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HÖLDER CONTINUITY RESULTS FOR PARAMETRIC SET OPTIMIZATION PROBLEMS VIA IMPROVEMENT SETS*

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Abstract: In this paper, we consider a class of parametric set optimization problems, where both objective functions and constraint functions are perturbed by different parameters. Firstly, upper and lower set orderings with respect to improvement sets are introduced and used to define solution mappings. Then, some assumptions including strong domination properties are proposed to study the Hölder continuity of solution mappings and corresponding optimal value mappings. Our results generalize the upper Hölder continuity of efficient solution mappings for parametric vector optimization problems.

Key words: *Hölder continuity, upper and lower set orderings, solution mappings, optimal value mappings, improvement sets*

Mathematics Subject Classification: *90C31, 49K30*

1 Introduction

Set optimization problems are a class of generalized problems analogous to vector optimization problems. When the objective functions or/and constraint functions of set optimization problems are perturbed, the study of stability and sensitivity of solution sets are significant. For relevant research, we refer to [10, 18, 2, 12, 6, 19] and the references therein. As far as we know, there are two ways to define solution sets for set optimization problems. One is the vector approach based on Pareto orderings, the other is the set approach based on upper and lower set orderings. The former might not be appropriate for applications, some examples are given in [6]. So, researchers turn their attention to the set approach. Up to now, works on stability of solution mappings defined by set orderings for parametric set optimization problems (PSOPs for short) primarily focuses on the semicontinuity, see, for example, [18, 14, 19].

Hölder continuity of solution mappings for PSOPs reflects the perturbation of solution sets accurately than the semicontinuity. There has been some works on Hölder continuity of solution mappings for parametric vector optimization or multivalued equilibrium problems under appropriate assumptions, and these assumptions have also been continuously improved to make them easier to satisfy, see [8, 9, 7, 3, 4, 5, 15, 13, 17, 16] for more details. Up to now, limited work has been done on Hölder continuity for PSOPs, although Anh et.al [1]

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have made some contribution to this subject with respect to approximate solution mappings, recently. The main reason is that under different orderings, solution mappings of PSOPs are different, namely, when the solution mapping under Pareto orderings is Hölder continuous, the solution mapping under upper or lower set orderings may not be Hölder continuous. So, we need to introduce new assumptions different from those under Pareto orderings to characterize Hölder continuity under upper and lower set orderings.

Motivated by literatures [11, 12, 6, 13, 16], we consider a class of parametric set optimization problems, where the objective function and constraint function are perturbed by different parameters. Firstly, since improvement sets are helpful to unify various approximate solution sets, we define the solution mappings by upper and lower set orderings with respect to improvement sets. This class of solution mappings coincide with the effective solution mappings under Pareto orderings when set optimization problems reduce to vector optimization problems. Then, we propose new strong domination properties and some other assumptions to study the Hölder continuity of solution mappings and corresponding optimal value mappings.

The rest of the paper is organized as follows. In Section 2, we recall some notations for PSOPs. In Section 3, we discuss the Hölder continuity of solution mappings and corresponding optimal value mappings defined by the lower set ordering, and then we apply this method to that defined by the upper set ordering. Finally, we give conclusions in Section 4.

2 Notation and Preliminaries

Let X, Y, Z, Π_1, Π_2 be norm topological vector spaces, $\Lambda \subset \Pi_1$ and $\Omega \subset \Pi_2$ be nonempty sets, $C \subset Y$ be a pointed, closed and convex cone with nonempty interior. When there is no fear of confusion, we always denote by $\|\cdot\|$ and $U(\cdot)$ the norm and the neighborhood of a point in different spaces, respectively. We specify $d(x, y) := \|x - y\|$ for any $x, y \in X$. The distance of two points in other spaces is also defined in this way. For a nonempty set $A \subset X$, $\operatorname{int} A$ denotes the topological interior of A. For a set-valued mapping $F : X \rightrightarrows Y$, the domain, graph, and image set of F are given by, respectively,

$$\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\},\ \operatorname{gr} F := \{(x, y) \in X \times Y \mid y \in F(x)\},\ F(X) := \bigcup_{x \in X} F(x).$$

Now, we consider the following parametric set optimization problem (PSOP for short):

$$\text{(PSOP)} \quad \min \quad F(x, \lambda) \quad \text{s.t.} \quad x \in \Phi(\mu),$$

where $\lambda \in \Lambda, \mu \in \Omega$ are parameters and $F : X \times \Lambda \rightrightarrows Y$, $\Phi : \Omega \rightrightarrows X$ are set-valued mappings. To avoid the triviality, we always assume that $F(\cdot, \lambda)$ is nonempty-valued for each $\lambda \in \Lambda$.

Definition 2.1 ([12])**.** A nonempty set $E \subset Y$ is said to be an improvement set with respect to C if $0 \notin E$ and E is free disposal, i.e. $E + C = E$.

