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ON THE STABILITY OF APPROXIMATE SOLUTION MAPPINGS TO GENERALIZED KY FAN INEQUALITY*

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Abstract: This paper is concerned with the stability for a generalized Ky Fan inequality when it is perturbed by vector-valued bifunction sequence and constrain set sequence. By removing the assumptions of the strictly proper quasi C-convexity and the continuous convergence, we establish the Painlevé-Kuratowski convergence of the approximate solution mapping of a family for perturbed problems to the corresponding solution mapping of the original problem. The obtained results are new and improve the corresponding ones in the literatures ([21, 23]). Some examples are also given to illustrate the results.

Key words: *generalized Ky Fan inequality, Painlevé-Kuratowski convergence, approximate solution mappings, continuous convergence, gamma convergence*

Mathematics Subject Classification: *9K40, 90C29, 90C3*

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1 Introduction

It is well known that the Ky Fan Inequality (briefly, KFI) is a very general mathematical format, which embraces the formats of several disciplines, as those for equilibrium problems of Mathematical Physics, those from Game Theory, those from Optimization and Variational Inequalities, and so on (see [6, 10, 11]). Since Ky Fan Inequality was introduced in [10, 11], it has been extended and generalized to vector-valued mappings. The Ky Fan Inequality for a vector-valued mapping is known as the generalized Ky Fan Inequality (briefly, GKFI). In the literature, existence results for various types of (generalized) Ky Fan Inequalities have been investigated intensively, see [13, 14, 20] and the references therein.

The stability analysis of solution mappings for the KFI is one of the most interesting topics in optimization theory and applications. In general, it is concerned with the study of the behavior of the solution of problems when their data are subject to change. The main goal of this kind is to provide qualitative and quantitative information on the problem itself. In last years, many authors have intensively studied the stability of the solution (mapping) for variational inequalities or Ky Fan inequalities when the objective function are perturbed by parameters; see [1-5, 13-17, 27-31, 34-35]. Anh and Khanh [2] studied the

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stability of solution mappings for two class of parametric quasi-KFIs. Huang et al. [17] discussed the upper semicontinuity and lower semicontinuity of the solution mapping for a parametric implicit KFI. By virtue of a density result and scalarization technique, Gong and Yao [15] first discussed the lower semicontinuity of the set of efficient solutions for a GKFI. Li et al. [22] obtained the sufficient conditions for the lower semicontinuity of the solution mapping to a generalized PKFI with set-valued mappings. Chen and Huang [7] studied the continuity of solution mappings for parametric weak KFIs. Recently, Anh et al. [3] discuss Hölder continuity of approximate solution mappings to parametric KFIs under the concavity and convexity of bifunctions. Peng et al. [31] obtained the connectedness of solution sets for weak generalized symmetric KFIs via addition-invariant set. Some results also can see [27, 30, 34] and the references therein.

On the other hand, on the stability of the solution set under perturbations, either of the feasible set or the objective function, has also been of great interest in this field. Attouch and Riahi [1] studied stability of a vector optimization problem based on the notion of convergence of epigraphs. Huang [19] discussed the stability of the set of efficient points of vector-valued and set-valued optimization problems. Lucchetti and Miglierina [25] investigated the Painlevé-Kuratowski set convergence of the solution set for a convex vector optimization problem. Using continuous convergence, Lalitha and Chatterjee [21] established the stability of (weak) efficient sets of proper quasiconvex vector optimization problems, which improved the results of [25]. However, to the best of our knowledge, there are few papers concerning with the stability of the solution mapping to (G)KFI when it is perturbed by bifunction sequence and set sequence. Durea [9] considered the GKFI with perturbations of the multifunction and obtained the Painlevé-Kuratowski upper convergence of the solution set. Under the C-strict monotonicity, Fang and Li [12] obtained the Painlevé-Kuratowski convergence of efficient solution sets and proper efficient solution sets for a GKFI. Zhao [36] et al. obtained the convergence of the weak and global efficient solution sets for the GKFI. Recently, Li et al. [23] establish the Painlevé-Kuratowski convergence of the approximate solution mapping of a family of perturbed problems to the corresponding solution mapping of the original problem under the conditions of the strictly proper quasi C-convexity and the continuous convergence of objective functions. Very Recently, Peng et al. [28] obtained the Painlevé-Kuratowski stability of approximate efficient solutions for perturbed semi-infinite vector optimization problems. Han [18] discussed the Painlevé-Kuratowski convergence of the solution sets for set optimization problems with cone-quasiconnectedness.

Motivated by the work reported above ([18, 21, 23, 28, 33, 36] and the references therein), this paper aims at further concerning with the stability analysis of approximate solution mappings for GKFI. Without using the strictly proper quasi C-convexity and the continuous convergence, we aim to establish the stability results of the approximate solution mapping for a class of generalized Ky Fan inequality, when it is perturbed by vector-valued bifunction sequences and set sequences. Our consequences are new and different from the corresponding ones in the literature ([21, 23, 32, 36]).

2 Preliminaries

Throughout this paper, unless otherwise specified, let $A \subset \mathbb{R}^m$ be a nonempty closed set and C be a pointed closed convex cone in $\mathbb{R}^l$ with nonempty topological interior $\text{int}C$. Let $F : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^l$ be a vector-valued bifunction. We consider the following generalized Ky Fan inequalities:

$$\text{GKFI}(A, F)_1 \quad \text{Find } x \in A \text{ such that } F(x, y) \notin -C\backslash\{0\}, \quad \forall y \in A$$

and

$$\text{GKFI}(A,F)_2 \quad \text{Find } x \in A \text{ such that } F(x,y) \notin -\text{int}C, \quad \forall y \in A.$$

A solution $x \in A$ of the problem $\text{GKF}(A,F)_1$ (resp. $\text{GKF}(A,F)_2$) is said to be a generalized Ky Fan's efficient (resp. weak efficient) point of F in A. We denote by $\text{Sol}(A,F)$ and $\text{WSol}(A,F)$ the set of generalized Ky Fan's efficient and weak efficient points of F in A, respectively.

