Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 18, No. 2, pp. 281–297, 2022

TIGHTER EIGENVALUE LOCALIZATION SETS FOR FOURTH-ORDER PARTIALLY SYMMETRIC TENSOR AND ITS APPLICATIONS*

Kaiping Liu and Haitao Che†

Abstract: M-eigenvalues of the fourth-order partially symmetric tensors play an important role in the analysis of nonlinear elastic materials. In this paper, M-identity tensor is introduced to establish some new eigenvalue localization sets for fourth-order partially symmetric tensor. It is revealed that the new eigenvalue localization sets are tighter than some existing results. Numerical examples demonstrate the effectiveness of the results obtained. As applications, some bound estimations for the M-spectral radius and some checkable sufficient conditions for the positive definiteness of the fourth-order partially symmetric tensor are obtained based on the new eigenvalue localization sets.

Key words: *M-eigenvalue, fourth-order partially symmetric tensors, M-positive definiteness*

Mathematics Subject Classification: *15A06, 74B20, 47J25*

1 Introduction

Considering the following homogeneous polynomial optimization problems

$$\begin{cases} \max f(\mathbf{x},\mathbf{y}) = \mathcal{A}\mathbf{xxyy} = \sum\limits_{i,k\in[m]}\sum\limits_{j,l\in[n]} a_{ijkl}x_ix_ky_jy_l, \\ \text{s.t. } \ \mathbf{x}^{\mathrm{T}}\mathbf{x} = 1,\ \mathbf{y}^{\mathrm{T}}\mathbf{y} = 1, \\ \qquad \mathbf{x}\in\mathbb{R}^m,\ \mathbf{y}\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where $[m] = \{1, 2, \ldots, m\}$ and $[n] = \{1, 2, \ldots, n\}$, the coefficients a_{ijkl} are invariant under the following property

$$a_{ijkl} = a_{kjil} = a_{ilkj} = a_{klij},\ i,k\in[m], j,l\in[n].$$

In this sense, $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ is called a fourth-order partially symmetric tensor. To obtain the optimal solution of (1.1), Han et al. [4] introduced the following definition.

*This work was supported by the Natural Science Foundation of China (No. 11401438, 11601261, 11571120) and Shandong Provincial Natural Science Foundation (ZR2019MA022, ZR2020MA027).

†Corresponding author

Definition 1.1. Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric real tensor. If there is $\lambda \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^m,\ \mathbf{y} \in \mathbb{R}^n$, such that

$$\begin{cases} \mathscr{A} \cdot yxy = \lambda x, \\ \mathscr{A} xyx\cdot = \lambda y, \\ \mathbf{x}^T\mathbf{x} = 1, \\ \mathbf{y}^T\mathbf{y} = 1, \end{cases}$$

where

$$(\mathscr{A} \cdot yxy)_i = \sum_{k\in[m]} \sum_{j,l\in[n]} a_{ijkl} y_j x_k y_l,$$

$$(\mathscr{A} xyx\cdot)_l = \sum_{i,k\in[m]} \sum_{j\in[n]} a_{ijkl} x_i y_j x_k,$$

then λ is called an M-eigenvalue of $\mathscr{A}$, and $\mathbf{x},\ \mathbf{y}$ is called the left and right eigenvectors with respect to the M-eigenvalues.

Denote $\sigma_M(\mathscr{A})$ as the set of all M-eigenvalues of $\mathscr{A}$. Then, the M-spectral radius of $\mathscr{A}$ is denoted by $\rho_M(\mathscr{A}) = max\{|\lambda| : \lambda \in \sigma_M(\mathscr{A})\}$. Note that $f_{\mathscr{A}}(x, y)$ is positive definite if and only if M-eigenvalues of $\mathscr{A}$ are positive [10]. There are some works on fourth-order partially symmetric tensor have been implemented [2,3,7,12–14,16–18]. Due to the complexity of the tensor eigenvalue problem [6, 9], it is not easy to calculate all M-eigenvalues. Hence, some researchers turned to investigating the inclusion sets of M-eigenvalue, Che et al. [1] proposed a Gershgorin-type M-inclusion set for fourth-order partially symmetric tensors. He et al. [5] presented some new M-eigenvalue inclusion sets for fourth-order partially symmetric tensors and gave some veritable sufficient conditions of the M-positive definiteness.

In this paper, we present some M-eigenvalue localization sets with m parameter to locate all M-eigenvalues of fourth-order partially symmetric tensors. It is revealed that the new eigenvalue localization sets are tighter than some existing results [5, 8]. Numerical examples demonstrate the effectiveness of the results obtained. As applications, we present some checkable sufficient conditions for the positive definiteness and establish some bound estimations for the M-spectral radius of the fourth-order partially symmetric tensor.

2 Tighter Eigenvalue Localization Sets

In this section, tighter eigenvalue localization sets with parameter for fourth-order partially symmetric tensors are demonstrated.

To continue, we need the following definitions and technical results.

Theorem 2.1 ([8]). *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor. If λ is an M-eigenvalue of $\mathscr{A}$, then*

$$\lambda \in \Gamma_1(\mathscr{A}) = \{z \in \mathbb{R} : |z| \le \min\{\max_{1\le i\le m}\{R_i(\mathscr{A})\},\ \max_{1\le l\le n}\{C_l(\mathscr{A})\}\}\},$$

where

$$R_i(\mathscr{A}) = \sum_{k=1}^{m} \sum_{j,l=1}^{n} |a_{ijkl}|,\ C_l(\mathscr{A}) = \sum_{i,k=1}^{m} \sum_{j=1}^{n} |a_{ijkl}|.$$

Theorem 2.2 ([8]). *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor. If λ is an M-eigenvalue of $\mathscr{A}$, then*

$$\lambda \in \Theta(\mathscr{A}) = U(\mathscr{A}) \bigcap V(\mathscr{A}),$$

where

$$U(\mathscr{A}) = \bigcup_{s,p=1,s\neq p}^{m} \{z \in \mathbb{R} : (|z| - \sum_{j,l=1}^{n} |a_{pjpl}|)|z| \leq (R_p(\mathscr{A}) - \sum_{j,l=1}^{n} |a_{pjpl}|)R_s(\mathscr{A})\},$$

and

$$V(\mathscr{A}) = \bigcup_{t,q=1,t\neq q}^{n} \{z \in \mathbb{R} : (|z| - \sum_{i,k=1}^{m} |a_{iqkq}|)|z| \leq (C_q(\mathscr{A}) - \sum_{i,k=1}^{m} |a_{iqkq}|)C_t(\mathscr{A})\}.$$

Theorem 2.3 ([5]). *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor. If λ is an M-eigenvalue of $\mathscr{A}$, then*

$$\lambda \in \Delta(\mathscr{A}) = \Delta_1(\mathscr{A}) \bigcap \Delta_2(\mathscr{A}),$$

where

$$\Delta_1(\mathscr{A}) = (\bigcup_{i,k\in[m],i\neq k} \{z \in \mathbb{R} : (|z| - r_i^i(\mathscr{A}))(|z| - r_k^k(\mathscr{A})) \leq r_i(\mathscr{A})r_k(\mathscr{A})\}) \bigcup (\bigcup_{i\in[m]} \{z \in \mathbb{R} : |z| \leq r_i^i(\mathscr{A})\}),$$

$$\Delta_2(\mathscr{A}) = (\textstyle\bigcup_{j,l\in[n],j\neq l}\{z \in \mathbb{R} : (|z| - c_l^l(\mathscr{A}))(|z| - c_j^j(\mathscr{A})) \leq c_l(\mathscr{A})c_j(\mathscr{A})\}) \bigcup (\bigcup_{l\in[n]} \{z \in \mathbb{R} : |z| \leq c_l^l(\mathscr{A})\}),$$

$$r_i^i(\mathscr{A}) = \sum_{j,l=1,j\neq l}^{n} |a_{ijil}| + \tau_i,\ \ \tau_i = \max_{l\in[n]}\{|a_{ilil}|\},$$

$$r_i(\mathscr{A}) = \sum_{k=1,k\neq i}^{m} \sum_{j,l=1,j\neq l}^{n} |a_{ijkl}| + \eta_i,\ \ \eta_i = \max_{l\in[n]}\{\sum_{k=1,k\neq i}^{m} |a_{ilkl}|\},$$

$$c_l^l(\mathscr{A}) = \sum_{i,k=1,i\neq k}^{m} |a_{ilkl}| + \nu_l,\ \ \nu_l = \max_{i\in[m]}\{|a_{ilil}|\},$$

$$c_l(\mathscr{A}) = \sum_{i,k=1,i\neq k}^{m} \sum_{j=1,j\neq l}^{n} |a_{ijkl}| + \varsigma_l,\ \ \varsigma_l = \max_{i\in[m]}\{\sum_{j=1,j\neq l}^{m} |a_{ijil}|\}.$$

Motivated by [11], we begin our work by introducing the definition of M-identity tensor.

