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A CLARKE-LEDYAEV MULTIDIRECTIONAL MEAN VALUE INEQUALITY FOR CONVEX FUNCTIONS*

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Abstract: We establish a Clarke-Ledyaev multidirectional mean value inequality for a proper, convex, lower semicontinuous and bounded below function.

Key words: *convex function, multidirectional mean value inequality*

Mathematics Subject Classification: *49K27, 49J53, 49J52*

1 Introduction

The pioneering multidirectional inequality is due to F. Clarke and Y. Ledyaev and appeared in 1994 in [4]. It compares the values of a locally Lipschitz function on two bounded, closed and convex subsets of a Banach space, one of which is compact, see [4, Theorem 4.1], stated here as Theorem 2.1. After then, the theme of multidirectional mean value inequalities of different types and in different settings was developed in number of publications, see e.g. [1, 2, 5, 6, 8, 11, 14, 15, 16, 17].

In this short note we prove a multidirectional mean value inequality of the type found in [4] for a proper convex, lower semicontinuous and bounded below function. As such a function is not necessarily locally Lipschitz, [4, Theorem 4.1] could not be directly applied. Instead, we use Hausdorff approximations of the function, which are Lipschitz functions.

Our main result is the following

Theorem 1.1. *Let $(X, \|\cdot\|)$ be a Banach space and let $A, B \subseteq X$ be non-empty, closed, bounded and convex sets such that A is compact. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function, which is bounded below on X. In addition,*

(a) *if f is bounded on A, then for any $\varepsilon > 0$ and any $\delta > 0$ one can find $z \in [A, B] + \delta\mathbb{B}$ and $p \in \partial f(z)$ such that*

$$\inf_A f - \sup_B f < \langle p, a - b\rangle + \varepsilon, \quad \forall a \in A,\ \forall b \in B.$$

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(b) *if f is bounded on B, then for any $\varepsilon > 0$ and any $\delta > 0$ one can find $w \in [A,B] + \delta\mathbb{B}$ and $q \in \partial f(w)$ such that*

$$\underline{\inf_B} f - \sup_A f < \langle q, b-a\rangle + \varepsilon, \quad \forall a \in A,\ \forall b \in B,$$

where $\underline{\inf_B} f := \lim_{\gamma \downarrow 0} \inf_{b \in B_\gamma} f(b)$.

Let us note that in [4, Theorem 4.1] the function f is assumed locally Lipschitz; i.e., to each $z \in [A,B]$ there corresponds an open neighbourhood of z in X in which f satisfies a Lipschitz condition. However, it is not difficult to find examples of a convex function and a couple of sets satisfying the assumptions of Theorem 1.1 that do not satisfy the assumptions of [4, Theorem 4.1], see Section 4 below.

The paper is organized as follows. After a short Section 2 on notations and preliminaries, in Section 3 we give the proof of Theorem 1.1. In the final Section 4 two examples are provided.

2 Preliminaries and Notations

The notation used throughout the paper is standard. As usual, $(X, \|\cdot\|)$ denotes a real Banach space, that is, complete normed space over $\mathbb{R}$. Its closed unit ball is denoted by $\mathbb{B}$. The dual space X^* of X is the Banach space of all continuous linear functionals p from X to $\mathbb{R}$. The natural norm of X^* is again denoted by $\|\cdot\|$. The value of $p \in X^*$ at $x \in X$ is denoted by $\langle p, x\rangle$.

For two non-empty sets $A, B \subseteq X$, the *segment* $[A,B]$ is defined by

$$[A,B] := \{z : z = \lambda x + (1-\lambda)y,\ x \in A,\ y \in B,\ \lambda \in [0,1]\}.$$

The δ-*enlargement* of a non-empty set $C \subset X$ is $C_\delta := C + \delta\mathbb{B}$.

