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AN EFFICIENT ITERATIVE METHOD FOR SOLVING QUADRATIC INVERSE EIGENVALUE PROBLEM OVER HERMITIAN CENTROSKEW MATRICES WITH A SUBMATRIX CONSTRAINT*

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Abstract: In this paper, a nonlinear conjugate gradient method is constructed to find the least squares solution of the quadratic inverse eigenvalue problem under a prescribed submatrix constraint. The convergence analysis of the proposed method is given. A method for choosing the initial matrix group to obtain the least Frobenius norm solution is provided and the unique optimal approximation solution to a given matrix group is derived. Numerical examples are provided to illustrate the efficiency of the proposed method and testify the conclusions suggested in this paper.

Key words: *quadratic inverse eigenvalue problem, Hermitian centroskew matrix, submatrix constraint*

Mathematics Subject Classification: *15A24, 15A57, 65F10, 65F30*

1 Introduction

Throughout this paper, the following notations and concepts are needed. Let $\mathbb{C}^{m\times n}$ be the set of all complex $m\times n$ matrices. For $A\in\mathbb{C}^{m\times n}$, we write $A^T, A^H, \mathrm{tr}(A)$ and $\mathcal{R}(A)$ to denote the transpose, the conjugate transpose, the trace and the column space of matrix A, respectively. Let $I_n=(e_1,e_2,\dots,e_n)$ and $S_n=(e_n,e_{n-1},\dots,e_1)$ be the $n\times n$ identity matrix and reverse identity matrix, respectively, where e_i denotes the ith column of the identity matrix I_n. In the space $\mathbb{C}^{m\times n}$ over the real number field $\mathbb{R}$, the inner product for A,B is $\langle A,B\rangle=\mathrm{Re}[\mathrm{tr}(B^HA)]$. Then $\|A\|^2=\langle A,A\rangle=\mathrm{Re}[\mathrm{tr}(A^HA)]=\mathrm{tr}(A^HA)=\|A\|_F^2$ and $\|A\pm B\|^2=\|A\|^2+\|B\|^2\pm2\langle A,B\rangle$ for $A,B\in\mathbb{C}^{m\times n}$. We say A and B are orthogonal if $\langle A,B\rangle=0$. Let $D_{p,n}=\{d=(d_1,d_2,\dots,d_p):1\le d_1<d_2<\cdots<d_p\le n\}$ be the strictly increasing sequences of p elements from $1,2,\dots,n$. For $s=(s_1,s_2,\dots,s_p)\in D_{p,n}$, $t=(t_1,t_2,\dots,t_q)\in D_{q,n}$ and $u=(u_1,u_2,\dots,u_r)\in D_{r,n}$, we denote $E_s=(e_{s_1},e_{s_2},\dots,e_{s_p})\in\mathbb{C}^{n\times p}$, $E_t=(e_{t_1},e_{t_2},\dots,e_{t_q})\in\mathbb{C}^{n\times q}$ and $E_u=(e_{u_1},e_{u_2},\dots,e_{u_r})\in\mathbb{C}^{n\times r}$. Let $A[s|t]$ be the $p\times q$ submatrix of $A\in\mathbb{C}^{m\times n}$ determined by rows indexed by s and columns indexed by t, and let $A[\overline{s}|\overline{t}]$ be the $(m-p)\times(n-q)$ submatrix of $A\in\mathbb{C}^{m\times n}$ determined by deleting rows indexed by s and columns indexed by t. A matrix $A\in\mathbb{C}^{n\times n}$ is said to be a Hermitian centroskew matrix if $\overline{a}_{ij}=a_{ji}$ and $a_{ij}=-a_{n+1-i,n+1-j}$ for all $1\le i,j\le n$. Let $HSC^{n\times n}$

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be the set of all $n \times n$ Hermitian centroskew matrices. It is well known that Hermitian centroskew matrices include symmetric centroskew-symmetric and symmetric Toeplitz matrices. Hermitian centroskew matrices have practical applications in information theory, linear system theory, linear estimate theory and numerical analysis [11, 12].

In this paper, we are interested in the quadratic inverse eigenvalue problem which consists in finding matrices $A, B, C \in \mathbb{C}^{n\times n}$ such that

$$AX\Lambda^2 + BX\Lambda + CX = O, \tag{1.1}$$

where $X = (x_1,\ x_2,\ \ldots,\ x_m) \in \mathbb{C}^{n\times m}$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{C}^{m\times m}$ are given matrices. The quadratic inverse eigenvalue problem appears in some practical applications [16]. Such as, setting the pole assignment problem can be seen as a quadratic inverse eigenvalue problem [14]. The behavior of a time invariant second order differential system

$$A\ddot{x}(t) + B\dot{x}(t) + Cx(t) = f(t) \tag{1.2}$$

can be described by solving a quadratic inverse eigenvalue problem [35].

- If $A = O$, then the quadratic inverse eigenvalue problem (1.1) reduces to a generalized inverse eigenvalue problem
$$BX\Lambda + CX = O. \tag{1.3}$$
- If $A = O$ and $B = I$, then the quadratic inverse eigenvalue problem (1.1) reduces to a standard inverse eigenvalue problem
$$X\Lambda + CX = O. \tag{1.4}$$

In general, the inverse eigenvalue problem is as important as the eigenvalue problem [16, 30]. The inverse eigenvalue problem has wide applications in engineering and scientific computation [39, 17, 19]. For instance, the inverse eigenvalue problem of centro-symmetric matrices with a submatrix constraint initially occurs in the design of Hopfield neural network, civil engineering and aviation [13, 3]. The parameterized generalized inverse eigenvalue problem (PGIEP) [17] arises in the discrete analogue of inverse Sturm-Liouville problem [27, 28], factor analysis [28] and structural design [22, 29, 36, 38]. The problem of designing the truss structure with specified natural frequency may be recast as a PGIEP.

The literature on solvability condition and numerical method for the inverse eigenvalue problem is large and still growing rapidly [19, 40, 9, 10, 17, 20, 21]. For instance, Bai [4] and Zhang et al. [41] derived some necessary and sufficient conditions for the inverse eigenvalue problem with Hermitian generalized skew-Hamiltonian matrices and Hermitian generalized Hamiltonian matrices, respectively. Ghanbari [19] studied the explicit expression solution of a generalized inverse eigenvalue problem, in which C is a semi-infinite Jacobi matrix with positive off-diagonal entries and B is a nonsingular diagonal matrix. Liu et al. [31] considered an existence condition of a solution and an analytic expression solution of the generalized inverse eigenvalue problem for centrohermitian matrices. Moghaddam et al. [33] proposed an algorithm for reconstructing penta-diagonal coefficient matrices of a generalized inverse eigenvalue problem. Wei et al. [37] discussed a generalized inverse eigenvalue problem with Hermitian generalized Hamiltonian matrices and derived an analytic expression solution by the matrix decomposition theory and Hilbert space approximation theory. Aishima [1] introduced a quadratically convergent algorithm for the inverse symmetric eigenvalue problem. Dai and Liang [18] considered the generalized inverse eigenvalue problem for the (P, Q)-conjugate matrix and the associated approximation problem by using the generalized singular value decomposition and canonical correlation decomposition. Recently, Cai and Chen [9] proposed an iterative method for solving the generalized inverse eigenvalue problem and its optimal approximation problem over partially bisymmetric matrices. They also studied [10] the least squares solution of a generalized inverse eigenvalue problem over Hermitian Hamiltonian matrices with a submatrix constraint.

Inspired by the work mentioned above, we focus on the following problems.

Problem 1.1. Given $X \in \mathbb{C}^{n\times m}$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{C}^{m\times m}$, $s = (s_1, s_2, \ldots, s_p, n+1-s_p, \ldots, n+1-s_2, n+1-s_1) \in D_{2p,n}$, $t = (t_1, t_2, \ldots, t_q, n+1-t_q, \ldots, n+1-t_2, n+1-t_1) \in D_{2q,n}$, $u = (u_1, u_2, \ldots, u_r, n+1-u_r, \ldots, n+1-u_2, n+1-u_1) \in D_{2r,n}$, $A_p \in \mathbb{C}^{2p\times 2p}$, $B_q \in \mathbb{C}^{2q\times 2q}$ and $C_r \in \mathbb{C}^{2r\times 2r}$. Let $S_1 = \{Y|Y[s|s] = A_p, Y[\overline{s}|\overline{s}] \in HSC^{(n-2p)\times(n-2p)}\}$, $S_2 = \{Y|Y[t|t] = B_q, Y[\overline{t}|\overline{t}] \in HSC^{(n-2q)\times(n-2q)}\}$ and $S_3 = \{Y|Y[u|u] = C_r, Y[\overline{u}|\overline{u}] \in HSC^{(n-2r)\times(n-2r)}\}$. We want to find $A_* \in S_1$, $B_* \in S_2$ and $C_* \in S_3$ such that

$$\|A_*X\Lambda^2 + B_*X\Lambda + C_*X\| = \min_{(A,B,C)\in S_1\times S_2\times S_3} \|AX\Lambda^2 + BX\Lambda + CX\|.$$

Problem 1.2. Let S_E be the set of solutions of Problem 1.1. For given $\overline{A}, \overline{B}, \overline{C} \in \mathbb{C}^{n\times n}$, we want to find $(\widehat{A}, \widehat{B}, \widehat{C}) \in S_E$ such that

$$\|\widehat{A} - \overline{A}\|^2 + \|\widehat{B} - \overline{B}\|^2 + \|\widehat{C} - \overline{C}\|^2 = \min_{(A,B,C)\in S_E} \left[\|A - \overline{A}\|^2 + \|B - \overline{B}\|^2 + \|C - \overline{C}\|^2\right].$$

Although Hajarian [23, 25] constructed the conjugate direction and BCR methods to solve some special quadratic inverse eigenvalue problems. Hajarian and Hassan [24] established the CGNR method for finding the least squares solution of a quadratic inverse eigenvalue problem with partially bisymmetric matrices under a prescribed submatrix constraint. In fact, Hajarian [23, 25] and Hajarian and Hassan [24] extended the subspace method for linear system to the setting of quadratic inverse eigenvalue problem from the perspective of numerical algebra. Naturally, we want to know whether the quadratic inverse eigenvalue problem can be solved from the perspective of optimization or not. This is one of our motivations in this paper.

Since matrices X and Λ are known in the system (1.1), the investigated problem (Problem 1.1) can be transformed into finding the least squares solution of the generalized Sylvester matrix equation (1.1). Such a problem has been investigated in many papers. For example, Beik and Salkuyeh [5, 6] established some modified conjugate gradient type methods. However, the proposed methods in [5, 6] are extensions of the classical subspace methods for linear system and feasible only when the considered generalized Sylvester matrix equation is consistent. They can not be directly applied for the matrix equation with a submatrix constraint. Naturally, one may ask :“does the nonlinear conjugate gradient method can be extended to solve the quadratic inverse eigenvalue problem?” As another motivation in this paper, we are interested in studying a nonlinear conjugate gradient method for solving the quadratic inverse eigenvalue problem with a submatrix constraint. The contributions of this paper are as follows.

- We establish a nonlinear conjugate gradient method for finding the least squares solution of the quadratic inverse eigenvalue problem.
- We give the convergence analysis of the proposed nonlinear conjugate gradient method. A method for choosing the initial matrices to obtain the least Frobenius norm least squares solution of the quadratic inverse eigenvalue problem is given.

- By reformulating Problem 1.2 as another quadratic inverse eigenvalue problem with a submatrix constraint, we apply the nonlinear conjugate gradient method to solve it.
- Experimental results demonstrate that our proposed method yields superior performance for the quadratic inverse eigenvalue problem.

The remainder of this paper is organized as follows. In Section 2, we give an equivalent characterization of Problem 1.1 and establish a nonlinear conjugate gradient method to solve it. In Section 3, we present an iterative method for solving Problem 1.2. In Section 4, we report some numerical results to illustrate the feasibility and efficiency of the proposed method. This paper ends up with some concluding remarks in Section 5.

2 Iterative Method for Problem 1.1

Notice that the solution set of Problem 1.1 is merely a linear manifold rather than a linear subspace. First, we give an equivalent characterization of Problem 1.1. Let

$$\begin{aligned} S_1 &= \{Y|Y[s|s]=A_p, Y[\overline{s}|\overline{s}]=O\}\oplus\widetilde{S}_1,\\ S_2 &= \{Y|Y[t|t]=B_q, Y[\overline{t}|\overline{t}]=O\}\oplus\widetilde{S}_2,\\ S_3 &= \{Y|Y[u|u]=C_r, Y[\overline{u}|\overline{u}]=O\}\oplus\widetilde{S}_3,\end{aligned}$$

where

$$\widetilde{S}_1 = \{Y|Y\in HSC^{n\times n}, Y[s|s]=O\}, \tag{2.1a}$$
$$\widetilde{S}_2 = \{Y|Y\in HSC^{n\times n}, Y[t|t]=O\}, \tag{2.1b}$$
$$\widetilde{S}_3 = \{Y|Y\in HSC^{n\times n}, Y[u|u]=O\}. \tag{2.1c}$$

Then Problem 1.1 has the following equivalent form.

