A}^\dagger\mathcal{X}_1\mathcal{E}_1^H + \mathcal{E}_2(\mathcal{B}^\dagger + \mathcal{X}_2^H\mathcal{A}^\dagger\mathcal{X}_1)\mathcal{E}_1^H) \\ &= \mathcal{A}^\dagger\mathcal{A}\mathcal{A}^\dagger + \mathcal{E}_2\mathcal{Y}_2^H\mathcal{A}^\dagger - \mathcal{A}^\dagger\mathcal{A}\mathcal{E}_2\mathcal{X}^H\mathcal{A}^H - \mathcal{E}_2\mathcal{Y}_2^H\mathcal{E}_2\mathcal{X}^H\mathcal{A}^H - \mathcal{A}^\dagger\mathcal{A}\mathcal{A}^\dagger\mathcal{X}_1\mathcal{E}_1^H - \\ &\quad \mathcal{E}\mathcal{Y}_2^H\mathcal{A}^\dagger\mathcal{X}_1\mathcal{E}_1^H + \mathcal{A}^\dagger\mathcal{A}\mathcal{E}_2(\mathcal{B}^\dagger + \mathcal{X}_2^H\mathcal{A}^\dagger\mathcal{X}_1)\mathcal{E}_1^H + \mathcal{E}_2\mathcal{Y}_2^H\mathcal{E}_2(\mathcal{B}^\dagger + \mathcal{X}_2^H\mathcal{A}^\dagger\mathcal{X}_1)\mathcal{E}_1^H \\ &\stackrel{2}{=} \mathcal{A}^\dagger - \mathcal{E}_2\mathcal{X}_2^H\mathcal{A}^\dagger - \mathcal{A}^\dagger\mathcal{X}_1\mathcal{E}_1^H + \mathcal{E}_2(\mathcal{B}^\dagger + \mathcal{X}_2^H\mathcal{A}^\dagger\mathcal{X}_1)\mathcal{E}_1^H = \mathcal{S}^\dagger \end{aligned} \tag{4.15}$$

where we apply $\mathcal{A}\mathcal{A}^{\dagger}\mathcal{X}_1 = \mathcal{X}_1$, $\mathcal{A}\mathcal{Y}_2 = \mathcal{O}$, $\mathcal{Y}_2^H\mathcal{A}^{\dagger} = \mathcal{O}$ and (3) of the fourth condition at this Theorem 4.1, i.e., $\mathcal{E}_2\mathcal{Y}_2^H\mathcal{E}_2 = \mathcal{E}_2$ at the equality $\stackrel{2}{=}$ in simplification.

Since all the requireents in the definition of Moore-Penrose inverse are shown to be satisfied, we complete the prrof of the teorem. □

If tensors $\mathcal{B}$ and $(\mathcal{Y}_i^H\mathcal{Y}_i)$ for $i = 1, 2$ are invertible, then the constraints in Theorem 4.1 can be relazed, as in the following.

Corollary 4.2. *Given tensors $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_N}$, and the invertible tensor $\mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_K\times I_1\times\cdots\times I_K}$, $\mathcal{U} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_K}$ and $\mathcal{V} \in \mathbb{C}^{I_1\times\cdots\times I_K\times I_1\times\cdots\times I_N}$, if following conditions are valid :*

1. *$\mathcal{U} = \mathcal{X}_1 + \mathcal{Y}_1$, where $\mathcal{X}_1 \in \mathfrak{C}(\mathcal{A})$ and $\mathcal{Y}_1$ is orthgonal to $\mathfrak{C}(\mathcal{A})$;*
2. *$\mathcal{V}^H = \mathcal{X}_2 + \mathcal{Y}_2$, where $\mathcal{X}_2 \in \mathfrak{C}(\mathcal{A}^H)$ and $\mathcal{Y}_2$ is orthgonal to $\mathfrak{C}(\mathcal{A}^H)$;*
3. *tensors $(\mathcal{Y}_i^H\mathcal{Y}_i)$ for $i = 1, 2$ are invertible.*

Then the tensor

$$\begin{aligned}\mathcal{S} &= \mathcal{A} + \mathcal{U} \star_K \mathcal{B} \star_K \mathcal{V} \\ &= \mathcal{A} + (\mathcal{X}_1 + \mathcal{Y}_1) \star_K \mathcal{B} \star_K (\mathcal{X}_2 + \mathcal{Y}_2)^H,\end{aligned} \tag{4.16}$$

has the following Moore-Penrose generalized inverse identiy:

$$\begin{aligned}\mathcal{S}^{\dagger} &= \mathcal{A}^{\dagger} - \mathcal{E}_2 \star_K \mathcal{X}_2^H \star_N \mathcal{A}^{\dagger} - \mathcal{A}^{\dagger} \star_M \mathcal{X}_1 \star_K \mathcal{E}_1^H \\ &\quad + \mathcal{E}_2 \star_K (\mathcal{B}^{\dagger} + \mathcal{X}_2^H \star_N \mathcal{A}^{\dagger} \star_M \mathcal{X}_1) \star_K \mathcal{E}_1^H,\end{aligned} \tag{4.17}$$

where $\mathcal{E}_i \stackrel{def}{=} \mathcal{Y}_i(\mathcal{Y}_i^H\mathcal{Y}_i)^{-1}$ for $i = 1, 2$.

Proof. Because $(\mathcal{Y}_i^H\mathcal{Y}_i)$ for $i = 1, 2$ are invertible, then

$$\mathcal{E}_2\mathcal{B}^{\dagger}\mathcal{E}_1^H\mathcal{Y}_1\mathcal{B} = \mathcal{E}_2\mathcal{B}^{-1}\left[\mathcal{Y}_1(\mathcal{Y}_1^H\mathcal{Y}_1)^{-1}\right]^H\mathcal{Y}_1\mathcal{B} = \mathcal{E}_2. \tag{4.18}$$

By similar arguments, all following conditions are valid:

- $\mathcal{X}_1\mathcal{E}_1^H\mathcal{Y}_1\mathcal{B} = \mathcal{X}_1\mathcal{B}$,
- $\mathcal{Y}_1\mathcal{E}_1^H\mathcal{Y}_1 = \mathcal{Y}_1$,
- $\mathcal{B}\mathcal{Y}_2^H\mathcal{E}_2\mathcal{B}^{-1}\mathcal{E}_1^H = \mathcal{E}_1^H$,
- $\mathcal{B}\mathcal{Y}_2^H\mathcal{E}_2\mathcal{X}_2^H = \mathcal{B}\mathcal{X}_2^H$,
- $\mathcal{E}_2\mathcal{Y}_2^H\mathcal{E}_2 = \mathcal{E}_2$.