Definition 2.4. We call $\mathscr{F}_M \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ an M-identity tensor, if its entries are

$$(\mathscr{F}_M)_{ijkl} = \begin{cases} 1, & \text{if } i = k, j = l, \\ 0, & \textit{otherwise}, \end{cases}$$

where $i, k \in [m],\ j, l \in [n]$.

Now we are in a position to exhibit our results.

Theorem 2.5. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor, and $\mathscr{F}_M$ an M-identity tensor. For any $\alpha = (\alpha_1, \dots, \alpha_m)^T \in \mathbb{R}^m$ and $\beta = (\beta_1, \dots, \beta_n)^T \in \mathbb{R}^n$, if λ is the M-eigenvalue of $\mathscr{A}$, then*

$$\lambda \in \Gamma(\mathscr{A}) = \{z \in \mathbb{R} : |z - b_\pi| \le G_\pi(\mathscr{A})\},$$

where

$$G_\pi(\mathscr{A}) = \min\{\max_{1\le i\le m}\{\mathrm{D_i}(\mathscr{A})\},\ \ \max_{1\le l\le n}\{\mathrm{F_l}(\mathscr{A})\}\},$$

$$b = \{\ \alpha, \beta\ \ |\ \ \min\{\max_{1\le i\le m}\{\mathrm{D_i}(\mathscr{A})\},\ \ \max_{1\le l\le n}\{\mathrm{F_l}(\mathscr{A})\}\}\},$$

$$\pi = \{\ i, l\ \ |\ \ \min\{\max_{1\le i\le m}\{\mathrm{D_i}(\mathscr{A})\},\ \ \max_{1\le l\le n}\{\mathrm{F_l}(\mathscr{A})\}\}\},$$

$$D_i(\mathscr{A}) = \sum_{k=1}^{m}\sum_{j,l=1}^{n} |a_{ijkl} - \alpha_i(\mathscr{F}_M)_{ijkl}|,$$

$$F_l(\mathscr{A}) = \sum_{i,k=1}^{m}\sum_{j=1}^{n} |a_{ijkl} - \beta_l(\mathscr{F}_M)_{ijkl}|.$$

Proof. Assume that λ is an M-eigenvalue of $\mathscr{A}$, $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$ are the corresponding nonzero left and right M-eigenvectors. Let

$$|x_t| = \max_{1\le i\le m}\{|x_i|\},\ |y_s| = \max_{1\le l\le n}\{|y_l|\}.$$

Since $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{y}^T\mathbf{y} = 1$, one has

$$\mathscr{A} \cdot yxy = \lambda x = \lambda(\mathscr{F}_M) \cdot yxy,\ \mathscr{A}xyx\cdot = \lambda y = \lambda(\mathscr{F}_M)xyx\,\cdot.$$

Noting the tth equation of $\mathscr{A} \cdot yxy = \lambda x = \lambda(\mathscr{F}_M) \cdot yxy$, it yields

$$\sum_{k=1}^{m}\sum_{j,l=1}^{n} \lambda(\mathscr{F}_M)_{tjkl} y_j x_k y_l = \sum_{k=1}^{m}\sum_{j,l=1}^{n} a_{tjkl} y_j x_k y_l.$$

Hence, for any α_t, it follows that

$$\begin{aligned}(\lambda - \alpha_t)x_t &= \sum_{k\in[m],j,l\in[n]} (\lambda - \alpha_t)(\mathscr{F}_M)_{tjkl} y_j x_k y_l \\ &= \sum_{k\in[m],j,l\in[n]} (a_{tjkl} - \alpha_t(\mathscr{F}_M)_{tjkl}) y_j x_k y_l.\end{aligned} \tag{2.1}$$

From (2.1), we obtain

$$\begin{aligned}|\lambda - \alpha_t||x_t| &= |\sum_{k\in[m],j,l\in[n]} (\lambda - \alpha_t)(\mathscr{F}_M)_{tjkl} y_j x_k y_l| \\ &\le \sum_{k\in[m],j,l\in[n]} |a_{tjkl} - \alpha_t(\mathscr{F}_M)_{tjkl}||x_t|.\end{aligned}$$

Furthermore,

$$|\lambda - \alpha_t| \le \sum_{k\in[m],j,l\in[n]} |a_{tjkl} - \alpha_t(\mathscr{F}_M)_{tjkl}| = D_t(\mathscr{A}) \le \max_{1\le i\le m}\{D_i(\mathscr{A})\}. \tag{2.2}$$

Similarly, from

$$\sum_{i,k=1}^{m}\sum_{j=1}^{n}\lambda(\mathscr{F}_M)_{ijks}x_iy_jx_k = \sum_{i,k=1}^{m}\sum_{j=1}^{n}a_{ijks}x_iy_jx_k,$$

for any β_s, we have

$$\begin{aligned}(\lambda-\beta_s)y_s &= \sum_{i,k\in[m],j\in[n]}(\lambda-\beta_s)(\mathscr{F}_M)_{ijks}x_iy_jx_k\\ &= \sum_{i,k\in[m],j\in[n]}(a_{ijks}-\beta_s(\mathscr{F}_M)_{ijks})x_iy_jx_k.\end{aligned}$$

Moreover,

$$\begin{aligned}|\lambda-\beta_s||y_s| &= |\sum_{i,k\in[m],j\in[n]}(\lambda-\beta_s)(\mathscr{F}_M)_{ijks}x_iy_jx_k|\\ &\le \sum_{i,k\in[m],j\in[n]}|a_{ijks}-\beta_s(\mathscr{F}_M)_{ijks}||y_s|,\end{aligned}$$

which means

$$|\lambda-\beta_s| \le \sum_{i,k\in[m],j\in[n]}|a_{ijks}-\beta_s(\mathscr{F}_M)_{ijks}| = F_s(\mathscr{A}) \le \max_{1\le l\le n}\{F_l(\mathscr{A})\}. \tag{2.3}$$

From (2.2) and (2.3), we have

$$|z-b_\pi| \le G_\pi(\mathscr{A}),$$

which implies that $\lambda\in\Gamma(\mathscr{A})$, and the desired result holds. □

Theorem 2.6. *Let $\mathscr{A}=(a_{ijkl})\in\mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor, and $\mathscr{F}_M$ an M-identity tensor. For any $\alpha=(\alpha_1,\dots,\alpha_m)^T\in\mathbb{R}^m$ and $\beta=(\beta_1,\dots,\beta_n)^T\in\mathbb{R}^n$, if λ is the M-eigenvalue of $\mathscr{A}$, then*

$$\lambda\in\Lambda(\mathscr{A}) = U_1(\mathscr{A})\bigcap U_2(\mathscr{A}),$$

where

$$U_1(\mathscr{A}) = \bigcup_{s\ne p,s,p=1}^{m}\{z\in\mathbb{R} : (|z-\alpha_p|-D_p^p(\mathscr{A}))|z-\alpha_s| \le (D_p(\mathscr{A})-D_p^p(\mathscr{A}))D_s(\mathscr{A})\},$$

$$U_2(\mathscr{A}) = \bigcup_{t\ne q,t,q=1}^{n}\{z\in\mathbb{R} : (|z-\beta_q|-F_q^q(\mathscr{A}))|z-\beta_t| \le (F_q(\mathscr{A})-F_q^q(\mathscr{A}))F_t(\mathscr{A})\},$$

$$D_p^p(\mathscr{A}) = \sum_{j,l=1}^{n}|a_{pjpl}-\alpha_p(\mathscr{F}_M)_{pjpl}|,$$

$$F_q^q(\mathscr{A}) = \sum_{i,k=1}^{m}|a_{iqkq}-\beta_q(\mathscr{F}_M)_{iqkq}|.$$