Problem 2.1. Given $X\in\mathbb{C}^{n\times m}$, $\Lambda=\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_m)\in\mathbb{C}^{m\times m}$, $s=(s_1,s_2,\ldots,s_p,n+1-s_p,\ldots,n+1-s_2,n+1-s_1)\in D_{2p,n}$, $t=(t_1,t_2,\ldots,t_q,n+1-t_q,\ldots,n+1-t_2,n+1-t_1)\in D_{2q,n}$, $u=(u_1,u_2,\ldots,u_r,n+1-u_r,\ldots,n+1-u_2,n+1-u_1)\in D_{2r,n}$, $A_p\in\mathbb{C}^{2p\times 2p}$, $B_q\in\mathbb{C}^{2q\times 2q}$ and $C_r\in\mathbb{C}^{2r\times 2r}$. We want to find $\widetilde{A}_*\in\widetilde{S}_1$, $\widetilde{B}_*\in\widetilde{S}_2$ and $\widetilde{C}_*\in\widetilde{S}_3$ such that

$$\|\widetilde{A}_*X\Lambda^2+\widetilde{B}_*X\Lambda+\widetilde{C}_*X-\widetilde{Z}\| = \min_{(\widetilde{A},\widetilde{B},\widetilde{C})\in\widetilde{S}_1\times\widetilde{S}_2\times\widetilde{S}_3}\|\widetilde{A}X\Lambda^2+\widetilde{B}X\Lambda+\widetilde{C}X-\widetilde{Z}\|,$$

where $\widetilde{Z}=-\widetilde{A}_pX\Lambda^2-\widetilde{B}_qX\Lambda-\widetilde{C}_rX$, in which $\widetilde{A}_p$, $\widetilde{B}_q$ and $\widetilde{C}_r$ denote the matrices satisfying $\widetilde{A}_p[s|s]=A_p$, $\widetilde{B}_q[t|t]=B_q$, $\widetilde{C}_r[u|u]=C_r$ and zeros elsewhere.

It is easy to see that $(\widetilde{A}_*,\widetilde{B}_*,\widetilde{C}_*)$ is a solution of Problem 2.1 if and only if $(A,B,C)=(\widetilde{A}_*+\widetilde{A}_p,\widetilde{B}_*+\widetilde{B}_q,\widetilde{C}_*+\widetilde{C}_r)$ is a solution of Problem 1.1.

In the sequel, we illustrate that the solution to Problem 2.1 exists. First, the property of Hermitian centroskew matrix is needed.

Lemma 2.1 ([27])**.** *A matrix $X\in HSC^{n\times n}$ if and only if $X=X^H=-S_nXS_n$.*

The following result is important for the proof of the existence of the solution to Problem 2.1.

Lemma 2.2 ([26]). *If a quadratic function f has a lower bound in a nonempty polyhedron $\mathbb{X}$, then f must have a global minimum in $\mathbb{X}$.*

Proposition 2.3. *Let $L_1(\widetilde{A},\widetilde{B},\widetilde{C}) = \widetilde{A}X\Lambda^2 + \widetilde{B}X\Lambda + \widetilde{C}X - \widetilde{Z}$, $L_2(\widetilde{A},\widetilde{B},\widetilde{C}) = \widetilde{A}^HX\Lambda^2 + \widetilde{B}^HX\Lambda + \widetilde{C}^HX - \widetilde{Z}$, $L_3(\widetilde{A},\widetilde{B},\widetilde{C}) = S_n\widetilde{A}S_nX\Lambda^2 + S_n\widetilde{B}S_nX\Lambda + S_n\widetilde{C}S_nX + \widetilde{Z}$ and $L_4(\widetilde{A},\widetilde{B},\widetilde{C}) = S_n\widetilde{A}^HS_nX\Lambda^2 + S_n\widetilde{B}^HS_nX\Lambda + S_n\widetilde{C}^HS_nX + \widetilde{Z}$. Then*

$$\min_{\{\widetilde{A}\in\mathbb{C}^{n\times n},\widetilde{B}\in\mathbb{C}^{n\times n},\widetilde{C}\in\mathbb{C}^{n\times n}:\widetilde{A}[s|s]=O,\widetilde{B}[t|t]=O,\widetilde{C}[u|u]=O\}} \left\| \frac{L_1(\widetilde{A},\widetilde{B},\widetilde{C})}{4} + \frac{L_2(\widetilde{A},\widetilde{B},\widetilde{C})}{4} - \frac{L_3(\widetilde{A},\widetilde{B},\widetilde{C})}{4} - \frac{L_4(\widetilde{A},\widetilde{B},\widetilde{C})}{4} \right\|^2 \quad (2.2)$$

takes a global minimum value at $(\widetilde{A}_*,\widetilde{B}_*,\widetilde{C}_*)$, where $\widetilde{A}_*[s|s] = O$, $\widetilde{B}_*[t|t] = O$ and $\widetilde{C}_*[u|u] = O$.

Proof. Let

$$\begin{aligned} & h(\mathrm{vec}(\widetilde{A}),\mathrm{vec}(\widetilde{B}),\mathrm{vec}(\widetilde{C})) \\ = & \left\| \frac{1}{4}\{\mathrm{vec}[L_1(\widetilde{A},\widetilde{B},\widetilde{C})] + \mathrm{vec}[L_2(\widetilde{A},\widetilde{B},\widetilde{C})] - \mathrm{vec}[L_3(\widetilde{A},\widetilde{B},\widetilde{C})] - \mathrm{vec}[L_4(\widetilde{A},\widetilde{B},\widetilde{C})]\} \right\|^2. \end{aligned}$$

Then h is a quadratic function about variable $(\mathrm{vec}(\widetilde{A}),\mathrm{vec}(\widetilde{B}),\mathrm{vec}(\widetilde{C}))$. Let

$$\mathbb{X} = \{\mathrm{vec}(\widetilde{A}) \in \mathbb{C}^{n^2}, \mathrm{vec}(\widetilde{B}) \in \mathbb{C}^{n^2}, \mathrm{vec}(\widetilde{C}) \in \mathbb{C}^{n^2} : \widetilde{A}[s|s] = O, \widetilde{B}[t|t] = O, \widetilde{C}[u|u] = O\}.$$

Obviously, $\mathbb{X}$ is a nonempty polyhedron. It then follows from Lemma 2.2 that the problem (2.2) takes the global minimum value at some point $(\mathrm{vec}(\widetilde{A}_*),\mathrm{vec}(\widetilde{B}_*),\mathrm{vec}(\widetilde{C}_*))$, where $\widetilde{A}_*[s|s] = O$, $\widetilde{B}_*[t|t] = O$ and $\widetilde{C}_*[u|u] = O$. By the inverse vec operator, the result is established. The proof is completed. □

Proposition 2.4. *The solution to* Problem 2.1 *exists.*

Proof. For arbitrary $\widetilde{A},\widetilde{B},\widetilde{C} \in HSC^{n\times n}$, it follows from Lemma 2.1 that

$$\begin{aligned} & \|\widetilde{A}X\Lambda^2 + \widetilde{B}X\Lambda + \widetilde{C}X - \widetilde{Z}\|^2 \\ & = \Big\| \frac{\widetilde{A} + \widetilde{A}^H - S_n\widetilde{A}S_n - S_n\widetilde{A}^HS_n}{4} X\Lambda^2 \\ & \quad + \frac{\widetilde{B} + \widetilde{B}^H - S_n\widetilde{B}S_n - S_n\widetilde{B}^HS_n}{4} X\Lambda + \frac{\widetilde{C} + \widetilde{C}^H - S_n\widetilde{C}S_n - S_n\widetilde{C}^HS_n}{4} X - \widetilde{Z} \Big\|^2 \\ & = \left\| \frac{L_1(\widetilde{A},\widetilde{B},\widetilde{C})}{4} + \frac{L_2(\widetilde{A},\widetilde{B},\widetilde{C})}{4} - \frac{L_3(\widetilde{A},\widetilde{B},\widetilde{C})}{4} - \frac{L_4(\widetilde{A},\widetilde{B},\widetilde{C})}{4} \right\|^2. \end{aligned}$$

Assume that the problem (2.2) takes the global minimum value at $(\widetilde{A}_*,\widetilde{B}_*,\widetilde{C}_*)$, where $\widetilde{A}_*[s|s] = O$, $\widetilde{B}_*[t|t] = O$ and $\widetilde{C}_*[u|u] = O$. Then, for arbitrary $(\widetilde{A},\widetilde{B},\widetilde{C}) \in \widetilde{S}_1 \times \widetilde{S}_2 \times \widetilde{S}_3$, we have

$$\begin{aligned}
&\|\widetilde{A}X\Lambda^2+\widetilde{B}X\Lambda+\widetilde{C}X-\widetilde{Z}\|^2\\
&=\Big\|\frac{L_1(\widetilde{A},\widetilde{B},\widetilde{C})}{4}+\frac{L_2(\widetilde{A},\widetilde{B},\widetilde{C})}{4}-\frac{L_3(\widetilde{A},\widetilde{B},\widetilde{C})}{4}-\frac{L_4(\widetilde{A},\widetilde{B},\widetilde{C})}{4}\Big\|^2\\
&\geq \min_{\{\widetilde{A}\in\mathbb{C}^{n\times n},\widetilde{B}\in\mathbb{C}^{n\times n},\widetilde{C}\in\mathbb{C}^{n\times n}:\widetilde{A}[s|s]=O,\widetilde{B}[t|t]=O,\widetilde{C}[u|u]=O\}}\Big\|\frac{L_1(\widetilde{A},\widetilde{B},\widetilde{C})}{4}+\frac{L_2(\widetilde{A},\widetilde{B},\widetilde{C})}{4}\\
&\qquad-\frac{L_3(\widetilde{A},\widetilde{B},\widetilde{C})}{4}-\frac{L_4(\widetilde{A},\widetilde{B},\widetilde{C})}{4}\Big\|^2\\
&=\Big\|\frac{L_1(\widetilde{A}_*,\widetilde{B}_*,\widetilde{C}_*)}{4}+\frac{L_2(\widetilde{A}_*,\widetilde{B}_*,\widetilde{C}_*)}{4}-\frac{L_3(\widetilde{A}_*,\widetilde{B}_*,\widetilde{C}_*)}{4}-\frac{L_4(\widetilde{A}_*,\widetilde{B}_*,\widetilde{C}_*)}{4}\Big\|^2\\
&=\Big\|\frac{\widetilde{A}_*+\widetilde{A}_*^H-S_n\widetilde{A}_*S_n-S_n\widetilde{A}_*^HS_n}{4}X\Lambda^2+\frac{\widetilde{B}_*+\widetilde{B}_*^H-S_n\widetilde{B}_*S_n-S_n\widetilde{B}_*^HS_n}{4}X\Lambda\\
&\qquad+\frac{\widetilde{C}_*+\widetilde{C}_*^H-S_n\widetilde{C}_*S_n-S_n\widetilde{C}_*^HS_n}{4}X-\widetilde{Z}\Big\|^2. \qquad (2.3)
\end{aligned}$$

Let

$$\begin{aligned}
\widetilde{\mathcal{A}}_*&=\frac{\widetilde{A}_*+\widetilde{A}_*^H-S_n\widetilde{A}_*S_n-S_n\widetilde{A}_*^HS_n}{4},\\
\widetilde{\mathcal{B}}_*&=\frac{\widetilde{B}_*+\widetilde{B}_*^H-S_n\widetilde{B}_*S_n-S_n\widetilde{B}_*^HS_n}{4},\\
\widetilde{\mathcal{C}}_*&=\frac{\widetilde{C}_*+\widetilde{C}_*^H-S_n\widetilde{C}_*S_n-S_n\widetilde{C}_*^HS_n}{4}.
\end{aligned}$$

Using Lemma 2.1 again and the fact that $\widetilde{A}_*[s|s]=O$, $\widetilde{B}_*[t|t]=O$ and $\widetilde{C}_*[u|u]=O$, we obtain

$$\widetilde{\mathcal{A}}_*\in\widetilde{S}_1,\widetilde{\mathcal{B}}_*\in\widetilde{S}_2,\widetilde{\mathcal{C}}_*\in\widetilde{S}_3.$$

It then follows from (2.3) that

$$\min_{(\widetilde{A},\widetilde{B},\widetilde{C})\in\widetilde{S}_1\times\widetilde{S}_2\times\widetilde{S}_3}\|\widetilde{A}X\Lambda^2+\widetilde{B}X\Lambda+\widetilde{C}X-\widetilde{Z}\|=\|\widetilde{\mathcal{A}}_*X\Lambda^2+\widetilde{\mathcal{B}}_*X\Lambda+\widetilde{\mathcal{C}}_*X-\widetilde{Z}\|,$$

which implies that $(\widetilde{\mathcal{A}}_*,\widetilde{\mathcal{B}}_*,\widetilde{\mathcal{C}}_*)$ is a solution to Problem 2.1. The proof is completed. □

Remark 2.5. From Proposition 2.4, we have that the solution to Problem 1.1 exists.