Therefore, this corollary is proved from Theorem 4.1 because all conditions required at Theorem 4.1 are satisfied. □

When the column space projections of the tensors $\mathcal{U}$ and $\mathcal{V}$ are zero, i.e., $\mathcal{X}_1 = \mathcal{O}$ and $\mathcal{X}_2 = \mathcal{O}$. Theorem 4.1 can be simplfied as following corollary.

Corollary 4.3. *Given tensors $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_N}$,
$\mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_K\times I_1\times\cdots\times I_K}$, $\mathcal{U} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_K}$ and $\mathcal{V} \in \mathbb{C}^{I_1\times\cdots\times I_K\times I_1\times\cdots\times I_N}$, and suppose the following items hold:*

1. $\mathcal{U} = \mathcal{Y}_1$, *where* $\mathcal{Y}_1$ *is orthgonal to* $\mathfrak{C}(\mathcal{A})$*;*

2. $\mathcal{V}^H = \mathcal{Y}_2$, *where* $\mathcal{Y}_2$ *is orthgonal to* $\mathfrak{C}(\mathcal{A}^H)$*;*

3. *(1)* $\mathcal{E}_2\mathcal{B}^\dagger\mathcal{E}_1^H\mathcal{Y}_1\mathcal{B} = \mathcal{E}_2$, *(2)* $\mathcal{Y}_1\mathcal{E}_1^H\mathcal{Y}_1 = \mathcal{Y}_1$*;*

4. *(1)* $\mathcal{B}\mathcal{Y}_2^H\mathcal{E}_2\mathcal{B}^\dagger\mathcal{E}_1^H = \mathcal{E}_1^H$, *(2)* $\mathcal{E}_2\mathcal{Y}_2^H\mathcal{E}_2 = \mathcal{E}_2$.

Then the tensor

$$\begin{aligned} \mathcal{S} &= \mathcal{A} + \mathcal{U} \star_K \mathcal{B} \star_K \mathcal{V} \\ &= \mathcal{A} + \mathcal{Y}_1 \star_K \mathcal{B} \star_K \mathcal{Y}_2^H, \end{aligned} \tag{4.19}$$

has the following Moore-Penrose generalized inverse identiy:

$$\mathcal{S}^\dagger = \mathcal{A}^\dagger + \mathcal{E}_2 \star_K \mathcal{B}^\dagger \star_K \mathcal{E}_1^H, \tag{4.20}$$

where $\mathcal{E}_i \stackrel{def}{=} \mathcal{Y}_i(\mathcal{Y}_i^H\mathcal{Y}_i)^\dagger$ *for* $i = 1, 2$.

Proof. The proof can be obtained by replacing tensors $\mathcal{X}_1$ and $\mathcal{X}_2$ as zero tensor $\mathcal{O}$ in the proof of Theorem 4.1. □

If the tensor $\mathcal{A}$ is a Hermitian tensor, we can have following corollary.

Corollary 4.4. *Given tensors* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$, $\mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_K\times I_1\times\cdots\times I_K}$, $\mathcal{U} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_K}$ *and* $\mathcal{V} \in \mathbb{C}^{I_1\times\cdots\times I_K\times I_1\times\cdots\times I_M}$, *and suppose the following items hold:*

1. $\mathcal{A}^H = \mathcal{A}$, *and* $\mathcal{U}^H = \mathcal{V}$.

2. $\mathcal{U} = \mathcal{X} + \mathcal{Y}$, *where* $\mathcal{X} \in \mathfrak{C}(\mathcal{A})$ *and* $\mathcal{Y}$ *is orthgonal to* $\mathfrak{C}(\mathcal{A})$*;*

3. *(1)* $\mathcal{E}\mathcal{B}^\dagger\mathcal{E}^H\mathcal{Y}\mathcal{B} = \mathcal{E}$, *(2)* $\mathcal{X}\mathcal{E}^H\mathcal{Y}\mathcal{B} = \mathcal{X}\mathcal{B}$, *(3)* $\mathcal{Y}\mathcal{E}^H\mathcal{Y} = \mathcal{Y}$*;*

4. *(1)* $\mathcal{B}\mathcal{Y}^H\mathcal{E}\mathcal{B}^\dagger\mathcal{E}^H = \mathcal{E}^H$, *(2)* $\mathcal{B}\mathcal{Y}^H\mathcal{E}\mathcal{X}^H = \mathcal{B}\mathcal{X}^H$, *(3)* $\mathcal{E}\mathcal{Y}^H\mathcal{E} = \mathcal{E}$.

Then the tensor

$$\begin{aligned} \mathcal{S} &= \mathcal{A} + \mathcal{U} \star_K \mathcal{B} \star_K \mathcal{V} \\ &= \mathcal{A} + (\mathcal{X} + \mathcal{Y}) \star_K \mathcal{B} \star_K (\mathcal{X} + \mathcal{Y})^H, \end{aligned} \tag{4.21}$$

has the following Moore-Penrose generalized inverse identiy:

$$\begin{aligned} \mathcal{S}^\dagger &= \mathcal{A}^\dagger - \mathcal{E} \star_K \mathcal{X}^H \star_N \mathcal{A}^\dagger - \mathcal{A}^\dagger \star_M \mathcal{X} \star_K \mathcal{E}^H \\ &\quad + \mathcal{E} \star_K (\mathcal{B}^\dagger + \mathcal{X}^H \star_N \mathcal{A}^\dagger \star_M \mathcal{X}) \star_K \mathcal{E}^H, \end{aligned} \tag{4.22}$$

where $\mathcal{E} \stackrel{def}{=} \mathcal{Y}(\mathcal{Y}^H\mathcal{Y})^\dagger$.

Proof. Because $\mathcal{A}^H = \mathcal{A}$, and $\mathcal{U}^H = \mathcal{V}$, the proof from Theorem 4.1 can be applied here by removing those subscript indices, 1 and 2. □

4.2 Illustrative Examples

In this section, we will provide two examples to demonstarte the validity of Theorem 4.1 and Corollary 4.3. We have to use following tensor equation in this section.

$$\mathcal{A} \star_N \mathcal{Z} \star_M \mathcal{B} = (\mathcal{A} \otimes \mathcal{B}^H) \star_{(N+M)} \mathcal{Z}, \tag{4.23}$$

where $\otimes$ is the Kronecker product of tensors [55].

