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DIRECTIONAL OPTIMALITY CONDITIONS FOR MULTIOBJECTIVE PROGRAM CONSTRAINED BY PARAMETERIZED VARIATIONAL INEQUALITIES*

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Abstract: In this paper, we discuss the multiobjective program constrained by parameterized variational inequalities. By employing the directional limiting coderivative, which depicts the local behavior of set-valued mappings along relevant directions, a weaker constraint qualification condition is imposed on the optimization problem. Under the metric subregularity constraint qualification of constraint functions and some assumptions on the problem data, the calculus for the directional limiting coderivatives of investigated set-valued mappings is obtained. With the weaker constraint qualification condition, the directionally necessary condition for the multiobjective program with respect to the critical direction is established.

Key words: *parameterized variational inequalities, multiobjective program, directional limiting coderivative*

Mathematics Subject Classification: *49J53, 65K10, 90C29, 90C31*

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1 Introduction

In this paper, we consider the multiobjective program constrained by parameterized variational inequalities

$$\text{(MP-PVI)} \qquad \begin{array}{ll} \min & \varphi(x,y) \\ \text{s.t.} & y \in \Omega, \langle F(x,y), y-z\rangle \leq 0, \forall z \in \Omega, \end{array}$$

where $\varphi : R^n \times R^m \to R^l$ is directionally Lipschitz continuous, $F : R^n \times R^m \to R^m$ is continuously differentiable, Ω is a subset of R^m. $\Omega = \{y \in R^m \mid g_i(y) \leq 0, i = 1, 2, \cdots, q\}$, $g_i : R^m \to R$ is twice continuously differentiable.

Multiobjective programs constrained by parameterized variational inequalities are important parts of multiobjective bilevel optimization problems, where the objective functions of both levels are vector-valued. Nowadays, multiobjective bilevel optimization problems have been investigated intensively in the literature, refer to [3, 5, 6, 27, 27, 28, 29]. Under some proper assumptions, the Karush-Kuhn-Tucker (KKT) condition is necessary and sufficient for global optimality of all lower-level problems, and, in this case, the multiobjective bilevel

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optimization problems can be transferred into the multiobjefctive optimization problems with equilibrium constraints, which are described by some parametric variational inequality or complementarity problem. During last twenty years, much progress has been made for solving multiobjefctive optimization problems with equilibrium constraints [14, 20, 21, 30]. In the problem (MP-PVI), if the set Ω is convex, then the problem (MP-PVI) can be equivalently written as a multiobjective program with a generalized equation constraint

$$\text{(MP-GEC)} \qquad \begin{array}{ll} \min & \varphi(x,y) \\ \text{s.t.} & 0 \in F(x,y) + N_\Omega(y). \end{array}$$

Furthermore, if the Mangasarian-Fromovitz constraint qualification (MFCQ) or other similar constraint qualification holds for the constraint $g_i(y) \leq 0$, then $N_\Omega(y) = \nabla g(y)^T N_{R^q_-}(g(y))$, and the problem (MP-GEC) becomes the multiobjective program with complementarity constraint

$$\text{(MP-CC)} \qquad \begin{array}{ll} \min & \varphi(x,y) \\ \text{s.t.} & F(x,y) + \nabla g(y)^T \lambda = 0, \\ & 0 \geq g(y) \perp \lambda \geq 0. \end{array}$$

In this case, problems (MP-PVI), (MP-GEC) and (MP-CC) are equivalent if the multiplier λ is unique for each y. When the multiplier is not unique or the constraint set Ω is not convex, the problem (MP-PVI) can not be transformed into the problems (MP-GEC) and (MP-CC) due to the changing feasible region. In order to better handle the problem (MP-PVI), researchers propose many qualification conditions to deal with the problem (MP-PVI), especially when deriving optimality conditions, referring to [4, 25, 31]. Ye[26] provided several constraint qualifications for multiobjective optimization problems with variational inequality constraints, including the error bound constraint qualification, the no nonzero abnormal multiplier constraint qualification, the generalized MFCQ and the linear constraint qualification, which are commonly used in mathematical programs with equilibrium constraints. Mordukhovich [15] gave the calmness conditions for the auxiliary set-valued mappings to ensure the stability of parameterized variational inequalities. In [16], the coderivative qualification condition was imposed on the multiobjective optimization problem with equilibrium constraints. All above qualification conditions are very strong and the equivalence of the problems (MP-PVI),(MP-GEC) and (MP-CC) is guaranteed under these qualification conditions. When some properties of the constraint functions are not satisfied, these qualification conditions may be invalid. One of reasons for causing this situation is that some of these qualification conditions are described by standard coderivatives (also called Mordukhovich coderivatives). This kind of standard coderivatives are rough and some characters of the constraint system are not considered, especially in the cases where the standard coderivatives disappeared.