We also need the following lemmas in order to derive the normal equation of Problem 2.1.

Lemma 2.6. *Let $A\in\mathbb{C}^{n\times n}$ and $X\in HSC^{n\times n}$. Then*

$$\Big\langle X,\frac{A-S_nAS_n}{2}\Big\rangle=\Big\langle X,\frac{1}{4}\big(A+A^H-S_n(A+A^H)S_n\big)\Big\rangle=\langle X,A\rangle.$$

Proof. For any $A\in\mathbb{C}^{n\times n}$ and $X\in HSC^{n\times n}$, we have

$$\begin{aligned}
\Big\langle X,\frac{A-S_nAS_n}{2}\Big\rangle &= \frac{1}{2}\langle X,A\rangle-\frac{1}{2}\langle X,S_nAS_n\rangle=\frac{1}{2}\langle X,A\rangle-\frac{1}{2}\langle S_nXS_n,A\rangle\\
&= \frac{1}{2}\langle X,A\rangle-\frac{1}{2}\langle -X,A\rangle=\frac{1}{2}\langle X,A\rangle+\frac{1}{2}\langle X,A\rangle=\langle X,A\rangle.
\end{aligned}$$

Then

$$\Big\langle X, \frac{(A+A^H)-S_n(A+A^H)S_n}{4}\Big\rangle = \frac{1}{2}\Big(\big\langle X, \frac{A-S_nAS_n}{2}\big\rangle + \big\langle X, \frac{A^H-S_nA^HS_n}{2}\big\rangle\Big)$$
$$= \frac{1}{2}\Big(\big\langle X, \frac{A-S_nAS_n}{2}\big\rangle + \big\langle X, \frac{A-S_nAS_n}{2}\big\rangle\Big) = \frac{1}{2}\big(\langle X, A\rangle + \langle X, A\rangle\big) = \langle X, A\rangle.$$

The proof is completed. □

Lemma 2.7. *Suppose that* $X \in \mathbb{C}^{n\times n}$, $W_1 \in \widetilde{S}_1$, $W_2 \in \widetilde{S}_2$ *and* $W_3 \in \widetilde{S}_3$. *Then* $\langle E_sE_s^TXE_sE_s^T, W_1\rangle = 0$, $\langle E_tE_t^TXE_tE_t^T, W_2\rangle = 0$ *and* $\langle E_uE_u^TXE_uE_u^T, W_3\rangle = 0$.

Proof. Since $W_1 \in \widetilde{S}_1$, $W_2 \in \widetilde{S}_2$ and $W_3 \in \widetilde{S}_3$, we immediately have $E_s^TW_1E_s = 0$, $E_t^TW_2E_t = 0$ and $E_u^TW_3E_u = 0$. Hence

$$\begin{aligned}\langle E_sE_s^TXE_sE_s^T, W_1\rangle &= \mathrm{Re}\{\mathrm{tr}(W_1^HE_sE_s^TXE_sE_s^T)\}\\ &= \mathrm{Re}\{\mathrm{tr}(W_1E_sE_s^TXE_sE_s^T)\}\\ &= \mathrm{Re}\{\mathrm{tr}(E_sE_s^TW_1E_sE_s^TX)\} = 0.\end{aligned}$$

Similarly, we have $\langle E_tE_t^TXE_tE_t^T, W_2\rangle = 0$ and $\langle E_uE_u^TXE_uE_u^T, W_3\rangle = 0$. The proof is completed. □

Lemma 2.8 ([15]). *Let* $\mathfrak{X}$ *be a finite dimensional inner product space and let* $\mathfrak{G}$ *be the subspace of* $\mathfrak{X}$. *Assume that* $\mathfrak{G}^\perp$ *is the orthogonal complement of* $\mathfrak{G}$. *Then, for any* $X \in \mathfrak{X}$, *there exists an unique* $G_0 \in \mathfrak{G}$ *such that*

$$\|X - G_0\| \le \|X - G\|, \quad \forall G \in \mathfrak{G}.$$

In this case, we say G_0 *is the projection of* X *onto the subspace* $\mathfrak{G}$, *denoted by* $G_0 = \mathcal{P}_{\mathfrak{G}}(X)$. *Moreover,* $G_0 = \mathcal{P}_{\mathfrak{G}}(X)$ *if and only if* $(X - G_0) \perp \mathfrak{G}$, i.e., $(X - G_0) \in \mathfrak{G}^\perp$.

Lemma 2.9 ([27]). *Let* $f(X)$ *be a continuous and differentiable function on a linear subspace* $\mathcal{L}$ *of* $\mathbb{C}^{n\times n}$. *Then there exists* $X_* \in \mathcal{L}$ *such that* $f(X_*) = \min_{X\in\mathcal{L}} f(X)$ *if and only if* $\mathcal{P}_{\mathcal{L}}(\nabla f(X_*)) = O$.

Theorem 2.10. *Let* $R = \widetilde{Z} - \widetilde{A}X\Lambda^2 - \widetilde{B}X\Lambda - \widetilde{C}X$. *Then* $(\widetilde{A}_*, \widetilde{B}_*, \widetilde{C}_*)$ *is a solution of* Problem 2.1 *if and only if it is a solution of the following system of matrix equations*

$$\begin{cases} R(X\Lambda^2)^H + X\Lambda^2R^H - S_n(R(X\Lambda^2)^H + X\Lambda^2R^H)S_n \\ -E_sE_s^T[R(X\Lambda^2)^H + X\Lambda^2R^H - S_n(R(X\Lambda^2)^H + X\Lambda^2R^H)S_n]E_sE_s^T = O, \\ R(X\Lambda)^H + X\Lambda R^H - S_n(R(X\Lambda)^H + X\Lambda R^H)S_n \\ -E_tE_t^T[R(X\Lambda)^H + X\Lambda R^H - S_n(R(X\Lambda)^H + X\Lambda R^H)S_n]E_tE_t^T = O, \\ RX^H + XR^H - S_n(RX^H + XR^H)S_n \\ -E_uE_u^T[RX^H + XR^H - S_n(RX^H + XR^H)S_n]E_uE_u^T = O. \end{cases} \tag{2.4}$$

Proof. We first define the following function on $\widetilde{S}_1 \times \widetilde{S}_2 \times \widetilde{S}_3$:

$$f(\widetilde{A}, \widetilde{B}, \widetilde{C}) = \|\widetilde{Z} - \widetilde{A}X\Lambda^2 - \widetilde{B}X\Lambda - \widetilde{C}X\|^2.$$

It is easy to see that $(\widetilde{A}_*, \widetilde{B}_*, \widetilde{C}_*)$ is a solution of Problem 2.1 if and only if

$$f(\widetilde{A}_*, \widetilde{B}_*, \widetilde{C}_*) = \min_{(\widetilde{A},\widetilde{B},\widetilde{C})\in\widetilde{S}_1\times\widetilde{S}_2\times\widetilde{S}_3} f(\widetilde{A}, \widetilde{B}, \widetilde{C}). \tag{2.5}$$

Lemma 2.9 implies that (2.5) is equivalent to

$$\mathcal{P}_{\widetilde{S}_1}\Big(\frac{\partial f}{\partial \widetilde{A}}\Big)|_{\widetilde{A}=\widetilde{A}^*}=O,\ \ \mathcal{P}_{\widetilde{S}_1}\Big(\frac{\partial f}{\partial \widetilde{B}}\Big)|_{\widetilde{B}=\widetilde{B}^*}=O,\ \ \mathcal{P}_{\widetilde{S}_1}\Big(\frac{\partial f}{\partial \widetilde{C}}\Big)|_{\widetilde{C}=\widetilde{C}^*}=O. \tag{2.6}$$

Now we derive $\frac{\partial f}{\partial \widetilde{A}}$, $\frac{\partial f}{\partial \widetilde{B}}$ and $\frac{\partial f}{\partial \widetilde{C}}$. For any $(M_1,\ M_1,\ M_3)\in\mathbb{C}^{n\times n}\times\mathbb{C}^{n\times n}\times\mathbb{C}^{n\times n}$ and $\xi\in\mathbb{R}$, by the Taylor series expansion, we have

$$f(\widetilde{A}+\xi M_1,\widetilde{B}+\xi M_2,\widetilde{C}+\xi M_3)=f(\widetilde{A},\widetilde{B},\widetilde{C})+\xi\Big\langle\frac{\partial f}{\partial \widetilde{A}},M_1\Big\rangle+\xi\Big\langle\frac{\partial f}{\partial \widetilde{B}},M_2\Big\rangle+\xi\Big\langle\frac{\partial f}{\partial \widetilde{C}},M_3\Big\rangle+o(\xi). \tag{2.7}$$

According to the basic properties of Frobenius norm and matrix inner product, we get the expression

$$\begin{aligned}
&f(\widetilde{A}+\xi M_1,\widetilde{B}+\xi M_2,\widetilde{C}+\xi M_3)\\
&=\|\widetilde{Z}-(\widetilde{A}+\xi M_1)X\Lambda^2-(\widetilde{B}+\xi M_2)X\Lambda-(\widetilde{C}+\xi M_3)X\|^2\\
&=\|\widetilde{Z}-\widetilde{A}X\Lambda^2-\widetilde{B}X\Lambda-\widetilde{C}X\|^2-2\xi\langle\widetilde{Z}-\widetilde{A}X\Lambda^2-\widetilde{B}X\Lambda-\widetilde{C}X,M_1X\Lambda^2+M_2X\Lambda+M_3X\rangle\\
&\quad+\xi^2\|M_1X\Lambda^2+M_2X\Lambda+M_3X\|^2\\
&=f(\widetilde{A},\widetilde{B},\widetilde{C})-2\xi\langle R,M_1X\Lambda^2+M_2X\Lambda+M_3X\rangle+\xi^2\|M_1X\Lambda^2+M_2X\Lambda+M_3X\|^2\\
&=f(\widetilde{A},\widetilde{B},\widetilde{C})-2\xi\langle R(X\Lambda^2)^H,M_1\rangle-2\xi\langle R(X\Lambda)^H,M_2\rangle-2\xi\langle RX^H,M_3\rangle+o(\xi),
\end{aligned}$$

which, together with (2.7), yields

$$\frac{\partial f}{\partial \widetilde{A}}=-2R(X\Lambda^2)^H,\ \frac{\partial f}{\partial \widetilde{B}}=-2R(X\Lambda)^H,\ \frac{\partial f}{\partial \widetilde{C}}=-2RX^H. \tag{2.8}$$

For any $T_1\in\widetilde{S}_1$, by Lemmas 2.6, 2.7 and 2.8, we obtain

$$\begin{aligned}
&\langle-2R(X\Lambda^2)^H,T_1\rangle=-\frac{1}{2}\langle R(X\Lambda^2)^H+X\Lambda^2R^H-S_n(R(X\Lambda^2)^H+X\Lambda^2R^H)S_n,T_1\rangle\\
=\ &-\frac{1}{2}\langle R(X\Lambda^2)^H+X\Lambda^2R^H-S_n(R(X\Lambda^2)^H+X\Lambda^2R^H)S_n-E_sE_s^T[R(X\Lambda^2)^H+X\Lambda^2R^H\\
&-S_n(R(X\Lambda^2)^H+X\Lambda^2R^H)S_n]E_sE_s^T,T_1\rangle.
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
\langle-2R(X\Lambda^2)^H,T_1\rangle&=\langle-2R(X\Lambda^2)^H-\mathcal{P}_{\widetilde{S}_1}\big(-2R(X\Lambda^2)^H\big),T_1\rangle+\langle\mathcal{P}_{\widetilde{S}_1}\big(-2R(X\Lambda^2)^H\big),T_1\rangle\\
&=\langle\mathcal{P}_{\widetilde{S}_1}\big(-2R(X\Lambda^2)^H\big),T_1\rangle.
\end{aligned} \tag{2.10}$$

Together (2.9) and (2.10) yields

$$\begin{aligned}
&\langle\mathcal{P}_{\widetilde{S}_1}\big(-2R(X\Lambda^2)^H\big),T_1\rangle=-\frac{1}{2}\langle R(X\Lambda^2)^H+X\Lambda^2R^H-S_n(R(X\Lambda^2)^H+X\Lambda^2R^H)S_n\\
&-E_sE_s^T[R(X\Lambda^2)^H+X\Lambda^2R^H-S_n(R(X\Lambda^2)^H+X\Lambda^2R^H)S_n]E_sE_s^T,T_1\rangle.
\end{aligned}$$