Following example is provided to verify Corollary 4.3.

Example 1. Given tensor $\mathcal{A} \in \mathbb{R}^{2\times2\times2\times2}$

$$\mathcal{A} \;=\; \left[\begin{array}{cc:cc} a_{11,11} & a_{12,11} & a_{11,12} & a_{12,12} \\ a_{21,11} & a_{22,11} & a_{21,12} & a_{22,12} \\ \hdashline a_{11,21} & a_{12,21} & a_{11,22} & a_{12,22} \\ a_{21,21} & a_{22,21} & a_{11,22} & a_{22,22} \end{array}\right] = \left[\begin{array}{cc:cc} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hdashline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \tag{4.24}$$

Then, the column tensor $\mathbf{a}_{11}$ of tensor $\mathcal{A}$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, the column tensor $\mathbf{a}_{12}$ of tensor $\mathcal{A}$ is $\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, the column tensor $\mathbf{a}_{21}$ of tensor $\mathcal{A}$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and the column tensor $\mathbf{a}_{22}$ of tensor $\mathcal{A}$ is $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

If we take Hermition for the tensor $\mathcal{A}$, we have

$$\mathcal{A}^H \;=\; \left[\begin{array}{cc:cc} a_{11,11} & a_{11,12} & a_{12,11} & a_{12,12} \\ a_{11,21} & a_{11,22} & a_{12,21} & a_{12,22} \\ \hdashline a_{21,11} & a_{21,12} & a_{22,11} & a_{22,12} \\ a_{21,21} & a_{21,22} & a_{22,21} & a_{22,22} \end{array}\right] = \left[\begin{array}{cc:cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \tag{4.25}$$

Then, the column tensor $\mathbf{a}_{11}$ of tensor $\mathcal{A}^H$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, the column tensor $\mathbf{a}_{12}$ of tensor $\mathcal{A}^{\mathcal{H}}$ is $\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$, the column tensor $\mathbf{a}_{21}$ of tensor $\mathcal{A}^H$ is $\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$, and the column tensor $\mathbf{a}_{22}$ of tensor $\mathcal{A}^H$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

The tensor $\mathcal{B} \in \mathbb{R}^{1\times1\times1\times1}$ has only one entry with value 1, the value in this tensor $\mathcal{B}$ is denoted as $1 \in \mathbb{R}^{1\times1\times1\times1}$. The tensor $\mathcal{U} \in \mathbb{R}^{2\times2\times1\times1}$ is

$$\mathcal{U} \;=\; \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.26}$$

and the tensor $\mathcal{V} \in \mathbb{R}^{1\times1\times2\times2}$ is

$$\mathcal{V} \;=\; \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}. \tag{4.27}$$

Then, we have the tensor $\mathcal{S}$ expressed as

$$\begin{aligned}
\mathcal{S} &= \mathcal{A}+\mathcal{U}\mathcal{B}\mathcal{V} \\
&= \left[\begin{array}{cc:cc} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hdashline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] + \mathcal{U} \otimes \mathcal{V}^H \star_4 \mathcal{B} \\
&= \left[\begin{array}{cc:cc} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hdashline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] + \left[\begin{array}{cc:cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc:cc} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ \hdashline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]. \qquad (4.28)
\end{aligned}$$

The goal is to verify Sherman-Morrison-Woodbury identiy for the inverse of the tensor $\mathcal{S}$.

Because the tensor $\mathcal{A}$ is not invertible, the Moore-Penrose inverse of the tensor $\mathcal{A}$ becomes

$$\mathcal{A}^{\dagger} = \left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hdashline 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right]. \qquad (4.29)$$

Before applying Theorem 4.1, we have to decompose the tensors $\mathcal{U}$ and $\mathcal{V}^H$ according to the column-tensor spaces $\mathfrak{C}(\mathcal{A})$ and $\mathfrak{C}(\mathcal{A}^H)$. They are decomposed as following:

$$\begin{aligned}
\mathcal{U} &= \mathcal{Y}_1 \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \qquad (4.30)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{V} &= \mathcal{Y}_2^H \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}. \qquad (4.31)
\end{aligned}$$

Therefore, the tensors $\mathcal{U}$ and $\mathcal{V}$ are in orthogonal spaces of $\mathfrak{C}(\mathcal{A})$ and $\mathfrak{C}(\mathcal{A}^H)$, respectively.

Since we define $\mathcal{E}_i \stackrel{\text{def}}{=} \mathcal{Y}_i(\mathcal{Y}_i^H\mathcal{Y}_i)^{\dagger}$ for $i = 1, 2$, the tensors $\mathcal{E}_i$ can be evaluated as

$$\begin{aligned}
\mathcal{E}_1 &= \mathcal{Y}_1(\mathcal{Y}_1^H\mathcal{Y}_1)^{\dagger} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \star_2 (1 \in \mathbb{R}^{1\times1\times1\times1}) \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times2\times1\times1}; \qquad (4.32)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_2 &= \mathcal{Y}_2(\mathcal{Y}_2^H\mathcal{Y}_2)^{\dagger} \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \star_2 (\frac{1}{2} \in \mathbb{R}^{1\times1\times1\times1}) \\
&= \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \in \mathbb{R}^{2\times2\times1\times1}, \qquad (4.33)
\end{aligned}$$

where $\frac{1}{2} \in \mathbb{R}^{1\times1\times1\times1}$ is a single entry tensor with value $\frac{1}{2}$ with tensor dimension $1\times1\times1\times1$ (order 4).