The directional limiting coderivative is a subset of standard coderivatives, and it describes the local behavior of the set-valued mappings along the relevant directions. Based on the directional limiting coderivatives, many types of qualification conditions can be restated. For example, the metric regularity (subregularity) of a set-valued mapping in some directions can be described by its directional limiting coderivative [7]. In recent years, many researchers apply this kind of directional limiting coderivatives to the theoretical analysis and stability analysis for optimization problems. Gfrerer made a systematically study on the sufficient conditions for the calmness and Aubin property of implicit multifunctions, which includes many types equilibrium constraints, such as the parameter-dependent variational inequalities with non-polyhedral constraint sets and parameterized generalized equations with conic constraints, referring to [1, 7, 8] etc. In [9], the author established a M-stationarity condition with respect to every critical direction under the subregularity of the constraint mapping. In [24], the authors built the sum rules for the directionally coderivatives of multifunctions and derived the directionally necessary conditions for a set-valued optimization problem with equilibrium constraints. More approaches and results for directional limiting coderivatives can be found in [2, 10, 11, 12].

Inspired by the ideas from [8, 9], we intend to establish the directional optimality conditions of multiobjective program constrained by parameterized variational inequalities, which is the directional version of the literature [21]. Compared to [21], the optimality conditions and the qualification conditions in this paper are both weaken under some proper assumptions. The contribution of this paper is as follows.

- We employ a new generalized differential, i.e., the directional limiting coderivative, to analyze the behavior of the constraint mappings along relevant directions and it overcomes the shortcoming that the standard coderivatives are invalid.
- The more data of the original problem are considered instead of imposing strong conditions when establishing the calculus of directionally limiting coderivatives for the set-valued mappings in parameterized variational inequalities.
- The directional optimality condition for the multiobjective program is built. The obtained optimality condition is detailed and it shows the local character of the optimal solution along the critical direction, especially for these nonsmooth optimal solutions.

The paper is organized as follows. Section 2 provides some preliminaries about variational analysis and generalized differentials. Section 3 gives the calculus for directionally limiting coderivatives of set-valued mappings under some mild assumptions. In section 4, the directional necessary optimality condition for the considered optimization problem is built. Section 5 is a conclusion.

$\boxed{2}$ Preliminaries

In this section we give some generalized differentials from variational analysis which will be used throughout the whole paper. Detailed discussions on these subjects can be found in books written by Rockafellar [22, 23] and Mordukhovich [17, 18]. For a subset $\Theta \subset R^n$, denote by Θ° the polar cone of Θ, that is $\Theta^\circ = \{v \,|\, \langle v, u\rangle \leq 0, \forall u \in \Theta\}$. $\Theta^\perp$ denotes the orthogonal complement to Θ. The symbol $\mathcal{B}$ denotes the closed unit ball in R^n. Given a point $\bar{x}$, $\mathbf{d}(\bar{x}, \Gamma)$ denotes the distance from $\bar{x}$ to set Γ. Denote $\times_{i=1}^{q}\Gamma := \Gamma_1 \times \cdots \times \Gamma_q$. For $x, y \in R^n$, $x \preceq y$ represents $x_i \leq y_i, i = 1, \ldots, n$. For a mapping $F : R^n \to R^m$, $JF(x)$ denotes the Jacobian of F at x.

Given a direction $u \in R^n$ and positive numbers $\rho, \delta > 0$, consider the set $V_{\rho,\delta}(u)$ given by

$$V_{\rho,\delta}(u) := \{z \in \rho\mathcal{B} \,|\, \| \|u\|z - \|z\|u \| \leq \delta\|z\|\|u\|\}.$$

It is obvious that the directional neighborhood of direction $u = 0$ is the closed unit ball $\rho\mathcal{B}$ and the set of the directional neighborhood of a nonzero direction $u \neq 0$ is smaller than $\rho\mathcal{B}$. Hence we can take use of the directional version of regularity conditions, which is weaker than the usual nondirectional one.

Definition 2.1 (Tangent cone and normal cone [23]). Given a set $\Gamma \subset R^n$ locally closed around $\bar{x} \in \Gamma$, define

(1) Contingent (Bouligand-Severi tangent) cone

$$T_\Gamma(\bar{x}) = \limsup_{t\to 0} \frac{\Gamma - \bar{x}}{t}.$$

(2) Regular (Frechet) normal cone

$$\widehat{N}_\Gamma(\bar{x}) = \{v \in R^n \mid \limsup_{x\to\bar{x}} \frac{\langle v, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leq 0\}.$$

(3) Limiting (Mordukhovich) normal cone

$$N_\Gamma(\bar{x}) = \limsup_{x\to\bar{x}} \widehat{N}_\Gamma(x).$$

Definition 2.2 (Directional normal cone [9])**.** Let $\Gamma \in R^n$, $x \in \Gamma$ and $u \in R^n$ be given. The limiting normal cone to Γ in direction u at x is defined by

$$N_\Gamma(\bar{x}; u) = \{x^* \mid \exists t_k \downarrow 0, u_k \to u, x_k^* \to x^* : x_k^* \in \widehat{N}_\Gamma(\bar{x} + t_k u_k)\}. \tag{2.1}$$

If Γ is convex, then $\widehat{N}_\Gamma(\bar{x}) = N_\Gamma(\bar{x})$ becomes the classical normal cone in the sense of convex analysis and we will write $N_\Gamma(\bar{x})$. By the definition, the limiting normal cone coincides with the directional limiting normal cone in direction 0, i.e., $N_\Gamma(\bar{x}) = N_\Gamma(\bar{x}; 0)$. Furthermore, $N_\Gamma(\bar{x}; u) \subset N_\Gamma(\bar{x})$ for all u, $N_\Gamma(\bar{x}; d) = \emptyset$ whenever $d \notin T_\Gamma(\bar{x})$.