For a sequence of bifunctions $F_n : A_n \times A_n \to \mathbb{R}^l, n = 1, 2, \cdots$, we now, respectively, approximate the above problems by the sequence of problems:

$$\text{GKFI}(A_n,F_n)_1 \quad \text{Find } x \in A_n \text{ such that } F_n(x,y) + \varepsilon_n e \notin -C\backslash\{0\}, \quad \forall y \in A_n$$

and

$$\text{GKFI}(A_n,F_n)_2 \quad \text{Find } x \in A_n \text{ such that } F_n(x,y) + \varepsilon_n e \notin -\text{int}C, \quad \forall y \in A_n.$$

where $\{A_n\}$ is a nonempty set sequence of $\mathbb{R}^m$, $\{F_n\}$ is a vector-valued bifunction sequence from $\mathbb{R}^m \times \mathbb{R}^m$ to $\mathbb{R}^l$, $e \in \text{int}C$ is a given point and $\{\varepsilon_n\}$ is a nonnegative sequence. Similarly, we denote by $\text{Sol}(A_n, F_n, \varepsilon_n)$ and $\text{WSol}(A_n, F_n, \varepsilon_n)$ the set of approximate solutions of problems $\text{GKFI}(A_n, F_n)_1$ and $\text{GKFI}(A_n, F_n)_2$, respectively. In this paper, using some weaker assumptions, we mainly analyze the behavior of $\text{Sol}(\cdot,\cdot,\cdot)$ and $\text{WSol}(\cdot,\cdot,\cdot)$ when the data vary around a given point. Thus, we always assume $\text{Sol}(\cdot,\cdot,\cdot)$ and $\text{WSol}(\cdot,\cdot,\cdot)$ are nonempty around the considered point.

Special cases: When $A_n \equiv A$ and $F_n \equiv F$ for any n, then we get $\text{Sol}(A_n, F_n, 0) = \text{Sol}(A,F)$ and $\text{WSol}(A_n, F_n, 0) = \text{WSol}(A,F)$.

We shall use the following notations. Let $\{\varepsilon_n\}$ be a scalar-valued sequence, we denote by $\varepsilon_n \searrow \varepsilon$ (resp. $\varepsilon_n \xrightarrow{\mathbb{R}_+} \varepsilon$) when $\varepsilon_n > \varepsilon$ (resp. $\varepsilon_n \in \mathbb{R}_+$) for all n and $\varepsilon_n \to \varepsilon$.

Now, we recall some notions and results which will be used in the sequel.

Definition 2.1. Let A be a nonempty convex subset of $\mathbb{R}^m$. Let f be a mapping from $\mathbb{R}^m$ to $\mathbb{R}^l$. We say that

(i) f is C-convex on A, if $\forall x_1, x_2 \in A, \lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) \in \lambda f(x_1) + (1-\lambda) f(x_2) - C;$$

(ii) f is properly quasi C-convex on A, if $\forall x_1, x_2 \in A$, $\forall \lambda \in [0,1]$, either

$$f(\lambda x_1 + (1-\lambda)x_2) \in f(x_1) - C \text{ or } f(\lambda x_1 + (1-\lambda)x_2) \in f(x_2) - C;$$

(iii) f is strictly properly quasi C-convex on A, if $\forall x_1, x_2 \in A$, $x \neq y$, $\forall \lambda \in (0,1)$, either

$$f(\lambda x_1 + (1-\lambda)x_2) \in f(x_1) - \text{int}C$$

or

$$f(\lambda x_1 + (1-\lambda)x_2) \in f(x_2) - \text{int}C.$$

Definition 2.2. [34] Let E be a nonempty convex subset of $\mathbb{R}^m$. Let f be a mapping from $\mathbb{R}^m$ to $\mathbb{R}^l$. We say f is

(i) naturally quasi C-convex on E, if for every $x_1, x_2 \in E$ and $\lambda \in [0,1]$, there exists $\mu \in [0,1]$, such that

$$f(\lambda x_1 + (1-\lambda)x_2) \in \mu f(x_1) + (1-\mu) f(x_2) - C;$$

(ii) strictly (resp. semistrictly) natural quasi C-convex on E, if for every $x_1, x_2 \in E$ with $x_1 \neq x_2$ (resp.$f(x_1) \neq f(x_2)$) and $\lambda \in (0,1)$, there exists $\mu \in (0,1)$, such that

$$f(\lambda x_1 + (1-\lambda)x_2) \in \mu f(x_1) + (1-\mu)f(x_2) - \text{int}C.$$

Definition 2.3. Let E be a nonempty convex subset of $\mathbb{R}^m$. Let f be a mapping from $\mathbb{R}^m$ to $\mathbb{R}^l$. We say f is

(i) C-quasiconvex on E, if for each $z \in \mathbb{R}^l, x_1, x_2 \in E$ and $\lambda \in [0,1]$,

$$f(x_1), f(x_2) \in z - C \;\text{ implies }\; f(\lambda x_1 + (1-\lambda)x_2) \in z - C;$$

(ii) strictly (resp. semistrictly) C-quasiconvex on E, if for each $z \in \mathbb{R}^l, x_1, x_2 \in E$ with $x_1 \neq x_2$ (resp. $f(x_1) \neq f(x_2)$), and for each $\lambda \in (0,1)$,

$$f(x_1), f(x_2) \in z - C \;\text{ implies }\; f(\lambda x_1 + (1-\lambda)x_2) \in z - \text{int}C.$$

Obviously, the strict C-quasiconvexity implies the semistrict C-quasiconvexity.

Remark 2.4. From the definitions, we can obtain immediately the following implications for the mapping f:

$$\begin{array}{ccccc}
\text{proper quasi } C\text{-convexity} & \Longrightarrow & \text{natural quasi } C\text{-convexity} & \Longrightarrow & C\text{-quasiconvexity} \\
\cup & & \cup & & \cup \\
\text{st.proper quasi } C\text{-convexity} & \Longrightarrow & \text{st.natural quasi } C\text{-convexity} & \Longrightarrow & \text{st.}C\text{-quasiconvexity} \\
\cap & & \cap & & \cap \\
\text{sst.proper quasi } C\text{-convexity} & \Longrightarrow & \text{sst.natural quasi } C\text{-convexity} & \Longrightarrow & \text{sst.}C\text{-quasiconvexity}
\end{array}$$

(for simplicity, st./sst. stands for strict(ly)/semistrict(ly), respevtively)

However, the converse implications are generally not valid.