In the sequel, we always assume that $\operatorname{int} E \neq \emptyset$. Here, we recall some useful definitions. For a set A and an improvement set E, $a \in A$ is called an E-minimal element of A if $(a - E) \cap A = \emptyset$. As we know, the E-minimal solution mapping $S : \Lambda \times \Omega \rightrightarrows X$ for (PSOP) can be defined as

$$S(\lambda, \mu) := \{x \in \Phi(\mu) \mid \exists z \in F(x, \lambda) \text{ s.t. } (z - E) \cap F(y, \lambda) = \emptyset, \forall y \in \Phi(\mu)\}.$$

However, the solution sets of set optimization problems defined by Pareto orderings may not be appropriate for some applications. In this case, researchers tend to choose upper and lower set orderings to define solutions sets. Dhingra and Lalitha [12] extended upper and lower set orderings to the following E-upper and E-lower set orderings via an improvement set E, respectively:

$$A \leq_E^u B \Leftrightarrow A \subset B - E, \ A \leq_E^l B \Leftrightarrow B \subset A + E.$$

Based on this, the E-l-minimal solution mapping and E-u-minimal solution mapping for (PSOP) are defined by $S_l : \Lambda \times \Omega \rightrightarrows X$ and $S_u : \Lambda \times \Omega \rightrightarrows X$, respectively,

$$S_l(\lambda, \mu) := \big\{x \in \Phi(\mu) \mid F(y, \lambda) \leq_E^l F(x, \lambda) \Rightarrow F(x, \lambda) \leq_E^l F(y, \lambda), \forall y \in \Phi(\mu)\big\},$$

$$S_u(\lambda, \mu) := \big\{x \in \Phi(\mu) \mid F(y, \lambda) \leq_E^u F(x, \lambda) \Rightarrow F(x, \lambda) \leq_E^u F(y, \lambda), \forall y \in \Phi(\mu)\big\}.$$

The corresponding E-l optimal value mapping and E-u optimal value mapping for (PSOP) are defined by $V_l : \Lambda \times \Omega \rightrightarrows Y$ and $V_u : \Lambda \times \Omega \rightrightarrows Y$, respectively,

$$V_l(\lambda, \mu) := F(S_l(\lambda, \mu), \lambda) = \bigcup_{x \in S_l(\lambda, \mu)} F(x, \lambda),$$

$$V_u(\lambda, \mu) := F(S_u(\lambda, \mu), \lambda) = \bigcup_{x \in S_u(\lambda, \mu)} F(x, \lambda).$$

Lemma 2.2. *Let $E \subset C \backslash \{0\}$. Suppose that $F(x, \lambda)$ is compact-valued for any $\lambda \in \Lambda$ and $x \in X$. Then, the following statements are true.*

(i) *$x \in S_l(\lambda, \mu)$ if and only if there is no $y \in \Phi(\mu)$ such that $F(y, \lambda) \leq_E^l F(x, \lambda)$,*

(ii) *$x \in S_u(\lambda, \mu)$ if and only if there is no $y \in \Phi(\mu)$ such that $F(y, \lambda) \leq_E^u F(x, \lambda)$.*

Proof. The proof is similar to Proposition 3.3 of [12] and we omit it here. □

We conclude this section by recalling the definitions of Hölder continuity for set-valued mappings. Let $l, l_1, l_2 \geq 0$, $\alpha, \alpha_1, \alpha_2 > 0$ be constants and $\mathbb{B}_Y$ be an unit ball of Y.

Definition 2.3 ([8])**.** A set-valued mapping $F : X \rightrightarrows Y$ is said to be $l.\alpha$-Hölder continuous at $x_0 \in \operatorname{dom} F$ if there exists $U(x_0)$ such that $F(x_1) \subset F(x_2) + l d^{\alpha}(x_1, x_2)\mathbb{B}_Y$ for all $x_1, x_2 \in U(x_0)$.

Definition 2.4 ([8])**.** A set-valued mapping $F : X \times Y \rightrightarrows Z$ is said to be, respectively,

(i) $(l_1.\alpha_1, l_2.\alpha_2)$-Hölder continuous at $(x_0, y_0) \in \operatorname{dom} F$ if there exist $U(x_0)$ and $U(y_0)$ such that $F(x_1, y_1) \subset \big\{z \in Z \mid \exists w \in F(x_2, y_2), d(z, w) \leq l_1 d^{\alpha_1}(x_1, x_2) + l_2 d^{\alpha_2}(y_1, y_2)\big\}$ for all $x_1, x_2 \in U(x_0)$ and $y_1, y_2 \in U(y_0)$.

(ii) $(l_1.\alpha_1, l_2.\alpha_2)$-pseudo-Hölder continuous at $(x_0, y_0, z_0) \in \operatorname{gr} F$ if there exist $U(x_0)$, $U(y_0)$ and zero neighborhood U such that $F(x_1, y_1) \cap (z_0 + U) \subset \big\{z \in Z \mid \exists w \in F(x_2, y_2), d(z, w) \leq l_1 d^{\alpha_1}(x_1, x_2) + l_2 d^{\alpha_2}(y_1, y_2)\big\}$ for all $x_1, x_2 \in U(x_0)$ and $y_1, y_2 \in U(y_0)$.

For (PSOP), the following example shows that the E-minimal solution mapping is Hölder continuous, while the E-u-minimal solution mapping is not. In some degree, it inspires us to study Hölder continuity of solution mappings defined by set orderings.