We are ready to evalute following terms $\mathcal{E}\mathcal{X}_2^H\mathcal{A}^\dagger$, $\mathcal{A}^\dagger\mathcal{X}_1\mathcal{E}_1^H$, and $\mathcal{E}_2(\mathcal{B}^\dagger + \mathcal{X}_2^H\mathcal{A}^\dagger\mathcal{X}_1)\mathcal{E}_1^H$. But tensors $\mathcal{E}\mathcal{X}_2^H\mathcal{A}^\dagger$ and $\mathcal{A}^\dagger\mathcal{X}_1\mathcal{E}_1^H$ are zero tensors since $\mathcal{X}_1$ and $\mathcal{X}_2$ are zero tensors. Because we also have following:

$$\mathcal{X}_2^H\mathcal{A}^\dagger\mathcal{X}_1 = 0 \in \mathbb{R}^{1\times1\times1\times1}, \tag{4.34}$$

we have

$$\begin{aligned}\mathcal{E}_2(\mathcal{B}^\dagger + \mathcal{X}_2^H\mathcal{A}^\dagger\mathcal{X}_1)\mathcal{E}_1^H &= \mathcal{E}_2 \star_2 (1 \in \mathbb{R}^{1\times1\times1\times1}) \star_2 \mathcal{E}_1^H \\ &= \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{array}\right]\end{aligned} \tag{4.35}$$

Finally, from the Eq. (4.35), we have

$$\begin{aligned}\mathcal{S}^\dagger &= \mathcal{A}^\dagger + \mathcal{E}_2(\mathcal{B}^\dagger + \mathcal{X}_2^H\mathcal{A}^\dagger\mathcal{X}_1)\mathcal{E}_1^H \\ &= \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right],\end{aligned} \tag{4.36}$$

which is the inverse of the tensor $\mathcal{S}$.

The following example, which is more complicated. See Theorem 4.1.

Example 2. Given same tensors $\mathcal{A} \in \mathbb{R}^{2\times2\times2\times2}$ and $\mathcal{B} \in \mathbb{R}^{1\times1\times1\times1}$ as ExampleF 1, the tensor $\mathcal{U} \in \mathbb{R}^{2\times2\times1\times1}$ is

$$\mathcal{U} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \tag{4.37}$$

and the tensor $\mathcal{V} \in \mathbb{R}^{1\times1\times2\times2}$ is

$$\mathcal{V} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}. \tag{4.38}$$

Then, we have the tensor $\mathcal{S}$ expressed as

$$\begin{aligned}\mathcal{S} &= \mathcal{A} + \mathcal{U}\mathcal{B}\mathcal{V} \\ &= \left[\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] + \mathcal{U} \otimes \mathcal{V}^H \star_4 \mathcal{B} \\ &= \left[\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] + \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right].\end{aligned} \tag{4.39}$$

We wish to show Sherman-Morrison-Woodbury identiy for the inverse of the tensor $\mathcal{S}$.

Before applying Theorem 4.1, we have to decompose the tensors $\mathcal{U}$ and $\mathcal{V}^H$ according to the column-tensor spaces $\mathfrak{C}(\mathcal{A})$ and $\mathfrak{C}(\mathcal{A}^H)$. They are decomposed as following:

$$\begin{aligned}\mathcal{U} &= \mathcal{X}_1 + \mathcal{Y}_1 \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\end{aligned} \tag{4.40}$$

and

$$\begin{aligned}\mathcal{V} &= \mathcal{X}_2^H + \mathcal{Y}_2^H \\ &= \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.\end{aligned} \tag{4.41}$$

Under this decomposition, the subtensors $\mathcal{X}_1$ and $\mathcal{X}_2$ are in the column-tensor spaces $\mathfrak{C}(\mathcal{A})$ and $\mathfrak{C}(\mathcal{A}^H)$, respectively. Moreover, the subtensors $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are orthgonal to the column-tensor spaces $\mathfrak{C}(\mathcal{A})$ and $\mathfrak{C}(\mathcal{A}^H)$, respectively. Since we define $\mathcal{E}_i \stackrel{\text{def}}{=} \mathcal{Y}_i(\mathcal{Y}_i^H \mathcal{Y}_i)^\dagger$ for $i = 1, 2$, the tensors $\mathcal{E}_i$ are evaluated at Eqs. (4.32) and (4.33) since $\mathcal{Y}_i$ are same with the previous example.

We are ready to evalute following terms $\mathcal{E}\mathcal{X}_2^H \mathcal{A}^\dagger$, $\mathcal{A}^\dagger \mathcal{X}_1 \mathcal{E}_1^H$, and $\mathcal{E}_2(\mathcal{B}^\dagger + \mathcal{X}_2^H \mathcal{A}^\dagger \mathcal{X}_1)\mathcal{E}_1^H$.

$$\begin{aligned}\mathcal{A}^\dagger \mathcal{X}_1 \mathcal{E}_1^H &= \left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hdashline 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right] \star_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hdashline 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right] \star_2 \left[\begin{array}{cc:cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc:cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].\end{aligned} \tag{4.42}$$

$$\begin{aligned}\mathcal{E}_2 \mathcal{X}_2^H \mathcal{A}^\dagger &= \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \star_2 \left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hdashline 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc:cc} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} \\ \hdashline 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{array}\right] \star_2 \left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hdashline 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right] = \left[\begin{array}{cc:cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right].\end{aligned} \tag{4.43}$$

Since we have following:

$$\begin{aligned}\mathcal{X}_2^H \mathcal{A}^\dagger \mathcal{X}_1 &= \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \star_2 \left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hdashline 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right] \star_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \star_2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0 \in \mathbb{R}^{1\times 1\times 1\times 1},\end{aligned} \tag{4.44}$$

we have

$$\begin{aligned}\mathcal{E}_2(\mathcal{B}^\dagger + \mathcal{X}_2^H \mathcal{A}^\dagger \mathcal{X}_1)\mathcal{E}_1^H &= \mathcal{E}_2 \star_2 (1 \in \mathbb{R}^{1\times 1\times 1\times 1}) \star_2 \mathcal{E}_1^H \\ &= \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{array}\right]\end{aligned} \tag{4.45}$$

Finally, from the Eqs. (4.42), (4.43), (4.45), we have

$$\begin{aligned}\mathcal{S}^\dagger &= \mathcal{A}^\dagger - \mathcal{A}^\dagger \mathcal{X}_1 \mathcal{E}_1^H - \mathcal{E}_2 \mathcal{X}_2^H \mathcal{A}^\dagger + \mathcal{E}_2(\mathcal{B}^\dagger + \mathcal{X}_2^H \mathcal{A}^\dagger \mathcal{X}_1)\mathcal{E}_1^H \\ &= \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & \frac{1}{2} \end{array}\right],\end{aligned} \tag{4.46}$$

which is the inverse of the tensor $\mathcal{S}$.