Proof. Assume that λ is an M-eigenvalue of $\mathscr{A}$, $\mathbf{x}\in\mathbb{R}^m$ and $\mathbf{y}\in\mathbb{R}^n$ are the corresponding nonzero left and right M-eigenvectors. Let

$$|x_p|\ge|x_s|\ge\max_{k\ne p,s,1\le k\le m}\{|x_k|\},\ \ |y_q|\ge|y_t|\ge\max_{k\ne q,t,1\le k\le n}\{|y_k|\}.$$

Since $\mathscr{A}\cdot yxy=\lambda x=\lambda(\mathscr{F}_M)\cdot yxy,\ \mathscr{A}xyx\cdot=\lambda y=\lambda(\mathscr{F}_M)xyx\cdot$,

$$\begin{aligned}(\lambda-\alpha_p)x_p &= \sum_{k\in[m],j,l\in[n]}(\lambda-\alpha_p)(\mathscr{F}_M)_{pjkl}y_jx_ky_l\\ &= \sum_{j,l\in[n]}(\lambda-\alpha_p)(\mathscr{F}_M)_{pjpl}y_jx_py_l\\ &+ \sum_{k\neq p,k\in[m]}\sum_{j,l\in[n]}(\lambda-\alpha_p)(\mathscr{F}_M)_{pjkl}y_jx_ky_l\\ &= \sum_{j,l\in[n]}(a_{pjpl}-\alpha_p(\mathscr{F}_M)_{pjpl})y_jx_py_l\\ &+ \sum_{k\neq p,k\in[m]}\sum_{j,l\in[n]}(a_{pjkl}-\alpha_p(\mathscr{F}_M)_{pjkl})y_jx_ky_l,\end{aligned}$$

which implies

$$|\lambda-\alpha_p||x_p|\leq\sum_{j,l\in[n]}|a_{pjpl}-\alpha_p(\mathscr{F}_M)_{pjpl})||x_p|+(D_p(\mathscr{A})-D_p^p(\mathscr{A}))|x_s|.$$

Therefore, we have

$$(|\lambda-\alpha_p|-D_p^p(\mathscr{A}))|x_p|\leq(D_p(\mathscr{A})-D_p^p(\mathscr{A}))|x_s|. \tag{2.4}$$

From Theorem 2.1, one has

$$|\lambda-\alpha_s||x_s|\leq D_s(\mathscr{A})|x_p|. \tag{2.5}$$

If $|x_s|>0$, multiplying (2.4) with (2.5), we have

$$(|\lambda-\alpha_p|-D_p^p(\mathscr{A}))|\lambda-\alpha_s|\leq(D_p(\mathscr{A})-D_p^p(\mathscr{A}))D_s(\mathscr{A}).$$

If $|x_s|=0$, then $|\lambda-\alpha_p|-D_p^p(\mathscr{A})\leq 0$. Therefore, we have $\lambda\in U_1(\mathscr{A})$.

Similarly, for any β_q, we obtain

$$|\lambda-\beta_q||y_q|\leq\sum_{i,k\in[m]}|a_{iqkq}-\beta_q(\mathscr{F}_M)_{iqkq})||y_q|+(F_q(\mathscr{A})-F_q^q(\mathscr{A}))|y_t|.$$

Therefore, we have

$$(|\lambda-\beta_q|-F_q^q(\mathscr{A}))|y_q|\leq(F_q(\mathscr{A})-F_q^q(\mathscr{A}))|y_t|. \tag{2.6}$$

From Theorem 2.1, it holds

$$|\lambda-\beta_t||y_t|\leq F_t(\mathscr{A})|y_q|. \tag{2.7}$$

If $|y_t|>0$, multiplying (2.6) with (2.7), we have

$$(|\lambda-\beta_t|-F_q^q(\mathscr{A}))|\lambda-\beta_t|\leq(F_q(\mathscr{A})-F_q^q(\mathscr{A}))F_t(\mathscr{A}).$$

If $|y_t|=0$, then $|\lambda-\beta_q|-F_q^q(\mathscr{A})\leq 0$. Therefore, we have $\lambda\in U_2(\mathscr{A})$.

From $\lambda\in U_1(\mathscr{A})$ and $\lambda\in U_2(\mathscr{A})$, we obtain $\lambda\in U_1(\mathscr{A})\bigcap U_2(\mathscr{A})$, and the desired result follows. □

Theorem 2.7. *Let $\mathscr{A}=(a_{ijkl})\in\mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor, and $\mathscr{F}_M$ an M-identity tensor. For any $\alpha=(\alpha_1,\ldots,\alpha_m)^T\in\mathbb{R}^m$ and $\beta=(\beta_1,\ldots,\beta_n)^T\in\mathbb{R}^n$, if λ is the M-eigenvalue of $\mathscr{A}$, then*

$$\lambda\in\Phi(\mathscr{A})=\Phi_1(\mathscr{A})\bigcap\Phi_2(\mathscr{A}),$$

where

$$\Phi_1(\mathscr{A}) \quad = (\bigcup_{i,k\in[m],i\neq k} \{z \in \mathbb{R} : (|z-\alpha_i| - X_i^i(\mathscr{A}))(|z-\alpha_k| - X_k^k(\mathscr{A})) \leq X_i(\mathscr{A})X_k(\mathscr{A})\}) \bigcup (\bigcup_{i\in[m]} \{z \in \mathbb{R} : |z-\alpha_i| \leq X_i^i(\mathscr{A})\}),$$

$$\Phi_2(\mathscr{A}) \quad = (\bigcup_{j,l\in[n],j\neq l} \{z \in \mathbb{R} : (|z-\beta_l| - Y_l^l(\mathscr{A}))(|z-\beta_j| - Y_j^j(\mathscr{A})) \leq Y_l(\mathscr{A})Y_j(\mathscr{A})\} \bigcup (\bigcup_{l\in[n]} \{z \in \mathbb{R} : |z-\beta_l| \leq Y_l^l(\mathscr{A})\}),$$

$$X_i^i(\mathscr{A}) = \sum_{j,l=1,j\neq l}^{n} |a_{ijil}| + \varepsilon_i,\ \ \varepsilon_i = \max_{l\in[n]}\{|a_{ilil} - \alpha_i(\mathscr{F})_{ilil}|\},$$

$$X_i(\mathscr{A}) = \sum_{k=1,k\neq i}^{m} \sum_{j,l=1,j\neq l}^{n} |a_{ijkl}| + \xi_i,\ \ \xi_i = \max_{l\in[n]}\{\sum_{k=1,k\neq i}^{m} |a_{ilkl}|\},$$

$$Y_l^l(\mathscr{A}) = \sum_{i,k=1,i\neq k}^{m} |a_{ilkl}| + \delta_l,\ \ \delta_l = \max_{i\in[m]}\{|a_{ilil} - \beta_l(\mathscr{F})_{ilil}|\},$$

$$Y_l(\mathscr{A}) = \sum_{i,k=1,i\neq k}^{m} \sum_{j=1,j\neq l}^{n} |a_{ijkl}| + \omega_l,\ \ \omega_l = \max_{i\in[m]}\{\sum_{j=1,j\neq l}^{m} |a_{ijil}|\}.$$

Proof. Assume that λ is an M-eigenvalue of $\mathscr{A}$, $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$ are the corresponding nonzero left and right M-eigenvectors. Let

$$|x_p| \geq |x_s| \geq \max_{k\neq p,s,1\leq k\leq m}\{|x_k|\},\ \ |y_q| \geq |y_t| \geq \max_{k\neq q,t,1\leq k\leq n}\{|y_k|\}.$$