It follows from the relation (2.8) and the arbitrariness of T_1 that

$$\begin{aligned}
\mathcal{P}_{\widetilde{S}_1}\Big(\frac{\partial f}{\partial \widetilde{A}}\Big)&=\mathcal{P}_{\widetilde{S}_1}\big(-2R(X\Lambda^2)^H\big)\\
&=-\frac{1}{2}\Big\{R(X\Lambda^2)^H+X\Lambda^2R^H+S_n(R(X\Lambda^2)^H+X\Lambda^2R^H)S_n\\
&\qquad-E_sE_s^T[R(X\Lambda^2)^H+X\Lambda^2R^H+S_n(R(X\Lambda^2)^H+X\Lambda^2R^H)S_n]E_sE_s^T\Big\}.
\end{aligned}$$

Similarly

$$\begin{cases} \mathcal{P}_{\widetilde{S}_2}\left(\frac{\partial f}{\partial \widetilde{B}}\right) = -\mathcal{P}_{\widetilde{S}_2}\left(2R(X\Lambda)^H\right) = -\frac{1}{2}\Big\{R(X\Lambda)^H + X\Lambda R^H + S_n(R(X\Lambda)^H + X\Lambda R^H)S_n \\ \qquad - E_tE_t^T[R(X\Lambda)^H + X\Lambda R^H + S_n(R(X\Lambda)^H + X\Lambda R^H)S_n]E_tE_t^T\Big\}, \\ \mathcal{P}_{\widetilde{S}_3}\left(\frac{\partial f}{\partial \widetilde{C}}\right) = -\mathcal{P}_{\widetilde{S}_3}\left(2RX^H\right) = -\frac{1}{2}\Big\{RX^H + XR^H + S_n(RX^H + XR^H)S_n \\ \qquad - E_uE_u^T[RX^H + XR^H + S_n(RX^H + XR^H)S_n]E_uE_u^T\Big\}. \end{cases}$$

By the relation (2.6), the result is established. The proof is completed. □

It is well known that many iterative methods for matrix equations are based on iterative methods for linear system $Ax = b$, where $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. For the least squares problem

$$\min_{x\in\mathbb{C}^n} \|b - Ax\| \quad \text{for} \quad A \in \mathbb{C}^{m\times n} \quad \text{and } b \in \mathbb{C}^m, \tag{2.11}$$

the LSQR and CGLS methods [8, 34] are all efficient. More importantly, CG-type method overcomes the shortcoming of slow convergence of steepest descent method, and avoids storing and computing Hessian matrix and its inverse of Newton method. Many researchers have investigations into this direction. For instance, Zhou et al. [42] established a gradient-based iterative algorithm for solving the coupled matrix equation. Based on CG method, Lv and Zhang [32] raised a neat iterative algorithm in the least squares sense for solving a periodic Sylvester matrix equation.

Motivated by the work mentioned above, we propose a nonlinear conjugate gradient method for solving Problem 2.1.

Algorithm 2.11. (Nonlinear conjugate gradient method (NCG) for Problem 2.1)

Step 0 Input $X \in \mathbb{C}^{n\times m}$, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{C}^{m\times m}$, $s = (s_1, s_2, \ldots, s_p, n+1-s_p, \ldots, n+1-s_2, n+1-s_1) \in D_{2p,n}$, $t = (t_1, t_2, \ldots, t_q, n+1-t_q, \ldots, n+1-t_2, n+1-t_1) \in D_{2q,n}$, $u = (u_1, u_2, \ldots, u_r, n+1-u_r, \ldots, n+1-u_2, n+1-u_1) \in D_{2r,n}$, $A_p \in \mathbb{C}^{2p\times 2p}$, $B_q \in \mathbb{C}^{2q\times 2q}$, $C_r \in \mathbb{C}^{2r\times 2r}$, $\widetilde{A}_p \in \mathbb{C}^{n\times n}$, $\widetilde{B}_q \in \mathbb{C}^{n\times n}$, $\widetilde{C}_r \in \mathbb{C}^{n\times n}$ and $\widetilde{Z} = -\widetilde{A}_pX\Lambda^2 - \widetilde{B}_qX\Lambda - \widetilde{C}_rX$. Choose the initial matrices $\widetilde{A}(1) \in \widetilde{S}_1$, $\widetilde{B}(1) \in \widetilde{S}_2$ and $\widetilde{C}(1) \in \widetilde{S}_3$.

Step 1 Compute

$$\begin{aligned} R(1) &= \widetilde{Z} - \widetilde{A}(1)X\Lambda^2 - \widetilde{B}(1)X\Lambda - \widetilde{C}(1)X, \\ P_1(1) &= R(1)(X\Lambda^2)^H + X\Lambda^2R^H(1) - S_n(R(1)(X\Lambda^2)^H + X\Lambda^2R^H(1))S_n \\ &\quad -E_sE_s^T[R(1)(X\Lambda^2)^H + X\Lambda^2R^H(1) - S_n(R(1)(X\Lambda^2)^H + X\Lambda^2R^H(1))S_n]E_sE_s^T, \\ P_2(1) &= R(1)(X\Lambda)^H + X\Lambda R^H(1) - S_n(R(1)(X\Lambda)^H + X\Lambda R^H(1))S_n \\ &\quad -E_tE_t^T[R(1)(X\Lambda)^H + X\Lambda R^H(1) - S_n(R(1)(X\Lambda)^H + X\Lambda R^H(1))S_n]E_tE_t^T, \\ P_3(1) &= R(1)X^H + XR(1)^H - S_n(R(1)X^H + XR^H(1))S_n \\ &\quad -E_uE_u^T[R(1)X^H + XR^H(1) - S_n(R(1)X^H + XR^H(1))S_n]E_uE_u^T. \end{aligned}$$

Set $M_1(1) = -P_1(1)$, $M_2(1) = -P_2(1)$ and $M_3(1) = -P_3(1)$. Denote

$$P(1) = \begin{pmatrix} P_1(1) & O & O \\ O & P_2(1) & O \\ O & O & P_3(1) \end{pmatrix} \quad \text{and} \quad M(1) = \begin{pmatrix} M_1(1) & O & O \\ O & M_2(1) & O \\ O & O & M_3(1) \end{pmatrix}.$$

Set $k = 1$.

Step 2 If $\|R(k)\| = 0$ or $\|P(k)\| = 0$, stop. Otherwise, go to Step 3.
Step 3 Compute $Q(k) = M_1(k)X\Lambda^2 + M_2(k)X\Lambda + M_3(k)X$ and

$$\alpha_k = \frac{\langle M(k), P(k)\rangle}{4\|Q(k)\|^2}. \tag{2.12}$$

Update the sequences

$$\begin{aligned}
&\widetilde{A}(k+1) = \widetilde{A}(k) + \alpha_k M_1(k), \\
&\widetilde{B}(k+1) = \widetilde{B}(k) + \alpha_k M_2(k), \\
&\widetilde{C}(k+1) = \widetilde{C}(k) + \alpha_k M_3(k), R(k+1) = R(k) - \alpha_k Q(k),
\end{aligned}$$

$$\begin{aligned}
P_1(k+1) = {} & R(k+1)(X\Lambda^2)^H + X\Lambda^2 R^H(k+1) \\
& - S_n(R(k+1)(X\Lambda^2)^H + X\Lambda^2 R^H(k+1))S_n \\
& - E_s E_s^T[R(k+1)(X\Lambda^2)^H + X\Lambda^2 R^H(k+1) \\
& - S_n(R(k+1)(X\Lambda^2)^H + X\Lambda^2 R^H(k+1))S_n]E_s E_s^T, \\
P_2(k+1) = {} & R(k+1)(X\Lambda)^H + X\Lambda R^H(k+1) - S_n(R(k+1)(X\Lambda)^H \\
& + X\Lambda R^H(k+1))S_n - E_t E_t^T[R(k+1)(X\Lambda)^H + X\Lambda R^H(k+1) \\
& - S_n(R(k+1)(X\Lambda)^H + X\Lambda R^H(k+1))S_n]E_t E_t^T, \\
P_3(k+1) = {} & R(k+1)X^H + XR(k+1)^H - S_n(R(k+1)X^H \\
& + XR^H(k+1))S_n - E_u E_u^T[R(k+1)X^H + XR^H(k+1) \\
& - S_n(R(k+1)X^H + XR^H(k+1))S_n]E_u E_u^T.
\end{aligned}$$

Denote

$$P(k+1) = \begin{pmatrix} P_1(k+1) & O & O \\ O & P_2(k+1) & O \\ O & O & P_3(k+1) \end{pmatrix}.$$

Compute

$$\beta_k = \frac{\|P(k+1)\|^2}{\|P(k)\|^2}. \tag{2.13}$$

Update the sequences

$$M_j(k+1) = -P_j(k+1) + \beta_k M_j(k)$$

for $j = 1, 2, 3$. Denote

$$M(k+1) = \begin{pmatrix} M_1(k+1) & O & O \\ O & M_2(k+1) & O \\ O & O & M_3(k+1) \end{pmatrix}.$$

Step 4 Set $k = k + 1$, go to Step 2.

Remark 2.12. According to Lemma 2.1 and Algorithm 2.11, it follows that matrix sequences generated by Algorithm 2.11 satisfy $\{\widetilde{A}(k)\}$, $\{P_1(k)\}$, $\{M_1(k)\} \subseteq \widetilde{S}_1$, $\{\widetilde{B}(k)\}$, $\{P_2(k)\}$, $\{M_2(k)\} \subseteq \widetilde{S}_2$ and $\{\widetilde{C}(k)\}$, $\{P_3(k)\}$, $\{M_3(k)\} \subseteq \widetilde{S}_3$.

Remark 2.13. If $\|R(k)\| = 0$, then $(\widetilde{A}(k), \widetilde{B}(k), \widetilde{C}(k))$ is a solution of matrix equation $\widetilde{A}X\Lambda^2 + \widetilde{B}X\Lambda + \widetilde{C}X - \widetilde{Z} = O$. That is, $(\widetilde{A}(k), \widetilde{B}(k), \widetilde{C}(k))$ is a solution of Problem 2.1 and

$$\min_{(\widetilde{A},\widetilde{B},\widetilde{C})\in\widetilde{S}_1\times\widetilde{S}_2\times\widetilde{S}_3} \|\widetilde{A}X\Lambda^2 + \widetilde{B}X\Lambda + \widetilde{C}X - \widetilde{Z}\| = 0.$$

If $\|P(k)\| = 0$, then $(\widetilde{A}(k), \widetilde{B}(k), \widetilde{C}(k))$ is a solution of Problem 2.1 and

$$\min_{(\widetilde{A},\widetilde{B},\widetilde{C})\in\widetilde{S}_1\times\widetilde{S}_2\times\widetilde{S}_3} \|\widetilde{A}X\Lambda^2 + \widetilde{B}X\Lambda + \widetilde{C}X - \widetilde{Z}\| > 0.$$

Now, we list some basic properties of Algorithm 2.11.

Lemma 2.14. *Let the sequences* $\{P(k)\}$ *and* $\{M(k)\}$ *be generated by* Algorithm 2.11. *Then*

$$\langle M(k), P(k+1)\rangle = 0.$$

Proof. We first define $G_1(k)$, $G_2(k)$ and $G_3(k)$ as follows:

$$\begin{cases} G_1(k) = Q(k)(X\Lambda^2)^H + X\Lambda^2 Q^H(k) - S_n(Q(k)(X\Lambda^2)^H + X\Lambda^2 Q^H(k))S_n \\ \qquad - E_sE_s^T[Q(k)(X\Lambda^2)^H + X\Lambda^2 R^H(k) - S_n(Q(k)(X\Lambda^2)^H + X\Lambda^2 Q^H(k))S_n]E_sE_s^T, \\ G_2(k) = Q(k)(X\Lambda)^H + X\Lambda Q^H(k) - S_n(Q(k)(X\Lambda)^H + X\Lambda Q^H(k))S_n \\ \qquad - E_tE_t^T[Q(k)(X\Lambda)^H + X\Lambda R^H(1) - S_n(Q(1)(X\Lambda)^H + X\Lambda Q^H(1))S_n]E_tE_t^T, \\ G_3(k) = Q(1)X^H + XQ(k)^H - S_n(Q(k)X^H + XQ^H(k))S_n \\ \qquad - E_uE_u^T[Q(k)X^H + XQ^H(k) - S_n(Q(k)X^H + XQ^H(k))S_n]E_uE_u^T. \end{cases}$$

Since $R(k+1) = R(k) - \alpha_k Q(k)$, it follows that

$$\begin{aligned} P_1(k+1) &= R(k+1)(X\Lambda^2)^H + X\Lambda^2 R^H(k+1) - S_n(R(k+1)(X\Lambda^2)^H \\ &\quad + X\Lambda^2 R^H(k+1))S_n - E_sE_s^T[R(k+1)(X\Lambda^2)^H + X\Lambda^2 R^H(k+1) \\ &\quad - S_n(R(k+1)(X\Lambda^2)^H + X\Lambda^2 R^H(k+1))S_n]E_sE_s^T \\ &= P_1(k) - \alpha_k[Q(k)(X\Lambda^2)^H + X\Lambda^2 Q^H(k) - S_n(Q(k)(X\Lambda^2)^H + X\Lambda^2 Q^H(k))S_n \\ &\quad - E_sE_s^T[Q(k)(X\Lambda^2)^H + X\Lambda^2 R^H(k) - S_n(Q(k)(X\Lambda^2)^H + X\Lambda^2 Q^H(k))S_n]E_sE_s^T \\ &= P_1(k) - \alpha_k G_1(k). \end{aligned}$$