5 Application: Sensitivity Analysis for Multilinear Systems

In this section, we will apply the results obtained in Section 4 to perform sensitivity analysis for a multilinear system of equations, i.e. $\mathcal{A} \star \mathcal{X} = \mathcal{D}$, by deriving the normalized upper bound for the error in the solution when coefficient tensors are perturbed in Section 5.1. In Section 5.2, we investigate the effects of perturbation values ϵ_A, ϵ_D to the normalized solution error $\frac{\|\mathcal{Y}-\mathcal{X}\|}{\|\mathcal{X}\|}$, denoted as E_n. All norms discussed in this paper are based on the Frobenius norm definition.

5.1 Sensitivity Analysis

Serveral preparation lemmas will be given before presenting our results asscoaited to sensitivity analysis for multilinear systems.

Lemma 5.1. *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$ and $\mathcal{B} \in \mathbb{C}^{J_1\times\cdots\times J_N\times K_1\times\cdots\times K_L}$, we have following inequality for Frobenius norm of tensors:*

$$\|\mathcal{A} \star_N \mathcal{B}\| \leq \|\mathcal{A}\| \, \|\mathcal{B}\| \,. \tag{5.1}$$

Proof. Because

$$
\begin{aligned}
\|\mathcal{A} \star_N \mathcal{B}\|^2 &= \sum_{i_1,\cdots,i_M,k_1\times\cdots\times k_L} |\sum_{j_1,\cdots,j_N} a_{i_1,\cdots,i_M,j_1,\cdots,j_N} b_{j_1,\cdots,j_N,k_1,\cdots,k_L}|^2 \\
&\leq \sum_{i_1,\cdots,i_M,k_1\times\cdots\times k_L} \left[\left(\sum_{j_1,\cdots,j_N} |a_{i_1,\cdots,i_M,j_1,\cdots,j_N}|^2\right) \times \right. \\
&\quad \left.\left(\sum_{j_1,\cdots,j_N} |b_{j_1,\cdots,j_N,k_1,\cdots,k_L}|^2\right)\right] \\
&= \left(\sum_{i_1,\cdots,i_M} \left(\sum_{j_1,\cdots,j_N} |a_{i_1,\cdots,i_M,j_1,\cdots,j_N}|^2\right)\right) \times \\
&\quad \left(\sum_{j_1,\cdots,j_N} \left(\sum_{k_1\times\cdots\times k_L} |b_{j_1,\cdots,j_N,k_1,\cdots,k_L}|^2\right)\right) = \|\mathcal{A}\|^2 \|\mathcal{B}\|^2 \qquad (5.2)
\end{aligned}
$$

where the inequality is based on Cauchy-Schwarz inequality. By taking square root of both sides, the lemma is established. □

Lemma 5.2. *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$ and $\mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$, we have following inequality for Frobenius norm of tensors:*

$$
\|\mathcal{A} + \mathcal{B}\| \leq \|\mathcal{A}\| + \|\mathcal{B}\|. \qquad (5.3)
$$

Proof. Because

$$
\begin{aligned}
\|\mathcal{A} + \mathcal{B}\|^2 &\leq \|\mathcal{A}\|^2 + \|\mathcal{B}\|^2 + 2 \sum_{i_1,\cdots,i_M,j_1,\cdots,j_N} |a_{i_1,\cdots,i_M,j_1,\cdots,j_N}||b_{i_1,\cdots,i_M,j_1,\cdots,j_N}| \\
&\overset{1}{\leq} \|\mathcal{A}\|^2 + \|\mathcal{B}\|^2 + 2\|\mathcal{A}\|\|\mathcal{B}\| \\
&= (\|\mathcal{A}\| + \|\mathcal{B}\|)^2 \qquad (5.4)
\end{aligned}
$$

where the ineqsuality $\overset{1}{\leq}$ is based on Cauchy-Schwarz inequality. By taking square root of both sides, the lemma is established. □

Given a multilinear system of equations, the exact solution expressed by tensor inverse or Moore-Penrose inverse is given by following Theorem. The proof can be found at [5].

Theorem 5.3. *For given tensors $\mathcal{A} \in \mathbb{C}^{K_1\times\cdots\times K_P\times I_1\times\cdots\times I_M}$, $\mathcal{D} \in \mathbb{C}^{K_1\times\cdots\times K_P\times L_1\times\cdots\times L_Q}$, the tensor equation*

$$
\mathcal{A} \star_M \mathcal{X} = \mathcal{D}, \qquad (5.5)
$$

has a solution if and only if $\mathcal{A} \star_M \mathcal{A}^\dagger \star_P \mathcal{D} = \mathcal{D}$. The solution can be expressed as

$$
\mathcal{X} = \mathcal{A}^\dagger \star_P \mathcal{D} + (\mathcal{I} - \mathcal{A}^\dagger \star_P \mathcal{A}) \star_M \mathcal{U}, \qquad (5.6)
$$

where $\mathcal{I}$ is the identiy tensor in $\mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ and $\mathcal{U}$ is an arbitrary tensor in $\mathbb{C}^{I_1\times\cdots\times I_M\times L_1\times\cdots\times L_Q}$.

If the tensor $\mathcal{A}$ is invertible, then the Eq. (5.6) *can be further reduced as*

$$
\mathcal{X} = \mathcal{A}^{-1} \star_P \mathcal{D}. \qquad (5.7)
$$

We are ready to present our theorem about sensitvity analysis for solution of a multilinear system.

Theorem 5.4. *The original multilinear system of equations is*

$$\mathcal{A} \star_M \mathcal{X} = \mathcal{D} \tag{5.8}$$

where $\mathcal{A} \in \mathbb{C}^{K_1 \times \cdots \times K_P \times I_1 \times \cdots \times I_M}$, $\mathcal{D} \in \mathbb{C}^{K_1 \times \cdots \times K_P \times L_1 \times \cdots \times L_Q}$, *and* $\mathcal{O} \neq \mathcal{D} \in \mathbb{C}^{K_1 \times \cdots \times K_P \times L_1 \times \cdots \times L_Q}$. *The perturbed system can be expressed as*

$$(\mathcal{A} + \delta\mathcal{A}) \star_M \mathcal{Y} = (\mathcal{D} + \delta\mathcal{D}), \tag{5.9}$$