Next we introduce the critical cone of a convex polyhedral set Γ, given a vector $d \in R^n$, the cone

$$\mathcal{K}_\Gamma(\bar{x}, d) := T_\Gamma(\bar{x}) \cap \{d\}^\perp$$

is called the critical cone to Γ at $\bar{x}$ with respect to d. Recall that $\mathcal{F}$ is a face of a polyhedral convex cone C provided for some vector $z^* \in C^\circ$ one has

$$\mathcal{F} = C \cap [z^*]^\perp.$$

With the above tangent cone and normal cones, the corresponding generalized derivatives are given as follows. Let $S : R^n \rightrightarrows R^m$ be a multifunction with a closed graph and $(\bar{x}, \bar{y}) \in \mathrm{gph}S := \{(x, y) \in R^n \times R^m \mid y \in S(x)\}$.

Definition 2.3 (Graphical derivative and (regular) limiting coderivative [2])**.** Consider a point $(\bar{x}, \bar{y}) \in \mathrm{gph}S$. Then

(i) the multifunction $DS(\bar{x}, \bar{y}) : R^n \rightrightarrows R^m$, defined by

$$DS(\bar{x}, \bar{y})(u) := \{v \in R^n \mid (u, v) \in T_{\mathrm{gph}S}(\bar{x}, \bar{y})\}, u \in R^n \tag{2.2}$$

is called the graphical derivative of S at $(\bar{x}, \bar{y})$;

(ii) the multifunction $\widehat{D}^*S(\bar{x}, \bar{y}) : R^m \rightrightarrows R^n$, defined by

$$\widehat{D}^*S(\bar{x}, \bar{y})(y^*) := \{x^* \in R^n \mid (x^*, -y^*) \in \widehat{N}_{\mathrm{gph}S}(\bar{x}, \bar{y})\}, y^* \in R^m \tag{2.3}$$

is called the regular coderivative of S at $(\bar{x}, \bar{y})$;

(iii) the multifunction $D^*S(\bar{x},\bar{y}) : R^m \rightrightarrows R^n$, defined by

$$D^*S(\bar{x},\bar{y})(y^*) := \{x^* \in R^n \mid (x^*,-y^*) \in N_{gphS}(\bar{x},\bar{y})\}, y^* \in R^m \tag{2.4}$$

is called the limiting coderivative of S at $(\bar{x},\bar{y})$.

Definition 2.4 (Directional limiting coderivative [2]). Given a pair of directions $(u,v) \in R^n \times R^m$, the multifunction $D^*S((\bar{x},\bar{y});(u,v))(y^*) : R^m \rightrightarrows R^n$, defined by

$$D^*S((\bar{x},\bar{y});(u,v))(y^*) := \{x^* \in R^n \mid (x^*,-y^*) \in N_{gphS}((\bar{x},\bar{y});(u,v))\}, y^* \in R^m \tag{2.5}$$

is called the directional limiting coderivative of S at $(\bar{x},\bar{y})$ in direction (u,v).

If S is single-valued at $\bar{x}$, we drop $\bar{y}$ in the notation of (2.2)-(2.5). In the case of smooth single-valued mappings, for all $u \in R^n$, $y^* \in R^m$ and the direction $u \in R^n$, we have the representation

$$DS(\bar{x})(u) = \{\nabla S(\bar{x})u\}, \text{ and } \widehat{D}^*S(\bar{x})(y^*) = D^*S(\bar{x})(y^*) = D^*S(\bar{x};u)(y^*) = \{\nabla S(\bar{x})^T y^*\}.$$

The mapping S is outer semicontinuous (osc) at $\bar{x}$ if the existence of sequences $x_k \to \bar{x}$ and $y_k \to y$ with $y_k \in S(x_k)$ implies $y \in S(\bar{x})$ and we say that S is osc if it is osc at every point, which is equivalent to the closedness of gph$S$, see [23, Theorem 5.7].

Consider the mappings $S_1 : R^n \rightrightarrows R^m$, $S_2 : R^m \rightrightarrows R^s$ and associate with them the intermediate multifunction $\Xi : R^n \times R^s \rightrightarrows R^m$ defined by

$$\Xi(x,u) = \{w \in S_1(x) \mid u \in S_2(w)\}.$$

The following theorem gives the directional coderivative chain rule of composite mappings. This rule is very important to establish the directional subdifferentials of constraint functions and objective functions, which is essential in directional optimality conditions.

Theorem 2.5 (Directional coderivative chain rule [2]). *Suppose $S = S_2 \circ S_1$ for osc mappings S_1, S_2. Let $\bar{x} \in domS, y \in S(\bar{x})$ and $(h,l) \in R^n \times R^s$ be two given directions. Assume that*

(a) *there is a directional neighborhood U of (h,l) such that $\Xi((\bar{x},\bar{u}) + U)$ is bounded;*

(b) *the mapping*

$$\mathcal{W}(x,w,u) := \begin{pmatrix} gphS_1 - (x,w) \\ gphS_2 - (w,u) \end{pmatrix}$$

is metrically subregular at $(\bar{x},w,\bar{u},0,0)$ for all $w \in \Xi(\bar{x},\bar{u})$ in directions (h,k,l) with k such that $(h,k) \in T_{gphS_1}(\bar{x},w)$,$(k,l) \in T_{gphS_2}(w,\bar{u})$.