For any α_p, it follows from

$$\mathscr{A} \cdot yxy = \lambda x = \lambda \mathscr{F}_M \cdot yxy,\ \ \mathscr{A}xyx\cdot = \lambda y = \lambda \mathscr{F}_M xyx\cdot$$

that

$$\begin{aligned}(\lambda-\alpha_p)x_p &= \sum_{k\in[m],j,l\in[n]} (\lambda-\alpha_p)(\mathscr{F}_M)_{pjkl} y_j x_k y_l \\ &= \sum_{k=1}^{m} \sum_{j,l=1,j\neq l}^{n} (\lambda-\alpha_p)(\mathscr{F}_M)_{pjkl} y_j x_k y_l + \sum_{k=1}^{m}\sum_{l=1}^{n} (\lambda-\alpha_p)(\mathscr{F}_M)_{plkl} x_k y_l^2 \\ &= \sum_{k=1}^{m} \sum_{j,l=1,j\neq l}^{n} (a_{pjkl} - \alpha_p(\mathscr{F}_M)_{pjkl}) y_j x_k y_l + \sum_{k=1}^{m}\sum_{l=1}^{n} (a_{plkl} - \alpha_p(\mathscr{F}_M)_{plkl}) x_k y_l^2.\end{aligned} \tag{2.8}$$

From (2.8), it reduces that

$$
\begin{aligned}
|\lambda-\alpha_p||x_p| \leq & \sum_{k=1}^{m}\sum_{j,l=1,j\neq l}^{n}|a_{pjkl}-\alpha_p(\mathscr{F}_M)_{pjkl}||y_j||x_k||y_l| \\
& +\sum_{k=1}^{m}\sum_{l=1}^{n}|a_{plkl}-\alpha_p(\mathscr{F}_M)_{plkl}||x_k||y_l^2| \\
& +\sum_{l=1}^{n}|a_{plpl}-\alpha_p(\mathscr{F}_M)_{plpl}||x_p||y_l^2| \\
& +\sum_{k=1,k\neq p}^{m}\sum_{l=1}^{n}|a_{plkl}-\alpha_p(\mathscr{F}_M)_{plkl}||x_k||y_l^2| \\
\leq & \Big(\sum_{j,l=1,j\neq l}^{n}|a_{pjpl}-\alpha_p(\mathscr{F}_M)_{pjpl}| \\
& +\sum_{l=1}^{n}|a_{plpl}-\alpha_p(\mathscr{F}_M)_{plpl}|\Big)|x_p| \\
& +\Big(\sum_{k=1,k\neq p}^{m}\sum_{j,l=1,j\neq l}^{n}|a_{pjkl}-\alpha_p(\mathscr{F}_M)_{pjkl}| \\
& +\sum_{k=1,k\neq p}^{m}\sum_{l=1}^{n}|a_{plkl}-\alpha_p(\mathscr{F}_M)_{plkl}|\Big)|x_s| \\
\leq & X_p^p(\mathscr{A})|x_p|+X_p(\mathscr{A})|x_s|.
\end{aligned} \tag{2.9}
$$

Then, (2.9) can be rewritten as

$$
(|\lambda-\alpha_p|-X_p^p(\mathscr{A}))|x_p| \leq X_p(\mathscr{A})|x_s|. \tag{2.10}
$$

Meanwhile,

$$
(|\lambda-\alpha_s|-X_s^s(\mathscr{A}))|x_s| \leq X_s(\mathscr{A})|x_p|. \tag{2.11}
$$

If $|\lambda-\alpha_p|-X_p^p(\mathscr{A}) \leq 0$, then $|\lambda-\alpha_p| \leq X_p^p(\mathscr{A})$. If $|\lambda-\alpha_p|-X_p^p(\mathscr{A}) > 0$, multiplying (2.10) with (2.11), then we have

$$
(|\lambda-\alpha_p|-X_p^p(\mathscr{A}))(|\lambda-\alpha_s|-X_s^s(\mathscr{A})) \leq X_p(\mathscr{A})X_s(\mathscr{A}).
$$

Thus, we obtain $\lambda \in \Phi_1(\mathscr{A})$.

Next, we will prove $\lambda \in \Phi_2(\mathscr{A})$. Similarly,

$$
\begin{aligned}
|\lambda-\beta_q||y_q| \leq & \sum_{i,k=1,i\neq k}^{m}|a_{iqkq}-\beta_q(\mathscr{F}_M)_{iqkq}||x_i||y_q||x_k| \\
& +\sum_{i,k=1,i\neq k}^{m}\sum_{j=1,j\neq q}^{n}|a_{ijkq}-\beta_q(\mathscr{F}_M)_{ijkq}||x_i||y_j||x_k| \\
& +\sum_{i=1}^{m}|a_{iqiq}-\beta_q(\mathscr{F}_M)_{iqiq}||x_i^2||y_q|+\sum_{i=1}^{m}\sum_{j=1,j\neq q}^{n}|a_{ijiq}-\beta_q(\mathscr{F}_M)_{ijiq}||x_i^2||y_j| \\
& \leq Y_q^q(\mathscr{A})|y_q|+Y_q(\mathscr{A})|y_t|.
\end{aligned} \tag{2.12}
$$

(2.12) means

$$(|\lambda - \beta_q| - Y_q^q(\mathscr{A}))|y_q| \le Y_q(\mathscr{A})|y_t|. \tag{2.13}$$

Meanwhile,

$$(|\lambda - \beta_t| - Y_t^t(\mathscr{A}))|y_t| \le Y_t(\mathscr{A})|y_q|. \tag{2.14}$$

If $|\lambda - \beta_q| - Y_q^q(\mathscr{A}) \le 0$, then $|\lambda - \beta_q| \le Y_q^q(\mathscr{A})$. If $|\lambda - \beta_q| - Y_q^q(\mathscr{A}) > 0$, multiplying (2.13) with (2.14), then we have

$$(|\lambda - \beta_q| - Y_q^q(\mathscr{A}))(|\lambda - \beta_t| - Y_t^t(\mathscr{A})) \le Y_q(\mathscr{A})Y_t(\mathscr{A}).$$

Thus, we obtain $\lambda \in \Phi_2(\mathscr{A})$. Furthermore, $\lambda \in \Phi(\mathscr{A}) = \Phi_1(\mathscr{A}) \bigcap \Phi_2(\mathscr{A})$. And the proof is completed. □

The following conclusion exhibits the relationship between $\Phi(\mathscr{A})$, $\Lambda(\mathscr{A})$ and $\Gamma(\mathscr{A})$.

Corollary 2.8. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor, and $\mathscr{F}_M$ an M-identity tensor. For any $\alpha = (\alpha_1, \ldots, \alpha_m)^T \in \mathbb{R}^m$ and $\beta = (\beta_1, \ldots, \beta_n)^T \in \mathbb{R}^n$, if λ is the M-eigenvalue of $\mathscr{A}$, then $\lambda \in \Phi(\mathscr{A}) \subseteq \Lambda(\mathscr{A}) \subseteq \Gamma(\mathscr{A})$.*

Proof. By Theorem 2.5 and Theorem 2.6, it is sufficient to prove $\Lambda(\mathscr{A}) \subseteq \Gamma(\mathscr{A})$. For any $\lambda \in \Lambda(\mathscr{A})$, without loss of generality, there exists $t \in [m]$, such that $\lambda \in U_1(\mathscr{A})$, for all $s \neq t$, $(|\lambda - \alpha_t| - D_t^t(\mathscr{A}))|\lambda - \alpha_s| \le (D_t(\mathscr{A}) - D_t^t(\mathscr{A}))D_s(\mathscr{A})$. We now break up the argument into two cases.