where $\delta\mathcal{A} \in \mathbb{C}^{K_1 \times \cdots \times K_P \times I_1 \times \cdots \times I_M}$ *and* $\delta\mathcal{D} \in \mathbb{C}^{K_1 \times \cdots \times K_P \times L_1 \times \cdots \times L_Q}$. *If the tensor* $\delta\mathcal{A}$ *is decomposed as (for example, by SVD decomposition when* $\delta\mathcal{A}$ *is a square tensor, see [55])*

$$\begin{aligned} \delta\mathcal{A} &= \mathcal{U} \star_P \mathcal{B} \star_M \mathcal{V} \\ &= (\mathcal{X}_1 + \mathcal{Y}_1) \star_P \mathcal{B} \star_M (\mathcal{X}_2 + \mathcal{Y}_2)^H, \end{aligned} \tag{5.10}$$

where $\mathcal{X}_1 \in \mathfrak{C}(\mathcal{A})$, $\mathcal{Y}_1$ *is orthgonal to* $\mathfrak{C}(\mathcal{A})$, $\mathcal{X}_2 \in \mathfrak{C}(\mathcal{A}^H)$ *and* $\mathcal{Y}_2$ *is orthgonal to* $\mathfrak{C}(\mathcal{A}^H)$.

We further assume that $\|\mathcal{X}_i\| \leq \epsilon_A \|\mathcal{A}\|$ *for* $1 \leq i \leq 2$, $\|\mathcal{E}_i\| \leq \epsilon_A \|\mathcal{A}\|$ *for* $1 \leq i \leq 2$ *(Recall* $\mathcal{E}_i \stackrel{def}{=} \mathcal{Y}_i(\mathcal{Y}_i^H \mathcal{Y}_i)^\dagger$*) and* $\|\delta\mathcal{D}\| \leq \epsilon_D \|\mathcal{B}\|$, *then*

$$\begin{aligned} \frac{\|\mathcal{Y} - \mathcal{X}\|}{\|\mathcal{X}\|} &\leq (1 + \epsilon_D) \|\mathcal{A}\|^3 (2\epsilon_A^2 \left\|\mathcal{A}^\dagger\right\| + \epsilon_A^3 \|\mathcal{A}\| + \epsilon_A^4 \|\mathcal{A}\|^2 \left\|\mathcal{A}^\dagger\right\|) + \\ &\quad \epsilon_D \|\mathcal{A}\| \left\|\mathcal{A}^\dagger\right\|. \end{aligned} \tag{5.11}$$

Proof. From Theorem 5.3 and the Eq. 5.6, the solution for the Eq. (5.8) is

$$\mathcal{X} = \mathcal{A}^\dagger \star_P \mathcal{D} + (\mathcal{I} - \mathcal{A}^\dagger \star_P \mathcal{A}) \star_M \mathcal{U}, \tag{5.12}$$

and, similarly, the solution for the Eq. (5.9) is

$$\mathcal{Y} = (\mathcal{A} + \delta\mathcal{A})^\dagger \star_P (\mathcal{D} + \delta\mathcal{D}) + (\mathcal{I} - (\mathcal{A} + \delta\mathcal{A})^\dagger \star_P (\mathcal{A} + \delta\mathcal{A}) \star_M \mathcal{U}. \tag{5.13}$$

Since the tensor $\mathcal{U}$ can be chosen arbitraryly, we can set $\mathcal{U}$ as zero tensor and we have

$$\begin{aligned} \mathcal{Y} - \mathcal{X} &= (\mathcal{A} + \delta\mathcal{A})^\dagger \star_P (\mathcal{D} + \delta\mathcal{D}) - \mathcal{A}^\dagger \star_P \mathcal{D} \\ &= \left[\mathcal{A} + (\mathcal{X}_1 + \mathcal{Y}_1) \star_P \mathcal{B} \star_M (\mathcal{X}_2 + \mathcal{Y}_2)^H\right]^\dagger \star_P (\mathcal{D} + \delta\mathcal{D}) - \mathcal{A}^\dagger \star_P \mathcal{D} \\ &\stackrel{1}{=} \big(\mathcal{A}^\dagger - \mathcal{E}_2 \star_M \mathcal{X}_2^H \star_M \mathcal{A}^\dagger - \mathcal{A}^\dagger \star_P \mathcal{X}_1 \star_P \mathcal{E}_1^H \\ &\quad + \mathcal{E}_2 \star_M (\mathcal{B}^\dagger + \mathcal{X}_2^H \star_M \mathcal{A}^\dagger \star_P \mathcal{X}_1) \star_P \mathcal{E}_1^H\big) \star_P (\mathcal{D} + \delta\mathcal{D}) - \mathcal{A}^\dagger \star_P \mathcal{D} \\ &= \mathcal{E}_2 \star_M \mathcal{X}_2^H \star_M \mathcal{A}^\dagger \star_P \mathcal{D} - \mathcal{A}^\dagger \star_P \mathcal{X}_1 \star_P \mathcal{E}_1^H \star_P \mathcal{D} \\ &\quad + \mathcal{E}_2 \star_M (\mathcal{B}^\dagger + \mathcal{X}_2^H \star_M \mathcal{A}^\dagger \star_P \mathcal{X}_1) \star_P \mathcal{E}_1^H \star_P \mathcal{D} + \mathcal{A}^\dagger \star_P \delta\mathcal{D} \\ &\quad - \mathcal{E}_2 \star_M \mathcal{X}_2^H \star_M \mathcal{A}^\dagger \star_P \delta\mathcal{D} - \mathcal{A}^\dagger \star_P \mathcal{X}_1 \star_P \mathcal{E}_1^H \star_P \delta\mathcal{D} \\ &\quad + \mathcal{E}_2 \star_M (\mathcal{B}^\dagger + \mathcal{X}_2^H \star_M \mathcal{A}^\dagger \star_P \mathcal{X}_1) \star_P \mathcal{E}_1^H \star_P \delta\mathcal{D}, \end{aligned} \tag{5.14}$$

where we apply Theorem 4.1 at $\stackrel{1}{=}$. If we take Frobenius norm at both sides of Eq. (5.14), we get