Remark 2.5. From Remark 2.4, the class of C-quasiconvexity is strictly larger than the naturally quasi C-convexity, meanwhile, the naturally quasi C-quasiconvexity is strictly larger than the properly quasi C-convexity.

The following examples show that there exists C-quasiconvex functions are not necessary naturally quasi C-convex functions, and naturally quasi C-convex functions are not necessary properly quasi C-convex functions.

Example 2.6. (i) Let $X = \mathbb{R}, A = [0, \sqrt{2}] \subset X, Y = \mathbb{R}^2, C = \mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}$. Define $f : A \to Y$ by

$$f(x) = (\frac{x^2}{2},\; -\frac{x^2}{2} + 1).$$

(ii) Let $X = \mathbb{R}, E = [0,2] \subset X, Y = \mathbb{R}^2, C = \mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}$. Define $f : E \to Y$ by

$$h(x) = \big(\frac{3}{2}\cos(\frac{\pi x}{4}),\; \frac{3}{2}\sin(\frac{\pi x}{4})\big).$$

By virtue of Remark 2.1 of [34] and Definitions 2.1-2.3 above, we can verity f is naturally quasi C-convex on A, but it is not properly quasi C-convex on A, simultaneously, h is C-quasiconvex on E, but it is not naturally quasi C-convex on E.

Now, we recall some notions of convergence.

Definition 2.7. [29] Let $\{A_n\}(n \in \mathcal{N})$ be a sequence of sets $\mathbb{R}^m$ and A be a subset of $\mathbb{R}^m$.

(i) $\liminf_{n\to\infty} A_n := \{x \in \mathbb{R}^m | \exists (x_n), x_n \in A_n, \forall n \in \mathcal{N}, x_n \to x\}$ is its inner limit;

(ii) $\limsup_{n\to\infty} A_n := \{x \in \mathbb{R}^m | \exists (n_k), \exists (x_{n_k}), x_{n_k} \in A_{n_k}, \forall k \in \mathcal{N}, x_{n_k} \to x\}$ is its outer limit;

(iii) $\limsup_n^\infty A_n := \{0\} \cup \{x \in \mathbb{R}^m | \exists x_{n_k} \in A_{n_k}, \mu_{n_k} \searrow 0, \mu_{n_k} x_{n_k} \to x\}$ is its horizon outer limit;

(iv) $\{A_n : n \in \mathcal{N}\}$ is said to converges in the sense of Painlevé-Kuratowski (P.K.) to A (denoted as $A_n \xrightarrow{P.K.} A$) if and only if $\limsup_{n\to\infty} A_n \subset A \subset \liminf_{n\to\infty} A_n$.

The relations $\limsup_{n\to\infty} A_n \subset A$ and $A \subset \liminf_{n\to\infty} A_n$ are, respectively, referred as the upper part and the lower part of the (Painlevé-Kuratowski) convergence. Clearly, $\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$.

Definition 2.8. [24] Let $F_n : A_n \to \mathbb{R}^l$ and $F : A \to \mathbb{R}^l (n \in \mathcal{N})$ be vector-valued mappings. We say that $F_n|_{A_n}$ continuous converges to $F|_A$ (denoted as $F_n|_{A_n} \xrightarrow{c} F|_A$), iff the following conditions are satisfied:

(a) $A_n \xrightarrow{P.K.} A$;

(b) for every $x \in A$ and for every sequence $\{x_n\}$ in A_n, $F_n(x_n) \to F(x)$ for all $x_n \to x$.

Definition 2.9. [26] Let $F_n : A_n \times A_n \to \mathbb{R}^l$ and $F : A \times A \to \mathbb{R}^l (n \in \mathcal{N})$ be vector-valued mappings and let $\mathcal{U}(x) \times \mathcal{U}(y)$ be the family of neighborhoods of (x, y). We say that $(F_n|_{A_n})_{n\in\mathcal{N}}$ gamma converges to $F|_A$(denoted as $F_n|_{A_n} \xrightarrow{\Gamma} F|_A$), if for every $(x, y) \in A\times A$:

(a) $A_n \xrightarrow{P.K.} A$;

(b) $\forall\, U \in \mathcal{U}(x) \times \mathcal{U}(y), \forall \zeta \in \text{int}C, \exists n_{\zeta,U} \in \mathcal{N}$ such that $\forall n \geq n_{\zeta,U}, \exists (x_n, y_n) \in U \bigcap (A_n \times A_n)$ such that

$$F_n(x_n, y_n) \in F(x, y) + \zeta - C;$$

(c) $\forall \zeta \in \text{int}C, \exists U_\zeta \in \mathcal{U}(x) \times \mathcal{U}(y), k_\zeta \in \mathcal{N}$ such that $\forall (x', y') \in U_\zeta \bigcap (A_n \times A_n), \forall n \geq k_\zeta$,

$$F_n(x', y') \in F(x, y) - \zeta + C.$$

Remark 2.10. Clearly, continuous convergence implies gamma convergence, but the converse is generally not true. We can give an example to illustrate the case.

Example 2.11. Let $n \in \mathcal{N}$. Defined $F_n : R \to R$ by

$$F_n(x) = \begin{cases} n^2x^2 - 2nx, & -\frac{1}{n} \leq x < 0; \\ nx + 1, & -\frac{2}{n} \leq x < -\frac{1}{n}; \\ n^2x^2 - 2nx, & 0 < x \leq \frac{1}{n}; \\ nx - 2, & \frac{1}{n} < x < \frac{2}{n}; \\ 0, & x < -\frac{2}{n} \text{ or } x > \frac{2}{n}; \\ -1, & x = 0. \end{cases}$$

It is easy to verify that $F_n \xrightarrow{\Gamma} F$, where

$$F(x) = \begin{cases} 0, & x \neq 0; \\ -1, & x = 0. \end{cases}$$

However, the sequence $\{F_n\}_{n\in\mathcal{N}}$ is not continuous converges to F. In fact, take two sequences $\{x_n : x_n = \frac{1}{3n}\}$ and $\{x'_n : x'_n = -\frac{2}{n}\}$, it is obvious that both of them converge to zero, but $F_n(x_n)$ converges to $-\frac{5}{9}$ and $F_n(x'_n)$ converges to -1.