Similarly, we have $P_2(k+1) = P_2(k) - \alpha_k G_2(k)$ and $P_3(k+1) = P_3(k) - \alpha_k G_3(k)$. Then

$$\langle M(k), P(k+1)\rangle = \sum_{j=1}^{3}\langle M_j(k), P_j(k) - \alpha_k G_j(k)\rangle = \langle M(k), P(k)\rangle - \alpha_k\langle M(k), G(k)\rangle. \quad (2.14)$$

On the other hand, by Algorithm 2.11, Lemmas 2.1, 2.6 and 2.7 and Remark 2.12, it follows that

$$\begin{aligned} \langle M_1(k), G_1(k)\rangle &= \langle M_1(k), Q(k)(X\Lambda^2)^H + X\Lambda^2 Q^H(k) \\ &\quad -S_n(Q(k)(X\Lambda^2)^H + X\Lambda^2 Q^H(k))S_n \\ &\quad -E_sE_s^T[Q(k)(X\Lambda^2)^H + X\Lambda^2 Q^H(k) \\ &\quad -S_n(Q(k)(X\Lambda^2)^H + X\Lambda^2 Q^H(k))S_n]E_sE_s^T\rangle \\ &= \langle M_1(k), Q(k)(X\Lambda^2)^H + X\Lambda^2 Q^H(k) \\ &\quad -S_n(Q(k)(X\Lambda^2)^H + X\Lambda^2 Q^H(k))S_n\rangle \\ &= 4\langle M_1(k), Q(k)(X\Lambda^2)^H\rangle = 4\langle M_1(k)X\Lambda^2, Q(k)\rangle. \end{aligned}$$

Similarly, we have

$$\langle M_2(k), G_2(k)\rangle = 4\big\langle M_2(k)X\Lambda, Q(k)\big\rangle \quad \text{and} \quad \langle M_3(k), G_3(k)\rangle = 4\big\langle M_3(k)X, Q(k)\big\rangle.$$

Hence

$$\langle M(k), G(k)\rangle = \sum_{j=1}^{3}\langle M_j(k), G_j(k)\rangle = 4\big\langle Q(k), Q(k)\big\rangle,$$

which, together with (2.12) and (2.14), yields

$$\begin{aligned}\langle M(k), P(k+1)\rangle &= \langle M(k), P(k)\rangle - \alpha_k\langle M(k), G(k)\rangle = \langle M(k), P(k)\rangle - 4\alpha_k\langle Q(k), Q(k)\rangle \\ &= \langle M(k), P(k)\rangle - 4\times\frac{\langle M(k), P(k)\rangle}{4\|Q(k)\|^2}\|Q(k)\|^2 = 0.\end{aligned}$$

The proof is completed. □

Lemma 2.15. *Let the sequences* $\{M(k)\}$ *and* $\{P(k)\}$ *be generated by* Algorithm 2.11. *Then*

$$\langle M(k), P(k)\rangle = -\|P(k)\|^2.$$

Proof. For $k = 1$, since $M_j(1) = -P_j(1)$, we immediately have

$$\langle M(1), P(1)\rangle = \sum_{j=1}^{3}\langle M_j(1), P_j(1)\rangle = -\sum_{j=1}^{3}\langle P_j(1), P_j(1)\rangle = -\|P(1)\|^2.$$

For $k \geq 2$, it follows from Lemma 2.14 that

$$\begin{aligned}\langle M(k+1), P(k+1)\rangle &= \sum_{j=1}^{3}\langle -P_j(k+1) + \beta_k M_j(k), P_j(k+1)\rangle \\ &= -\|P(k+1)\|^2 + \beta_k\langle M(k), P(k+1)\rangle = -\|P(k+1)\|^2.\end{aligned}$$

The proof is completed. □

Lemma 2.16. *Let the sequences* $\{M(k)\}$ *and* $\{P(k)\}$ *be generated by* Algorithm 2.11. *Then*

$$\sum_{k=1}^{\infty}\frac{\|P(k)\|^4}{\|M(k)\|^2} < \infty. \tag{2.15}$$

Proof. By direct calculations, we have

$$\begin{aligned}\|Q(k)\|^2 &= \|M_1(k)X\Lambda^2 + M_2(k)X\Lambda + M_3(k)X\|^2 \\ &= \big\|\big((X\Lambda^2)^T\otimes I\big)\mathrm{vec}(M_1(k)) + \big((X\Lambda)^T\otimes I\big)\mathrm{vec}(M_2(k)) + \big(X^T\otimes I\big)\mathrm{vec}(M_3(k))\big\|^2 \\ &= \left\|\begin{pmatrix}(X\Lambda^2)^T\otimes I & (X\Lambda)^T\otimes I & X^T\otimes I\end{pmatrix}\begin{pmatrix}\mathrm{vec}(M_1(k))\\ \mathrm{vec}(M_2(k))\\ \mathrm{vec}(M_3(k))\end{pmatrix}\right\|^2 \\ &\leq \left\|\begin{pmatrix}(X\Lambda^2)^T\otimes I & (X\Lambda)^T\otimes I & X^T\otimes I\end{pmatrix}\right\|^2\left\|\begin{pmatrix}\mathrm{vec}(M_1(k))\\ \mathrm{vec}(M_2(k))\\ \mathrm{vec}(M_3(k))\end{pmatrix}\right\|^2 \\ &= \theta\sum_{j=1}^{3}\|\mathrm{vec}(M_j(k))\|^2 = \theta\sum_{j=1}^{3}\|M_j(k)\|^2 = \theta\|M(k)\|^2, \end{aligned} \tag{2.16}$$

where

$$\theta = \left\| \begin{pmatrix} (X\Lambda^2)^T \otimes I & (X\Lambda)^T \otimes I & X^T \otimes I \end{pmatrix} \right\|^2 .$$

According to Algorithm 2.11, Lemmas 2.6 and 2.7, we obtain

$$\begin{aligned}
&\|R(k+1)\|^2 = \|R(k) - \alpha_k Q(k)\|^2 = \|R(k)\|^2 - 2\alpha_k \langle R(k), Q(k)\rangle + \alpha_k^2 \|Q(k)\|^2 \\
&= \|R(k)\|^2 - 2\alpha_k \langle R(k), M_1(k)X\Lambda^2 + M_2(k)X\Lambda + M_3(k)X\rangle + \alpha_k^2\|Q(k)\|^2 \\
&= \|R(k)\|^2 - 2\alpha_k \Big[\langle R(k)(X\Lambda^2)^H, M_1(k)\rangle + \langle R(k)(X\Lambda)^H, M_2(k)\rangle + \langle R(k)X^H, M_3(k)\rangle\Big] \\
&\quad + \alpha_k^2\|Q(k)\|^2 \\
&= \|R(k)\|^2 - \frac{\alpha_k}{2}\Big\{\big\langle R(k)(X\Lambda^2)^H + X\Lambda^2R^H(k) - S_n(R(k)(X\Lambda^2)^H + X\Lambda^2R^H(k))S_n \\
&\quad - E_sE_s^T[R(k)(X\Lambda^2)^H + X\Lambda^2R^H(k) - S_n(R(k)(X\Lambda^2)^H + X\Lambda^2R^H(k))S_n]E_sE_s^T, M_1(k)\big\rangle \\
&\quad + \big\langle R(k)(X\Lambda)^H + X\Lambda R^H(k) - S_n(R(k)(X\Lambda)^H + X\Lambda R^H(k))S_n \\
&\quad - E_tE_t^T[R(k)(X\Lambda)^H + X\Lambda R^H(k) - S_n(R(k)(X\Lambda)^H + X\Lambda R^H(k))S_n]E_tE_t^T, M_2(k)\big\rangle \\
&\quad + \big\langle R(k)X^H + XR^H(k) - S_n(R(k)X^H + XR^H(k))S_n \\
&\quad - E_uE_u^T[R(k)X^H + XR^H(k) - S_n(R(k)X^H + XR^H(k))S_n]E_uE_u^T, M_3(k)\big\rangle\Big\} \\
&\quad + \alpha_k^2\|Q(k)\|^2 \\
&= \|R(k)\|^2 - \frac{\alpha_k}{2}\left[\langle P_1(k), M_1(k)\rangle + \langle P_2(k), M_2(k)\rangle + \langle P_3(k), M_3(k)\rangle\right] + \alpha_k^2\|Q(k)\|^2 \\
&= \|R(k)\|^2 - \frac{\alpha_k}{2}\langle P(k), M(k)\rangle + \alpha_k \frac{\langle M(k), P(k)\rangle}{4\|Q(k)\|^2}\|Q(k)\|^2 \\
&= \|R(k)\|^2 - \frac{\alpha_k}{4}\langle P(k), M(k)\rangle.
\end{aligned}$$

This implies that

$$\|R(k+1)\|^2 - \|R(k)\|^2 = -\frac{\alpha_k}{4}\langle P(k), M(k)\rangle = -\frac{1}{16}\frac{\langle M(k), P(k)\rangle^2}{\|Q(k)\|^2} \le 0. \tag{2.17}$$

Then the sequence $\{\|R(k)\|^2\}$ is decreasing and its limit exists. Hence, by Lemma 2.15 and the relations (2.16) and (2.17), it follows that

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{\|P(k)\|^4}{\|M(k)\|^2} &= \sum_{k=1}^{\infty}\frac{\langle M(k), P(k)\rangle^2}{\|M(k)\|^2} \le \sum_{k=1}^{\infty}\theta\frac{\langle M(k), P(k)\rangle^2}{\|Q(k)\|^2} \\
&= 16\theta\sum_{k=1}^{\infty}\Big[\|R(k)\|^2 - \|R(k+1)\|^2\Big] = 16\theta\Big[\|R(1)\|^2 - \lim_{k\to\infty}\|R(k)\|^2\Big] < \infty.
\end{aligned}$$

The proof is completed. □

Theorem 2.17. *Let* $\{P_j(k)\}\,(j = 1,2,3)$ *be generated by* Algorithm 2.11. *Then* $\lim\limits_{k\to\infty}\|P_j(k)\| = 0$ *for* $j = 1,2,3$.

Proof. By Lemma 2.14, it follows that

$$\begin{aligned}\|M(k+1)\|^2 &= \sum_{j=1}^{3} \big\|M_j(k+1)\big\|^2 = \sum_{j=1}^{3} \big\| - P_j(k+1) + \beta_k M_j(k)\big\|^2 \\ &= \big\|P(k+1)\big\|^2 - 2\beta_k \langle P(k+1), M(k)\rangle + \beta_k^2 \big\|M(k)\big\|^2 \\ &= \big\|P(k+1)\big\|^2 + \beta_k^2 \big\|M(k)\big\|^2 = \big\|P(k+1)\big\|^2 + \frac{\big\|P(k+1)\big\|^4}{\big\|P(k)\big\|^4}\big\|M(k)\big\|^2.\end{aligned}$$

Then

$$\frac{\|M(k+1)\|^2}{\|P(k+1)\|^4} = \frac{1}{\|P(k+1)\|^2} + \frac{\|M(k)\|^2}{\|P(k)\|^4}. \tag{2.18}$$

Let $t_k = \frac{\|M(k)\|^2}{\|P(k)\|^4}$. According to the relation (2.18), we immediately have

$$t_{k+1} = t_k + \frac{1}{\|P(k+1)\|^2}. \tag{2.19}$$

We now prove this result by contradiction. Assume that $\lim\limits_{k\to\infty} \|P(k)\| \neq 0$. Then there exists a positive number $\delta > 0$ such that $\|P(k)\| \geq \delta$ for all $k \geq 1$. It follows from the relation (2.19) that

$$t_{k+1} \leq t_0 + \frac{k+1}{\delta},$$

which implies that

$$\sum_{k=1}^{\infty} \frac{1}{t_k} \geq \sum_{k=1}^{\infty} \frac{\delta}{\delta t_0 + k + 1} = \infty. \tag{2.20}$$

However, by Lemma 2.16, we get

$$\sum_{k=1}^{\infty} \frac{1}{t_k} = \sum_{k=1}^{\infty} \frac{\|P(k)\|^4}{\|M(k)\|^2} < \infty,$$

which obtains a contradiction. Hence $\lim\limits_{k\to\infty} \|P(k)\| = 0$, and then $\lim\limits_{k\to\infty} \|P_j(k)\| = 0$ for $j = 1, 2, 3$. The proof is completed. □

Finally, we consider the least Frobenius norm solution of Problem 2.1. The following lemma is needed for the main result.