Example 2.5. Based on Example 2.2 of [19], suppose that $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $\Lambda = \Omega = [0,1]$, $C = \mathbb{R}^2_+$ and $E = \mathbb{R}^2_+\backslash\{0\}$. Let $\Phi(\mu) = [0,1]$ and

$$F(x,\lambda) = \begin{cases} [0,1+x]\times[0,1+x], & \lambda = 0, \\ \{\lambda\}\times[0,1], & \lambda \neq 0. \end{cases}$$

For any $\mu \in \Omega$, it is easy to get that

$$S_u(\lambda,\mu) = \begin{cases} \{0\}, & \lambda = 0, \\ [0,1], & \lambda \in (0,1], \end{cases}$$

and $S(\lambda,\mu) = [0,1]$ for any $\lambda \in [0,1]$. Let $l_1 = l_2 = \alpha_1 = \alpha_2 = \frac{1}{2}$, $\lambda_0 = \mu_0 = \frac{1}{2}$, $U(\lambda_0) = U(\mu_0) = [0,1]$, we observe that S is $(l_1.\alpha_1, l_2.\alpha_2)$-Hölder continuous at (λ_0,μ_0). However, if we take $\lambda_1 = 1$, $\lambda_2 = 0$, $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{3}$, there exists $x_1 = 1 \in S_u(\lambda_1,\mu_1)$ such that $d(1,0) = 1 > \frac{1}{2}\cdot 1^{\frac{1}{2}} + \frac{1}{2}\cdot\left(\frac{1}{6}\right)^{\frac{1}{2}}$. This shows that S_u is not $(l_1.\alpha_1, l_2.\alpha_2)$-Hölder continuous at (λ_0,μ_0).

3 Hölder Continuity of $S_l(S_u)$ and $V_l(V_u)$

Let $(\lambda_0,\mu_0) \in \Lambda\times\Omega$ be given, $(\lambda,\mu) \in U(\lambda_0)\times U(\mu_0)$ be any given, $S_l(\lambda,\mu)$ and $S_u(\lambda,\mu)$ be nonempty for all $(\lambda,\mu) \in U(\lambda_0)\times U(\mu_0)$, $\varepsilon > 0$, $\beta \geq 1$, $l \geq 0$, $\alpha > 0$ be constants. Firstly, we introduce the following assumptions for (PSOP) for further study.

- (A_1) For each $x \in \Phi(\mu)$ and $z \in F(x,\lambda)$, there exist $\tilde{x} \in S_l(\lambda,\mu)$ and $\tilde{z} \in F(\tilde{x},\lambda)$ $(\tilde{z} \neq z)$ that satisfy $z - \tilde{z} + \varepsilon d^\beta(x,\tilde{x})\mathbb{B}_Y \subset E$.
- (A_2) There exists a neighborhood $U(x_0)$ of $x_0 \in X$ such that for each $x \in \Phi(\mu)\cap U(x_0)$ and $z \in F(x,\lambda)$, there exist $\tilde{x} \in S_l(\lambda,\mu)\cap U(x_0)$ and $\tilde{z} \in F(\tilde{x},\lambda)$ $(\tilde{z} \neq z)$ that satisfy $z - \tilde{z} + \varepsilon d^\beta(x,\tilde{x})\mathbb{B}_Y \subset E$.
- (A_3) Replace the $S_l(\lambda,\mu)$ in (A_1) with $S_u(\lambda,\mu)$.
- (A_4) Replace the $S_l(\lambda,\mu)$ in (A_2) with $S_u(\lambda,\mu)$.
- (A_5) $E \subset C\backslash\{0\}$, $F(x,\lambda)$ is compact-valued for every $x \in X$.
- (A_6) Φ is $l.\alpha$-Hölder continuous at $\mu_0 \in \operatorname{dom}\Phi$.
- (A_6') Φ is $l.\alpha$-pseudo-Hölder continuous at $(\mu_0, x_0) \in \operatorname{gr}\Phi$.
- (A_7) There exist constants $m,\gamma > 0$ such that

$$\sup_{z_1\in F(x_1,\lambda)}\ \sup_{z_2\in F(x_2,\lambda)} d(z_1,z_2) \leq m d^\gamma(x_1,x_2),\ \forall x_1,x_2 \in \Phi(U(\mu_0)), \forall\lambda \in U(\lambda_0),$$

and there exist constants $n,\delta > 0$ such that

$$\sup_{w_1\in F(x,\lambda_1)}\ \sup_{w_2\in F(x,\lambda_2)} d(w_1,w_2) \leq n d^\delta(\lambda_1,\lambda_2),\ \forall\lambda_1,\lambda_2 \in U(\lambda_0), \forall x \in \Phi(\mu), \mu \in U(\mu_0).$$

Remark 3.1. Assumptions (A_1) and (A_3) generalize Definition 2.3 of [13], which are called strong domination properties of S_l and S_u, respectively. Here, it is reasonable to suppose that $z \neq \tilde{z}$. Indeed, if $z = \tilde{z}$, we have $\varepsilon d^\beta(x,\tilde{x})\mathbb{B}_Y \subset E$, which implies that $0 \in E$. This is a contradiction with that E is an improvement set. Moreover, we denote $\|y\|_+ := d(y, Y\backslash E)$ for every $y \in Y$. Then, $z - \tilde{z} + \varepsilon d^\beta(x,\tilde{x})\mathbb{B}_Y \subset E$ becomes $\varepsilon d^\beta(x,\tilde{x}) \leq \|z-\tilde{z}\|_+$.