3 Main Results

In this section, we attempt to establish the upper and lower Painlevé-Kuratowski convergence behavior of $\mathrm{Sol}(\cdot,\cdot,\cdot)$ and $\mathrm{WSol}(\cdot,\cdot,\cdot)$ without the continuous convergence and strictly proper quasi C-convexity, when the data vary around a given point.

Firstly, only using gamma convergence, we establish the upper Painlevé-Kuratowski convergence of $\mathrm{Sol}(\cdot,\cdot,\cdot)$.

Theorem 3.1. *Assume that* $-F_n(\cdot,\cdot)\mid_{A_n}\xrightarrow{\Gamma} -F(\cdot,\cdot)\mid_A$ *and* $\varepsilon_n \searrow 0$. *Then*

$$\limsup_{n\to\infty} WSol(A_n, F_n, \varepsilon_n) \subset WSol(A, F).$$

Proof. Take any $x \in \mathrm{limsup}_n \mathrm{WSol}(A_n, F_n, \varepsilon_n)$. Then, there exists a subsequence $\{x_{n_k}\}$ in $\mathrm{WSol}(A_{n_k}, F_{n_k}, \varepsilon_{n_k})$ such that $x_{n_k} \to x$. As $A_n \xrightarrow{P.K.} A$, we have $x_{n_k} \to x \in A$. For any $y \in A$, there exists $y_n \in A_n$ such that $y_{n_k} \to y$ since $A_n \xrightarrow{P.K.} A$. It follows from $\{x_{n_k}\} \subset \mathrm{WSol}(A_{n_k}, F_{n_k}, \varepsilon_{n_k})$ and $y_{n_k} \in A_{n_k}$ that

$$F_{n_k}(x_{n_k}, y_{n_k}) + \varepsilon_{n_k} e \notin -\mathrm{int}C. \tag{3.1}$$

Because $-F_n(\cdot,\cdot)\mid_{A_n}\xrightarrow{\Gamma} -F(\cdot,\cdot)\mid_A$, from Definition 2.9, for any $\zeta \in \mathrm{int}C, \exists U_\zeta \in \mathcal{U}(x) \times \mathcal{U}(y), k_\zeta \in \mathcal{N}$ such that $\forall (x', y') \in U_\zeta \bigcap (A_n \times A_n), \forall n \geq k_\zeta$,

$$-F_n(x', y') \in -F(x, y) - \zeta + C.$$

Without loss of generality, let $\zeta = \varepsilon_{n_k} e$, there exists $k_\zeta \leq N_\zeta \in \mathcal{N}$ such that $(x_{n_k}, y_{n_k}) \in U_\zeta$ and

$$-F_{n_k}(x_{n_k}, y_k) \in -F(x, y) - \varepsilon_{n_k} e + C, \quad \forall\ n_k \geq N_\zeta,$$

that is,

$$F(x, y) \in F_{n_k}(x_{n_k}, y_k) - \varepsilon_{n_k} e + C, \quad \forall\ n_k \geq N_\zeta. \tag{3.2}$$

Together with (3.1)-(3.2) and the closedness of $Y \setminus -\mathrm{int}C$ yields that

$$F(x, y) \in F_{n_k}(x_{n_k}, y_{n_k}) + \varepsilon_{n_k} e + C - 2\varepsilon_{n_k} e \subset Y \setminus -\mathrm{int}C - 2\varepsilon_{n_k} e.$$

Then, from $\varepsilon_n \searrow 0$ and the closedness of $Y \setminus -\mathrm{int}C$, we can get

$$F(x, y) \notin -\mathrm{int}C.$$

As $y \in A$ is arbitrary, we conclude that $x \in \mathrm{WSol}(A, F)$. Thus, the proof is complete. □

Remark 3.2. By using gamma convergence, which is weaker than continuous converge (from Remark 2.10), we establish the upper Painlevé-Kuratowski convergence of $\mathrm{WSol}(\cdot,\cdot,\cdot)$. Thus, Theorem 3.1 extends and improves Theorem 3.1 of [23].

The following example is given to illustrate the case.

Example 3.3. Let $A_n := A = \mathbb{R}$ and $C := \mathbb{R}^2_+ = \{x = (x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$. Defined $F_n, F : A \times A \to \mathbb{R}^2$ such that for any $x, y \in A$,

$$F_n(x, y) := \big(-x, -nxe^{-2n^2x^2}\big),$$

and

$$F(x,y) := \begin{cases} (-x,\ 0), & x \neq 0; \\ (0, \frac{1}{2}e^{-\frac{1}{2}}), & x = 0. \end{cases}$$

It follows from direct computations that

$$\text{WSol}(A,F) = (-\infty,+\infty),$$

and for any $\varepsilon_n \searrow 0$,

$$\text{WSol}(A_n,F_n,\varepsilon_n) = (-\infty,+\varepsilon_n].$$

By virtue of [26], we can easily verify that $-F_n \mid_{A_n} \xrightarrow{\Gamma} -F \mid_A$ when $n \to \infty$. From Theorem 3.1, one has

$$\limsup_{n\to\infty} \text{WSol}(A_n,F_n,\varepsilon_n) \subset \text{WSol}(A,F),$$

for any $\varepsilon_n \searrow 0$. Obviously, Theorem 3.1 is applicable.

However, the sequence $\{F_n\}|_{n\in\mathcal{N}}$ is not continuous converges to F. In fact, we consider the sequence: $x'_n = (\frac{1}{3n}), x''_n = (-\frac{1}{3n})$, both converging to zero, but

$$F_n(x'_n,y'_n) \to (0,-\frac{1}{3}e^{-\frac{2}{9}}) \neq (0,\frac{1}{3}e^{-\frac{2}{9}}) \leftarrow F_n(x''_n,y''_n).$$

Therefore, Theorem 3.1 in [23] is not applicable here.