$$\begin{aligned}\|\mathcal{Y}-\mathcal{X}\| &= \big\|\mathcal{E}_2 \star_M \mathcal{X}_2^H \star_M \mathcal{A}^\dagger \star_P \mathcal{D} - \mathcal{A}^\dagger \star_P \mathcal{X}_1 \star_P \mathcal{E}_1^H \star_P \mathcal{D} \\ &\quad + \mathcal{E}_2 \star_M (\mathcal{B}^\dagger + \mathcal{X}_2^H \star_M \mathcal{A}^\dagger \star_P \mathcal{X}_1) \star_P \mathcal{E}_1^H \mathcal{D} + \mathcal{A}^\dagger \star_P \delta\mathcal{D} \\ &\quad - \mathcal{E}_2 \star_M \mathcal{X}_2^H \star_M \mathcal{A}^\dagger \star_P \delta\mathcal{D} - \mathcal{A}^\dagger \star_P \mathcal{X}_1 \star_P \mathcal{E}_1^H \star_P \delta\mathcal{D} \\ &\quad + \mathcal{E}_2 \star_M (\mathcal{B}^\dagger + \mathcal{X}_2^H \star_M \mathcal{A}^\dagger \star_P \mathcal{X}_1) \star_P \mathcal{E}_1^H \star_P \delta\mathcal{D}\big\| \\ &\overset{2}{\leq} \|\mathcal{E}_2\| \left\|\mathcal{X}_2^H\right\| \left\|\mathcal{A}^\dagger\right\| \|\mathcal{D}\| + \left\|\mathcal{A}^\dagger\right\| \|\mathcal{X}_1\| \left\|\mathcal{E}_1^H\right\| \|\mathcal{D}\| + \|\mathcal{E}_2\| \left\|\mathcal{B}^\dagger\right\| \left\|\mathcal{E}_1^H\right\| \|\mathcal{D}\| + \\ &\quad \left\|\mathcal{X}_2^H\right\| \left\|\mathcal{A}^\dagger\right\| \|\mathcal{X}_1\| \left\|\mathcal{E}_1^H\right\| \|\mathcal{D}\| + \left\|\mathcal{A}^\dagger\right\| \|\delta\mathcal{D}\| + \|\mathcal{E}_2\| \left\|\mathcal{X}_2^H\right\| \left\|\mathcal{A}^\dagger\right\| \|\delta\mathcal{D}\| + \\ &\quad \left\|\mathcal{A}^\dagger\right\| \|\mathcal{X}_1\| \left\|\mathcal{E}_1^H\right\| \|\delta\mathcal{D}\| + \|\mathcal{E}_2\| \left\|\mathcal{B}^\dagger\right\| \left\|\mathcal{E}_1^H\right\| \|\delta\mathcal{D}\| + \\ &\quad \left\|\mathcal{X}_2^H\right\| \left\|\mathcal{A}^\dagger\right\| \|\mathcal{X}_1\| \left\|\mathcal{E}_1^H\right\| \|\delta\mathcal{D}\| \end{aligned} \tag{5.15}$$

where we apply Lemma 5.1 and Lemma 5.2 to the inequality $\overset{2}{\leq}$. Because we have that $\|\mathcal{X}_i\| \leq \epsilon_A \|\mathcal{A}\|$ for $1 \leq i \leq 2$, $\|\mathcal{E}_i\| \leq \epsilon_A \|\mathcal{A}\|$ for $1 \leq i \leq 2$ (Recall $\mathcal{E}_i \overset{\text{def}}{=} \mathcal{Y}_i(\mathcal{Y}_i^H \mathcal{Y}_i)^\dagger$) and $\|\delta\mathcal{D}\| \leq \epsilon_D \|\mathcal{D}\|$, then the Eq. (5.15) can be further reduced as:

$$\begin{aligned}\|\mathcal{Y}-\mathcal{X}\| &= (1+\epsilon_D)\|\mathcal{D}\| \,(2\epsilon_A^2 \|\mathcal{A}\|^2 \left\|\mathcal{A}^\dagger\right\| + \epsilon_A^3 \|\mathcal{A}\|^3 + \epsilon_A^4 \|\mathcal{A}\|^4 \left\|\mathcal{A}^\dagger\right\|) + \\ &\quad \epsilon_D \left\|\mathcal{A}^\dagger\right\| \|\mathcal{D}\|, \end{aligned} \tag{5.16}$$

and since $\|\mathcal{D}\| = \|\mathcal{A}\mathcal{X}\| \leq \|\mathcal{A}\| \|\mathcal{X}\|$, we have

$$\begin{aligned}\frac{\|\mathcal{Y}-\mathcal{X}\|}{\|\mathcal{X}\|} &\leq (1+\epsilon_D)\|\mathcal{A}\|^3 \,(2\epsilon_A^2 \left\|\mathcal{A}^\dagger\right\| + \epsilon_A^3 \|\mathcal{A}\| + \epsilon_A^4 \|\mathcal{A}\|^2 \left\|\mathcal{A}^\dagger\right\|) + \\ &\quad \epsilon_D \|\mathcal{A}\| \left\|\mathcal{A}^\dagger\right\| \end{aligned} \tag{5.17}$$

The theorem is proved. □

5.2 Numerical Evaluation

In this section, we will apply the normalized error bound results derived in Section 5.1 to the multilinear equation $\mathcal{A} \star_2 \mathcal{X} = \mathcal{D}$ with following tensors:

$$\mathcal{A} = \left[\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right], \tag{5.18}$$

$$\mathcal{A}^\dagger = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 0 & -0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \end{array}\right], \tag{5.19}$$

and

$$\mathcal{D} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}. \tag{5.20}$$

In Fig. 1, the normalized error bound for the multilinear system $\mathcal{A} \star_2 \mathcal{X} = \mathcal{D}$ is presented against the change of the Frobenius norm of the tesnor $\mathcal{A}$ according to the Theorem 5.4.

Fig. 1 delineates the normalized error bound with respect to three different perturbation values $\epsilon_A = 0.09, 0.05, 0.01$ of the tensor $\mathcal{A}$ subject to the perturbation value $\epsilon_D = 0.01$ of the tensor $\mathcal{D}$. The way we change the Frobenius norm of the tesnor $\mathcal{A}$ is by scaling the tensor $\mathcal{A}$ with some positive number α, i.e., $\alpha\mathcal{A}$ is a tensor obtained by multiplying the value α to each entries of the tensor $\mathcal{A}$. We observe that the normalization error E_n increases with the increase of the perturbation value ϵ_A. Given the same perturbaiton value ϵ_A, the normalized error bound E_n can achieve its minimum by scaling the tesnor $\mathcal{A}$ properly. For example, when the value ϵ_A is 0.09, the minimum error bound happens when the value of $\|\mathcal{A}\|$ is about 2.5.