Theorem 3.2. *If* (A_1), (A_5), (A_6) *and* (A_7) *are satisfied, then* S_l *is* $(l_1.\alpha_1, l_2.\alpha_2)$*-Hölder continuous at* $(\lambda_0, \mu_0) \in \operatorname{dom} S_l$*, where* $l_1 = \left(\frac{2n}{\varepsilon}\right)^{\frac{1}{\beta}}$, $\alpha_1 = \frac{\delta}{\beta}$, $l_2 = \left(\frac{2ml^\gamma}{\varepsilon}\right)^{\frac{1}{\beta}} + l$, $\alpha_2 = \min\left\{\alpha, \frac{\alpha\gamma}{\beta}\right\}$ *if* $d(\mu_1, \mu_2) \le 1$ *or* $\alpha_2 = \max\left\{\alpha, \frac{\alpha\gamma}{\beta}\right\}$ *if* $d(\mu_1, \mu_2) > 1$.

Proof. Let $\lambda_1, \lambda_2 \in U(\lambda_0)$ and $\mu_1, \mu_2 \in U(\mu_0)$ be arbitrary. For any $x(\lambda_1, \mu_1) \in S_l(\lambda_1, \mu_1)$, in view of $x(\lambda_1, \mu_1) \in \Phi(\mu_1)$ and (A_6), there exists $x(\mu_2) \in \Phi(\mu_2)$ such that

$$d(x(\lambda_1, \mu_1), x(\mu_2)) \le l d^\alpha(\mu_1, \mu_2). \tag{3.1}$$

By (A_1), for $x(\mu_2) \in \Phi(\mu_2)$ and each $z(\mu_2) \in F(x(\mu_2), \lambda_2)$, there exist $x(\lambda_2, \mu_2) \in S_l(\lambda_2, \mu_2)$ and $z(\lambda_2, \mu_2) \in F(x(\lambda_2, \mu_2), \lambda_2)$ with $z(\lambda_2, \mu_2) \ne z(\mu_2)$ such that

$$z(\mu_2) - z(\lambda_2, \mu_2) + \varepsilon d^\beta(x(\mu_2), x(\lambda_2, \mu_2))\mathbb{B}_Y \subset E. \tag{3.2}$$

If $x(\mu_2) = x(\lambda_2, \mu_2)$, from (3.1), we get that $d(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)) \le l d^\alpha(\mu_1, \mu_2)$ and S_l is $(0.\alpha_1, l.\alpha)$-Hölder continuous at (λ_0, μ_0). Without loss of generality, we assume that $x(\mu_2) \ne x(\lambda_2, \mu_2)$. For $x(\lambda_2, \mu_2) \in \Phi(\mu_2)$, using (A_6) again, there exists $x(\mu_1) \in \Phi(\mu_1)$ such that

$$d(x(\lambda_2, \mu_2), x(\mu_1)) \le l d^\alpha(\mu_1, \mu_2). \tag{3.3}$$

For any $z(\lambda_1, \mu_1) \in F(x(\lambda_1, \mu_1), \lambda_1)$, there exist $z(\mu_1) \in F(x(\mu_1), \lambda_1)$, $z'(\lambda_2, \mu_2) \in F(x(\lambda_2, \mu_2), \lambda_1)$ and $z'(\mu_2) \in F(x(\mu_2), \lambda_1)$ such that

$$z(\mu_1) - z(\lambda_1, \mu_1) = z(\lambda_2, \mu_2) - z(\mu_2) + \vartheta, \tag{3.4}$$

where $\vartheta = z(\mu_1) - z'(\lambda_2, \mu_2) + z'(\lambda_2, \mu_2) - z(\lambda_2, \mu_2) + z(\mu_2) - z'(\mu_2) + z'(\mu_2) - z(\lambda_1, \mu_1)$. It follows from (A_7) that

$$\begin{aligned} \|\vartheta\| &\le m d^\gamma(x(\mu_1), x(\lambda_2, \mu_2)) + 2n d^\delta(\lambda_1, \lambda_2) + m d^\gamma(x(\mu_2), x(\lambda_1, \mu_1)) \\ &\le 2ml^\gamma d^{\alpha\gamma}(\mu_1, \mu_2) + 2n d^\delta(\lambda_1, \lambda_2). \end{aligned} \tag{3.5}$$

The second inequality of (3.5) is due to (3.1) and (3.3). Here, we prove that $\|z(\mu_2) - z(\lambda_2, \mu_2)\|_+ \le \|\vartheta\|$. Suppose that $\|z(\mu_2) - z(\lambda_2, \mu_2)\|_+ > \|\vartheta\|$, then

$$z(\mu_2) - z(\lambda_2, \mu_2) + \|\vartheta\|\mathbb{B}_Y \subset \operatorname{int} E. \tag{3.6}$$

If $\|\vartheta\| = 0$, we have $z(\lambda_1, \mu_1) - z(\mu_1) \in \operatorname{int} E$ by (3.4) and (3.6). Considering the arbitrariness of $z(\lambda_1, \mu_1) \in F(x(\lambda_1, \mu_1), \lambda_1)$ and the existence of $z(\mu_1) \in F(x(\mu_1), \lambda_1)$, we know for $x(\mu_1) \in \Phi(\mu_1)$ that $F(x(\lambda_1, \mu_1), \lambda_1) \subset F(x(\mu_1), \lambda_1) + E$, namely, $F(x(\mu_1), \lambda_1) \le_E^l F(x(\lambda_1, \mu_1), \lambda_1)$. This contradicts with $x(\lambda_1, \mu_1) \in S_l(\lambda_1, \mu_1)$ by virtue of (A_5) and Lemma 2.2. If $\|\vartheta\| > 0$, take $-\frac{\vartheta}{\|\vartheta\|} \in \mathbb{B}_Y$, we have $z(\mu_2) - z(\lambda_2, \mu_2) + \|\vartheta\|\left(-\frac{\vartheta}{\|\vartheta\|}\right) \in \operatorname{int} E$ and $z(\lambda_1, \mu_1) - z(\mu_1) \in \operatorname{int} E$, which leads to a contradiction, similarly. Therefore, it follows from (3.2) and Remark 3.1 that