Recall that the generalized directional derivative of a locally Lipschitz function $f : X \to \mathbb{R}$ at x in direction h is given by

$$f^\circ(x,h) := \limsup_{\substack{x' \to x \\ t \downarrow 0}} \frac{f(x'+th) - f(x')}{t},$$

and the Clarke subdifferential of f at x is the non-empty set

$$\partial^C f(x) := \{\xi \in X^* : f^\circ(x,h) \geq \langle \xi, h\rangle,\ \forall h \in X\}.$$

Our main tool is the following fundamental Clarke-Ledyaev multidirectional mean value theorem.

Theorem 2.1. *[4, [Theorem 4.1]]*
Let $(X, \|\cdot\|)$ be a Banach space. Let A, B be non-empty closed convex bounded subsets of X, such that at least one of them is compact, and let $\varepsilon > 0$ be given. Let $f : X \to \mathbb{R}$ be a locally Lipschitz function on $[A,B]$. Then there exists a point $z \in [A,B]$ and $\xi \in \partial^C f(z)$, such that

$$\inf_A f - \sup_B f < \langle \xi, a-b\rangle + \varepsilon, \quad \forall a \in A,\ \forall b \in B.$$

The *effective domain* $\operatorname{dom} f$ of an extended real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$ is the set of points x where $f(x) \in \mathbb{R}$. The function f is *proper* if $\operatorname{dom} f \neq \varnothing$. It is *lower semicontinuous* if $f(\bar{x}) \leq \liminf_{x \to \bar{x}} f(x)$ for all $\bar{x} \in X$.

Let us recall that for $\varepsilon \geq 0$, the ε-subdifferential of a proper, convex and lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ at $x \in \operatorname{dom} f$ is the set

$$\partial_\varepsilon f(x) = \{p \in X^* : \langle p, y - x\rangle \leq f(y) - f(x) + \varepsilon,\ \forall y \in X\},$$

and $\partial_\varepsilon f = \varnothing$ on $X \setminus \operatorname{dom} f$. Of course, for $\varepsilon = 0$, $\partial_0 f(x)$ coincides with the subdifferential $\partial f(x)$ of f at x in the sense of Convex Analysis. The latter coincides with the Clarke subdifferential of f at x when f is continuous at x.

The *domain* $\operatorname{dom} \partial_\varepsilon f$ of $\partial_\varepsilon f$ consists of all points $x \in X$ such that $\partial_\varepsilon f(x)$ is non-empty. Note that for $\varepsilon > 0$ the sets $\partial_\varepsilon f(x)$ are always non-empty, while $\partial f(x)$ could be empty at some points. Furthermore, for any real numbers ε_1 and ε_2 such that $0 < \varepsilon_1 \leq \varepsilon_2$ one has $\partial_{\varepsilon_1} f(x) \subset \partial_{\varepsilon_2} f(x)$ and $\partial f(x) = \bigcap_{\varepsilon > 0} \partial_\varepsilon f(x)$.

From now on, $f : X \to \mathbb{R} \cup \{+\infty\}$ will always be a proper, lower semicontinuous and convex function. For $n \in \mathbb{N}$ define the inf-convolutions $\{f_n\}$ by

$$f_n(x) := \inf_{y \in X} \{f(y) + n\|x - y\|\}.$$

The approximating sequence $\{f_n\}$ was originally introduced by Hausdorff [9] for any lower bounded lower semicontinuous function f of a real variable. It is clear that for sufficiently large n the function f_n is finite valued and we will always consider this case even if it is not stated explicitly. Some well-known properties of these inf-convolutions of f (see, for instance, [13, 10, 7]) are listed in next

Lemma 2.2. *For n large enough,*

(i) *f_n is convex and n-Lipschitzian;*

(ii) *$f_n(x) \leq f_{n+1}(x) \leq f(x)$ for all $x \in X$ and all $n \in \mathbb{N}$;*

(iii) *$f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in X$.*

Let us denote the set of the ε-minima of the function $f(\cdot) + n\|x - \cdot\|$ by

$$M^n_\varepsilon(x) := \{y \in X : f(y) + n\|x - y\| \leq f_n(x) + \varepsilon\}.$$

It is clear that for $\varepsilon > 0$ the sets $M^n_\varepsilon(x)$ are non-empty.