Under certain assumptions, we discuss the relationships between $\text{Sol}(\cdot,\cdot)$ and $\text{WSol}(\cdot,\cdot)$.

Theorem 3.4. *Assume that*

- (i) *A is a nonempty convex subset of $\mathbb{R}^m$;*
- (ii) $\forall x \in A, F(x,x) = 0$;
- (iii) $\forall x \in A, y \mapsto F(x,y)$ *is semistrictly C-quasiconvex on A.*

Then,

$$\mathit{Sol}(A,F) = \mathit{WSol}(A,F).$$

Proof. By the definition, $\text{Sol}(A,F) \subset \text{WSol}(A,F)$. We need only to prove $\text{WSol}(A,F) \subset \text{Sol}(A,F)$. Suppose to the contrary, there exists $x_0 \in \text{WSol}(A,F)$ such that $x_0 \notin \text{Sol}(A,F)$. Hence, there exists $y_0 \in A$ such that

$$F(x_0,y_0) \in -C \setminus \{0\}. \tag{3.3}$$

Together with (ii), one has

$$F(x_0,y_0) \in F(x_0,x_0) - C \setminus \{0\}. \tag{3.4}$$

Since $F(x_0,\cdot)$ is semistrictly C-quasiconvex, combing (3.3) and (3.4), for every $\lambda \in (0,1)$ we have

$$F(x_0,\lambda x_0 + (1-\lambda)y_0) \in F(x_0,x_0) - \text{int}C.$$

Using the assumptions (i)-(ii), for $\lambda x_0 + (1-\lambda)y_0 \in A$, one has

$$F(x_0,\lambda x_0 + (1-\lambda)y_0) \in -\text{int}C,$$

which contradicts $x_0 \in \text{WSol}(A,F)$. Thus, the proof is complete. ∎

From Theorem 3.4 and Remark 2.4, we can get Corollary 3.5.

Corollary 3.5. *Assume that*

(i) *A is a nonempty convex subset of $\mathbb{R}^m$;*

(ii) $\forall x \in A, F(x,x) = 0$;

(iii) $\forall x \in A, y \mapsto F(x,y)$ *is semistrictly natural quasi C-convex on A.*

Then, we have

$$Sol(A,F) = WSol(A,F).$$

Combing Theorems 3.1 and 3.4 (resp. Corollary 3.5), we obtain the following results easily.

Theorem 3.6. *Suppose that* $-F_n(\cdot,\cdot)\mid_{A_n} \xrightarrow{\Gamma} -F(\cdot,\cdot)\mid_A$ *and* $\varepsilon_n \searrow 0$. *Assume that*

(i) *A is a nonempty convex subset of $\mathbb{R}^m$;*

(ii) $\forall x \in A, F(x,x) = 0$;

(iii) $\forall x \in A, y \mapsto F(x,y)$ *is semistrictly C-quasiconvex on A.*

Then

$$\limsup_{n\to\infty} Sol(A_n, F_n, \varepsilon_n) \subset Sol(A,F).$$

Corollary 3.7. *Suppose that* $-F_n(\cdot,\cdot)\mid_{A_n} \xrightarrow{\Gamma} -F(\cdot,\cdot)\mid_A$ *and* $\varepsilon_n \searrow 0$. *Assume that*

(i) *A is a nonempty convex subset of $\mathbb{R}^m$;*

(ii) $\forall x \in A, F(x,x) = 0$;

(iii) $\forall x \in A, y \mapsto F(x,y)$ *is semistrictly natural quasi C-convex on A.*

Then

$$\limsup_{n\to\infty} Sol(A_n, F_n, \varepsilon_n) \subset Sol(A,F).$$

Remark 3.8. In [23], by virtue of continuous convergence and strictly proper quasi C-convexity, Li et. al obtain the upper Painlev*é*-Kuratowski convergence of Sol$(\cdot,\cdot,\cdot)$ (see [23, Theorem 3.2]). In Theorem 3.6, by using more weaker conditions (gamma convergence and semistrictly C-quasiconvexity), we obtain the same result. Therefore, Theorem 3.6 extends and improves Theorem 3.2 of [23].

Now, we give an example to illustrate that our results (Theorem 3.6 and Corollary 3.7) are applicable, while the corresponding results in [21] and [23] (i.e., Theorem 3.2) may be not.

Example 3.9. Let $C := \mathbb{R}^2_+ = \{x = (x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$ and $A = [0,1], A_n = [\frac{1}{n}, 1 - \frac{1}{n}]$ $(n \geq 3)$. Define the mapping $F : A \times A \longrightarrow \mathbb{R}^2$ by

$$F(x, y) := (y - x, x - y),$$

and define $F_n : A_n \times A_n \longrightarrow \mathbb{R}^2$ by

$$F_n(x, y) := (y - x - \frac{1}{n^2},\ x - y + \frac{1}{n^2}).$$

It follows from direct computations that for any $\varepsilon_n \searrow 0$,

$$\mathrm{Sol}(A_n, F_n, \varepsilon_n) = (\frac{1}{n} - \varepsilon_n,\ 1 - \frac{1}{n}] \cap A_n \quad \text{and} \quad \mathrm{Sol}(A, F) = [0, 1].$$

We can verify that all assumptions of Theorem 3.6 and Corollary 3.7 are satisfied, and we also observe that

$$\limsup_{n\to\infty} \mathrm{Sol}(A_n, F_n, \varepsilon_n) \subset \mathrm{Sol}(A, F) = [0, 1],$$

for any $\varepsilon_n \searrow 0$. Surely, Theorem 3.6 and Corollary 3.7 are applicable.

However, from Definition 2.1, we can easily find that $F(x, \cdot)$ is not strictly properly quasi C-convex on A. Indeed, take $y_1 = \frac{1}{2}, y_2 = 1 (x \neq y)$ and $\lambda = \frac{1}{2}$, then it follows that

$$F(x, \lambda y_1 + (1 - \lambda)y_2) \notin F(x, y_1) - \mathrm{int}C,$$

and

$$F(x, \lambda y_1 + (1 - \lambda)y_2) \notin F(x, y_2) - \mathrm{int}C.$$

This means that F is not strictly properly quasi C-convex on A, therefore, Theorem 3.2 of [23] (and also Theorem 5.1 of [21]) are not applicable.

