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A PARAMETRIC APPROACH TO DESIGN OPTIMIZED VELOCITY-PLUS-ACCELERATION CONTROLLER FOR SECOND-ORDER QUASI-LINEAR SYSTEMS*

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Abstract: This paper investigates the design problem of velocity-plus-acceleration feedback law to a class of second-order quasi-linear systems whose coefficient matrices contain time-variant parameters and system state variables. Based on the solution to a type of generalized second-order Sylvester equations, we obtain the generally parameterized expressions of the velocity-plus-acceleration feedback gain matrices and right closed-loop eigenvector, and also a group of arbitrary parameters. With the proposed controller, the closed-loop system is converted into a linear constant one with expected eigenstructure. Furthermore, an effective and simple approach to realize robust optimization is provided. By using the degrees of freedom in arbitrary parameters, the overall eigenvalue sensitivity is formulated to measure the robustness of the closed-loop system. Finally, an example of attitude control of combined spacecraft is presented to prove the effectiveness of the proposed approach.

Key words: *second-order quasi-linear systems, parametric control, velocity-plus-acceleration feedback, robust optimization, degrees of freedom*

Mathematics Subject Classification: *93B51, 93D09*

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1 Introduction

Generally, second-order dynamical systems are widely used in mechanical and civil engineering, robotics, cybernetics, and decision science and other fields, such as double-link flexible-joint manipulator [1], motor control [2], attitude control of combined spacecraft [3, 4], spacecraft rendezvous problem [5] and so on. Thus, many scholars have paid more attention to second-order dynamical systems, in which feedback control is the most commonly used approach due to higher control accuracy and disturbance rejection. In recent studies, Maruki *et al.* presented an effective technology by combining proportional and differential (PD) feedback control with adaptive law and back-stepping, which applied to a wheeled inverted pendulum, and aimed for achieving a stabilized controller for the wheel angle under PD-positive feedback [6]. Zhang *et al.* proposed a class of both displacement-plus-acceleration feedback and velocity-plus-acceleration feedback under partial eigenstructure assignment, through this method, it can obtain a smaller norm solution of feedback

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gains [7, 8, 9]. Gu *et al.* also provided the parametric approach of partial eigenstructure assignment for high-order linear systems via proportional plus derivative state feedback [10]. Abdelaziz presented a parametric control method to implement velocity-plus-acceleration feedback by taking advantage of eigenstructure assignment and extended it to descriptor second-order systems [11, 12]. Zhou proposed a hybrid optimized method that combined measured receptance and system matrices under the condition that eigenvalue sensitivity and norms of feedback gains were minimum to solve partial quadratic eigenvalue assignment for velocity-plus-feedback control [13]. Furthermore, Rofooei [14] and Yang [15] put forward the optimization problems of the velocity-plus-acceleration controller under different algorithms, respectively. Unfortunately, the objects of the above research are all linear constant systems rather than linear time-varying systems and even nonlinear systems. In this paper, we design a type of velocity-plus-acceleration feedback controller for a class of second-order nonlinear systems, called second-order quasi-linear systems, by using the parametric method.

Robust control is one of the hottest topics in modern control theories. For robust design, a widely used method is LMI approach [16, 17, 18]. Moreover, Lindemann and Dimarogonas [19], Liu *et al.* [20] and other researchers also devoted themselves to achieving robustness criteria. However, the above methods are complicated to implement and produce a large computation load, and also control or state constraints. In this paper, we utilize the overall eigenvalue sensitivity function (see [21, 22] and references therein), to measure the robustness of the closed-loop system. In the procedure of optimization, the degrees of freedom in arbitrary parameters can be further used to simplify the computation.

The parametric control approach proposed by Prof. Duan can realize not only basic control design, but also performance optimization, which has developed a new research filed. Gu and his colleagues extended the Duan approach to output feedback of quasi-linear systems and multi-objective optimization [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34], which provided the fundamental to deal with the design of velocity-plus-acceleration control for second-order quasi-linear systems.

This paper considers a parametric design method for second-order quasi-linear systems by velocity-plus-acceleration feedback. The proposed approach applies the solution of second-order generalized Sylvester equations [35, 36], and aims at realizing a linear time-invariant closed-loop system with expected eigenstructure which is dependent on an arbitrary matrix F including expected eigenvalues. More specifically, the proposed method obtains the generally parametric expression of the right eigenvector matrix and gives a group of arbitrary parameters that provides flexibility in design. Then, the completely parameterized forms of the velocity-plus-acceleration feedback gain matrices are established concerning the matrix F, right eigenvector, and the arbitrary parameters. Note that the arbitrary parameters can provide the degrees of freedom to implement robust optimization.