The following is well-known, for proof see e.g. [18, Lemma 3].

Lemma 2.3. *For any $\varepsilon \geq 0$, and any $y \in M^n_\varepsilon(x)$ it holds that*

$$\partial f_n(x) \subset \partial_\varepsilon f(y) \cap \partial_\varepsilon n\| \cdot \|(x - y). \tag{2.1}$$

The result of Brøndsted and Rockafellar stating that the graph of $\partial_\varepsilon f$ is close to the graph of ∂f is also well known:

Theorem 2.4 (Brøndsted-Rockafellar [3])**.** *Let $f : X \to \mathbb{R} \cup \{+\infty\}$, be a proper, convex and lower semicontinuous function, let $x \in \operatorname{dom} f$, let $\varepsilon > 0$ and $p \in \partial_\varepsilon f(x)$. Then there exists $q \in \partial f(z)$ such that*

$$\|z - x\| \leq \sqrt{\varepsilon},\ \textit{and}\ \|q - p\| \leq \sqrt{\varepsilon}.$$

From Brøndsted-Rockafellar Theorem it easily follows that $\operatorname{dom} \partial f$ is f-graphically dense in $\operatorname{dom} f$, i.e., for any $a \in \operatorname{dom} f$ there exists a sequence $x_n \in \operatorname{dom} \partial f$ with $x_n \to a$ and $f(x_n) \to f(a)$.

3 Proof of Theorem 1.1

Let us fix any $\varepsilon > 0$ and $\delta > 0$. Since A and B are assumed bounded, $d := \operatorname{diam}(A - B) = \sup\{\|a - b\| : a \in A,\ b \in B\}$ is finite. Take $\varepsilon' > 0$ such that $\varepsilon' + d\sqrt{\varepsilon'} < \varepsilon$, and $\sqrt{\varepsilon'} < \delta/2$.

(a) We have to consider two cases.

Case 1. $\sup_B f = +\infty$. Let us note that since f is bounded on A, $A \subseteq \operatorname{dom} f$. Take any $a \in A$. By f-graphical density of $\operatorname{dom} \partial f$ in $\operatorname{dom} f$ there exists $z \in \operatorname{dom} \partial f$ such that $\|a - z\| \le \delta$. Pick any $p \in \partial f(z)$. Then, since $\sup_B f = +\infty$ and A, and B are assumed bounded, we obviously have

$$\inf_A f - \sup_B f < \langle p, a - b \rangle, \quad \forall a \in A,\ \forall b \in B,$$

and the claim follows.

Case 2. $\sup_B f < +\infty$.

For $n \in \mathbb{N}$ large enough, the Hausdorff approximation $f_n : X \to \mathbb{R}$ is n-Lipschitz continuous, see Lemma 2.2(i). We apply Theorem 2.1 to f_n, A and B to obtain $z'_n \in [A, B]$ and $p'_n \in \partial f_n(z'_n)$ such that

$$\inf_A f_n - \sup_B f_n < \langle p'_n, a - b \rangle + \varepsilon'/8, \quad \forall a \in A,\ \forall b \in B. \tag{3.1}$$

By compactness one can find $a_n \in A$ such that $f_n(a_n) = \inf_A f_n$. Lemma 2.2(ii) yields that $\sup_B f_n \le \sup_B f$. Incorporating these in (3.1), we have that

$$f_n(a_n) - \sup_B f < \langle p'_n, a - b \rangle + \varepsilon'/8, \quad \forall a \in A,\ \forall b \in B. \tag{3.2}$$

Take any $y(a_n) \in M^n_{\varepsilon'/8}(a_n)$, i.e.