Lemma 2.18. *Suppose that* $(\widetilde{A}_*, \widetilde{B}_*, \widetilde{C}_*)$ *is a solution of* Problem 2.1. *Then, an arbitrary solution* $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ *of* Problem 2.1 *can be expressed as*

$$(\widetilde{A}, \widetilde{B}, \widetilde{C}) = (\widetilde{A}_* + \mathcal{W}_1, \widetilde{B}_* + \mathcal{W}_2, \widetilde{C}_* + \mathcal{W}_3), \tag{2.21}$$

where $(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3) \in \widetilde{S}_1 \times \widetilde{S}_2 \times \widetilde{S}_3$ *satisfies*

$$\mathcal{W}_1 X \Lambda^2 + \mathcal{W}_2 X \Lambda + \mathcal{W}_3 X = O. \tag{2.22}$$

Proof. For an arbitrary solution $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ of Problem 2.1, we first define the following matrices

$$\mathcal{W}_1 = \widetilde{A} - \widetilde{A}_*, \quad \mathcal{W}_2 = \widetilde{B} - \widetilde{B}_*, \quad \mathcal{W}_3 = \widetilde{C} - \widetilde{C}_*.$$

Obviously $(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3) \in \widetilde{S}_1 \times \widetilde{S}_2 \times \widetilde{S}_3$. Let $R_* = \widetilde{A}_* X \Lambda^2 + \widetilde{B}_* X \Lambda + \widetilde{C}_* X - \widetilde{Z}$. Since $(\widetilde{A}_*, \widetilde{B}_*, \widetilde{C}_*)$ is a solution of Problem 2.1, it follows from Theorem 2.10 that

$$\begin{aligned}
& \langle \widetilde{A}_* X\Lambda^2 + \widetilde{B}_* X\Lambda + \widetilde{C}_* X - \widetilde{Z}, \mathcal{W}_1 X\Lambda^2 + \mathcal{W}_2 X\Lambda + \mathcal{W}_3 X\rangle \\
= \; & \langle R_*(X\Lambda^2)^H, \mathcal{W}_1\rangle + \langle R_*(X\Lambda)^H, \mathcal{W}_2\rangle + \langle R_* X^H, \mathcal{W}_3\rangle \\
= \; & \frac{1}{4}\Big\{ \langle R_*(X\Lambda^2)^H + X\Lambda^2 R_*^H - S_n(R_*(X\Lambda^2)^H + X\Lambda^2 R_*^H)S_n \\
& -E_sE_s^T[R_*(X\Lambda^2)^H + X\Lambda^2R_*^H - S_n(R_*(X\Lambda^2)^H + X\Lambda^2R_*^H)S_n]E_sE_s^T, \mathcal{W}_1\rangle \\
& +\langle R_*(X\Lambda)^H + X\Lambda R_*^H - S_n(R_*(X\Lambda)^H + X\Lambda R_*^H)S_n \\
& -E_tE_t^T[R_*(X\Lambda)^H + X\Lambda R_*^H - S_n(R_*(X\Lambda)^H + X\Lambda R_*^H)S_n]E_tE_t^T, \mathcal{W}_2\rangle \\
& +\langle R_*X^H + XR_*^H - S_n(R_*X^H + XR_*^H)S_n \\
& -E_uE_u^T[R_*X^H + XR_*^H - S_n(R_*X^H + XR_*^H)S_n]E_uE_u^T, \mathcal{W}_3\rangle \Big\} = 0.
\end{aligned}$$

Then

$$\begin{aligned}
& \|\widetilde{A}X\Lambda^2 + \widetilde{B}X\Lambda + \widetilde{C}X - \widetilde{Z}\|^2 \\
= \; & \|(\widetilde{A}_* + \mathcal{W}_1)X\Lambda^2 + (\widetilde{B}_* + \mathcal{W}_2)X\Lambda + (\widetilde{C}_* + \mathcal{W}_3)X - \widetilde{Z}\|^2 \\
= \; & \|\widetilde{A}_*X\Lambda^2 + \widetilde{B}_*X\Lambda + \widetilde{C}_*X - \widetilde{Z}\|^2 \\
& +2\langle \widetilde{A}_*X\Lambda^2 + \widetilde{B}_*X\Lambda + \widetilde{C}_*X - \widetilde{Z}, \mathcal{W}_1X\Lambda^2 + \mathcal{W}_2X\Lambda + \mathcal{W}_3X\rangle \\
& +\|\mathcal{W}_1X\Lambda^2 + \mathcal{W}_2X\Lambda + \mathcal{W}_3X\|^2 \\
= \; & \|\widetilde{A}_*X\Lambda^2 + \widetilde{B}_*X\Lambda + \widetilde{C}_*X - \widetilde{Z}\|^2 + \|\mathcal{W}_1X\Lambda^2 + \mathcal{W}_2X\Lambda + \mathcal{W}_3X\|^2.
\end{aligned}$$

This implies that

$$\mathcal{W}_1X\Lambda^2 + \mathcal{W}_2X\Lambda + \mathcal{W}_3X = O.$$

The proof is completed. □

Theorem 2.19. *If we choose the initial matrices* $\widetilde{A}(1)$, $\widetilde{B}(1)$ *and* $\widetilde{C}(1)$ *as follows*:

$$\begin{cases}
\widetilde{A}(1) = Y(1)(X\Lambda^2)^H + X\Lambda^2Y^H(1) - S_n(Y(1)(X\Lambda^2)^H + X\Lambda^2Y^H(1))S_n \\
\qquad - E_sE_s^T[Y(1)(X\Lambda^2)^H + X\Lambda^2Y^H(1) - S_n(Y(1)(X\Lambda^2)^H + X\Lambda^2Y^H(1))S_n]E_sE_s^T, \\
\widetilde{B}(1) = Y(1)(X\Lambda)^H + X\Lambda Y^H(1) - S_n(Y(1)(X\Lambda)^H + X\Lambda Y(1)^H)S_n \\
\qquad - E_tE_t^T[Y(1)(X\Lambda)^H + X\Lambda Y^H(1) - S_n(Y(1)(X\Lambda)^H + X\Lambda Y^H(1))S_n]E_tE_t^T, \\
\widetilde{C}(1) = Y(1)X^H + XY(1)^H - S_n(Y(1)X^H + XY^H(1))S_n \\
\qquad - E_uE_u^T[Y(1)X^H + XY^H(1) - S_n(Y(1)X^H + XY^H(1))S_n]E_uE_u^T,
\end{cases} \tag{2.23}$$

where $Y(1) \in \mathbb{C}^{n\times m}$ *is an arbitrary matrix* (*especially, take* $Y(1) = O$), *then the solution* $(\widetilde{A}_*, \widetilde{B}_*, \widetilde{C}_*)$ *given by* Algorithm 2.11 *is the unique least Frobenius norm least squares solution of* Problem 2.1.

Proof. Assume that the initial value $(\widetilde{A}(1), \widetilde{B}(1), \widetilde{C}(1))$ is chosen as in (2.23). According to Theorem 2.17, the solution $(\widetilde{A}_*, \widetilde{B}_*, \widetilde{C}_*)$ can be obtained by Algorithm 2.11 and it has the following form:

$$\begin{cases} \widetilde{A}_* = Y(X\Lambda^2)^H + X\Lambda^2 Y^H - S_n(Y(X\Lambda^2)^H + X\Lambda^2 Y^H)S_n \\ \quad - E_s E_s^T[Y(X\Lambda^2)^H + X\Lambda^2 Y^H - S_n(Y(X\Lambda^2)^H + X\Lambda^2 Y^H)S_n]E_s E_s^T, \\ \widetilde{B}_* = Y(X\Lambda)^H + X\Lambda Y^H - S_n(Y(X\Lambda)^H + X\Lambda Y^H(k))S_n \\ \quad - E_t E_t^T[Y(X\Lambda)^H + X\Lambda Y^H - S_n(Y(X\Lambda)^H + X\Lambda Y^H)S_n]E_t E_t^T, \\ \widetilde{C}_* = YX^H + XY^H - S_n(YX^H + XY^H)S_n \\ \quad - E_u E_u^T[YX^H + XY^H - S_n(YX^H + XY^H)S_n]E_u E_u^T. \end{cases} \tag{2.24}$$

Now suppose that $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ is an arbitrary solution of Problem 2.1. It follows from Lemma 2.18 that there exists $(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3) \in \widetilde{S}_1 \times \widetilde{S}_2 \times \widetilde{S}_3$ such that

$$(\widetilde{A}, \widetilde{B}, \widetilde{C}) = (\widetilde{A}_* + \mathcal{W}_1, \widetilde{B}_* + \mathcal{W}_2, \widetilde{C}_* + \mathcal{W}_3) \tag{2.25}$$

and

$$\mathcal{W}_1 X\Lambda^2 + \mathcal{W}_2 X\Lambda + \mathcal{W}_3 X = O. \tag{2.26}$$

By the relation (2.26) and Lemmas 2.6 and 2.7, we have

$$\begin{aligned} & \langle \widetilde{A}_*, \mathcal{W}_1\rangle + \langle \widetilde{B}_*, \mathcal{W}_2\rangle + \langle \widetilde{C}_*, \mathcal{W}_3\rangle = 4\langle Y(X\Lambda^2)^T, \mathcal{W}_1\rangle + 4\langle Y(X\Lambda)^T, \mathcal{W}_2\rangle + 4\langle YX^T, \mathcal{W}_3\rangle \\ = \; & 4\langle Y, \mathcal{W}_1 X\Lambda^2 + \mathcal{W}_2 X\Lambda + \mathcal{W}_3 X\rangle = 0, \end{aligned}$$

which, together with (2.25), yields

$$\begin{aligned} & \|\widetilde{A}\|^2 + \|\widetilde{B}\|^2 + \|\widetilde{C}\|^2 = \|\widetilde{A}_* + \mathcal{W}_1\|^2 + \|\widetilde{B}_* + \mathcal{W}_2\|^2 + \|\widetilde{C}_* + \mathcal{W}_3\|^2 \\ = \; & \|\widetilde{A}_*\|^2 + \|\widetilde{B}_*\|^2 + \|\widetilde{C}_*\|^2 + \|\mathcal{W}_1\|^2 + \|\mathcal{W}_2\|^2 + \|\mathcal{W}_3\|^2 \\ & +2\big[\langle \widetilde{A}_*, \mathcal{W}_1\rangle + \langle \widetilde{B}_*, \mathcal{W}_2\rangle + \langle \widetilde{C}_*, \mathcal{W}_3\rangle\big] \\ = \; & \|\widetilde{A}_*\|^2 + \|\widetilde{B}_*\|^2 + \|\widetilde{C}_*\|^2 + \|\mathcal{W}_1\|^2 + \|\mathcal{W}_2\|^2 + \|\mathcal{W}_3\|^2 \geq \|\widetilde{A}_*\|^2 + \|\widetilde{B}^*\|^2 + \|\widetilde{C}_*\|^2. \end{aligned}$$

This implies that the solution $(\widetilde{A}_*, \widetilde{B}_*, \widetilde{C}_*)$ is the least Frobenius norm solution of Problem 2.1. The proof is completed. □

3 Iterative Method for Problem 1.2

In this section, we will establish an iterative method for solving Problem 1.2. For given $\overline{A}, \overline{B}, \overline{C} \in \mathbb{C}^{n\times n}$ and an arbitrary solution $(A, B, C) = (\widetilde{A} + \widetilde{A}_p, \widetilde{B} + \widetilde{B}_q, \widetilde{C} + \widetilde{C}_r) \in S_E$. Let