Next, we establish sufficient conditions for the lower Painlevé-Kuratowski convergence of $\mathrm{Sol}(\cdot, \cdot, \cdot)$ and $\mathrm{WSol}(\cdot, \cdot, \cdot)$.

Theorem 3.10. *Assume that* $F_n(\cdot, \cdot) \mid_{A_n} \xrightarrow{\Gamma} F(\cdot, \cdot) \mid_A$ *and* $limsup_n^\infty A_n = \{0\}$. *Then, there exist* $\varepsilon_n \searrow 0$ *such that*

(a) $Sol(A, F) \subset \liminf_{n\to\infty} Sol(A_n, F_n, \varepsilon_n)$,

(b) $WSol(A, F) \subset \liminf_{n\to\infty} WSol(A_n, F_n, \varepsilon_n)$.

Proof. (a) Take any $x_0 \in \mathrm{Sol}(A, F)$. From $A_n \xrightarrow{P.K.} A$, it follows that there exists $x_n \in A_n$ such that $x_n \to x_0$. First, we shall prove the following property for sequence $\{x_n\}$:

$$\forall \varepsilon > 0, \exists n_\varepsilon, \forall n \geq n_\varepsilon,\ x_n \in \mathrm{Sol}(A_n, F_n, \varepsilon). \tag{3.5}$$

Indeed, suppose to the contrary that there exists $\varepsilon^0 > 0$, for all k, there exists $n_k \geq k$ such that

$$x_{n_k} \notin \mathrm{Sol}(A_{n_k}, F_{n_k}, \varepsilon^0).$$

Then, there exists $y_{n_k} \in A_{n_k}$ such that

$$F_{n_k}(x_{n_k}, y_{n_k}) + \varepsilon^0 e \in -C \setminus \{0\}. \tag{3.6}$$

Since $\limsup_n^\infty A_n = \{0\}$, by [29, Chapter 4, p123], then we conclude A_n is eventually bounded. Therefore, the sequence $\{y_{n_k}\}$ is eventually bounded. Without loss of generality, we assume that $y_{n_k} \to y_0$, then $y_0 \in A$ as $A_n \xrightarrow{P.K.} A$.

Since $x_0 \in \mathrm{Sol}(A, F)$ and $y_0 \in A$, one has

$$F(x_0, y_0) \in Y \setminus (-C \setminus \{0\}). \tag{3.7}$$

Because $F_n(\cdot,\cdot) \mid_{A_n} \xrightarrow{\Gamma} F(\cdot,\cdot) \mid_A$, for any $\zeta \in \mathrm{int}C, \exists U_\zeta \in \mathcal{U}(x_0) \times \mathcal{U}(y_0), \mu_\zeta \in \mathcal{N}$ such that $\forall (x', y') \in U_\zeta \bigcap (A_n \times A_n), \forall n \geq \mu_\zeta$, one has

$$F_n(x', y') \in F(x_0, y_0) - \zeta + C. \tag{3.8}$$

Without loss of generality in (3.8), let $\zeta = \varepsilon^0 e,\ (x', y') = (x_{n_k}, y_{n_k})$, we get

$$F_{n_k}(x_{n_k}, y_{n_k}) \in F(x_0, y_0) - \varepsilon^0 e + C. \tag{3.9}$$

Combing (3.7) and (3.9), when $n_k \geq \max\{k, \mu_\zeta\}$ we have

$$\begin{aligned} F_{n_k}(x_{n_k}, y_{n_k}) &\in F(x_0, y_0) - \varepsilon^0 e + C \\ &\subset Y \setminus (-C \setminus \{0\}) - \varepsilon^0 e + C \\ &\subset Y \setminus (-C \setminus \{0\}) - \varepsilon^0 e, \end{aligned}$$

that is

$$F_{n_k}(x_{n_k}, y_{n_k}) + \varepsilon^0 e \notin -C \setminus \{0\},$$

which contradicts (3.6). So, we conclude that the assertion (3.5) is correct. By the definition of $\liminf_{n\to\infty} \mathrm{Sol}(A_n, F_n, \varepsilon_n)$ and using the similar proof method of Theorem 3.3(a) in [23], with appropriate modifications, we can easily obtain that there exist $\varepsilon_n \searrow 0$ such that $x_0 \in \liminf_{n\to\infty} \mathrm{Sol}(A_n, F_n, \varepsilon_n)$. The proof is complete.

(b) The proof follows on similar technique as for (a). So, we omit it. □

Remark 3.11. Since the gamma convergence is weaker than continuous convergence, Theorem 3.10 extends and improves the corresponding ones in [23] ([23, Theorems 3.3-3.5]) and [21] ([21, Theorem 5.1]). We give Example 3.12 to illustrate the case.

Example 3.12. Let $A := [-1, 1], A_n := [-1 - \frac{1}{n}, 1] \subset \mathbb{R}$ and $C := \mathbb{R}^2_+ = \{x = (x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$. Defined $F_n : A_n \times A_n \to \mathbb{R}^2$ such that for any $x, y \in A_n$,

$$F_n(x, y) := \begin{cases} (x, 0), & -1 - \frac{1}{n} \leq x \leq 0; \\ (x, nx), & 0 < x \leq \frac{1}{n}; \\ (x, 1), & \frac{1}{n} \leq x < 1. \end{cases}$$

and defined $F : A \times A \to \mathbb{R}^2$ as

$$F(x, y) := \begin{cases} (x, 0), & -1 \leq x \leq 0; \\ (x, 1), & 0 < x \leq 1. \end{cases}$$

It is easy to verify that all assumptions of Theorem 3.10 are satisfied, and there exist $\varepsilon_n \searrow 0$ such that

$$\mathrm{Sol}(A, F) \subset \liminf_{n\to\infty} \mathrm{Sol}(A_n, F_n, \varepsilon_n), \text{and } \mathrm{WSol}(A, F) \subset \liminf_{n\to\infty} \mathrm{WSol}(A_n, F_n, \varepsilon_n).$$

Indeed, take $\varepsilon'_n = \frac{1}{3n} \searrow 0$. Then, we have

$$\mathrm{Sol}(A_n, F_n, \varepsilon'_n) = [-\frac{1}{3n}, 1], \quad \mathrm{WSol}(A_n, F_n, \varepsilon'_n) = [-1 - \frac{1}{n}, 1];$$

and

$$\mathrm{Sol}(A, F) = [0, 1], \quad \mathrm{WSol}(A, F) = [-1, 1].$$

Thus, Theorem 3.10 is true and applicable.