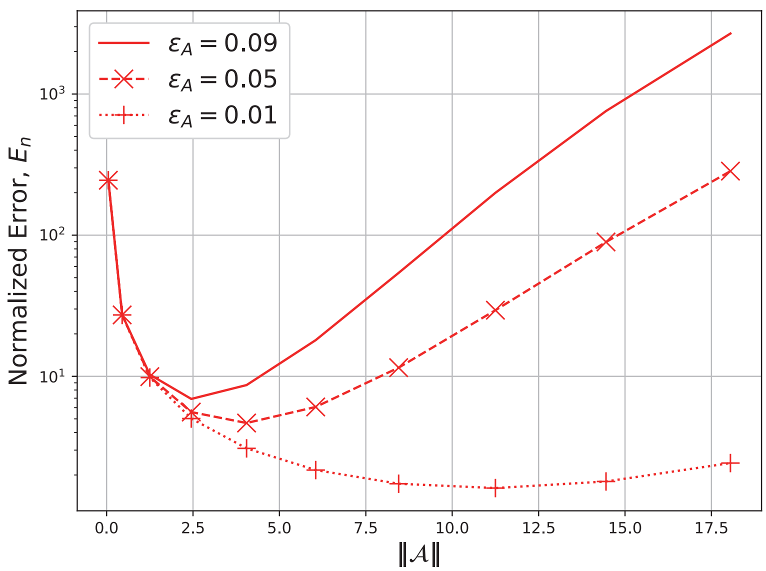

Figure 1: The normalized error bound E_n for the perturbed multilinear system $\mathcal{A}\mathcal{X} = \mathcal{D}$ with respect to the tesnor norm $\|\mathcal{A}\|$ for different ϵ_A values when the tesnor norm $\|\mathcal{D}\|$ is 7 and $\epsilon_D = 0.01$.

In Fig. 2, the normalized error bound for the multilinear system $\mathcal{A} \star_2 \mathcal{X} = \mathcal{D}$ is presented against the change of the Frobenius norm of the tesnor $\mathcal{A}$. Fig. 2 plots the normalized error bound with respect to three different perturbation values $\epsilon_D = 0.09, 0.05, 0.01$ of the tensor $\mathcal{D}$ subject to the perturbation value $\epsilon_A = 0.01$ of the tensor $\mathcal{A}$. We find that the normalization error E_n increases with the increase of the perturbation value ϵ_D. Given the same perturbaiton value ϵ_A, the bound E_n also can achieve its minimum by scaling the tesnor $\mathcal{A}$ properly. For example, when the value ϵ_D is 0.01, the minimum error bound happens when the value of $\|\mathcal{A}\|$ is about 1.25. Compared to Fig. 2, the error bounds difference between various perturbation values ϵ_A becomes more significant when the value of the Frobenius norm of the tensor $\mathcal{A}$ increases. On the other hand, the error bounds difference between various perturbation values ϵ_D becomes less significant when the value of the Frobenius norm of the tensor $\mathcal{A}$ increases. Both figures show that the error bound variation is more sensitive with respect to the Frobenius norm of the tensor $\mathcal{A}$ for smaller value range of $\|\mathcal{A}\|$.

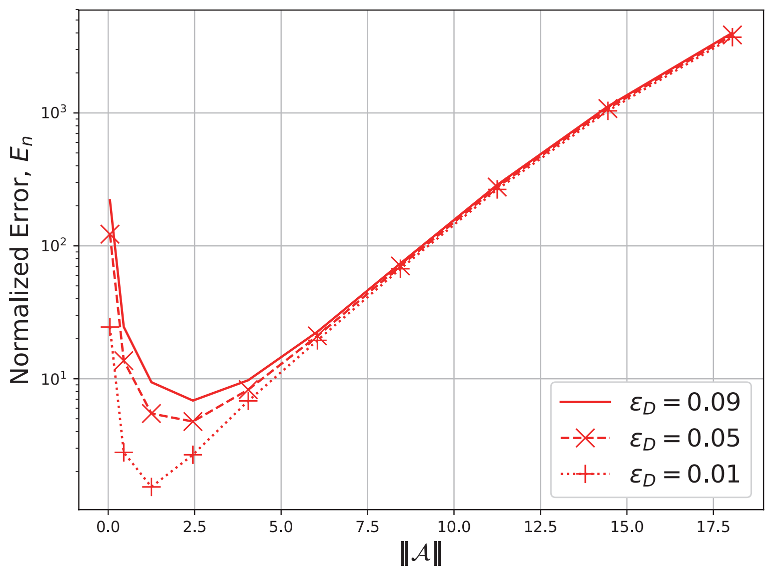

Figure 2: The normalized error E_n for the perturbed multilinear system equations $\mathcal{A}\mathcal{X} = \mathcal{D}$ with respect to the tesnor norm $\|\mathcal{A}\|$ for different ϵ_D when the tesnor norm $\|\mathcal{D}\|$ is 7.

6 Conclusions

Motivated by great applications of the Sherman-Morrison -Woodbury matrix identity, analogously, we developed the Sherman-Morrison-Woodbury identity for tensors to facilitate the tensor inversion computation with those benefits in the matrix inversion computation when the correction of the original tensors is required. We first established the Sherman-Morrison-Woodbury identity for invertible tensors. Furthermore, we generalized the Sherman-Morrison-Woodbury identity for tensor with Moore-Penrose inverse by using orthogonal projection of the correction tensor part into the original tensor and its Hermitian tensor. Finally, we applied the Sherman-Morrison-Woodbury identity to characterize the error bound for the solution of a multilinear system between the original system and the corrected system, i.e., the coefficient tensors are corrected by other tensors with same dimensions.

There are several possible future works that can be extended based on current work. Because we can quantify the normalized error bound with respect to perturbation values and the Frobenius norm of the coefficient tensor, the next question is how to design a robust multilinear system to have the minimum normalized solution error given perturbation values. Such robust design should be crucial in many engineering problems which are modeled by multilinear systems. We have to decompose the perturbed tensor in the Eq. (5.10) in order to apply our result, similar to the matrix case, how can we select low rank decomposition for the perturbed tensor is the second direction for the future research. Since we have developed a new Sherman-Morrison-Woodbury identity for tensor, it will be interested in finding more impactful applications based on this new identity. We expect this new identity will shed light on the development of more efficient tensor-based calculations in the near future.

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