Case 1. If $(D_t(\mathscr{A}) - D_t^t(\mathscr{A}))D_s(\mathscr{A}) = 0$, then

$$(|\lambda - \alpha_t| - D_t^t(\mathscr{A}))|\lambda - \alpha_s| \le 0.$$

Hence, we have $|\lambda - \alpha_t| \le D_t^t(\mathscr{A})$ or $\lambda = \alpha_s$.
Case 2. If $(D_t(\mathscr{A}) - D_t^t(\mathscr{A}))D_s(\mathscr{A}) > 0$, then

$$\frac{|\lambda - \alpha_t| - D_t^t(\mathscr{A})}{D_t(\mathscr{A}) - D_t^t(\mathscr{A})} \cdot \frac{|\lambda - \alpha_s|}{D_s(\mathscr{A})} \le 1,$$

which implies that $\frac{|\lambda - \alpha_t| - D_t^t(\mathscr{A})}{D_t(\mathscr{A}) - D_t^t(\mathscr{A})} \le 1$ or $\frac{|\lambda - \alpha_s|}{D_s(\mathscr{A})} \le 1$.
If there exists $q \in [n]$, such that $\lambda \in U_2(\mathscr{A})$, for all $p \neq q$, $(|\lambda - \beta_q| - F_q^q(\mathscr{A}))|\lambda - \beta_p| \le (F_q(\mathscr{A}) - F_q^q(\mathscr{A}))F_p(\mathscr{A})$. Similarly, we break up the argument into two cases.
Case 1. If $(F_q(\mathscr{A}) - F_q^q(\mathscr{A}))F_p(\mathscr{A}) = 0$, then

$$(|\lambda - \beta_q| - F_q^q(\mathscr{A}))|\lambda - \beta_p| \le 0.$$

Hence, we have $|\lambda - \beta_q| \le F_q^q(\mathscr{A})$ or $\lambda = \beta_p$.
Case 2. If $(F_q(\mathscr{A}) - F_q^q(\mathscr{A}))F_p(\mathscr{A}) > 0$, then

$$\frac{|\lambda - \beta_q| - F_q^q(\mathscr{A})}{F_q(\mathscr{A}) - F_q^q(\mathscr{A})} \cdot \frac{|\lambda - \beta_p|}{F_p(\mathscr{A})} \le 1,$$

which implies that $\frac{|\lambda - \beta_q| - F_q^q(\mathscr{A})}{F_q(\mathscr{A}) - F_q^q(\mathscr{A})} \le 1$ or $\frac{|\lambda - \beta_p|}{F_p(\mathscr{A})} \le 1$.
Hence, we have $\lambda \in \Gamma(\mathscr{A})$.

Next, we prove $\Phi(\mathscr{A}) \subseteq \Lambda(\mathscr{A})$. For any $\lambda \in \Lambda(\mathscr{A})$, we have $\lambda \in \Phi_1(\mathscr{A}) \cap \Phi_2(\mathscr{A})$. If $\lambda \in \Phi_1(\mathscr{A})$, then $|\lambda - \alpha_p| \le X_p^p(\mathscr{A})$, or $(|\lambda - \alpha_p| - X_p^p(\mathscr{A}))(|\lambda - \alpha_k| - X_k^k(\mathscr{A})) \le X_p(\mathscr{A})X_k(\mathscr{A})$. We break up the argument into two cases.

Case 1. From $|\lambda-\alpha_p|\le X_p^p(\mathscr{A})$, then

$$X_p^p(\mathscr{A})\le\sum_{p,k=1}^{m}|a_{plkl}-\alpha_p(\mathscr{F}_M)_{plkl}|,$$

which means

$$|\lambda-\alpha_p|\le\sum_{p,k=1}^{m}|a_{plkl}-\alpha_p(\mathscr{F}_M)_{plkl}|.$$

Then,

$$(|\lambda-\alpha_p|-D_p^p(\mathscr{A}))|\lambda-\alpha_s|\le 0\le(D_p(\mathscr{A})-D_p^p(\mathscr{A}))D_s(\mathscr{A}).$$

Hence, we have $\Phi_1(\mathscr{A})\subseteq U_1(\mathscr{A})$.

Case 2. From $|\lambda-\alpha_p|>X_p^p(\mathscr{A}),X_p(\mathscr{A})X_k(\mathscr{A})\neq 0$, one has

$$\frac{|\lambda-\alpha_p|-X_p^p(\mathscr{A})}{X_p(\mathscr{A})}\cdot\frac{|\lambda-\alpha_k|-X_k^k(\mathscr{A})}{X_k(\mathscr{A})}\le 1.$$

From (2.10), we have $\frac{|\lambda-\alpha_p|-X_p^p(\mathscr{A})}{X_p(\mathscr{A})}\le 1$. If $\frac{|\lambda-\alpha_k|-X_k^k(\mathscr{A})}{X_k(\mathscr{A})}\le 1$, then

$$|\lambda-\alpha_k|\le X_k^k(\mathscr{A})+X_k(\mathscr{A})\le D_k(\mathscr{A}). \tag{2.15}$$

$$|\lambda-\alpha_p|-D_p^p(\mathscr{A})\le X_p(\mathscr{A})\le D_p(\mathscr{A})-D_p^p(\mathscr{A}). \tag{2.16}$$

Multiplying (2.15) with (2.16), we obtain $\lambda\in U_1(\mathscr{A})$.

If $\frac{|\lambda-\alpha_k|-X_k^k(\mathscr{A})}{X_k(\mathscr{A})}>1$, then

$$\frac{|\lambda-\alpha_k|}{X_k^k(\mathscr{A})+X_k(\mathscr{A})}\le\frac{|\lambda-\alpha_k|-X_k^k(\mathscr{A})}{X_k(\mathscr{A})}.$$

Furthermore,

$$\begin{aligned}\frac{|\lambda-\alpha_p|-D_p^p(\mathscr{A})}{D_p(\mathscr{A})-D_p^p(\mathscr{A})}\cdot\frac{|\lambda-\alpha_k|}{D_k(\mathscr{A})}&\le\frac{|\lambda-\alpha_p|-X_p^p(\mathscr{A})}{X_p(\mathscr{A})}\cdot\frac{|\lambda-\alpha_k|}{X_k^k(\mathscr{A})+X_k(\mathscr{A})}\\&\le\frac{|\lambda-\alpha_p|-X_p^p(\mathscr{A})}{X_p(\mathscr{A})}\cdot\frac{|\lambda-\alpha_k|-X_k^k(\mathscr{A})}{X_k(\mathscr{A})}\\&\le 1,\end{aligned}$$

which implies that $\lambda\in U_1(\mathscr{A})$. Thus, $\Phi_1(\mathscr{A})\subseteq U_1(\mathscr{A})$. Similarly, if $\lambda\in\Phi_2(\mathscr{A})$, we also have $\Phi_2(\mathscr{A})\subseteq U_2(\mathscr{A})$. Hence, we have $\Phi(\mathscr{A})\subseteq\Lambda(\mathscr{A})$. The proof is completed. □

Remark 2.9. It is clear that Theorem 2.5 ,2.6 and Theorem 2.7 reduce to Theorem 2.1 ,2.2 and Theorem 2.3, if one takes $\alpha=\mathbf{0},\ \beta=\mathbf{0}$, respectively.

Now, we present the following examples to illustrate the M-eigenvalue inclusion sets in Theorem 2.1, 2.2, 2.3, 2.5, 2.6 and 2.7.

Example 2.10 ([11])**.** Consider the following partially symmetric tensor $\mathscr{A}=(a_{ijkl})\in(\mathbb{R})^{[2]\times[2]\times[2]\times[2]}$ defined by

$$a_{ijkl}=\begin{cases}a_{1111}=20,\ a_{1122}=a_{1221}=1,\ a_{1212}=8;\\ a_{2222}=10,\ a_{2112}=a_{2211}=1,\ a_{2121}=7;\\ a_{ijkl}=0,\ \ \textit{otherwise}.\end{cases}$$

The results are presented in the table, let $\alpha = \beta = (14,\ 8.5)^T$.