However, the sequence $\{F_n\}|_{n\in\mathcal{N}}$ is not continuous converges to F. In fact, we consider two sequences: $x'_n = (\frac{1}{6n}), x''_n = (\frac{4}{3n})$, it is obvious that both of them converge to zero, but

$$F_n(x'_n, y'_n) \to (0, \frac{1}{6}), F_n(x''_n, y''_n) \to (0, 1).$$

Then, the main results of [23] (e.g. [23, Theorems 3.3-3.5]) and Theorem 5.1 of [21] are not applicable here.

As a consequence of Theorems 3.1 and 3.10(b), we obtain the following result.

Theorem 3.13. *Assume that* $F_n(\cdot,\cdot) \mid_{A_n} \xrightarrow{\Gamma} F(\cdot,\cdot) \mid_A$, $-F_n(\cdot,\cdot) \mid_{A_n} \xrightarrow{\Gamma} -F(\cdot,\cdot) \mid_A$ *and* $limsup_n^\infty A_n = \{0\}$. *Then, there exist* $\varepsilon_n \searrow 0$ *such that*

$$WSol(A_n, F_n, \varepsilon_n) \xrightarrow{P.K.} WSol(A, F).$$

Similarly, by virtue of Theorem 3.4, Theorems 3.6 and 3.10(a), we obtain Theorem 3.14.

Theorem 3.14. *Assume that:*

(i) *A is a nonempty convex subset of* $\mathbb{R}^m$ *and* $\limsup_n^\infty A_n = \{0\}$;

(ii) $F_n(\cdot,\cdot) \mid_{A_n} \xrightarrow{\Gamma} F(\cdot,\cdot) \mid_A$, $-F_n(\cdot,\cdot) \mid_{A_n} \xrightarrow{\Gamma} -F(\cdot,\cdot) \mid_A$;

(iii) $\forall x \in A, F(x,x) = 0$;

(iv) $\forall x \in A, y \mapsto F(x,y)$ *is semistrictly C-quasiconvex on A.*

Then, there exists $\varepsilon_n \searrow 0$ *such that*

$$\limsup_{n\to\infty} Sol(A_n, F_n, \varepsilon_n) \subset Sol(A, F) \subset \liminf_{n\to\infty} Sol(A_n, F_n, \varepsilon_n).$$

Also, by Corollary 3.5, Corollary 3.7 and Theorem 3.10(a) (or from Theorem 3.14 and Remark 2.4), we can get the following Corollary.

Corollary 3.15. *Assume that:*

(i) *A is a nonempty convex subset of* $\mathbb{R}^m$ *and* $\limsup_n^\infty A_n = \{0\}$;

(ii) $F_n(\cdot,\cdot) \mid_{A_n} \xrightarrow{\Gamma} F(\cdot,\cdot) \mid_A$, $-F_n(\cdot,\cdot) \mid_{A_n} \xrightarrow{\Gamma} -F(\cdot,\cdot) \mid_A$;

(iii) $\forall x \in A, F(x,x) = 0$;

(iv) $\forall x \in A, y \mapsto F(x,y)$ *is semistrictly natural quasi C-convex on A.*

Then, there exists $\varepsilon_n \searrow 0$ *such that*

$$Sol(A_n, F_n, \varepsilon_n) \xrightarrow{P.K.} Sol(A, F).$$

Remark 3.16. If $A_n = A, F_n(x,y) = g(x,y,n)$ and $\varepsilon_n = \varepsilon$, GKFI$(A_n, F_n, \varepsilon_n)_2$ was investigated by Peng et al. [32]. They discussed the semicontinuity of approximate solution mappings for GKFI$(A_n, F_n, \varepsilon_n)_2$. Moreover, if $\varepsilon_n = \varepsilon$ is fixed, problem GKF$(A_n, F_n, \varepsilon_n)_2$ was also investigated by Anh and Khanh [2]. They also established the semicontinuity of solution mappings for GKFI$(A_n, F_n, \varepsilon_n)_2$. Comparing our results with the corresponding ones of Peng et al.[32], Anh and Khanh [2], one can find that the assumptions, the proof method (and the goal) of the paper are quite different from the references [32] and [2].

Remark 3.17. If $\varepsilon_n = \varepsilon = 0$, the problem GKFI$(A_n, F_n, \varepsilon_n)_1$ was investigated by Fang and Li [12] and Zhao et al.[36]. Under certain assumptions, they study the Painlev*é*-Kuratowski convergence of solution mappings for generalized Ky Fan inequality, respectively. By comparison, the assumptions and the proof method are different from the corresponding ones in Fang and Li [12], Zhao et al. [36].

Remark 3.18. In [23], GKFI$(A_n, F_n, \varepsilon_n)_1$ and GKFI$(A_n, F_n, \varepsilon_n)_2$ were investigated by Li et al. If $A_n = A, F_n(x,y) = f(x)$ and $\varepsilon_n = \varepsilon = 0$, GKFI$(A_n, F_n, \varepsilon_n)_1$ and GKFI$(A_n, F_n, \varepsilon_n)_2$ were also discussed by Lalitha et al. [21]. Since the assumptions in the paper are weaker than the strictly proper quasi C-convexity and the continuous convergence, our results improve and extend the corresponding ones in Lalitha et al. [21] and Li et al.[23].

Now, we give example 3.19 to illustrate the obtained results of Theorem 3.14 and Corollary 3.15.