$$f(y(a_n)) + n\|a_n - y(a_n)\| \le f_n(a_n) + \varepsilon'/8, \tag{3.3}$$

and observe by using Lemma 2.2(ii) again, that

$$f_n(a_n) = \inf_A f_n \le \inf_A f,$$

which is finite as f is bounded on A. Using the latter in (3.3) we obtain that

$$f(y(a_n)) + n\|a_n - y(a_n)\| \le \inf_A f + \varepsilon'/8. \tag{3.4}$$

Since f is bounded below on X, say by μ, the latter easily yields

$$\|a_n - y(a_n)\| \le \frac{\inf_A f - \mu + \varepsilon'/8}{n}.$$

Since the sequence $\{a_n\}$ is in the compact set A, there is a convergent to some $a_0 \in A$ subsequence, which we index in the same way for convenience. Hence, $\{y(a_n)\}$ also converges to a_0. By the lower semicontinuity of f at a_0, for large n,

$$f(a_0) \le f(y(a_n)) + \varepsilon'/4. \tag{3.5}$$

From (3.2), (3.3) and (3.5) we obtain that

$$f(a_0) - \sup_B f < \langle p'_n, a - b\rangle + \varepsilon'/2, \ \forall a \in A, \ \forall b \in B,$$

which yields

$$\begin{gathered}\inf_A f - \sup_B f < \langle p'_n, a-b\rangle + \varepsilon'/2, \\ \forall a \in A, \ \forall b \in B \text{ and } n \in \mathbb{N} \text{ large enough.}\end{gathered} \tag{3.6}$$

Recall that $p'_n \in \partial f_n(z'_n)$, $z'_n \in [A,B]$. By Lemma 2.3 it holds that for any $y(z'_n) \in M^n_{\varepsilon'}(z'_n)$ we have $p'_n \in \partial_{\varepsilon'} f(y(z'_n))$. Fix $y(z'_n) \in M^n_{\varepsilon'}(z'_n)$. By Brønsted-Rockafellar Theorem there exists $p_n \in \partial f(z_n)$ such that $\|z_n - y(z'_n)\| \le \sqrt{\varepsilon'}$ and $\|p_n - p'_n\| \le \sqrt{\varepsilon'}$. It easily follows that for any $a \in A$, and any $b \in B$,

$$\langle p'_n, a-b\rangle \le \langle p_n, a-b\rangle + \|p'_n - p_n\|\|a-b\| \le \langle p_n, a-b\rangle + \sqrt{\varepsilon'}d.$$

Using the latter in (3.6), for large n we have

$$\inf_A f - \sup_B f < \langle p_n, a-b\rangle + \varepsilon'/2 + \sqrt{\varepsilon'}d, \ \forall a \in A, \ \forall b \in B,$$

hence,

$$\inf_A f - \sup_B f < \langle p_n, a-b\rangle + \varepsilon, \ \forall a \in A, \ \forall b \in B,$$

by the choice of ε'.

We only need to show that for large n, $z_n \in [A,B] + \delta\mathbb{B}$. To this end, we use first that $\|z_n - y(z'_n)\| \le \sqrt{\varepsilon'} < \delta/2$, by the choice of ε', and second, that $y(z'_n) \in M^n_{\varepsilon'}(z'_n)$, hence

$$f(y(z'_n)) + n\|z'_n - y(z'_n)\| \le f_n(z'_n) + \varepsilon' \le f(z'_n) + \varepsilon', \tag{3.7}$$

where the last inequality follows by Lemma 2.2(ii). As in this case f is bounded above on A and on B, by convexity f is bounded above on $[A,B]$. Let f is bounded above on $[A,B]$ by some constant M. Using this and the fact that f is bounded below by μ on X we have by (3.7) that

$$\mu + n\|z'_n - y(z'_n)\| \le M + \varepsilon',$$

hence,

$$\|z'_n - y(z'_n)\| \le \frac{M - \mu + \varepsilon'}{n} < \delta/2,$$

for sufficiently large n.

So, $z'_n \in [A,B]$ and $\|z_n - z'_n\| \le \|z_n - y(z'_n)\| + \|y(z'_n) - z'_n\| \le \delta$, which means that $z_n \in [A,B] + \delta\mathbb{B}$ for large n.