The contributions of this research are summarized as follows. On the one hand, a parametric method is proposed to establish the completely parameterized forms of the velocity-plus-acceleration controller. On the other hand, the degrees of freedom in arbitrary parameters can be further utilized in applications to cope with robust optimization. Note that the significant advantages of the proposed optimized velocity-plus-acceleration feedback controller are listed as follows. Firstly, acceleration information measured by an accelerometer can be used directly such that it can reduce the accumulative error caused by numerical integration of displacement and velocity in PD feedback, which is more favorable and reliable. Secondly, acceleration feedback can effectively improve the anti-interference ability of the system, and it is beneficial to raise the natural frequency of the system, which can augment its ability of restraining mechanic resonance. Thirdly, a robust optimization problem is considered, the proposed approach utilizes the degrees of freedom in arbitrary parameters to optimize controller such that the closed-loop system is insensitive to perturbations, therefore enhance the robustness of the system. Finally, the proposed approach can locate the closed-loop eigenvalues in a particular region on the complex plane which is insensitive to perturbations and disturbances.

The remaining part of this paper is organized as follows. Section 2 establishes the problem formulation of parametric design to the second-order quasi-linear system by velocity-plus-acceleration feedback control and provides some preliminaries. In Section 3, we propose the generally parameterized expressions of second-order quasi-linear velocity-plus-acceleration feedback controller in two cases and realize the robust optimization, also give the general procedure of the proposed parametric design. In Section 4, an example of attitude control of combined spacecraft is presented to prove that the proposed method is effective. Section 5 draws the presented work.

2 Problem Formulation and Preliminaries

2.1 Problem statement

In this paper, we consider a type of second-order quasi-linear systems as follows

$$A_2(\theta, q, \dot{q})\ddot{q} + A_1(\theta, q, \dot{q})\dot{q} + A_0(\theta, q, \dot{q})q = B(\theta, q, \dot{q})u, \tag{2.1}$$

where $q \in \mathbb{R}^n$, $u \in \mathbb{R}^r$ are the state vector and the control vector, respectively; the matrices $A_2(\theta, q, \dot{q})$, $A_1(\theta, q, \dot{q})$, $A_0(\theta, q, \dot{q}) \in \mathbb{R}^{n\times n}$ and $B(\theta, q, \dot{q}) \in \mathbb{R}^{n\times r}$ are the system coefficient matrices which are piecewise continuous functions of θ, q and $\dot{q}$, where θ is a time-variant parameter vector which satisfies

$$\theta(t) = \begin{bmatrix} \theta_1(t) & \theta_2(t) & \cdots & \theta_l(t) \end{bmatrix}^{\mathrm{T}} \in \Omega \subset \mathbb{R}^l, t \geq 0.$$

Specifically, the above system (2.1) satisfies the following assumptions:

Assumption 2.1. rank $A_2(\theta, q, \dot{q}) = n$.

Assumption 2.2. $B(\theta, q, \dot{q})$ is uniformly bounded relating to q, $\dot{q}$ and $\theta(t) \in \Omega$.

For the above system (2.1), we choose the following velocity-plus-acceleration feedback control law

$$u = K_v(\theta, q, \dot{q})\dot{q} + K_a(\theta, q, \dot{q})\ddot{q}, \tag{2.2}$$

where $K_v(\theta, q, \dot{q}) \in \mathbb{R}^{r\times n}$ and $K_a(\theta, q, \dot{q}) \in \mathbb{R}^{r\times n}$ are the velocity and acceleration feedback gain matrices, respectively, which are also piecewise continuous functions relating to q, $\dot{q}$ and $\theta(t) \in \Omega$.

Under the controller (2.2), we can obtain the close-loop system as follows

$$A_2^c(\theta, q, \dot{q})\ddot{q} + A_1^c(\theta, q, \dot{q})\dot{q} + A_0(\theta, q, \dot{q}) = 0, \tag{2.3}$$

where

$$\begin{cases} A_2^c(\theta, q, \dot{q}) = A_2(\theta, q, \dot{q}) - B(\theta, q, \dot{q})K_a(\theta, q, \dot{q}), \\ A_1^c(\theta, q, \dot{q}) = A_1(\theta, q, \dot{q}) - B(\theta, q, \dot{q})K_v(\theta, q, \dot{q}). \end{cases}$$

Constraint 2.1. rank $A_2^c(\theta, q, \dot{q}) = n$.

Then, let

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix},$$

the system (2.3) can be transformed into the following first-order form

$$E_c(\theta, x)\dot{x} = A_c(\theta, x)x, \tag{2.4}$$

where

$$\begin{cases} E_c(\theta, x) = \operatorname{diag}\left(I_n, A_2^c(\theta, x)\right), \\ A_c(\theta, x) = \begin{bmatrix} 0 & I_n \\ -A_0(\theta, x) & -A_1^c(\theta, x) \end{bmatrix}. \end{cases}$$

Based on the above discussion, the parametric control of second-order quasi-linear system (2.1) via velocity-plus-acceleration feedback (2.2) can be stated as follows.

Problem 2.2 (VA)**.** Given the second-order quasi-linear system (2.1) satisfying Assumptions 2.1, 2.2 and Constraint 2.1, and an arbitrary constant matrix F, exist the non-singular right eigenvector matrix $V_c(\theta, x)$, and find the velocity and acceleration feedback gain matrices $K_v(\theta, x)$ and $K_a(\theta, x)$ such that

$$A_c(\theta, x)V_c(\theta, x) = E_c(\theta, x)V_c(\theta, x)F. \tag{2.5}$$

Remark 2.3. The condition (2.5) is equivalent to find a velocity-plus-acceleration controller (2.2) such that closed-loop system (2.4) is similar to a linear constant form with desired eigenstructure, that is, $\{E_c(\theta, x), A_c(\theta, x)\}$ is similar to a matrix F by the proposed parametric method.