Then one has

$$D^*S((\bar{x},\bar{u});(h,l)) \subset \bigcup_{\check{w}\in\Xi(\bar{x},\bar{u})} \bigcup_{\substack{k\in\{\xi\in DS_1(\bar{x},\check{w})(h)\,| \\ l\in DS_2(\bar{x},\check{w})(\xi)\}}} D^*S_1((\bar{x},\check{w});(h,k)) \circ D^*S_2((\check{w},\bar{u});(k,l)). \tag{2.6}$$

Definition 2.6 ([2]). Let $S : R^n \rightrightarrows R^m$ be a set-valued mapping and let $(\bar{x},\bar{y}) \in$ gph$S$ and $u \in R^n$.

(i) **(Directional metric regularity)** S is metrically regular in direction (u,v) at $(\bar{x},\bar{y})$, if there are positive numbers $\rho > 0, \delta > 0$, and $\kappa > 0$ such that

$$\mathbf{d}(x,S^{-1}(y)) \le \kappa\mathbf{d}(y,S(x)), \forall(x,y) \in (\bar{x},\bar{y}) + V_{\rho,\delta}(u,v).$$

(ii) **(Directional metric subregularity)** S is metrically subregular in direction u at $(\bar{x}, \bar{y})$, if there are positive numbers $\rho > 0, \delta > 0$, and $\kappa > 0$ such that

$$\mathbf{d}(x, S^{-1}(\bar{y})) \leq \kappa \mathbf{d}(\bar{y}, S(x)), \forall x \in \bar{x} + V_{\rho,\delta}(u).$$

It is easy to see that if S is metrically regular in direction (u, v) at $(\bar{x}, \bar{y})$, then it is metrically subregular in direction u at $\bar{x}$. If $(u, v) = (0, 0)$, then S is metric regular at $(\bar{x}, \bar{y})$.

One of great usage of the directional limiting coderivative is to characterize the directional metric regularity property of set-valued mappings, that is a directional extension of the Mordukhovich criterion [7, Theorem 2.9]: a multifunction $S : R^n \rightrightarrows R^m$ which has locally closed-graph around $(\bar{x}, \bar{y})$ is metric regular in direction $(u, v) \in R^n \times R^m$ if and only if

$$0 \in D^*S((\bar{x}, \bar{y}); (u, v))(v^*) \Rightarrow v^* = 0. \tag{2.7}$$

Let B be the feasible set of the problem (MP-PVI) and $K \subset R^l$ is a closed convex cone with nonempty interior. $(\bar{x}, \bar{y})$ is said to be the generalized Pareto efficient solution of (MP-PVI), if there exists $(\bar{x}, \bar{y}) \in B$ satisfying $\varphi(\bar{x}, \bar{y}) \in E(\varphi(B), B)$, that is

$$(\varphi(B) - \varphi(\bar{x}, \bar{y})) \cap (-K) = \{0\}.$$

$(\bar{x}, \bar{y})$ is said to be the generalized weakly Pareto efficient solution of (MP-PVI), if there exists $(\bar{x}, \bar{y}) \in B$ satisfying $\varphi(\bar{x}, \bar{y}) \in WE(\varphi(B), B)$, namely

$$(\varphi(B) - \varphi(\bar{x}, \bar{y})) \cap (-intK) = \emptyset. \tag{2.8}$$

Especially, when $K = R^l_+$, the generalized (weakly) Pareto efficient solution becomes the usually Pareto efficient solution in R^n.

We say that $f : R^n \to R^l$ is directionally Lipschitz continuous at $\bar{x}$ in direction u if there are positive numbers L, ρ, δ such that

$$\|f(x) - f(\bar{x})\| \leq L\|x - \bar{x}\|, x \in \bar{x} + V_{\rho,\delta}(u).$$

It is easy to see that if $f : R^n \to R^l$ is directionally Lipschitz continuous and directionally differentiable at $\bar{x}$ in direction u then for all sequence $\{u_k\}$ which converges to u, we have

$$f'(x; u) = \lim_{k\to\infty} \frac{f(\bar{x} + t_k u_k) - f(\bar{x})}{t_k}.$$

3 Calculus for directional limiting coderivatives

In this section, we introduce the calculus for the directionally limiting coderivatives of set-valued mappings in parameterized variational inequalities. The calculus is obtained just under the metric subregularity of the constraint function and some assumptions on the original data of the problem.

Definition 3.1 (Metric subregularity constraint qualification). Let $g(y) \in R^q_-$. We say that the metric subregularity constraint qualification (MSCQ) holds at $\bar{y}$ for the system $g(y) \in R^q_-$ if the set-valued map $G(y) := g(y) - R^q_-$ is metrically subregular at $(\bar{y}, 0)$, or equivalently the perturbed set-valued map $G^{-1}(w) := \{y \,|\, w \in g(y) - R^q_-\}$ is calm at $(0, \bar{y})$.