Table 1. The comparison of inclusion interval

Reference	Inclusion interval
Theorem 2.1	[-29,29]
Theorem 2.2	[-28.4081,28.4081]
Theorem 2.3	[-20,20.3852]
Theorem 2.5	[0,28]
Theorem 2.6	[0.7154,26.5539]
Theorem 2.7	[5.4385,20.3852]

By computation, its M-eigenvalues are 8, 20, 10, 8.9659, 7, 9.9815. Obviously, from Table 1, we obtain that the above results contain all the eigenvalues, and the new eigenvalue localization sets are tighter than some existing results [5, 8].

Example 2.11. Consider the following partially symmetric tensor $\mathscr{A} = (a_{ijkl}) \in (\mathbb{R})^{[2]\times[2]\times[2]\times[2]}$ defined by

$$a_{ijkl} = \begin{cases} a_{1111} = 2,\ a_{1211} = 3,\ a_{2111} = 6,\ a_{1121} = 6,\ a_{1112} = 3,\ a_{1212} = 2; \\ a_{2212} = 10,\ a_{1222} = 10,\ a_{2222} = 5; \\ a_{ijkl} = 0, \quad otherwise. \end{cases}$$

Let $\alpha = (2,\ 3)^T,\ \beta = (1,\ 3)^T$ the results are revealed in the Table 2.

Table 2. The comparison of inclusion interval

Reference	Inclusion interval
Theorem 2.1	[-26,26]
Theorem 2.2	[-24,24]
Theorem 2.3	[-16.3246,16.3246]
Theorem 2.5	[-20,24]
Theorem 2.6	[-19.1615,24]
Theorem 2.7	[-12.6119,16.3246]

By calculation, we can obtain that the M-eigenvalue of the fourth order partially symmetric tensor are $-7.6841,\ 13.8616,\ -4.2541,\ 6.6751$. From the above Table 2, it can be seen that our result range includes all eigenvalues and is more accurate.

Remark 2.12. If $\alpha = \beta = (2,\ 0)^T$, the result of Theorem 2.6 is $\lambda \in [-20.5853, 22.7617]$. Obviously, we can get different results by choosing different parameters α, we can find the appropriate parameter α to make the M-eigenvalue inclusion intervals for fourth-order partially symmetric tensor tighter.

3 Applications

As applications, we obtain some bound estimations on the spectral radius of a fourth-order partially symmetric tensor, and take these bounds as the parameter in the WQZ-algorithm [15] to show a more superior result. Finally, we introduce some checkable sufficient conditions for the M-positive definiteness of the fourth-order partially symmetric tensor.

From Theorem 2.5, 2.6 and 2.7, we propose three bound estimations on the spectral radius of nonnegative fourth-order partially symmetric tensors.

Corollary 3.1. *Let* $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric tensor, and* $\mathscr{F}_M$ *an M-identity tensor. Then, for real vector* $\alpha = (\alpha_1, \ldots, \alpha_m)^T \in \mathbb{R}^m$ *and* $\beta = (\beta_1, \ldots, \beta_n)^T \in \mathbb{R}^n$*, such that* $\rho(\mathscr{A}) \leq b_\pi + G_\pi(\mathscr{A})$.

Corollary 3.2. *Let* $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric tensor, and* $\mathscr{F}_M$ *an M-identity tensor. Then, for real vector* $\alpha = (\alpha_1, \ldots, \alpha_m)^T \in \mathbb{R}^m$ *and* $\beta = (\beta_1, \ldots, \beta_n)^T \in \mathbb{R}^n$*, such that* $\rho(\mathscr{A}) \leq min\{P_1(\mathscr{A}), P_2(\mathscr{A})\}$*,*
where

$$P_1(\mathscr{A}) = \max_{s,p\in[m],s\neq p}\{\frac{1}{2}(\alpha_p + D_p^p(\mathscr{A}) + \alpha_s + \Delta_{s,p}^{1/2})\},$$

$$\Delta_{s,p} = (\alpha_s + \alpha_p + D_p^p(\mathscr{A}))^2 - 4(\alpha_p\alpha_s + \alpha_s D_p^p(\mathscr{A}) + D_s(\mathscr{A})D_p^p(\mathscr{A}) - D_p(\mathscr{A})D_s(\mathscr{A})),$$

$$P_2(\mathscr{A}) = \max_{t,q\in[n],t\neq q}\{\frac{1}{2}(\beta_q + F_q^q(\mathscr{A}) + \beta_t + \Delta_{t,q}^{1/2})\},$$

$$\Delta_{t,q} = (\beta_t + \beta_q + F_q^q(\mathscr{A}))^2 - 4(\beta_q\beta_t + \beta_t F_q^q(\mathscr{A}) + F_t(\mathscr{A})F_q^q(\mathscr{A}) - F_q(\mathscr{A})F_t(\mathscr{A})).$$

Corollary 3.3. *Let* $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric tensor, and* $\mathscr{F}_M$ *an M-identity tensor. Then, for real vector* $\alpha = (\alpha_1, \ldots, \alpha_m)^T \in \mathbb{R}^m$ *and* $\beta = (\beta_1, \ldots, \beta_n)^T \in \mathbb{R}^n$*, such that* $\rho(\mathscr{A}) \leq min\{Q_1(\mathscr{A}), Q_2(\mathscr{A})\}$*,*
where

$$\begin{aligned}
Q_1(\mathscr{A}) &= \max_{i,k\in[m],k\neq i}\{\frac{1}{2}(\alpha_i + X_i^i(\mathscr{A}) + \alpha_k + X_k^k(\mathscr{A}) + \Delta_{i,k}^{1/2})\},\\
\Delta_{i,k} &= (\alpha_i + X_i^i(\mathscr{A}) + \alpha_k + X_k^k(\mathscr{A}))^2\\
&\quad - 4(\alpha_i\alpha_k + X_i^i(\mathscr{A})X_k^k(\mathscr{A}) + \alpha_k X_i^i(\mathscr{A}) + \alpha_i X_k^k(\mathscr{A}) - X_i(\mathscr{A})X_k(\mathscr{A})),\\
Q_2(\mathscr{A}) &= \max_{j,l\in[n],j\neq l}\{\frac{1}{2}(\beta_l + Y_l^l(\mathscr{A}) + \beta_j + Y_j^j(\mathscr{A}) + \Delta_{j,l}^{1/2})\},\\
\Delta_{j,l} &= (\beta_l + Y_l^l(\mathscr{A}) + \beta_j + Y_j^j(\mathscr{A}))^2\\
&\quad - 4(\beta_l\beta_j + Y_l^l(\mathscr{A})Y_j^j(\mathscr{A}) + \beta_l Y_j^j(\mathscr{A}) + \beta_j Y_l^l(\mathscr{A}) - Y_l(\mathscr{A})Y_j(\mathscr{A})).
\end{aligned}$$

Now, we give an example to show a more superior result by choosing these bounds as the parameter in WQZ-algorithm [15].