2.2 Preliminaries

There exists the following right coprime factorization (RCF) for system (2.1) (see [36])

$$\mathcal{A}(\theta, x, s)N(\theta, x, s) = B(\theta, x)D(\theta, x, s), \tag{2.6}$$

with

$$\mathcal{A}(\theta, x, s) = s^2 A_2(\theta, x) + sA_1(\theta, x) + A_0(\theta, x)\ ,$$

where $N(\theta, x, s) \in \mathbb{R}^{n\times r}[s]$ and $D(\theta, x, s) \in \mathbb{R}^{r\times r}[s]$ are a pair of polynomial matrices. Denote $N(\theta, x, s) = [n_{ij}(\theta, x, s)]_{n\times r}$ and $D(\theta, x, s) = [d_{ij}(\theta, x, s)]_{r\times r}$ and

$$\begin{cases} \tau_1 = \max\left\{\deg\left(n_{ij}(\theta, x, s)\right), i = 1, 2, \ldots, n, j = 1, 2, \ldots, r\right\}, \\ \tau_2 = \max\left\{\deg\left(d_{ij}(\theta, x, s)\right), i = 1, 2, \ldots, r, j = 1, 2, \ldots, r\right\}, \\ \ \tau = \max\left\{\tau_1, \tau_2\right\}, \end{cases}$$

where $\deg\left(n_{ij}(\theta, x, s)\right)$ and $\deg\left(d_{ij}(\theta, x, s)\right)$ represent the degrees of polynomial $n_{ij}(\theta, x, s)$ and $d_{ij}(\theta, x, s)$ in relation to s. Then, $N(\theta, x, s)$ and $D(\theta, x, s)$ can be written into the following forms

$$\begin{cases} N(\theta, x, s) = \sum_{i=0}^{\tau} N_i(\theta, x)s^i, \\ D(\theta, x, s) = \sum_{i=0}^{\tau} D_i(\theta, x)s^i. \end{cases} \tag{2.7}$$

3 Solution to Problem VA

3.1 Case of F arbitrary

Based on the above deduction, we provide the following theorem to solve Problem 2.2.

Theorem 3.1. *Let $N(\theta, x, s)$ and $D(\theta, x, s)$ in Equation (2.7) satisfy RCF (2.6), F is an arbitrary constant matrix, then*

1. The Problem 2.2 has a solution if and only if there exists an arbitrary parameter matrix $Z_c \in \mathbb{C}^{r \times 2n}$ satisfying

$$\det V_c(\theta, x) \neq 0, \tag{3.1}$$

where

$$V_c(\theta, x) = \begin{bmatrix} V(\theta, x) \\ V(\theta, x)F \end{bmatrix}, \tag{3.2}$$

and

$$V(\theta, x) = \sum_{i=0}^{\tau} N_i(\theta, x) Z_c F^i. \tag{3.3}$$

2. When the above condition is satisfied, the velocity and acceleration feedback gain matrices $K_v(\theta, x)$ and $K_a(\theta, x)$ can be solved as

$$\begin{bmatrix} K_v(\theta, x) & K_a(\theta, x) \end{bmatrix} = W_c(\theta, x)(V_c(\theta, x)F)^{-1}, \tag{3.4}$$

where

$$W_c(\theta, x) = \sum_{i=0}^{\tau} D_i(\theta, x) Z_c F^i. \tag{3.5}$$

Proof. Assume that there exist the velocity and acceleration feedback gain matrices $K_v(\theta, x)$, $K_a(\theta, x)$ and the non-singular right eigenvector matrix $V_c(\theta, x)$ satisfying Equation (2.5). Denote

$$V_c(\theta, x) = \begin{bmatrix} V_0(\theta, x) \\ V_1(\theta, x) \end{bmatrix},$$

then, we obtain

$$\begin{aligned} &A_c(\theta, x)V_c(\theta, x) \\ &= \begin{bmatrix} 0 & I_n \\ -A_0(\theta, x) & -A_1^c(\theta, x) \end{bmatrix} \begin{bmatrix} V_0(\theta, x) \\ V_1(\theta, x) \end{bmatrix} \\ &= \begin{bmatrix} V_1(\theta, x) \\ -A_0 V_0(\theta, x) - A_1^c(\theta, x)V_1(\theta, x) \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} &E_c(\theta, x)V_c(\theta, x)F \\ &= \begin{bmatrix} I_n & 0 \\ 0 & A_2^c(\theta, x) \end{bmatrix} \begin{bmatrix} V_0(\theta, x) \\ V_1(\theta, x) \end{bmatrix} F \\ &= \begin{bmatrix} V_0(\theta, x)F \\ A_2^c(\theta, x)V_1(\theta, x)F \end{bmatrix}, \end{aligned}$$

thus we have

$$V_1(\theta, x) = V_0(\theta, x)F,$$

and

$$-A_0 V_0(\theta, x) - A_1^c(\theta, x)V_1(\theta, x) = A_2^c(\theta, x)V_1(\theta, x)F, \tag{3.6}$$

let $V_0(\theta,x) = V(\theta,x)$, we obtain $V_1(\theta,x) = V(\theta,x)F$, hence Equation (3.2) is proven.