Finally, taking n large enough, and setting $p := p_n$ and $z := z_n$ we obtain that $p \in \partial f(z)$, $z \in [A,B] + \delta\mathbb{B}$, and

$$\inf_A f - \sup_B f < \langle p, a-b\rangle + \varepsilon, \ \forall a \in A, \ \forall b \in B.$$

The proof of (a) is then completed.

(b) The proof is quite similar to the proof of (a) but we prefer to present it for completeness. Since f is bounded on B, $B \subset \operatorname{dom} f$ and $\underline{\inf}_B f$ is finite.

We consider again two cases.

Case 1. $\sup_A f = +\infty$. Take any $b \in B \subseteq \operatorname{dom} f$. By f-graphical density of $\operatorname{dom}\partial f$ in $\operatorname{dom} f$ there exists $w \in \operatorname{dom}\partial f$ such that $\|b - w\| \le \delta$. Pick any $q \in \partial f(w)$. Then, since $\sup_A f = +\infty$ and the sets A, and B are assumed bounded, we obviously have

$$\underline{\inf}_B f - \sup_A f < \langle q, b - a\rangle, \quad \forall a \in A,\ \forall b \in B,$$

and the claim follows.

Case 2. $\sup_A f < +\infty$. Let $\gamma > 0$ be such that $\underline{\inf}_B f \le \inf_{B_\gamma} f + \varepsilon'/4$.

For large enough n we apply Theorem 2.1 for the n-Lipschitz continuous Hausdorff approximation $f_n : X \to \mathbb{R}$ and the sets A and B to obtain $w'_n \in [A, B]$ and $q'_n \in \partial f_n(w'_n)$ such that

$$\inf_B f_n - \sup_A f_n < \langle q'_n, b - a\rangle + \varepsilon'/4, \quad \forall a \in A,\ \forall b \in B. \tag{3.8}$$

For some $b_n \in B$ it holds that $f_n(b_n) \le \inf_B f_n + \varepsilon'/4$. Lemma 2.2(ii) yields that $\sup_A f_n \le \sup_A f$. Incorporating these in (3.8), we have that

$$f_n(b_n) - \sup_A f < \langle q'_n, b - a\rangle + \varepsilon'/2, \quad \forall a \in A,\ \forall b \in B. \tag{3.9}$$

Take any $y(b_n) \in M^n_{\varepsilon'/4}(b_n)$, i.e.

$$f(y(b_n)) + n\|b_n - y(b_n)\| \le f_n(b_n) + \varepsilon'/4, \tag{3.10}$$

and using Lemma 2.2(ii) again, we get that

$$f_n(b_n) \le \inf_B f_n + \varepsilon'/4 \le \inf_B f + \varepsilon'/4.$$

Note that $\inf_B f$ is finite as f is bounded on B. Using the latter we estimate the right hand side of (3.10) by

$$f(y(b_n)) + n\|b_n - y(b_n)\| \le \inf_B f + \varepsilon'/2. \tag{3.11}$$

Since f is bounded below on X by μ, the latter easily yields

$$\|b_n - y(b_n)\| \le \frac{\inf_B f - \mu + \varepsilon'/2}{n}.$$

Hence, for sufficiently large n, $\|b_n - y(b_n)\| \le \gamma$, and $y(b_n) \in B_\gamma$. From (3.9) and (3.10) we have that

$$f(y(b_n)) - \sup_A f < \langle q'_n, b - a\rangle + 3\varepsilon'/4,\ \forall a \in A,\ \forall b \in B. \tag{3.12}$$

Using that $f(y(b_n)) \ge \inf_{B_\gamma} f \ge \underline{\inf}_B f - \varepsilon'/4$ from (3.12) we get

$$\underline{\inf}_B f - \sup_A f < \langle q'_n, b - a\rangle + \varepsilon',\ \forall a \in A,\ \forall b \in B. \tag{3.13}$$