Example 3.4. Let the fourth-order partially symmetric tensor $\mathscr{A}$ satisfy:

$$\mathscr{A}(:,:,1,1) = \begin{bmatrix} -0.9727 & 0.3169 & -0.3437 \\ -0.6332 & -0.7866 & 0.4257 \\ -0.3350 & -0.9896 & -0.4323 \end{bmatrix},$$

$$\mathscr{A}(:,:,2,1) = \begin{bmatrix} -0.6332 & -0.7866 & 0.4257 \\ 0.7387 & 0.6873 & -0.3248 \\ -0.7986 & -0.5988 & -0.9485 \end{bmatrix},$$

$$\mathscr{A}(:,:,3,1) = \begin{bmatrix} -0.3350 & -0.9896 & -0.4323 \\ -0.7986 & -0.5988 & -0.9485 \\ 0.5853 & 0.5921 & 0.6301 \end{bmatrix},$$

$$\mathscr{A}(:,:,1,2) = \begin{bmatrix} 0.3169 & 0.6158 & -0.0184 \\ -0.7866 & 0.0160 & 0.0085 \\ -0.9896 & -0.6663 & 0.2559 \end{bmatrix},$$

$$\mathscr{A}(:,:,2,2) = \begin{bmatrix} -0.7866 & 0.0160 & 0.0085 \\ 0.6873 & 0.5160 & -0.0216 \\ -0.5988 & 0.0411 & 0.9857 \end{bmatrix},$$

$$\mathscr{A}(:,:,3,2) = \begin{bmatrix} -0.9896 & -0.6663 & 0.2559 \\ -0.5988 & 0.0411 & 0.9857 \\ 0.5921 & -0.2907 & -0.3881 \end{bmatrix},$$

$$\mathscr{A}(:,:,1,3) = \begin{bmatrix} -0.3437 & -0.0184 & 0.5649 \\ 0.4257 & 0.0085 & -0.1439 \\ -0.4323 & 0.2559 & 0.6162 \end{bmatrix},$$

$$\mathscr{A}(:,:,2,3) = \begin{bmatrix} 0.4257 & 0.0085 & -0.1439 \\ -0.3248 & -0.0216 & -0.0037 \\ -0.9485 & 0.9857 & -0.7734 \end{bmatrix},$$

$$\mathscr{A}(:,:,3,3) = \begin{bmatrix} -0.4323 & 0.2559 & 0.6162 \\ -0.9485 & 0.9857 & -0.7734 \\ 0.6301 & -0.3881 & -0.8526 \end{bmatrix}.$$

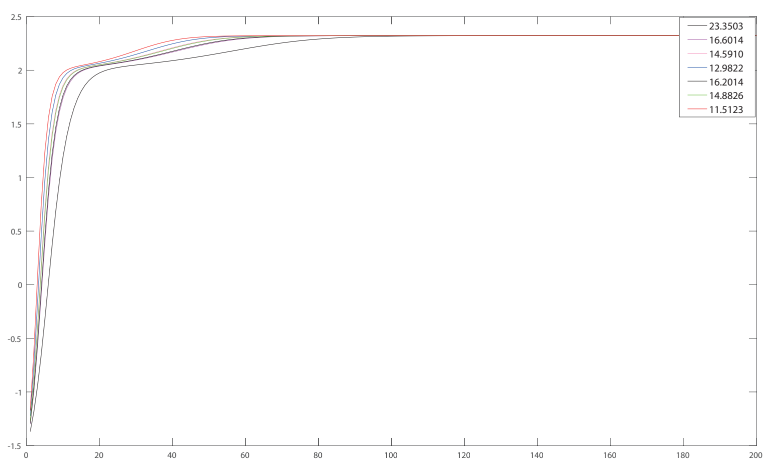

Figure 1: Analysis of convergence of WQZ algorithm for different parameters.

According to the WQZ-algorithm, we have $\sum_{1\le s\le t\le mn} |A_{st}| = 23.3503$. According to Example 2 in the literature [8] , we have $\rho(\mathscr{A}) \le 16.6014$, $\rho(\mathscr{A}) \le 14.5910$, $\rho(\mathscr{A}) \le 12.9822$. Let $\alpha = \beta = (-0.18,\ 0.3587,\ -0.2)^T$. By Corollary 3.1, we have $\rho(\mathscr{A}) \le 16.2014$. By Corollary 3.2, we have $\rho(\mathscr{A}) \le 14.8826$. By Corollary 3.3, we have $\rho(\mathscr{A}) \le 11.5123$. From Figure 1, when taking $\tau = 11.5123$, the sequence has more rapidly convergence in WQZ-algorithm.

Next, we introduce some checkable sufficient conditions for the M-positive definiteness of the fourth-order partially symmetric tensor.

Theorem 3.5. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor, and $\mathscr{F}_M$ be an M-identity tensor. If there exists a positive real vector $\alpha = (\alpha_1, \ldots, \alpha_m)^T \in \mathbb{R}^m$ or $\beta = (\beta_1, \ldots, \beta_n)^T \in \mathbb{R}^n$ such that $b_\pi > G_\pi(\mathscr{A})$, then $\mathscr{A}$ is positive definite.*

Proof. Suppose on the contrary that $\lambda \leq 0$. From Theorem 2.5, there exists $i = p \in [m]$, such that $|\lambda - b_\pi| = |\lambda - \alpha_p| \leq G_\pi(\mathscr{A}) = D_p(\mathscr{A})$. Furthermore, $\alpha_p > 0$ and $\lambda \leq 0$,

$$\alpha_p \leq |\lambda - \alpha_p| \leq D_p(\mathscr{A}),$$

or there exists $l = q \in [n]$, such that $|\lambda - b_\pi| = |\lambda - \beta_q| \leq G_\pi(\mathscr{A}) = F_q(\mathscr{A})$. Furthermore, $\beta_q > 0$ and $\lambda \leq 0$,

$$\beta_q \leq |\lambda - \beta_q| \leq F_q(\mathscr{A}).$$

We have $b_\pi \leq G_\pi(\mathscr{A})$, which contradicts the conditions. Therefore, we have $\lambda > 0$, and then $\mathscr{A}$ is M-positive definite. □

Theorem 3.6. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor, and $\mathscr{F}_M$ be an M-identity tensor. If there exists a positive real vector $\alpha = (\alpha_1, \ldots, \alpha_m)^T \in \mathbb{R}^m$ and $s \neq p$ such that*

$$(\alpha_p - D_p^p(\mathscr{A}))\alpha_s > (D_p(\mathscr{A}) - D_p^p(\mathscr{A}))D_s(\mathscr{A}),$$

or a positive real vector $\beta = (\beta_1, \ldots, \beta_n)^T \in \mathbb{R}^n$ and $t \neq q$ such that

$$(\beta_q - F_q^q(\mathscr{A}))\beta_t > (F_q(\mathscr{A}) - F_q^q(\mathscr{A}))F_t(\mathscr{A}),$$

then $\mathscr{A}$ is positive definite.

Proof. Suppose on the contrary that $\lambda \leq 0$. From Theorem 2.6, there exists $i = p \in [m]$, such that

$$(|\lambda - \alpha_p| - D_p^p(\mathscr{A}))|\lambda - \alpha_s| \leq (D_p(\mathscr{A}) - D_p^p(\mathscr{A}))D_s(\mathscr{A}).$$

Moreover, $\alpha_p > 0$ and $\lambda \leq 0$

$$(\alpha_p - D_p^p(\mathscr{A}))\alpha_s \leq (|\lambda - \alpha_p| - D_p^p(\mathscr{A}))|\lambda - \alpha_s| \leq (D_p(\mathscr{A}) - D_p^p(\mathscr{A}))D_s(\mathscr{A}),$$

which contradicts the conditions. Therefore, we have $\lambda > 0$, and then $\mathscr{A}$ is M-positive definite. The second conclusion can be obtained similarly. □

Theorem 3.7. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor, and $\mathscr{F}_M$ be an M-identity tensor. For any $i, k \in [m],\ k \neq i,\ j, l \in [n],\ j \neq l$, if there exists a positive real vector $\alpha = (\alpha_1, \ldots, \alpha_m)^T \in \mathbb{R}^m$ such that*

$$\alpha_i > X_i^i(\mathscr{A}),$$

$$(\alpha_i - X_i^i(\mathscr{A}))(\alpha_k - X_k^k(\mathscr{A})) > X_i(\mathscr{A})X_k(\mathscr{A}),$$

or a positive real vector $\beta = (\beta_1, \ldots, \beta_n)^T \in \mathbb{R}^n$ such that

$$\beta_l > Y_l^l(\mathscr{A}),$$

$$(\beta_l - Y_l^l(\mathscr{A}))(\beta_j - Y_j^j(\mathscr{A})) > Y_l(\mathscr{A})Y_j(\mathscr{A}),$$

then, $\mathscr{A}$ is M-positive definite.