Meanwhile, considering $A_2^c(\theta,x)$ and $A_1^c(\theta,x)$ in Equations (2.3) and (3.6) can be written as

$$\begin{aligned} &A_2(\theta,x)V(\theta,x)F^2 + A_1(\theta,x)V(\theta,x)F + A_0(\theta,x)V(\theta,x) \\ &= B(\theta,x)K_v(\theta,x)V(\theta,x)F + B(\theta,x)K_a(\theta,x)V(\theta,x)F^2 \ , \end{aligned} \tag{3.7}$$

let

$$\begin{aligned} W_c(\theta,x) &= K_v(\theta,x)V(\theta,x)F + K_a(\theta,x)V(\theta,x)F^2 \\ &= K(\theta,x)\bar{V}_c(\theta,x), \end{aligned} \tag{3.8}$$

where

$$K(\theta,x) = \left[\begin{array}{cc} K_v(\theta,x) & K_a(\theta,x) \end{array} \right],$$

and

$$\bar{V}_c(\theta,x) = \left[\begin{array}{c} V(\theta,x)F \\ V(\theta,x)F^2 \end{array} \right] = V_c(\theta,x)F.$$

Then, Equation (3.7) becomes the second-order generalized Sylvester equation

$$\sum_{i=0}^{2} A_i(\theta,x)V(\theta,x)F^i = B(\theta,x)W_c(\theta,x). \tag{3.9}$$

Therefore, using the general solution to the second-order generalized Sylvester matrix equation [35, 36], we can obtain the parametric solutions as given in Equations (3.3) and (3.5). Then, the velocity and acceleration feedback gain matrices $K_v(\theta,x)$ and $K_a(\theta,x)$ are solved by Equation (3.8) as shown in Equation (3.4).

With the above deduction, the proof is completed. □

3.2 Case of F diagonal

In practical applications, the matrix F chooses to be the following diagonal form

$$F = \mathrm{diag}\left\{\lambda_1, \lambda_2, \cdots, \lambda_{2n}\right\} \ , \tag{3.10}$$

where $\lambda_i \in \mathbb{C}^-, i = 1, 2, \ldots, 2n$ are a set of negative real poles. In this situation, we provide the following Corollary to deal with Problem 2.2.

Corollary 3.2. *Let $N(\theta,x,s)$ and $D(\theta,x,s)$ in Equation (2.7) satisfy RCF (2.6), F is a diagonal constant matrix as shown in Equation (3.10), then,*

1. The Problem 2.2 has a solution if and only if exists a group of arbitrary parameter vector $z_i^c \in \mathbb{C}^r, i = 1, 2, \ldots, 2n$ satisfying Equation (3.1), where

$$\begin{cases} V(\theta,x) = \left[\begin{array}{cccc} v_1(\theta,x) & v_2(\theta,x) & \cdots & v_{2n}(\theta,x) \end{array} \right], \\ v_i(\theta,x) = N(\theta,x,\lambda_i)z_i^c, \\ \qquad i = 1, 2, \ldots, 2n, \end{cases} \tag{3.11}$$

and

$$V_{ci} = \left[\begin{array}{c} N(\theta,x,\lambda_i)z_i^c \\ N(\theta,x,\lambda_i)z_i^c\lambda_i \end{array} \right] \ , \ i = 1, 2, \ldots, 2n. \tag{3.12}$$

2. When the above condition is satisfied, the velocity and acceleration feedback gain matrices $K_v(\theta, x)$ and $K_a(\theta, x)$ can be solved as Equation (3.4), where

$$\begin{cases} W_c(\theta, x) = \left[\begin{array}{cccc} w_1^c(\theta, x) & w_2^c(\theta, x) & \cdots & w_{2n}^c(\theta, x) \end{array} \right], \\ w_i^c(\theta, x) = D(\theta, x, \lambda_i) z_i^c, \\ \qquad i = 1, 2, \cdots, 2n. \end{cases} \tag{3.13}$$

In this case, $z_i^c \in \mathbb{C}^r, i = 1, 2, \ldots, 2n$ is a group of arbitrary parameter vectors satisfying

$$Z_c = \left[\begin{array}{cccc} z_1^c & z_2^c & \cdots & z_{2n}^c \end{array} \right],$$

which represents the degrees of freedom in solution.