When MSCQ is fulfilled for the constraint $g(y) \in R^q_-$ at $\bar{y}$, by definition, MSCQ also holds for all points $y \in \Omega$ near $\bar{y}$, then

$$N_\Omega(y) = \widehat{N}_\Omega(y) = \nabla g(y)^T N_{R^q_-}(g(y)).$$

Denote the mapping $M : R^n \times R^m \rightrightarrows R^m$ by

$$M(x,y) := F(x,y) + N_\Omega(y) = F(x,y) + \nabla g(y)^T N_{R^q_-}(g(y)). \tag{3.1}$$

Denote the mapping $Q : R^m \rightrightarrows R^m$ by

$$Q(y) := N_\Omega(y) = \nabla g(y)^T N_{R^q_-}(g(y)). \tag{3.2}$$

The following task is to establish the calculus for directionally limiting coderivative of composite set-value mapping Q under its graphical derivative. Firstly, we give the directional limiting coderivative of $N_{R^q_-} \circ g$. Denoting the mappings $H : R^m \rightrightarrows R^m$ and $\mathcal{H} : R^m \times R^q \rightrightarrows R^m \times R^q$ by

$$H(y) := N_{R^q_-}(g(y)) = (N_{R^q_-} \circ g)(y),$$

and

$$\mathcal{H}(y,\lambda) = (g(y),\lambda) - \mathrm{gph}N_{R^q_-}. \tag{3.3}$$

Denote $I := \{1,2,\ldots,q\}$, $I(\bar{y}) := \{i \in I \,|\, g_i(\bar{y}) = 0\}$, $I^+(\lambda) := \{i \in I \,|\, \lambda_i > 0\}$, then $I^+(\lambda) \subset I(\bar{y}) \subset I$.

Proposition 3.2. *Let $\lambda \in N_{R^q_-}(g(\bar{y}))$, let $v \in R^m$ be the direction satisfying $\nabla g(\bar{y})v \in \mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)$ and let $\eta \in N_{\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)}(\nabla g(\bar{y})v)$. Further assume that for every pair of faces $\mathcal{F}_1, \mathcal{F}_2$ of the critical cone $\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)$ with $\nabla g(\bar{y})v \in \mathcal{F}_2 \subset \mathcal{F}_1 \subset [\eta]^\perp$ there holds*

$$\nabla g(\bar{y})^T \xi = 0, \xi \in (\mathcal{F}_1 - \mathcal{F}_2)^\circ \Rightarrow \xi = 0.$$

Then the mapping $\mathcal{H}(y,\lambda)$ is metric regular at $(\bar{y},\lambda,0)$ in direction (v,η), and it holds

$$D^*H((\bar{y},\lambda);(v,\eta)) \subset \nabla g(\bar{y})^T D^* N_{R^q_-}((g(\bar{y}),\lambda);(\nabla g(\bar{y})v,\eta)). \tag{3.4}$$

Proof. From $0 \in \mathcal{H}(\bar{y},\lambda)$, it has $(g(\bar{y}),\lambda) \in \mathrm{gph}N_{R^q_-}$. According to the characterization of directional metric regularity in [11, Theorem 1], there is (ξ,γ) such that

$$\nabla g(\bar{y})^T \xi = 0, \gamma = 0, (\xi,\gamma) \in N_{\mathrm{gph}N_{R^q_-}}((g(\bar{y}),\lambda);(\nabla g(\bar{y})v,\eta)) \Rightarrow \gamma = \xi = 0. \tag{3.5}$$

By [7, Theorem 2.12], it has

$$N_{\mathrm{gph}N_{R^q_-}}((g(\bar{y}),\lambda);(\nabla g(\bar{y})v,\eta)) = (\mathcal{F}_1 - \mathcal{F}_2)^\circ \times (\mathcal{F}_1 - \mathcal{F}_2),$$

where $\mathcal{F}_1, \mathcal{F}_2$ are faces of the critical cone $\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)$ with $\nabla g(\bar{y})v \in \mathcal{F}_2 \subset \mathcal{F}_1 \subset [\eta]^\perp$. Hence, the mapping $\mathcal{H}(y,\lambda)$ is metric regular at $(\bar{y},\lambda,0)$ in direction (v,η).

Under the metric regularity of $\mathcal{H}$, from Theorem 2.5, we obtain

$$D^*H((\bar{y},\lambda);(v,\eta)) \subset D^*g((\bar{y},g(\bar{y}));(v,\nabla g(\bar{y})v)) \circ D^*N_{R^q_-}((g(\bar{y}),\lambda);(\nabla g(\bar{y})v,\eta)).$$

Since g is continous differential, one has $D^*g((\bar{y},g(\bar{y}));(v,\nabla g(\bar{y})v))) = \nabla g(\bar{y})^T$, then

$$D^*H((\bar{y},\lambda);(v,\eta)) \subset \{\nabla g(\bar{y})^T y^* \,|\, y^* \in D^*N_{R^q_-}((g(\bar{y}),\lambda);(\nabla g(\bar{y})v,\eta))\}.$$