$$\varepsilon d^\beta(x(\mu_2), x(\lambda_2, \mu_2)) \le \|z(\mu_2) - z(\lambda_2, \mu_2)\|_+ \le \|\vartheta\| \le 2ml^\gamma d^{\alpha\gamma}(\mu_1, \mu_2) + 2n d^\delta(\lambda_1, \lambda_2).$$

In view of $\beta \ge 1$, we get

$$d(x(\mu_2), x(\lambda_2, \mu_2)) \le \left(\frac{2ml^\gamma}{\varepsilon}\right)^{\frac{1}{\beta}} d^{\frac{\alpha\gamma}{\beta}}(\mu_1, \mu_2) + \left(\frac{2n}{\varepsilon}\right)^{\frac{1}{\beta}} d^{\frac{\delta}{\beta}}(\lambda_1, \lambda_2).$$

Further, there holds

$$\begin{aligned} d(x(\lambda_1,\mu_1),x(\lambda_2,\mu_2)) &\le d(x(\lambda_1,\mu_1),x(\mu_2)) + d(x(\mu_2),x(\lambda_2,\mu_2)) \\ &\le ld^{\alpha}(\mu_1,\mu_2) + \left(\frac{2ml^{\gamma}}{\varepsilon}\right)^{\frac{1}{\beta}} d^{\frac{\alpha\gamma}{\beta}}(\mu_1,\mu_2) + \left(\frac{2n}{\varepsilon}\right)^{\frac{1}{\beta}} d^{\frac{\delta}{\beta}}(\lambda_1,\lambda_2). \end{aligned}$$

Take $l_1 = \left(\frac{2n}{\varepsilon}\right)^{\frac{1}{\beta}}$, $\alpha_1 = \frac{\delta}{\beta}$, $l_2 = \left(\frac{2ml^{\gamma}}{\varepsilon}\right)^{\frac{1}{\beta}} + l$, $\alpha_2 = \min\left\{\alpha, \frac{\alpha\gamma}{\beta}\right\}$ if $d(\mu_1,\mu_2) \le 1$, $\alpha_2 = \max\left\{\alpha, \frac{\alpha\gamma}{\beta}\right\}$ if $d(\mu_1,\mu_2) > 1$. In a word, for arbitrary $x(\lambda_1,\mu_1) \in S_l(\lambda_1,\mu_1)$, there exists $x(\lambda_2,\mu_2) \in S_l(\lambda_2,\mu_2)$ such that $d(x(\lambda_1,\mu_1),x(\lambda_2,\mu_2)) \le l_1 d^{\alpha_1}(\lambda_1,\lambda_2) + l_2 d^{\alpha_2}(\mu_1,\mu_2)$. That is, S_l is $(l_1.\alpha_1, l_2.\alpha_2)$-Hölder continuous at (λ_0,μ_0). □

Remark 3.3. In [13], the authors studied a class of perturbed vector optimization problems. They defined the efficient solution mapping under Pareto orderings and obtained its upper Hölder continuity. By contrast, we consider set optimization problems and characterize the Hölder continuity of the E-l-minimal solution mapping. Since E-l-minimal solution sets for set optimization problems can be collapsed to efficient solution sets for vector problems and Hölder continuity implies upper Hölder continuity, our results are more generalized. Next, we give an example to illustrate Theorem 3.2.

Example 3.4. Suppose that $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $\Lambda = \Omega = [0,1]$, $C = \left\{(y_1,y_2) \in \mathbb{R}^2 \mid 2y_1 + y_2 \ge 0, \frac{1}{2}y_1 + y_2 \ge 0\right\}$ and $E = C\backslash\{0\}$. Let $\Phi(\mu) = [0,1]$ and $F(x,\lambda) = \left\{(y_1,y_2) \in \mathbb{R}^2 \mid (y_1+x)^2 + (y_2+x)^2 \le \lambda^2\right\}$. Take $\lambda_0 = \mu_0 = \frac{1}{2}$, $U(\lambda_0) = U(\mu_0) = [0,1]$. One easily see that $S_l(\lambda,\mu) = \{1\}$ for all $(\lambda,\mu) \in U(\lambda_0) \times U(\mu_0)$. The conditions in Theorem 3.2 are also satisfied. In fact, S_l is indeed Hölder continuous at (λ_0,μ_0).

Theorem 3.5. *If* (A_2), (A_5), (A_6') *and* (A_7) *are satisfied, then* S_l *is* $(l_1.\alpha_1, l_2.\alpha_2)$*-pseudo-Hölder continuous at* $(\lambda_0,\mu_0,x_0) \in \operatorname{gr} S_l$, *where* $l_1,\alpha_1,l_2,\alpha_2$ *are same as* Theorem 3.2.