Proof. According to Theorem 3.1, when the matrix F takes to be a diagonal form in Equation (3.10), $V_c(\theta, x)$ and $W_c(\theta, x)$ can be the form of columns provided by Equations (3.12) and (3.13). It is easy to prove Corollary 3.2. □

Remark 3.3. The completely parameterized expressions of the feedback gain matrices $K_v(\theta, x)$, $K_a(\theta, x)$ and the closed-loop right eigenvector are proposed in terms of a matrix F, which possesses the desired closed-loop eigenstructure, and the arbitrary parameter vector Z_c. In practical applications, both parameters matrices F and Z_c can be optimized simultaneously to achieve additional requirements of the closed-loop system.

3.3 Robust optimization by using the degrees of freedom in arbitrary parameters

This subsection is to derive completely parameterized representations of the system design specifications using the arbitrary parameter matrices F and Z_c. Once the system specification parametrization is obtained, the next step is to form the optimization problem.

To guarantee the robustness of the closed-loop system subject to parameter uncertainties and disturbances, we select the following overall eigenvalue sensitivity function (see [21, 22])

$$J(Z_c, F) = \|V_c(\theta, x)\|_2 \|V_c^{-1}(\theta, x)\|_2, \tag{3.14}$$

where $V_c(\theta, x)$ is the matrix of right eigenvector in (3.2) or (3.12). The measure attains the minimum value when the eigenproblem is perfectly conditioned and the assigned eigenvalues are as insensitive as possible.

Further, we usually intend to locate the eigenvalues $\lambda_i, i = 1, 2, \ldots, 2n$ in a desired region to fulfill the requirements of the practical control system. This leads to eigenvalue constraints, for example of the form $\underline{\lambda}_i \le \lambda_i \le \overline{\lambda}_i$, where $\underline{\lambda}_i, \overline{\lambda}_i \in \mathbb{R}$ are the lower and upper bounds. Then, the above constraints may be substituted by considering the change of variables given by

$$\lambda_i = \underline{\lambda}_i + (\overline{\lambda}_i - \underline{\lambda}_i)\sin^2(\|z_i^c\|_2), \tag{3.15}$$

where $z_i^c, i = 1, 2, \ldots, 2n$ are column vectors of the free parameter Z_c.

Based on the obtained specification parametrization (3.14), the optimization problem is formulated as

$$\begin{cases} \min J, \\ \text{s.t. } (3.1), (3.10), (3.15). \end{cases} \tag{3.16}$$

According to Equations (3.2), (3.14) and (3.15), we see that the optimization problem (3.16) depends on the matrix F and the arbitrary parameter Z_c. By seeking the proper F and Z_c, the robustness of closed-loop system can be improved.

3.4 General Procedure

According to Theorem 3.1 and Corollary 3.2, we present a general procedure to solve the parametric design problem for the second-order quasi-linear system (2.1) by velocity-plus-acceleration feedback control law (2.2).

Step 1 Design an arbitrary matrix F with desired eigenstructure.

Generally speaking, we choose the arbitrary matrix F in a diagonal form. Under the pole assignment theories [37, 38, 39, 40], it is required that F is a Hurwitz matrix, that is, all closed-loop eigenvalues are located in the left-half s-plane as follows

$$\lambda_i(F) \in \mathbb{R}^-, i = 1, 2, \ldots, 2n.$$

Step 2 Obtain a pair of RCF $\{N(\theta, x, s), D(\theta, x, s)\}$.

From the RCF (2.6), a pair of particular solutions can be given as

$$\begin{cases} N(s) = \operatorname{adj}\left(\mathcal{A}(\theta, x, s)\right) B(\theta, x), \\ D(s) = \det\left(\mathcal{A}(\theta, x, s)\right) I_r. \end{cases}$$

Step 3 Establish an optimization problem.

Once the condition (3.1) is satisfied, an optimization problem is formulated to improve the robustness of the closed-loop system as shown in Equation (3.16).

Step 4 Compute the velocity and acceleration feedback gain matrices $K_v(\theta, x)$ and $K_a(\theta, x)$.

By using the parameterized expressions of $V(\theta, x)$ and $W_c(\theta, x)$ in Equations (3.3), (3.5) or (3.11), (3.13), the velocity and acceleration feedback gain matrices $K_v(\theta, x)$ and $K_a(\theta, x)$ are computed as Equation (3.4).

4 Example—Attitude control of combined spacecrafts

4.1 System description

Consider the attitude motion of the extending space structures in Figure 1 (see [4]), the dynamic equations can be written in the following form

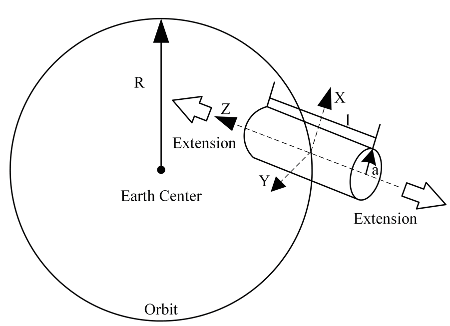

Figure 1: Coordinates systems

$$A_2(\theta, q)\ddot{q} + A_1(\theta, q)\dot{q} + A_0(\theta, q)q = B(\theta, q)u, \tag{4.1}$$

where $q = \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix}^{\mathrm{T}}$ represent pitch angle, roll angle and yaw angle, respectively, and