The proof of the proposition is completed. □

Next we give the graphical derivative of Q. Let $(y,y^*)\in N_\Omega(g(y))$, define by

$$\Lambda(y,y^*):=\{\lambda\in N_{R^q_-}(g(y))\,|\,\nabla g(y)^T\lambda=y^*\}$$

the set of Lagrange multipliers associated with (y,y^*). Moreover,

$$\Lambda(y,y^*;v):=\arg\max\{v^T\nabla(\nabla g(y)^T\lambda)v\,|\,\lambda\in\Lambda(y,y^*)\}$$

stands for the multiplier set in a direction $v\in\mathcal{K}_\Omega(y,y^*)$, where $\mathcal{K}_\Omega(y,y^*)=T_\Omega(y)\cap\{y^*\}^\perp$ is the critical cone of Ω .

Proposition 3.3. *Suppose that MSCQ holds for the system $g(y)\in R^q_-$ at $\bar{y}$. Let $v\in\mathcal{K}_\Omega(\bar{y},y^*),\lambda\in\Lambda(\bar{y},y^*)$. Then*

$$\begin{aligned}N_{\mathcal{K}_\Omega(\bar{y},y^*)}(v)&=\{\nabla g(\bar{y})^T\mu\,|\,\mu^T\nabla g(\bar{y})v=0,\mu\in T_{N_{R^q_-}}(g(\bar{y}))(\lambda)\}\\&=\nabla g(\bar{y})^TN_{\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)}(\nabla g(\bar{y})v),\end{aligned}\tag{3.6}$$

where $\mathcal{K}_\Omega(\bar{y},y^*)=\{v\,|\,\nabla g(\bar{y})v\in T_{R^q_-}(g(\bar{y})),v^Ty^*=0\}$.

Proof. From the definition of $\mathcal{K}_\Omega(y,y^*)$ and [11, Proposition 6], it has $\mathcal{K}_\Omega(\bar{y},y^*)=\{v\,|\,\nabla g(\bar{y})v\in T_{R^q_-}(g(\bar{y})),v^Ty^*=0\}$ and the first equation is given in [11, Lemma 1]. We prove that the second equation holds. Firstly, we intend to prove that if $\mu\in T_{N_{R^q_-}}(g(\bar{y}))(\lambda),\mu^T\nabla g(\bar{y})v=0$, then $\mu\in N_{\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)}(\nabla g(\bar{y})v)$. It is obvious that $\mu\in T_{N_{R^q_-}}(g(\bar{y}))(\lambda)\cap\{\nabla g(\bar{y})v\}^\perp$. There is $t>0$ satisfying $\lambda+t\mu\in N_{R^q_-}(g(\bar{y}))$, then $\mu\in-\frac{\lambda}{t}+N_{R^q_-}(g(\bar{y}))$, which means that $\mu\in T^\circ_{R^q_-}(g(\bar{y}))+R\lambda=(T_{R^q_-}(g(\bar{y})\cap\{\lambda\})^\circ=\mathcal{K}^\circ_{R^q_-}(g(\bar{y}),\lambda)$. Hence, $\mu\in\mathcal{K}^\circ_{R^q_-}(g(\bar{y}),\lambda)\cap\{\nabla g(\bar{y})v\}^\perp=N_{\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)}(\nabla g(\bar{y})v)$. Conversely, take $\mu\in N_{\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)}(\nabla g(\bar{y})v)$, from the previous proof, it has $\mu\in(T^\circ_{R^q_-}(g(\bar{y}))+R\lambda)\cap\{\nabla g(\bar{y})v\}^\perp$, which is $\mu^T\nabla g(\bar{y})v=0,\mu\in N_{R^q_-}(g(\bar{y}))+R\lambda$. Therefore, there is $\hat{\mu}\in N_{R^q_-}(g(\bar{y}))$ and $\alpha\in R$ satisfying $\mu=\hat{\mu}+\alpha\lambda$. Since $\lambda\in N_{R^q_-}(g(\bar{y}))$, then $\mu\in T_{N_{R^q_-}(g(\bar{y}))}(\lambda)$. The proof of the theorem is completed. ☐

Proposition 3.4. *Assume MSCQ holds at $\bar{y}$ for the system $g(y)\in R^q_-$. For given $y^*\in R^m,v\in R^m$, we have*

$$DQ(\bar{y},y^*)(v)=DN_\Omega(\bar{y},y^*)(v)=\nabla(\nabla g(\bar{y})^T\lambda)v+\nabla g(\bar{y})^TN_{\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)}(\nabla g(\bar{y})v),\tag{3.7}$$

where λ is the unique solution of the system $y^*=\nabla g(\bar{y})^T\lambda,\lambda\in N_{R^q_-}(g(\bar{y}))$.