$\overline{A}' = \overline{A} - \widetilde{A}_p$, $\overline{B}' = \overline{B} - \widetilde{B}_q$ and $\overline{C}' = \overline{C} - \widetilde{C}_r$. Then

$$
\begin{aligned}
& \|A-\overline{A}\|^2 + \|B-\overline{B}\|^2 + \|C-\overline{C}\|^2 = \|\widetilde{A}-\overline{A}'\|^2 + \|\widetilde{B}-\overline{B}'\|^2 + \|\widetilde{C}-\overline{C}'\|^2 \\
= \; & \Big\|\widetilde{A} - \frac{\overline{A}' + \overline{A}'^H}{2}\Big\|^2 + \Big\|\widetilde{B} - \frac{\overline{B}' + \overline{B}'^H}{2}\Big\|^2 + \Big\|\widetilde{C} - \frac{\overline{C}' + \overline{C}'^H}{2}\Big\|^2 + \Big\|\frac{\overline{A}' - \overline{A}'^H}{2}\Big\|^2 \\
& + \Big\|\frac{\overline{B}' - \overline{B}'^H}{2}\Big\|^2 + \Big\|\frac{\overline{C}' - \overline{C}'^H}{2}\Big\|^2 \\
= \; & \Big\|\widetilde{A} - \frac{\overline{A}' + \overline{A}'^H - S_n(\overline{A}' + \overline{A}'^H)S_n}{4}\Big\|^2 + \Big\|\widetilde{B} - \frac{\overline{B}' + \overline{B}'^H - S_n(\overline{B}' + \overline{B}'^H)S_n}{4}\Big\|^2 \\
& + \Big\|\widetilde{C} - \frac{\overline{C}' + \overline{C}'^H - S_n(\overline{C}' + \overline{C}'^H)S_n}{4}\Big\|^2 + \Big\|\frac{\overline{A}' - \overline{A}'^H}{2}\Big\|^2 \\
& + \Big\|\frac{\overline{B}' - \overline{B}'^H}{2}\Big\|^2 + \Big\|\frac{\overline{C}' - \overline{C}'^H}{2}\Big\|^2 \\
= \; & \Big\|\widetilde{A} - \frac{\overline{A}' + \overline{A}'^H - S_n(\overline{A}' + \overline{A}'^H)S_n - E_sE_s^T[\overline{A}' + \overline{A}'^H - S_n(\overline{A}' + \overline{A}'^H)S_n]E_sE_s^T}{4}\Big\|^2 \\
& + \Big\|\widetilde{B} - \frac{\overline{B}' + \overline{B}'^H - S_n(\overline{B}' + \overline{B}'^H)S_n - E_tE_t^T[\overline{B}' + \overline{B}'^H - S_n(\overline{B}' + \overline{B}'^H)S_n]E_tE_t^T}{4}\Big\|^2 \\
& + \Big\|\widetilde{C} - \frac{\overline{C}' + \overline{C}'^H - S_n(\overline{C}' + \overline{C}'^H)S_n - E_uE_u^T[\overline{C}' + \overline{C}'^H - S_n(\overline{C}' + \overline{C}'^H)S_n]E_uE_u^T}{4}\Big\|^2 \\
& + \Big\|\frac{E_sE_s^T[\overline{A}' + \overline{A}'^H - S_n(\overline{A}' + \overline{A}'^H)S_n]E_sE_s^T}{4}\Big\|^2 \\
& + \Big\|\frac{E_tE_t^T[\overline{B}' + \overline{B}'^H - S_n(\overline{B}' + \overline{B}'^H)S_n]E_tE_t^T}{4}\Big\|^2 \\
& + \Big\|\frac{E_uE_u^T[\overline{C}' + \overline{C}'^H - S_n(\overline{C}' + \overline{C}'^H)S_n]E_uE_u^T}{4}\Big\|^2 \\
& + \Big\|\frac{\overline{A}' - \overline{A}'^H}{2}\Big\|^2 + \Big\|\frac{\overline{B}' - \overline{B}'^H}{2}\|^2 + \|\frac{\overline{C}' - \overline{C}'^H}{2}\Big\|^2.
\end{aligned} \tag{3.1}
$$

Let

$$
\mathcal{A} = \frac{\overline{A}' + \overline{A}'^H - S_n(\overline{A}' + \overline{A}'^H)S_n - E_sE_s^T[\overline{A}' + \overline{A}'^H - S_n(\overline{A}' + \overline{A}'^H)S_n]E_sE_s^T}{4}, \tag{3.2}
$$

$$
\mathcal{B} = \frac{\overline{B}' + \overline{B}'^H - S_n(\overline{B}' + \overline{B}'^H)S_n - E_tE_t^T[\overline{A}' + \overline{B}'^H - S_n(\overline{B}' + \overline{B}'^H)S_n]E_tE_t^T}{4}, \tag{3.3}
$$

and

$$
\mathcal{C} = \frac{\overline{C}' + \overline{C}'^H - S_n(\overline{C}' + \overline{C}'^H)S_n - E_uE_u^T[\overline{A}' + \overline{C}'^H - S_n(\overline{C}' + \overline{C}'^H)S_n]E_uE_u^T}{4}. \tag{3.4}
$$

It is easy to see that $(\mathcal{A},\mathcal{B},\mathcal{C})\in\widetilde{S}_1\times\widetilde{S}_2\times\widetilde{S}_3$. And, it follows from (3.1)–(3.4) that

$$\begin{aligned}&\|A-\overline{A}\|^2+\|B-\overline{B}\|^2+\|C-\overline{C}\|^2\\ =\ &\|\widetilde{A}-\mathcal{A}\|^2+\|\widetilde{B}-\mathcal{B}\|^2+\|\widetilde{C}-\mathcal{C}\|^2+\Big\|\frac{E_sE_s^T[\overline{A}'+\overline{A}'^H-S_n(\overline{A}'+\overline{A}'^H)S_n]E_sE_s^T}{4}\Big\|^2\\ &+\Big\|\frac{E_tE_t^T[\overline{B}'+\overline{B}'^H-S_n(\overline{B}'+\overline{B}'^H)S_n]E_tE_t^T}{4}\Big\|^2\\ &+\Big\|\frac{E_uE_u^T[\overline{C}'+\overline{C}'^H-S_n(\overline{C}'+\overline{C}'^H)S_n]E_uE_u^T}{4}\Big\|^2\\ &+\Big\|\frac{\overline{A}'-\overline{A}'^H}{2}\Big\|^2+\Big\|\frac{\overline{B}'-\overline{B}'^H}{2}\Big\|^2+\Big\|\frac{\overline{C}'-\overline{C}'^H}{2}\Big\|^2. \qquad (3.5)\end{aligned}$$

By direct calculations, we have

$$\begin{aligned}&\min_{(\widetilde{A},\widetilde{B},\widetilde{C})\in\widetilde{S}_1\times\widetilde{S}_2\times\widetilde{S}_3}\|\widetilde{A}X\Lambda^2+\widetilde{B}X\Lambda+\widetilde{C}X-\widetilde{Z}\|\\ &=\min_{(\widetilde{A},\widetilde{B},\widetilde{C})\in\widetilde{S}_1\times\widetilde{S}_2\times\widetilde{S}_3}\|(\widetilde{A}-\mathcal{A})X\Lambda^2+(\widetilde{B}-\mathcal{B})X\Lambda+(\widetilde{C}-\mathcal{C})X-(\widetilde{Z}-\mathcal{A}X\Lambda^2-\mathcal{B}X\Lambda-\mathcal{C}X)\|.\end{aligned}$$

Denote $\widetilde{\mathcal{Z}}=\widetilde{Z}-\mathcal{A}X\Lambda^2-\mathcal{B}X\Lambda-\mathcal{C}X$. Then

$$\begin{aligned}&\min_{(\widetilde{A},\widetilde{B},\widetilde{C})\in\widetilde{S}_1\times\widetilde{S}_2\times\widetilde{S}_3}\|\widetilde{A}X\Lambda^2+\widetilde{B}X\Lambda+\widetilde{C}X-\widetilde{Z}\|\\ =\ &\min_{(\widetilde{A}-\mathcal{A},\widetilde{B}-\mathcal{B},\widetilde{C}-\mathcal{C})\in\widetilde{S}_1\times\widetilde{S}_2\times\widetilde{S}_3}\|(\widetilde{A}-\mathcal{A})X\Lambda^2+(\widetilde{B}-\mathcal{B})X\Lambda+(\widetilde{C}-\mathcal{C})X-\widetilde{\mathcal{Z}}\|. \qquad (3.6)\end{aligned}$$

According to (3.5) and (3.6), it follows that

$$\min_{(A,B,C)\in S_E}\left[\|A-\overline{A}\|^2+\|B-\overline{B}\|^2+\|C-\overline{C}\|^2\right]$$

is equivalent to finding a least Frobenius norm solution of the following problem

$$\min_{(\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}})\in\widetilde{S}_1\times\widetilde{S}_2\times\widetilde{S}_3}\|\widetilde{\mathcal{A}}X\Lambda^2+\widetilde{\mathcal{B}}X\Lambda+\widetilde{\mathcal{C}}X-\widetilde{\mathcal{Z}}\|, \qquad (3.7)$$

where

$$\begin{aligned}\widetilde{\mathcal{A}}&=\widetilde{A}-\mathcal{A}=\widetilde{A}-\frac{\overline{A}'+\overline{A}'^H-S_n(\overline{A}'+\overline{A}'^H)S_n-E_sE_s^T[\overline{A}'+\overline{A}'^H-S_n(\overline{A}'+\overline{A}'^H)S_n]E_sE_s^T}{4},\\ \widetilde{\mathcal{B}}&=\widetilde{B}-\mathcal{B}=\widetilde{B}-\frac{\overline{B}'+\overline{B}'^H-S_n(\overline{B}'+\overline{B}'^H)S_n-E_tE_t^T[\overline{B}'+\overline{B}'^H-S_n(\overline{B}'+\overline{B}'^H)S_n]E_tE_t^T}{4},\\ \widetilde{\mathcal{C}}&=\widetilde{C}-\mathcal{C}=\widetilde{C}-\frac{\overline{C}'+\overline{C}'^H-S_n(\overline{C}'+\overline{C}'^H)S_n-E_uE_u^T[\overline{C}'+\overline{C}'^H-S_n(\overline{C}'+\overline{C}'^H)S_n]E_uE_u^T}{4}.\end{aligned}$$

By applying Algorithm 2.11, we can obtain the unique least Frobenius norm solution $(\widetilde{\mathcal{A}}_*,\widetilde{\mathcal{B}}_*,\widetilde{\mathcal{C}}_*)$ of the problem (3.7). Then the unique solution of Problem 1.2 can be obtained as follows:

$$\begin{cases}\widehat{A}=\ \widetilde{\mathcal{A}}_*+\dfrac{\overline{A}'+\overline{A}'^H-S_n(\overline{A}'+\overline{A}'^H)S_n-E_sE_s^T[\overline{A}'+\overline{A}'^H-S_n(\overline{A}'+\overline{A}'^H)S_n]E_sE_s^T}{4}+\widetilde{A}_p,\\ \widehat{B}=\ \widetilde{\mathcal{B}}_*+\dfrac{\overline{B}'+\overline{B}'^H-S_n(\overline{B}'+\overline{B}'^H)S_n-E_tE_t^T[\overline{B}'+\overline{B}'^H-S_n(\overline{B}'+\overline{B}'^H)S_n]E_tE_t^T}{4}+\widetilde{B}_q,\\ \widehat{C}=\ \widetilde{\mathcal{C}}_*+\dfrac{\overline{C}'+\overline{C}'^H-S_n(\overline{C}'+\overline{C}'^H)S_n-E_uE_u^T[\overline{C}'+\overline{C}'^H-S_n(\overline{C}'+\overline{C}'^H)S_n]E_uE_u^T}{4}+\widetilde{C}_r.\end{cases}$$

4 Numerical Experiments

In this section, we report some numerical results to illustrate the efficiency of the proposed method and verify conclusions in this paper. All of the tests were run on the Intel (R) Core (TM), where the CPU is 2.40 GHz and the memory is 8.0 GB. The programming language is MATLAB R2015a. In view of the influence of round-off errors, we regard a matrix T as the zero matrix if $\langle T, T\rangle < 10^{-10}$.

Example 4.1. Consider the following quadratic inverse eigenvalue problem

$$AX\Lambda^2 + BX\Lambda + CX = O,$$

where

$$X = \begin{pmatrix} 0.8147+0.6948i & 0.2785+0.7655i & 0.9572+0.7094i & 0.7922+0.1190i & 0.6787+0.7513i & 0.7060+0.5472i \\ 0.9058+0.3171i & 0.5469+0.7952i & 0.4854+0.7547i & 0.9595+0.4984i & 0.7577+0.2551i & 0.0318+0.1386i \\ 0.1270+0.9502i & 0.9575+0.1869i & 0.8003+0.2760i & 0.6557+0.9597i & 0.7431+0.5060i & 0.2769+0.1493i \\ 0.9134+0.0344i & 0.9649+0.4898i & 0.1419+0.6797i & 0.0357+0.3404i & 0.3922+0.6991i & 0.0462+0.2575i \\ 0.6324+0.4387i & 0.1576+0.4456i & 0.4218+0.6551i & 0.8491+0.5853i & 0.6555+0.8909i & 0.0971+0.8407i \\ 0.0975+0.3816i & 0.9706+0.6463i & 0.9157+0.1626i & 0.9340+0.2238i & 0.1712+0.9593i & 0.8235+0.2543i \end{pmatrix}$$

and $\Lambda = \mathrm{diag}(4.6176 + 0.2816i, 1.1012 + 0.4456i, 0.4056 + 0.2760i, -0.4480 + 0.1190i, -0.0826 + 0.0344i, 0.1136 + 0.2238i)$. Let $s = \{2,5\}$, $t = \{2,5\}$, $u = \{3,4\}$, $A_p = \begin{pmatrix} 0.25+0.01i & -0.5+0.2i \\ -0.25-0.01i & 0.25+0.01i \end{pmatrix}$, $B_q = \begin{pmatrix} 0.25 & -0.5+0.2i \\ -0.25-0.01i & 0.25+0.01i \end{pmatrix}$ and $C_r = \begin{pmatrix} 0.05+0.01i & 0.15+0.03i \\ 0.2+0.02i & 0.1+0.01i \end{pmatrix}$. Then

$$\widetilde{A}_p = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.25+0.01i & 0 & 0 & -0.5+0.2i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.25-0.01i & 0 & 0 & 0.25+0.01i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{B}_q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 & -0.5+0.2i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.25-0.01i & 0 & 0 & 0.25+0.01i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{C}_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.05+0.01i & 0.15+0.03i & 0 & 0 \\ 0 & 0 & 0.2+0.02i & 0.1+0.01i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{Z} = -\widetilde{A}_p X\Lambda^2 - \widetilde{B}_q X\Lambda - \widetilde{C}_r X =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4.6417+0.6790i & 0.0901+0.0309i & 0.0834+0.1384i & -0.0790+0.0019i & -0.0316-0.0103i & -0.0847+0.0854i \\ -0.1328-0.0813i & -0.1760-0.1213i & -0.0381-0.1280i & -0.0183-0.1067i & -0.0700-0.1494i & -0.0116-0.0502i \\ -0.0974-0.2052i & -0.2794-0.1152i & -0.1619-0.1406i & -0.1121-0.2395i & -0.1707-0.1899i & -0.0544-0.0616i \\ 1.8788-0.5150i & 0.0682+0.3270i & -0.0054+0.0201i & -0.0072+0.0057i & 0.0021+0.0131i & 0.0477-0.0160i \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