$$
\begin{aligned}
A_2(\theta, q) &= \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix}, \\
A_1(\theta, q) &= \begin{bmatrix} \dot{J}_x & 0 & -\Omega(J_x - J_y + J_z) \\ 0 & \dot{J}_y & 0 \\ \Omega(J_x - J_y + J_z) & 0 & \dot{J}_z \end{bmatrix}, \\
A_0(\theta, q) &= \begin{bmatrix} 4\Omega^2(J_y - J_z) & 0 & -\Omega\dot{J}_x \\ 0 & 3\Omega^2(J_x - J_z) & 0 \\ \Omega\dot{J}_z & 0 & \Omega^2(J_y - J_x) \end{bmatrix}, \\
B(\theta, q) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\end{aligned}
$$

and let

$$
\begin{cases}
\lambda_1 \in [-0.1, 0), \\
\lambda_2 \in [-0.2, -0.1), \\
\lambda_3 \in [-0.3, -0.2), \\
\lambda_4 \in [-0.4, -0.3), \\
\lambda_5 \in [-0.5, -0.4), \\
\lambda_6 \in [-0.6, -0.5).
\end{cases}
$$

Meanwhile, a group of RCF satisfying Equation (4.1) can be obtained as

$$
\begin{cases}
N(\theta, q, s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
D(\theta, q, s) = \\
\begin{bmatrix} J_x s^2 + \dot{J}_x s + 4\Omega^2(J_y - J_z) & 0 & -\Omega(J_x - J_y + J_z)s - \Omega\dot{J}_x \\ 0 & J_y s^2 + \dot{J}_y s + 3\Omega^2(J_x - J_z) & 0 \\ \Omega(J_x - J_y + J_z)s + \Omega\dot{J}_z & 0 & J_z s^2 + \dot{J}_z s + \Omega^2(J_y - J_x) \end{bmatrix}.
\end{cases}
$$

$\boxed{4.2}$ Non-optimized solution

Choose arbitrary F and Z_c as

$$
F = \mathrm{diag}\left\{-0.1, -0.2, -0.3, -0.4, -0.5, -0.6\right\}, \tag{4.2}
$$

and

$$
Z_c = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \tag{4.3}
$$

then, according to Equations (3.3) and (3.5), we acquire $V(\theta, q)$

$$
V(\theta, q) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},
$$

and $W_c(\theta, q)$ as

$$W_c(\theta,q) = \left[\begin{array}{cc} 0.01J_x - 0.1\dot{J}_x + 4\Omega^2(J_y - J_z) & 0 \\ 0 & 0.04J_y - 0.2\dot{J}_y + 3\Omega^2(J_x - J_z) \\ \Omega\dot{J}_z - 0.1\Omega(J_x - J_y + J_z) & 0 \end{array}\right.$$

$$\begin{array}{cc} -\Omega\dot{J}_x + 0.3\Omega(J_x - J_y + J_z) & 0.16J_x - 0.4\dot{J}_x + 4\Omega^2(J_y - J_z) \\ 0 & 0 \\ 0.09J_z - 0.3\dot{J}_z + \Omega^2(J_y - J_x) & \Omega\dot{J}_z - 0.4\Omega(J_x - J_y + J_z) \end{array}$$

$$\left.\begin{array}{cc} 0 & -\Omega\dot{J}_x + 0.6\Omega(J_x - J_y + J_z) \\ 0.25J_y - 0.5\dot{J}_y + 3\Omega^2(J_x - J_z) & 0 \\ 0 & 0.36J_z - 0.6\dot{J}_z + \Omega^2(J_y - J_x) \end{array}\right].$$

Based on Equation (3.4), we can obtain the velocity and acceleration feedback gain matrices $K_v(\theta, q)$ and $K_a(\theta, q)$ as

$$K_v = \begin{bmatrix} \dot{J}_x - 50\Omega^2(J_y - J_z) & 0 & 5\Omega\dot{J}_x - \Omega(J_x - J_y + J_z) \\ 0 & \dot{J}_y - 21\Omega^2(J_x - J_z) & 0 \\ -12.5\Omega\dot{J}_z + \Omega(J_x - J_y + J_z) & 0 & \dot{J}_z - 5\Omega^2(J_y - J_x) \end{bmatrix},$$
$$K_a = \begin{bmatrix} J_x - 100\Omega^2(J_y - Jz) & 0 & \frac{50}{9}\Omega\dot{J}_x \\ 0 & J_y - 30\Omega^2(J_x - J_z) & 0 \\ -25\Omega\dot{J}_z & 0 & J_z - \frac{50}{9}\Omega^2(J_y - J_x) \end{bmatrix}, \tag{4.4}$$

by using the velocity-plus-acceleration feedback controller (4.4), the closed-loop system is obtained as

$$\begin{bmatrix} 100 & 0 & -\frac{50}{9} \\ 0 & 30 & 0 \\ 25 & 0 & \frac{50}{9} \end{bmatrix} \ddot{q} + \begin{bmatrix} 50 & 0 & -5 \\ 0 & 21 & 0 \\ 12.5 & 0 & 5 \end{bmatrix} \dot{q} + \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} q = 0.$$