Proof. Let $t_\Lambda, t_\Omega > 0$, $\mathbb{B}_\Lambda$ and $\mathbb{B}_\Omega$ be closed unit balls in Λ and Ω, respectively. For a zero neighborhood Q, suppose that U is an arbitrary zero neighborhood satisfying $U + 2^{\alpha} l t_\Omega^{\alpha} \mathbb{B}_X \subset Q$. Take $\lambda_1,\lambda_2 \in \lambda_0 + t_\Lambda\mathbb{B}_\Lambda$ and $\mu_1,\mu_2 \in \mu_0 + t_\Omega\mathbb{B}_\Omega$. For $x(\lambda_1,\mu_1) \in S_l(\lambda_1,\mu_1) \cap (x_0 + U)$, by (A_6'), there exists $x(\mu_2) \in \Phi(\mu_2)$ such that

$$d(x(\lambda_1,\mu_1),x(\mu_2)) \le ld^{\alpha}(\mu_1,\mu_2) \le l\big(d(\mu_1,\mu_0) + d(\mu_0,\mu_2)\big)^{\alpha} \le 2^{\alpha} l t_\Omega^{\alpha}.$$

Since $x(\mu_2) - x_0 = x(\mu_2) - x(\lambda_1,\mu_1) + x(\lambda_1,\mu_1) - x_0 \in 2^{\alpha} l t_\Omega^{\alpha}\mathbb{B}_X + U \subset Q$, we have $x(\mu_2) \in \Phi(\mu_2) \cap (x_0 + Q)$. By (A_2), for each $z(\mu_2) \in F(x(\mu_2),\lambda_2)$, there exist $x(\lambda_2,\mu_2) \in S_l(\lambda_2,\mu_2) \cap (x_0 + Q)$ and $z(\lambda_2,\mu_2) \in F(x(\lambda_2,\mu_2),\lambda_2)$ with $z(\lambda_2,\mu_2) \ne z(\mu_2)$ such that

$$z(\mu_2) - z(\lambda_2,\mu_2) + \varepsilon d^{\beta}(x(\mu_2),x(\lambda_2,\mu_2))\mathbb{B}_Y \subset E.$$

Without loss of generality, we assume that $x(\mu_2) \ne x(\lambda_2,\mu_2)$. To use (A_6') again, for $x(\lambda_2,\mu_2) \in \Phi(\mu_2)$, there exists $x(\mu_1) \in \Phi(\mu_1)$ such that

$$d(x(\lambda_2,\mu_2),x(\mu_1)) \le ld^{\alpha}(\mu_1,\mu_2).$$

In the sequel, we adopt the same discussion in the proof of Theorem 3.2 to obtain that S_l is $(l_1.\alpha_1, l_2.\alpha_2)$-pseudo-Hölder continuous at (λ_0,μ_0,x_0). □

Remark 3.6. We use mild conditions (A_2) and (A_6') to obtain the pseudo-Hölder continuity of S_l, although it also be got under the assumptions of Theorem 3.2.

Next, we turn to the Hölder continuity of optimal value mappings corresponding to E-l-minimal solution mappings.

Theorem 3.7. *Suppose that* (A_1), (A_5), (A_6) *and* (A_7) *are satisfied, and* $1 \le \gamma \le \beta$ *in* (A_7). *Then* V_l *is* $(l_1.\alpha_1, l_2.\alpha_2)$*-Hölder continuous at* $(\lambda_0, \mu_0) \in \operatorname{dom} V_l$, *where* $l_1 = n + \left(\frac{2m^{\frac{\beta}{\gamma}} n}{\varepsilon}\right)^{\frac{\gamma}{\beta}}$, $l_2 = ml^\gamma + \left(\frac{2m^{\frac{\beta}{\gamma}+1} l^\gamma}{\varepsilon}\right)^{\frac{\gamma}{\beta}}$, $\alpha_1 = \frac{\delta\gamma}{\beta}$ *if* $d(\lambda_1, \lambda_2) \le 1$ *or* $\alpha_1 = \delta$ *if* $d(\lambda_1, \lambda_2) > 1$, $\alpha_2 = \frac{\alpha\gamma^2}{\beta}$ *if* $d(\mu_1, \mu_2) \le 1$ *or* $\alpha_2 = \alpha\gamma$ *if* $d(\mu_1, \mu_2) > 1$.

Proof. Let $\lambda_1, \lambda_2 \in U(\lambda_0)$ and $\mu_1, \mu_2 \in U(\mu_0)$ be arbitrary. Take any $z(\lambda_1, \mu_1) \in V_l(\lambda_1, \mu_1)$, that is, take $z(\lambda_1, \mu_1) \in F(x(\lambda_1, \mu_1), \lambda_1)$ for $x(\lambda_1, \mu_1) \in S_l(\lambda_1, \mu_1)$. By (A_6), for $x(\lambda_1, \mu_1) \in \Phi(\mu_1)$, there exists $x(\mu_2) \in \Phi(\mu_2)$ such that

$$d(x(\lambda_1, \mu_1), x(\mu_2)) \le l d^\alpha(\mu_1, \mu_2). \tag{3.7}$$

By (A_1), for $x(\mu_2) \in \Phi(\mu_2)$ and each $z(\mu_2) \in F(x(\mu_2), \lambda_2)$, there exist $x(\lambda_2, \mu_2) \in S_l(\lambda_2, \mu_2)$ and $z(\lambda_2, \mu_2) \in F(x(\lambda_2, \mu_2), \lambda_2)$ with $z(\lambda_2, \mu_2) \ne z(\mu_2)$ such that

$$z(\mu_2) - z(\lambda_2, \mu_2) + \varepsilon d^\beta(x(\mu_2), x(\lambda_2, \mu_2))\mathbb{B}_Y \subset E. \tag{3.8}$$