Example 3.19. Let $C = R^2_+ = \{x = (x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$. Let $A = [0,1], A_n = [\frac{1}{n}, 1 + \frac{1}{n}]$ and define the mappings $F_n : A_n \times A_n \longrightarrow R^2$ by

$$F(x,y) = (-(1+e^x)(x^2 - y^2), -2(x^2 - y^2));$$

$$F_n(x,y) = (-(1+e^x)((x - \frac{2}{3n})^2 - (y - \frac{2}{3n})^2), -2((x - \frac{2}{3n})^2 - (y - \frac{2}{3n})^2)).$$

It follows from direct computations that

$$\text{WSol}(A_n, F_n) = \text{Sol}(A_n, F_n) = \{\frac{2}{3n}\}$$

and

$$\text{WSol}(A, F) = \text{Sol}(A, F) = \{0\}.$$

We can verify that conditions (i)(iii)-(iv) of Theorem 3.14 (Corollary 3.15) and $\limsup_n^\infty A_n = \{0\}$ are satisfied. The condition (ii) of Theorem 3.14 (Corollary 3.15) can be checked as follows. (without loss of generality, now we verify $-F_n(\cdot,\cdot) \mid_{A_n} \xrightarrow{\Gamma} -F(\cdot,\cdot) \mid_A$). In fact, we have

(a) $A_n \xrightarrow{P.K.} A$.

(b) $\forall\, U \in \mathcal{U}(x) \times \mathcal{U}(y), \forall \epsilon \in \text{int}C, \exists n_{\epsilon,U} \in \mathcal{N}$ such that $\forall n \geq n_{\epsilon,U}, \exists (x_n, y_n) := (x,y) \in U$ such that

$$-F_n(x_n, y_n) + F(x,y) = ((1+e^x)(\frac{4y}{3n} - \frac{4x}{3n}), \frac{8y}{3n} - \frac{8x}{3n}),$$

which implies
$$-F_n(x_n, y_n) \in -F(x, y) + \epsilon - C.$$

(c) $\forall \epsilon \in \mathrm{int}C, \exists U_\epsilon = (x - \frac{2}{3n}, x + \frac{2}{3n}) \times (y - \frac{2}{3n}, y + \frac{2}{3n}) \in \mathcal{U}(x) \times \mathcal{U}(y), \exists N_\epsilon$ such that $(x', y') \in U_\epsilon \bigcap (A_n \times A_n), \forall\ n \geq N_\epsilon$, we have

$$-F_n(x', y') = ((1 + e^{x'})(x'^2 - y'^2 + \frac{4y'}{3n} - \frac{4x'}{3n}), 2x'^2 - 2y'^2 + \frac{8y'}{3n} - \frac{8x'}{3n}),$$

$$F(x, y) = (-(1 + e^x)(x^2 - y^2), -2(x^2 - y^2)).$$

One can easily obtain that

$$-F_n(x', y') \in -F(x, y) - \epsilon + C.$$

Thus, all assumptions of Theorem 3.14 (or Corollary 3.15) are satisfied. From Theorem 3.14 (or Corollary 3.15), one can easily find there exists $\varepsilon_n \searrow 0$ such that

$$\mathrm{Sol}(A_n, F_n, \varepsilon_n) \xrightarrow{P.K.} \mathrm{Sol}(A, F) \quad (\ \mathrm{WSol}(A_n, F_n, \varepsilon_n) \xrightarrow{P.K.} \mathrm{WSol}(A, F)).$$

Thus, Theorem 3.14 and Corollary 3.15 are true and applicable.

Finally, Example 3.20 is given to illustrate that our main results extend and improve the corresponding ones in the literatures (i.e.,[21], [23]).

Example 3.20. Let $C := \mathbb{R}^2_+ = \{x = (x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$ and $A = [0, 1], A_n = [\frac{1}{n}, 1 - \frac{1}{n}]$ $(n \geq 2)$. Define the mappings F_n, F as the same as in Example 3.9. One can compute that for any $\varepsilon_n \searrow 0$,

$$\mathrm{WSol}(A_n, F_n, \varepsilon_n) = \mathrm{Sol}(A_n, F_n, \varepsilon_n) = (\frac{1}{n} - \varepsilon_n,\ 1 - \frac{1}{n}] \cap A_n$$

and
$$\mathrm{WSol}(A, F) = \mathrm{Sol}(A, F) = [0, 1].$$

We can also verify that all assumptions of Theorem 3.14 (or Corollary 3.15) are satisfied, by virtue of Theorem 3.14 (or Corollary 3.15), taking $\varepsilon'_n = \frac{4}{3n,}$ we obtain

$$\mathrm{Sol}(A_n, F_n, \varepsilon_n) \xrightarrow{P.K.} \mathrm{Sol}(A, F).$$

However, the assumption of strictly proper quasi C-convexity of $F(x, \cdot)$ on A is not satisfied. Indeed, take $y' = \frac{1}{4}, y'' = \frac{3}{4}(x \neq y)$ and $\lambda = \frac{1}{2}$, then it follows that

$$F(x, \lambda y_1 + (1 - \lambda) y_2) = (\frac{1}{2} - x, x - \frac{1}{2}) \notin F(x, y_1) - \mathrm{int}C,$$

and
$$F(x, \lambda y_1 + (1 - \lambda) y_2) = (\frac{3}{4} - x, x - \frac{3}{4}) \notin F(x, y_2) - \mathrm{int}C,$$

which illustrate that F is not strictly properly quasi C-convex on A. Thus, the main results in [21] (see, [21, Theorems 4.5 and 5.1]) and [23] (see, [23, Lemma 3.2, Theorems 3.2 and 3.5]) are all not applicable in this case.

4 Conclusion

This paper aims at concerning with the stability analysis of approximate solution mappings for GKFI. Without using the strictly proper quasi C-convexity and the continuous convergence, the Painlevé-Kuratowski convergence of the approximate solution mapping of a family for perturbed problems to the corresponding solution mapping of the original problem is explored, where it is perturbed by vector-valued bifunction sequences and set sequences. The relation between the convergence in the sense of Painlevé-Kuratowski and the convergence in the sense of Hausdorff is also very interesting and important, and we will study it later.

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