Recall that $q'_n \in \partial f_n(w'_n)$, $w'_n \in [A, B]$. By Lemma 2.3 it holds that for any $y(w'_n) \in M^n_{\varepsilon'}(w'_n)$, $q'_n \in \partial_{\varepsilon'} f(y(w'_n))$. Fix $y(w'_n) \in M^n_{\varepsilon'}(w'_n)$. By Brønsted-Rockafellar Theorem there exist $q_n \in \partial f(w_n)$ such that $\|w_n - y(w'_n)\| \le \sqrt{\varepsilon'}$ and $\|q_n - q'_n\| \le \sqrt{\varepsilon'}$. For any $a \in A$, and any $b \in B$,

$$\langle q'_n, b-a\rangle \le \langle q_n, b-a\rangle + \|q'_n - q_n\|\|b-a\| \le \langle q_n, b-a\rangle + \sqrt{\varepsilon'}d.$$

Using the latter in (3.13) we have for large n,

$$\underline{\inf}_B f - \sup_A f < \langle q_n, b-a\rangle + \varepsilon' + \sqrt{\varepsilon'}d, \ \forall a \in A, \ \forall b \in B,$$

hence,

$$\underline{\inf}_B f - \sup_A f < \langle q_n, b-a\rangle + \varepsilon, \ \forall a \in A, \ \forall b \in B,$$

by the choice of ε'.

Similarly as it was done in Case 2 of (a) we show that for large n, $w_n \in [A,B]_\delta$. We know that $\|w_n - y(w'_n)\| \le \sqrt{\varepsilon'} < \delta/2$, by the choice of ε', and that $y(w'_n) \in M^n_{\varepsilon'}(w'_n)$, hence

$$f(y(w'_n)) + n\|w'_n - y(w'_n)\| \le f_n(w'_n) + \varepsilon' \le f(w'_n) + \varepsilon', \tag{3.14}$$

where the last inequality follows by Lemma 2.2(ii). Note that, since f is convex and bounded above on A and B, it is bounded above on $[A,B]$ by some constant M. Using this and that f is bounded below by μ on X by (3.14) we have that

$$\mu + n\|w'_n - y(w'_n)\| \le M + \varepsilon',$$

hence,

$$\|w'_n - y(w'_n)\| \le \frac{M - \mu + \varepsilon'}{n} < \delta/2,$$

for sufficiently large n.

Thus, $w'_n \in [A,B]$ and $\|w_n - w'_n\| \le \|w_n - y(w'_n)\| + \|y(w'_n) - w'_n\| \le \delta$, which means that $w_n \in [A,B]_\delta$ for large n.

Finally, taking n large enough, and setting $q := q_n$ and $w := w_n$ we obtain that $q \in \partial f(w)$, $w \in [A,B]_\delta$, and

$$\underline{\inf}_B f - \sup_A f < \langle q_n, b-a\rangle + \varepsilon, \ \forall a \in A, \ \forall b \in B.$$

The proof of (b), as well as, the proof of the theorem are then completed. □

4 Examples

As we mentioned in the Introduction, it is not difficult to provide examples of a convex function and a couple of sets satisfying the assumptions of Theorem 1.1 which do not satisfy the assumptions of [4, Theorem 4.1].

First we give such an example in a finite dimensional case. Let $X := \mathbb{R}^2$, $A := \{(1,0)\}$, $B := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le \rho, 0 \le \rho < 1\}$, and

$$f(x,y) := \begin{cases} -\sqrt{1 - x^2 - y^2}, & \text{when } x^2 + y^2 \le 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Of course, in this example both sets A and B are compact.

Our second example is in a Banach space $(X, \|\cdot\|)$. Let $A := \{0\}$, $B := x_0 + \mathbb{B}$, where $\|x_0\| > 2$, and $x_0^* \in X^*$ is such that $\|x_0^*\| = 1$ and $\langle x_0^*, x_0\rangle = \|x_0\|$, and

$$f(x) := \begin{cases} -\sqrt{\langle x_0^*, x\rangle}, & \text{if } x \in [A,B], \\ +\infty, & \text{if not.} \end{cases}$$

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