Proof. Suppose on the contrary that $\lambda \leq 0$, From Theorem 2.7, $\lambda \in \Phi_1(\mathscr{A})$,

$$(|\lambda - \alpha_i| - X_i^i(\mathscr{A}))(|\lambda - \alpha_k| - X_k^k(\mathscr{A})) \leq X_i(\mathscr{A})X_k(\mathscr{A}),\ |\lambda - \alpha_i| \leq X_i^i(\mathscr{A}).$$

Further, it follows from $\alpha_i \geq 0$,

$$\begin{aligned} X_i(\mathscr{A})X_k(\mathscr{A}) &\geq (|\lambda - \alpha_i| - X_i^i(\mathscr{A}))(|\lambda - \alpha_k| - X_k^k(\mathscr{A})) \\ &\geq (\alpha_i - X_i^i(\mathscr{A}))(\alpha_k - X_k^k(\mathscr{A})),\ X_i^i(\mathscr{A}) \\ &\geq |\lambda - \alpha_i| \\ &\geq \alpha_i, \end{aligned}$$

which contradicts the conditions. Therefore, we have $\lambda > 0$, and then $\mathscr{A}$ is M-positive definite. The second conclusion can be obtained similarly. □

The following examples reveal that Theorem 3.5, 3.6 and 3.7 can judge the M-positive definiteness of the fourth-order partially symmetric tensor.

Example 3.8. Consider the following partially symmetric tensor $\mathscr{A} = (a_{ijkl}) \in (\mathbb{R})^{[2]\times[2]\times[2]\times[2]}$ defined by

$$a_{ijkl} = \begin{cases} a_{1111} = 10,\ a_{1221} = a_{1122} = -0.5,\ a_{1212} = 4; \\ a_{2121} = 5,\ a_{2112} = a_{2211} = -0.5,\ a_{2222} = 3; \\ a_{ijkl} = 0,\ \ \textit{otherwise}. \end{cases}$$

Set $\alpha = (8,\ 4)^T$. According to Theorem 3.5, we have $\alpha_1 = 8 > R_1(\mathscr{A}, \alpha_1) = 4,\ \alpha_2 = 4 > R_2(\mathscr{A}, \alpha_2) = 3$ then, $\mathscr{A}$ is positive definite. Or set $\beta = (6,\ 3)^T$. According to Theorem 3.6, we have $F_1^1(\mathscr{A}) = 5,\ F_2^2(\mathscr{A}) = 1,\ F_1(\mathscr{A}) = 6,\ F_2(\mathscr{A}) = 2$, then $\mathscr{A}$ is positive definite. By calculation, the M-eigenvalue of $\mathscr{A}$ are 4, 10, 5, 3, 0.028, 2.972 and all of them are positive, which verifies the validity of Theorem 3.5 and Theorem 3.6.

Example 3.9. Consider the following partially symmetric tensor $\mathscr{A} = (a_{ijkl}) \in (\mathbb{R})^{[2]\times[2]\times[2]\times[2]}$ defined by

$$a_{ijkl} = \begin{cases} a_{1111} = a_{1212} = 1.1,\ a_{1222} = -1; \\ a_{2121} = a_{2222} = 1,\ a_{2212} = -1; \\ a_{ijkl} = 0,\ \ \textit{otherwise}. \end{cases}$$

Set $\alpha = (1.1,\ 1)^T$. According to Theorem 3.3, we have

$$\alpha_1 = 1.1 > X_1^1(\mathscr{A}) = 0,\ \alpha_2 = 1 > X_2^2(\mathscr{A}) = 0,$$

$$(\alpha_1 - X_1^1(\mathscr{A}))(\alpha_2 - X_2^2(\mathscr{A})) = 1.1 > X_1(\mathscr{A})X_2(\mathscr{A}) = 1.$$

Then, $\mathscr{A}$ is M-positive definite. By calculation, the M-eigenvalue of $\mathscr{A}$ are 0.0488, 1.1, 2.0512, which verifies the validity of Theorem 3.7.

4 Conclusion

In this paper, M-identity tensor is introduced to establish some new eigenvalue localization sets for fourth-order partially symmetric tensor. It is revealed that the new eigenvalue localization sets are tighter than some existing results. As applications, some bound estimations for the M-spectral radius and some checkable sufficient conditions for the positive definiteness of the fourth-order partially symmetric tensor are obtained based on the new eigenvalue localization sets.

References

[1] H. Che, H. Chen and Y. Wang, On the M-eigenvalue estimation of fourth-order partially symmetric tensors, *Journal of Industrial and Management Optimization* 1 (2020) 309–324.

[2] H. Chen, Y. Chen, G. Li and L. Qi, A semidefinite program approach for computing the maximum eigenvalue of a class of structured tensors and its applications in hypergraphs and copositivity test, *Numerical Linear Algebra with Applications* 25 (2017): e2125.

[3] W. Ding, J. Liu, L. Qi and H. Yan, Elasticity M-tensors and the strong ellipticity condition, *Applied Mathematics and Computation* 373 (2020) 1–10.

[4] D. Han, H. Dai and L. Qi, Conditions for strong ellipticity of anisotropic elastic materials, *Journal of Elasticity* 1 (2009) 1–13.

[5] J. He, G. Xu and Y. Liu, New M-eigenvalue inclusion sets for fourth-order partially symmetric tensors with applications, *Bulletin of the Malaysian Mathematical Sciences Society* 1 (2021) 1–19.

[6] C. Hillar and L. H. Lim, Most tensor problems are NP hard, http://arxiv.org/abs/0911.1393.

[7] Z. Huang and L. Qi, Positive definiteness of paired symmetric tensors and elasticity tensors, *Journal of Computational and Applied Mathematics*, 338 (2018) 22–43.

[8] S. Li, C. Li and Y. Li, M-eigenvalue inclusion intervals for a fourth-order partially symmetric tensor, *Journal of Computational and Applied Mathematics* 356 (2019) 391–401.

[9] C. Ling, J. Nie and L. Qi, Bi-quadratic optimization over unit spheres and semidefinite programming relaxations, *SIAM Journal on Matrix Analysis on Optimization* 3 (2009) 1286-1310.

[10] L. Qi and Z. Luo, Tensor *Analysis: Spectral Aeory and Special Tensors*, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2017.

[11] G. Wang, L. Sun and L. Liu, M-Eigenvalues-based sufficient conditions for the positive definiteness of fourth-order partially symmetric tensors, *Complexity* 2020 (2020): Article ID 2474278.

[12] Y. Wang, L. Caccetta and G. Zhou, Convergence analysis of a block improvement method for polynomial optimization over unit spheres, *Numerical Linear Algebra with Applications* 6 (2015) 1059–1076.

[13] X. Wang, H. Chen and Y. Wang, Solution structures of tensor complementarity problem, *Frontiers of Mathematics in China* 4 (2018) 935–945.

[14] Y. Wang and L. Qi, On the successive supersymmetric rank-1 decomposition of higher-order supersymmetric tensors, *Numerical Linear Algebra with Applications* 6 (2007) 503–519.

[15] Y. Wang, L. Qi and X. Zhang, A practical method for computing the largest M-eigenvalue of a fourth-order partially symmetric tensor. *Numerical Linear Algebra with Applications* 16(2009) 589–601.

[16] Y. Wang, X. Sun and F. Meng, On the conditional and partial trade credit policy with capital constraints: a Stackelberg model, *Applied Mathematical Modelling* 40 (2016) 1–18.

[17] G. Zhou, G. Wang, L. Qi and M. Alqahtani, A fast algorithm for the spectral radii of weakly reducible nonnegative tensors, *Numerical Linear Algebra with Applications* 25 (2017): e2134.

[18] K. Zhang and Y. Wang, An H-tensor based iterative scheme for identifying the positive definiteness of multivariate homogeneous forms, *Journal of Computational and Applied Mathematics* 305 (2016) 1–10.

Manuscript received 12 June 2021
revised 3 September 2021
accepted for publication 9 September 2021

Kaiping Liu
School of Management Science, Qufu Normal University
Rizhao Shandong, 276800, China
E-mail address: kaipliu@163.com

Haitao Che
School of Mathematics and Information Science
Weifang University, Weifang Shandong, 261061, China
E-mail address: haitaoche@163.com