Moreover, it follows from (A_7) that $d(z(\mu_2), z(\lambda_2, \mu_2)) \le m d^\gamma(x(\mu_2), x(\lambda_2, \mu_2))$, which implies that

$$\varepsilon m^{-\frac{\beta}{\gamma}} d^{\frac{\beta}{\gamma}}(z(\mu_2), z(\lambda_2, \mu_2)) \le \varepsilon d^\beta(x(\mu_2), x(\lambda_2, \mu_2)).$$

Together with (3.8), we have

$$z(\mu_2) - z(\lambda_2, \mu_2) + \varepsilon m^{-\frac{\beta}{\gamma}} d^{\frac{\beta}{\gamma}}(z(\mu_2), z(\lambda_2, \mu_2))\mathbb{B}_Y \subset E. \tag{3.9}$$

Without loss of generality, we assume that $x(\mu_2) \ne x(\lambda_2, \mu_2)$. To use (A_6) again, for $x(\lambda_2, \mu_2) \in \Phi(\mu_2)$, there exists $x(\mu_1) \in \Phi(\mu_1)$ such that

$$d(x(\lambda_2, \mu_2), x(\mu_1)) \le l d^\alpha(\mu_1, \mu_2). \tag{3.10}$$

We observe that there exist $z(\mu_1) \in F(x(\mu_1), \lambda_1)$, $z'(\lambda_2, \mu_2) \in F(x(\lambda_2, \mu_2), \lambda_1)$ and $z'(\mu_2) \in F(x(\mu_2), \lambda_1)$ such that

$$z(\mu_1) - z(\lambda_1, \mu_1) = z(\lambda_2, \mu_2) - z(\mu_2) + \vartheta,$$

where $\vartheta = z(\mu_1) - z'(\lambda_2, \mu_2) + z'(\lambda_2, \mu_2) - z(\lambda_2, \mu_2) + z(\mu_2) - z'(\mu_2) + z'(\mu_2) - z(\lambda_1, \mu_1)$. By (A_7), (3.7) and (3.10), we get

$$\|\vartheta\| \le 2ml^\gamma d^{\alpha\gamma}(\mu_1, \mu_2) + 2nd^\delta(\lambda_1, \lambda_2). \tag{3.11}$$

Here, $\|z(\mu_2) - z(\lambda_2, \mu_2)\|_+ \le \|\vartheta\|$ is also true according to the proof of Theorem 3.2. Moreover, it follows from Remark 3.1, (3.9) and (3.11) that

$$\varepsilon m^{-\frac{\beta}{\gamma}} d^{\frac{\beta}{\gamma}}(z(\mu_2), z(\lambda_2, \mu_2)) \le 2ml^\gamma d^{\alpha\gamma}(\mu_1, \mu_2) + 2nd^\delta(\lambda_1, \lambda_2).$$

Since $\beta \ge \gamma$, one has

$$d(z(\mu_2), z(\lambda_2, \mu_2)) \le \left(\frac{2m^{\frac{\beta}{\gamma}+1} l^\gamma}{\varepsilon}\right)^{\frac{\gamma}{\beta}} d^{\frac{\alpha\gamma^2}{\beta}}(\mu_1, \mu_2) + \left(\frac{2m^{\frac{\beta}{\gamma}} n}{\varepsilon}\right)^{\frac{\gamma}{\beta}} d^{\frac{\delta\gamma}{\beta}}(\lambda_1, \lambda_2).$$

Hence, in terms of (A_7) and (3.7), there holds

$$d(z(\lambda_1,\mu_1),z(\lambda_2,\mu_2)) \le d(z(\lambda_1,\mu_1),z'(\mu_2)) + d(z'(\mu_2),z(\mu_2)) + d(z(\mu_2),z(\lambda_2,\mu_2))$$
$$\le m d^{\gamma}(x(\lambda_1,\mu_1),x(\mu_2)) + n d^{\delta}(\lambda_1,\lambda_2) + \left(\frac{2m^{\frac{\beta}{\gamma}+1}l^{\gamma}}{\varepsilon}\right)^{\frac{\gamma}{\beta}} d^{\frac{\alpha\gamma^2}{\beta}}(\mu_1,\mu_2) + \left(\frac{2m^{\frac{\beta}{\gamma}}n}{\varepsilon}\right)^{\frac{\gamma}{\beta}} d^{\frac{\delta\gamma}{\beta}}(\lambda_1,\lambda_2)$$
$$\le m l^{\gamma} d^{\alpha\gamma}(\mu_1,\mu_2) + n d^{\delta}(\lambda_1,\lambda_2) + \left(\frac{2m^{\frac{\beta}{\gamma}+1}l^{\gamma}}{\varepsilon}\right)^{\frac{\gamma}{\beta}} d^{\frac{\alpha\gamma^2}{\beta}}(\mu_1,\mu_2) + \left(\frac{2m^{\frac{\beta}{\gamma}}n}{\varepsilon}\right)^{\frac{\gamma}{\beta}} d^{\frac{\delta\gamma}{\beta}}(\lambda_1,\lambda_2).$$