4.3 Optimized solution

Consider optimization problem (3.16), use the *fminsearch* function in MATLAB® Optimization Toolbox, choose the initial conditions as Equations (4.2) and (4.3), then we have

$$F = \text{diag}\left\{-0.0077, -0.1071, -0.2089, -0.3477, -0.4450, -0.5391\right\},$$

and

$$Z_c = \begin{bmatrix} 1.2899 & 0 & 0 & 0.8088 & 0 & 0 \\ 0 & 1.3017 & 0 & 0 & 0.8351 & 0 \\ 0 & 0 & 1.2674 & 0 & 0 & 0.8953 \end{bmatrix},$$

then, according to Equations (3.3) and (3.5), we acquire $V(\theta, q)$

$$V = \begin{bmatrix} 1.2899 & 0 & 0 & 0.8088 & 0 & 0 \\ 0 & 1.3017 & 0 & 0 & 0.8351 & 0 \\ 0 & 0 & 1.2674 & 0 & 0 & 0.8953 \end{bmatrix},$$

and $W_c(\theta, q)$ as

$$W_c = \left[\begin{array}{cc} 0.0001\dot{J}_x - 0.0099J_x + 5.1596\Omega^2(J_y - J_z) & 0 \\ 0 & 0.0149J_y - 0.1394\dot{J}_y + 3.9051\Omega^2(J_x - J_z) \\ 1.2899\Omega\dot{J}_z - 0.0099\Omega(J_x - J_y + J_z) & 0 \end{array}\right.$$

$$\begin{array}{cc} 0.2648\Omega(J_x - J_y + J_z) - 1.2674\Omega\dot{J}_x & 0.0979J_x - 0.2812\dot{J}_x + 3.2352\Omega^2(J_y - J_z) \\ 0 & 0 \\ 0.0553J_z - 0.2648\dot{J}_z - 1.2674\Omega^2(J_x - J_y) & 0.8088\Omega\dot{J}_z - 0.2812\Omega(J_x - J_y + J_z) \end{array}$$

$$\left.\begin{array}{cc} 0 & 0.4827\Omega(J_x - J_y + J_z) - 0.8953\Omega\dot{J}_x \\ 0.1654J_y - 0.3716\dot{J}_y + 2.5053\Omega^2(J_x - J_z) & 0 \\ 0 & 0.2602J_z - 0.4827\dot{J}_z - 0.8953\Omega^2(J_x - J_y) \end{array}\right],$$

Based on Equation (3.4), we can obtain the velocity and acceleration feedback gain matrices $K_v(\theta, q)$ and $K_a(\theta, q)$ as

$$K_v = \begin{bmatrix} \dot{J}_x - 530.9847\Omega^2(J_y - J_z) & 0 & 6.6419\Omega\dot{J}_x - \Omega(J_x - J_y + J_z) \\ 0 & \dot{J}_y - 34.7528\Omega^2(J_x - J_z) & 0 \\ \Omega(J_x - J_y + J_z) - 132.7462\Omega\dot{J}_z & 0\Omega^2(J_y - J_x) & \dot{J}_z - 6.6419\Omega^2(J_y - J_x) \end{bmatrix},$$

$$K_a = \begin{bmatrix} J_x - 1494.0481\Omega^2(J_y - J_z) & 0 & 8.8796\Omega\dot{J}_x \\ 0 & J_y - 62.9465\Omega^2(J_x - J_z) & 0 \\ -373.5120\Omega\dot{J}_z & 0 & J_z - 8.8796\Omega^2(J_y - J_x) \end{bmatrix}, \tag{4.5}$$

by using the velocity-plus-acceleration feedback controller (4.5), the closed-loop system is obtained as

$$\begin{bmatrix} 1494.0481 & 0 & -8.8796 \\ 0 & 62.9465 & 0 \\ 373.5120 & 0 & 8.8796 \end{bmatrix} \ddot{q} + \begin{bmatrix} 530.9847 & 0 & -6.6419 \\ 0 & 34.7582 & 0 \\ 132.7642 & 0 & 6.6419 \end{bmatrix} \dot{q} + \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} q = 0.$$

4.4 Numerical simulation and comparison

Let J_n and J_o represent the non-optimized and optimized indices, respectively, then we have

$$J_n = 8.0423, \; J_o = 7.0944. \tag{4.6}$$

Choose initial conditions as

$$\begin{cases} q(0) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \text{rad}, \\ \dot{q}(0) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \text{rad/s}, \end{cases}$$

and add a disturbance as $g(t) = 0.5\sin(t - 100), t \in [100, 110]$, we can obtain Figures 2–4.