Proof. From the definition of graphical derivative in (2.2), we have

$$DN_\Omega(\bar{y},y^*)(v)=\{w\in R^m\,|\,(v,w)\in T_{\mathrm{gph}N_\Omega}(\bar{y},y^*)\}.$$

When MSCQ holds at $\bar{y}$, there is λ satisfying $y^*=\nabla g(\bar{y})^T\lambda,\lambda\in N_{R^q_-}(g(\bar{y}))$. Then the set of Lagrange multipliers $\Lambda(y,y^*)$ and the directional multiplier set $\Lambda(y,y^*;v)$ are unique, i.e., $\Lambda(y,y^*)=\Lambda(y,y^*;v)=\{\lambda\}$. From [11, Theorem 4], we obtain that

$$T_{\mathrm{gph}N_\Omega}(\bar{y},y^*)=\nabla(\nabla g(\bar{y})^T\lambda)v+N_{\mathcal{K}_\Omega(\bar{y},y^*)}(v).$$

From Proposition 3.3, we have $N_{\mathcal{K}_\Omega(\bar{y},y^*)}(v)=\nabla g(\bar{y})^TN_{\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)}(\nabla g(\bar{y})v)$. Combing the definition of the graphical derivative in (2.2), we obtain

$$DN_\Omega(\bar{y},y^*)(v)=\nabla(\nabla g(\bar{y})^T\lambda)v+\nabla g(\bar{y})^TN_{\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)}(\nabla g(\bar{y})v).$$

☐

With the help of directional coderivative chain rule, we establish the directional limiting coderivatives of the composite set-valued mapping Q. Rewrite the mapping $Q(y) = N_\Omega(y) = \nabla g(y)^T N_{R^q_-}(g(y))$ by

$$Q(y) = (S_2 \circ S_1)(y), \quad S_1(y) = \begin{pmatrix} y \\ N_{R^q_-}(g(y)) \end{pmatrix} = \begin{pmatrix} y \\ H(y) \end{pmatrix}, \quad S_2(y,d) = \nabla g(y)^T d,$$

where $S_1 : R^m \rightrightarrows R^m \times R^q$, $S_2 : R^m \times R^q \rightrightarrows R^m$. Define the intermediate mapping Θ as

$$\begin{aligned} \Theta(y, y^*) &= \{(y,d) \in S_1(y) \,|\, y^* = S_2(y,d)\} \\ &= \{(y,d) \,|\, d \in N_{R^q_-}(g(y)), \nabla g(y)^T d = y^*\}. \end{aligned}$$

Theorem 3.5. *Let $y^* \in R^m$ be given and let λ be (uniquely) given by $y^* = \nabla g(\bar{y})^T \lambda, \lambda \in N_{R^q_-}(g(\bar{y}))$. Assume that there is a directional neighborhood U of (v,l) such that $\Theta((\bar{y}, y^*) + U)$ is bounded. Let $(v,l) \in DQ(\bar{y}, y^*)$,i.e., there is η satisfying*

$$l = \nabla(\nabla g(\bar{y})^T \lambda)v + \nabla g(\bar{y})^T \eta, \eta \in N_{\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)}(\nabla g(\bar{y})v). \tag{3.8}$$

Assume that the mapping $\mathcal{H}(y,\lambda)$ is metric regular at $(\bar{y}, \lambda, 0)$ in direction (v, η). Then for $v^* \in R^m$, it has

$$D^*Q((\bar{y}, y^*); (v,l))(v^*) \subset \nabla(\nabla g(\bar{y})^T \lambda)v^* + \nabla g(\bar{y})^T D^* N_{R^q_-}((g(\bar{y}), \lambda); (\nabla g(\bar{y})v, \eta))(\nabla g(\bar{y})v^*). \tag{3.9}$$

Proof. Under MSCQ and the metric subregularity of $\mathcal{H}$, from [2, Corollary 5.1], we have

$$D^*Q((\bar{y}, y^*); (v,l))(v^*) \subset \bigcup_{\substack{\xi \in DS_1(\bar{y}, \bar{y}, \lambda) \\ l = \nabla S_2(\bar{y},\lambda)\xi}} D^* S_1((\bar{y}, \bar{y}, \lambda); (v, \xi)) \circ \nabla S_2(\bar{y}, \lambda)^T (v^*). \tag{3.10}$$

Since

$$(\bar{y}, \lambda) \in \mathrm{gph}(N_{R^q_-} \circ g) \Leftrightarrow (g(\bar{y}), \lambda) \in \mathrm{gph} N_{R^q_-},$$

due to the chain rule of tangent cone of constraint sets, it has

$$T_{\mathrm{gph}(N_{R^q_-} \circ g)}(\bar{y}, \lambda) \subset \mathcal{R} := \left\{ (v,\eta) \,\middle|\, \begin{pmatrix} \nabla g(\bar{y})v \\ \eta \end{pmatrix} \in T_{\mathrm{gph} N_{R^q_-}}(g(\bar{y}), \lambda) \right\}.$$