Take $l_1 = n + \left(\frac{2m^{\frac{\beta}{\gamma}}n}{\varepsilon}\right)^{\frac{\gamma}{\beta}}$, $l_2 = ml^{\gamma} + \left(\frac{2m^{\frac{\beta}{\gamma}+1}l^{\gamma}}{\varepsilon}\right)^{\frac{\gamma}{\beta}}$, $\alpha_1 = \frac{\delta\gamma}{\beta}$ if $d(\lambda_1,\lambda_2) \le 1$ or $\alpha_1 = \delta$ if $d(\lambda_1,\lambda_2) > 1$, $\alpha_2 = \frac{\alpha\gamma^2}{\beta}$ if $d(\mu_1,\mu_2) \le 1$ or $\alpha_2 = \alpha\gamma$ if $d(\mu_1,\mu_2) > 1$. In a word, we obtain that V_l is $(l_1.\alpha_1, l_2.\alpha_2)$-Hölder continuous at (λ_0,μ_0). □

The Hölder continuity of solution mappings and optimal value mappings defined by E-upper set orderings can be similarly characterized.

Theorem 3.8. *If* (A_3), (A_5), (A_6) *and* (A_7) *are satisfied, then* S_u *is* $(l_1.\alpha_1, l_2.\alpha_2)$*-Hölder continuous at* $(\lambda_0,\mu_0) \in \operatorname{dom} S_u$*, where* $l_1,\alpha_1,l_2,\alpha_2$ *are same as* Theorem 3.2.

Proof. The proof is similar to Theorem 3.2, we just reprove some different parts. Here, we need to take arbitrary $z(\mu_1) \in F(x(\mu_1),\lambda_1)$. One can see there also exist $z(\lambda_1,\mu_1) \in F(x(\lambda_1,\mu_1),\lambda_1)$, $z'(\lambda_2,\mu_2) \in F(x(\lambda_2,\mu_2),\lambda_1)$, $z'(\mu_2) \in F(x(\mu_2),\lambda_1)$ such that (3.4) holds. If $\|\vartheta\| = 0$, similarly, we have $z(\lambda_1,\mu_1) - z(\mu_1) \in \operatorname{int} E$. Considering the arbitrariness of $z(\mu_1) \in F(x(\mu_1),\lambda_1)$, we know there exists $x(\mu_1)$ such that $F(x(\mu_1),\lambda_1) \subset F(x(\lambda_1,\mu_1),\lambda_1) - E$. This contradicts with $x(\lambda_1,\mu_1) \in S_u(\lambda_1,\mu_1)$. We omit the rest of proof. □

Example 3.9. Based on Example 2.3 of [19], suppose that $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $\Lambda = \Omega = [0,1]$, $C = \mathbb{R}^2_+$ and $E = \mathbb{R}^2_+ \backslash \{0\}$. Let $\Phi(\mu) = [0,1]$ and

$$F(x,\lambda) = \begin{cases} [0,1+x] \times [0,1+x], & \lambda = 0 \\ [0,1+\lambda] \times [x,1+x], & \lambda \neq 0. \end{cases}$$

Through a simply calculation, we obtain that $S_u(\lambda,\mu) = \{0\}$ for any $\lambda,\mu \in [0,1]$. Take $\lambda_0 = \mu_0 = \frac{1}{2}$, $U(\lambda_0) = U(\mu_0) = [0,1]$. The assumptions in Theorem 3.8 are satisfied. In fact, S_u is indeed Hölder continuous at (λ_0,μ_0).

Theorem 3.10. *If* (A_4), (A_5), (A'_6) *and* (A_7) *are satisfied, then* S_u *is* $(l_1.\alpha_1, l_2.\alpha_2)$*-pseudo-Hölder continuous at* $(\lambda_0,\mu_0,x_0) \in \operatorname{gr} S_u$*, where* $l_1,\alpha_1,l_2,\alpha_2$ *are same as* Theorem 3.2.

Proof. The proof is similar to Theorem 3.5, so we omit it here. □

Theorem 3.11. *Suppose that* (A_3), (A_5), (A_6) *and* (A_7) *are satisfied, and* $1 \le \gamma \le \beta$ *in* (A_7). *Then* V_u *is* $(l_1.\alpha_1, l_2.\alpha_2)$*-Hölder continuous at* (λ_0,μ_0)*, where* $l_1,\alpha_1,l_2,\alpha_2$ *are same as* Theorem 3.7.

Proof. Combining the proofs of Theorems 3.7 and 3.8, the result is obtained immediately. □

Remark 3.12. It is worth mentioning that if $S_l(\lambda,\mu)$ and $S_u(\lambda,\mu)$ are directly defined by

$$S_l(\lambda,\mu) := \{x \in \Phi(\mu) \mid \text{there is no } y \in \Phi(\mu) \text{ such that } F(y,\lambda) \le^l_E F(x,\lambda)\},$$
$$S_u(\lambda,\mu) := \{x \in \Phi(\mu) \mid \text{there is no } y \in \Phi(\mu) \text{ such that } F(y,\lambda) \le^u_E F(x,\lambda)\},$$

respectively, then all results in this section still hold without using assumption (A_5).

4 Conclusion

In this article, we proposed some assumptions for (PSOP) and obtained the Hölder continuity of the E-l(E-u)-minimal solution mappings $S_l(S_u)$ and the corresponding optimal value mappings $V_l(V_u)$. Moreover, we used mild conditions to establish the pseudo-Hölder continuity of S_l and S_u.

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