First, we illustrate the efficiency of Algorithm 2.11 for solving the quadratic inverse eigenvalue problem. Let the initial matrices $\widetilde{A}(1) = \widetilde{B}(1) = \widetilde{C}(1) = \text{zeros}(6,6)$. By Algorithm 2.11, we obtain the least Frobenius norm solution of Problem 1.1 after 69 iterative steps as follows.

$$\widetilde{A}_{*,\mathrm{NCG}} = \widetilde{A}_{\mathrm{NCG}}(69) + \widetilde{A}_p =$$

$$\begin{pmatrix} 1.3332-0.0000i & -0.0959-0.6550i & 0.3883+0.4556i & -0.2133-0.1127i & -0.8091-0.3359i & 0.0000+0.6204i \\ -0.0959+0.6550i & \mathbf{0.25+0.01i} & -0.6988-0.2109i & 0.1047-0.0930i & \mathbf{-0.5+0.2i} & 0.8091-0.3359i \\ 0.3883-0.4556i & -0.6988+0.2109i & 0.0751-0.0000i & 0.0000-0.0439i & -0.1047-0.0930i & 0.2133-0.1127i \\ -0.2133+0.1127i & 0.1047+0.0930i & -0.0000+0.0439i & -0.0751+0.0000i & 0.6988-0.2109i & -0.3883+0.4556i \\ -0.8091+0.3359i & \mathbf{-0.25-0.01i} & -0.1047+0.0930i & 0.6988+0.2109i & \mathbf{0.25+0.01i} & 0.0959-0.6550i \\ -0.0000-0.6204i & 0.8091+0.3359i & 0.2133+0.1127i & -0.3883-0.4556i & 0.0959+0.6550i & -1.3332+0.0000i \end{pmatrix},$$

$$\widetilde{B}_{*,\mathrm{NCG}} = \widetilde{B}_{\mathrm{NCG}}(69) + \widetilde{B}_q =$$

$$\begin{pmatrix} -0.8631+0.0000i & 0.5177+0.5353i & -0.1306-0.2102i & -0.5186-0.8008i & 0.4984+0.3361i & -0.0000-0.8201i \\ 0.5177-0.5353i & \mathbf{0.25} & 0.1804-0.8582i & -0.0705+0.4471i & \mathbf{-0.5+0.2i} & -0.4984+0.3361i \\ -0.1306+0.2102i & 0.1804+0.8582i & -0.6414+0.0000i & -0.0000-0.1609i & 0.0705+0.4471i & 0.5186-0.8008i \\ -0.5186+0.8008i & -0.0705-0.4471i & 0.0000+0.1609i & 0.6414-0.0000i & -0.1804-0.8582i & 0.1306-0.2102i \\ 0.4984-0.3361i & \mathbf{-0.25-0.01i} & 0.0705-0.4471i & -0.1804+0.8582i & \mathbf{0.25+0.01i} & -0.5177+0.5353i \\ 0.0000+0.8201i & -0.4984-0.3361i & 0.5186+0.8008i & 0.1306+0.2102i & -0.5177-0.5353i & 0.8631-0.0000i \end{pmatrix},$$

$$\widetilde{C}_{*,\mathrm{NCG}} = \widetilde{C}_{\mathrm{NCG}}(69) + \widetilde{C}_r =$$

$$\begin{pmatrix} -0.1761+0.0000i & 0.0071-0.0632i & 0.0298+0.0820i & -0.0113+0.1228i & 0.1681-0.2018i & 0.0000+0.1091i \\ 0.0071+0.0632i & 0.3339-0.0000i & -0.0686+0.0594i & -0.1465+0.1472i & 0.0000+0.0033i & -0.1681-0.2018i \\ 0.0298-0.0820i & -0.0686-0.0594i & \mathbf{0.05+0.01i} & \mathbf{0.15+0.03i} & 0.1465+0.1472i & 0.0113+0.1228i \\ -0.0113-0.1228i & -0.1465-0.1472i & \mathbf{0.2+0.02i} & \mathbf{0.1+0.01i} & 0.0686+0.0594i & -0.0298+0.0820i \\ 0.1681+0.2018i & -0.0000-0.0033i & 0.1465-0.1472i & 0.0686-0.0594i & -0.3339+0.0000i & -0.0071-0.0632i \\ -0.0000-0.1091i & -0.1681+0.2018i & 0.0113-0.1228i & -0.0298-0.0820i & -0.0071+0.0632i & 0.1761-0.0000i \end{pmatrix}.$$

At this moment, the norms of $R(k)$ and $P(k)$ are $\|R(k)\| = 2.1513$ and $\|P(k)\| = 7.4137e{-}11$, respectively. The relationship between the number of iterations and the norm of $P(k)$ is shown in Figure 1. Figure 1 illustrates that Algorithm 2.11 is efficient for solving the constrained quadratic inverse eigenvalue problem.

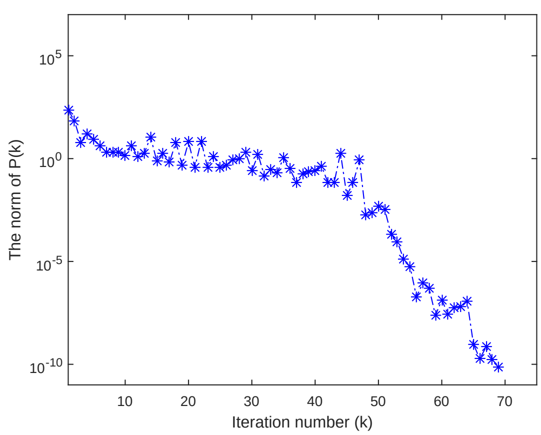

Figure 1: The relationship between the number of iterations and the norm of $P(k)$ for Example 4.1

Second, we compare Algorithm 2.11 with the CGNR method developed by Hajarian and Abbas [24]. For fairness, we also choose the zero matrices as the initial values. By the CGNR method, we obtain the solution of Problem 2.1 after 67 iterative steps as follows.

$$\widetilde{A}_{\rm CGNR}(67) =$$
$$\begin{pmatrix}
1.2846+0.0000i & -0.1124-0.6407i & 0.3666+0.4325i & -0.2154-0.0908i & -0.7429-0.3443i & 0.0000+0.6495i \\
-0.1124+0.6407i & \mathbf{0.0206+0.0000i} & -0.6837-0.2260i & 0.0929-0.1350i & \mathbf{0.0000+0.0742i} & 0.7429-0.3443i \\
0.3666-0.4325i & -0.6837+0.2260i & 0.0645+0.0000i & 0.0000-0.0401i & -0.0929-0.1350i & 0.2154-0.0908i \\
-0.2154+0.0908i & 0.0929+0.1350i & 0.0000+0.0401i & -0.0645+0.0000i & 0.6837-0.2260i & -0.3666+0.4325i \\
-0.7429+0.3443i & \mathbf{0.0000-0.0742i} & -0.0929+0.1350i & 0.6837+0.2260i & \mathbf{-0.0206+0.0000i} & 0.1124-0.6407i \\
0.0000-0.6495i & 0.7429+0.3443i & 0.2154+0.0908i & -0.3666-0.4325i & 0.1124+0.6407i & -1.2846+0.0000i
\end{pmatrix}$$

$$\widetilde{B}_{\rm CGNR}(67) =$$
$$\begin{pmatrix}
-0.8897+0.0000i & 0.5512+0.5673i & -0.1055-0.1817i & -0.4905-0.8084i & 0.5023+0.3001i & 0.0000-0.8655i \\
0.5512-0.5673i & \mathbf{0.0728+0.0000i} & 0.1803-0.8202i & -0.0926+0.5108i & \mathbf{0.0000+0.0396i} & -0.5023+0.3001i \\
-0.1055+0.1817i & 0.1803+0.8202i & -0.6147+0.0000i & 0.0000-0.1681i & 0.0926+0.5108i & 0.4905-0.8084i \\
-0.4905+0.8084i & -0.0926-0.5108i & 0.0000+0.1681i & 0.6147+0.0000i & -0.1803-0.8202i & 0.1055-0.1817i \\
0.5023-0.3001i & \mathbf{0.0000-0.0396i} & 0.0926-0.5108i & -0.1803+0.8202i & \mathbf{-0.0728+0.0000i} & -0.5512+0.5673i \\
0.0000+0.8655i & -0.5023-0.3001i & 0.4905+0.8084i & 0.1055+0.1817i & -0.5512-0.5673i & 0.8897+0.0000i
\end{pmatrix}$$

$$\widetilde{C}_{\rm CGNR}(67) =$$
$$\begin{pmatrix}
-0.1764+0.0000i & 0.0234-0.0624i & 0.0200+0.0848i & -0.0243+0.1200i & 0.1750-0.1994i & 0.0000+0.1028i \\
0.0234+0.0624i & 0.3266+0.0000i & -0.0635+0.0406i & -0.1431+0.1476i & 0.0000+0.0137i & -0.1750-0.1994i \\
0.0200-0.0848i & -0.0635-0.0406i & \mathbf{-0.0015+0.0000i} & \mathbf{0.0000-0.0213i} & 0.1431+0.1476i & 0.0243+0.1200i \\
-0.0243-0.1200i & -0.1431-0.1476i & \mathbf{0.0000+0.0213i} & \mathbf{0.0015+0.0000i} & 0.0635+0.0406i & -0.0200+0.0848i \\
0.1750+0.1994i & 0.0000-0.0137i & 0.1431-0.1476i & 0.0635-0.0406i & -0.3266+0.0000i & -0.0234-0.0624i \\
0.0000-0.1028i & -0.1750+0.1994i & 0.0243-0.1200i & -0.0200-0.0848i & -0.0234+0.0624i & 0.1764+0.0000i
\end{pmatrix}$$

It is easy to see that $\widetilde{A}_{\rm CGNR}(67)[s|s] \neq O$, $\widetilde{B}_{\rm CGNR}(67)[t|t] \neq O$ and $\widetilde{C}_{\rm CGNR}(67)[u|u] \neq O$. The reason is that the iterative sequences $\{\widetilde{A}_{\rm CGNR}(k)\}$, $\{\widetilde{B}_{\rm CGNR}(k)\}$ and $\{\widetilde{C}_{\rm CGNR}(k)\}$ generated by CGNR method can not satisfy

$$\widetilde{A}_{\rm CGNR}(k)[s|s] = O, \widetilde{B}_{\rm CGNR}(k)[t|t] = O, \widetilde{C}_{\rm CGNR}(k)[u|u] = O.$$

In other words, the submatrix constrained solution generated by CGNR method can only be approximated.

By contrast, our proposed method projects the iterative sequences $\{\widetilde{A}_{\rm NCG}(k)\}$, $\{\widetilde{B}_{\rm NCG}(k)\}$ and $\{\widetilde{C}_{\rm NCG}(k)\}$ to linear subspaces $\widetilde{S}_1$, $\widetilde{S}_2$ and $\widetilde{S}_3$, respectively. Hence, we always have

$$\widetilde{A}_{\rm NCG}(k)[s|s] = O, \widetilde{B}_{\rm NCG}(k)[t|t] = O, \widetilde{C}_{\rm NCG}(k)[u|u] = O.$$

This illustrates that our proposed method can solve the submatrix constrained quadratic inverse eigenvalue problem more accurately.

5 Concluding Remarks

This paper is concerned with the least squares solution of a class of constrained quadratic inverse eigenvalue problem and its optimal approximation problem. We propose a nonlinear conjugate gradient method for finding the solution over Hermitian centroskew matrices with a submatrix constraint. The convergence analysis of the proposed method is given. Numerical results illustrate that the proposed method is efficient for the quadratic inverse eigenvalue problem.

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