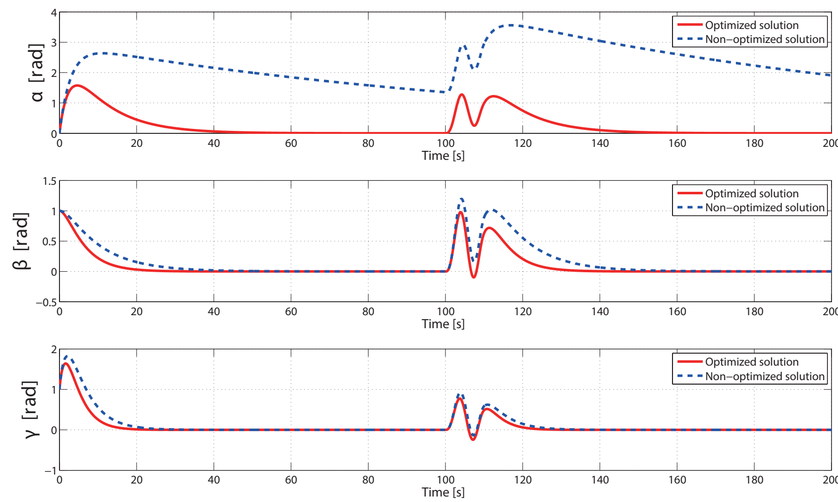

Figure 2: Variation diagram of attitude angle in the process of on-orbit refueling

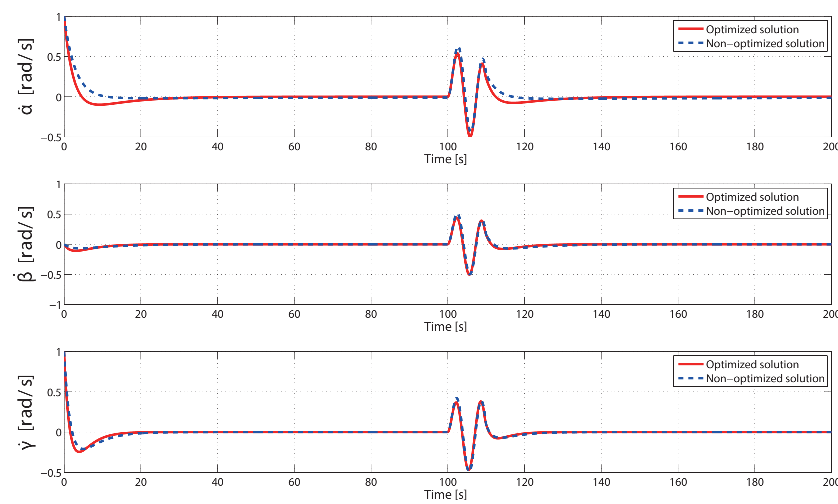

Figure 3: Variation diagram of attitude velocity in the process of on-orbit refueling

In Figure 2, the optimized controller reduces the maximum amplitude of attitude angles and results in better dynamical performance, and in Figure 3, two controllers possess the same attitude velocity within errors permissibility. From Figure 4, we can see that the bounds of input of the optimized controller are smaller than the non-optimized one, which

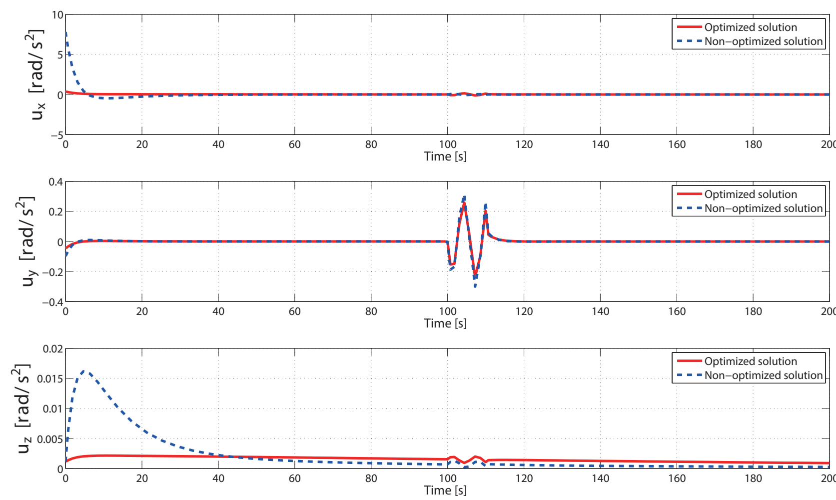

Figure 4: Variation diagram of control signals

means that the optimized controller leads to better control performances such that the effectiveness of the proposed method can be illustrated effectively. Meanwhile, through Equation (4.6) yields $J_n < J_o$, which implies the overfull sensitivity is reduced such that the robustness of the system can be improved.

5 Conclusion

In this paper, a parametric method is proposed for a type of quasi-linear systems, which provides the parameterized forms of the robust optimized controller under velocity-plus-acceleration feedback and also the right closed-loop eigenvector that is dependent on an arbitrary matrix F with desired closed-loop eigenvalues. With this controller, the closed-loop system is a linear time-invariant one with desired eigenstructure. Meanwhile, a group of parameters Z_c can be exploited to optimize the controller and achieve regional pole assignment via its degrees of freedom to realize practical control requirements of the closed-loop system. The contribution of such a control law for automobile suspension, robotic control, vibration control, and other practical applications can be momentous.

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