For given $(v,\eta) \in \mathcal{R}$, there is $t_k \downarrow 0$ such that $\mathbf{d}((g(\bar{y}) + t_k \nabla g(\bar{y})v, \lambda + t_k \eta), \mathrm{gph} N_{R^q_-}) = o(t_k)$, hence, $\mathbf{d}((g(\bar{y} + t_k v), \lambda + t_k \eta), \mathrm{gph} N_{R^q_-}) = o(t_k)$. According to the metric subregularity of $\mathcal{H}(\bar{y}, \lambda)$, there is (y_k, λ_k) satisfying

$$\|(y_k, \lambda_k) - (\bar{y} + t_k v, \lambda + t_k \eta)\| = o(t_k), \forall (g(y_k), \lambda_k) \in \mathrm{gph} N_{R^q_-},$$

then $(v,\eta) \in T_{\mathrm{gph}(N_{R^q_-} \circ g)}(\bar{y}, \lambda)$ and $\mathcal{R} \subset T_{\mathrm{gph}(N_{R^q_-} \circ g)}(\bar{y}, \lambda)$. Therefore, we have $\mathcal{R} = T_{\mathrm{gph}(N_{R^q_-} \circ g)}(\bar{y}, \lambda)$. Furthermore, there is $t \downarrow 0$ such that $(g(\bar{y}) + t\nabla g(\bar{y})v, \lambda + t\eta) \in \mathrm{gph} N_{R^q_-}$. From [4, Lemma 2E.4], we obtain that

$$(g(\bar{y}) + t\nabla g(\bar{y})v, \lambda + t\eta) \in \mathrm{gph} N_{R^q_-} \Leftrightarrow (t\nabla g(\bar{y})v, t\eta) \in \mathrm{gph} N_{\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)}$$

for $(t\nabla g(\bar{y})v, t\eta)$ sufficiently near $(0,0)$, then

$$T_{\mathrm{gph}N_{R^q_-}}(g(\bar{y}),\lambda) = \mathrm{gph}N_{\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)},$$

therefore,

$$T_{\mathrm{gph}(N_{R^q_-}\circ g)}(\bar{y},\lambda) = \{(v,\eta)\,|\,\eta \in N_{\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)}(\nabla g(\bar{y})v)\}.$$

By a simple computation, we obtain that $DS_1(\bar{y},\bar{y},\lambda)(v) = \{(v,\eta)\,|\,\eta \in N_{\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)}(\nabla g(\bar{y})v)\}$, $\nabla S_2(\bar{y},\lambda) = (\nabla(\nabla g(\bar{y})^T\lambda), \nabla g(\bar{y})^T)$. From the formula $\xi \in DS_1(\bar{y},\bar{y},\lambda), l = \nabla S_2(\bar{y},\lambda)\xi$ in (3.10), the relation (3.8) holds. Since

$$D^*S_1((\bar{y},\bar{y},\lambda);(v,\xi)) = (I, D^*H((\bar{y},\lambda);(v,\xi))), \tag{3.11}$$

and

$$D^*S_2((\bar{y},\bar{y},\lambda);(v,\xi))(v^*) = \nabla S_2(y,\lambda)^T v^* = \begin{pmatrix} [\nabla(\nabla g(\bar{y})^T\lambda)]^T v^* \\ \nabla g(\bar{y})v^* \end{pmatrix}, \tag{3.12}$$

from (3.10), (3.11) and (3.12) , one has

$$D^*Q((\bar{y},y^*);(v,l))(v^*) \subset [\nabla(\nabla g(\bar{y}))^T\lambda]^T v^* + D^*H((\bar{y},\lambda);(v,\eta))(\nabla g(\bar{y})v^*).$$

Combing Proposition 3.2, we obtain

$$\begin{aligned} &D^*Q((\bar{y},y^*);(v,l))(v^*) \\ &\qquad \subset [\nabla(\nabla g(\bar{y}))^T\lambda]^T v^* + \nabla g(\bar{y})^T D^*N_{R^q_-}((g(\bar{y}),\lambda);(\nabla g(\bar{y})v,\eta))(\nabla g(\bar{y})v^*). \end{aligned}$$

The proof of the theorem is completed. □

4 Directional necessary optimality conditions

In this section, under the directional generalized differentials of set-valued mappings, we establish the directional necessary optimality condition of the problem (MP-PVI). Before giving directional optimality conditions, the scalar function of the multiobjective problem is provided.

Lemma 4.1 ([13])**.** *Let $K \subset R^l$ be a closed convex cone with nonempty interior. Then for every $e \in intK$, the function $\phi_e : R^l \to R$ given by*

$$\phi_e(w) = \inf\{\lambda \in R\,|\,w \in \lambda e - K\}$$

is continuous, Lipschitz, sublinear, strictly-intK-monotone, in addition,

(i) *for every $\lambda \in R$, one has*

$$\{w\,|\,\phi_e(w) \le \lambda\} = \lambda e - K, \quad \{w\,|\,\phi_e(w) < \lambda\} = \lambda e - intK;$$

(ii) *$\partial\phi_e(w) \subset K^*$;*

(iii) *for every $w \in R^l$, $\partial\phi_e(w) \ne \emptyset$, and*

$$\partial\phi_e(w) = \{w^* \in K^*\,|\,\langle w^*, e\rangle = 1, \langle w^*, w\rangle = 1\}.$$

The scalarization of (MP-PVI) is provided by

$$\text{(SMP-PVI)} \qquad \begin{array}{ll} \min & (\phi_e \circ \varphi)(x,y) \\ \text{s.t.} & 0 \in M(x,y). \end{array}$$

Definition 4.2. Let $(\bar{x},\bar{y})$ be feasible for the problem (MP-PVI). We call $(u,v) \in R^n \times R^m$ a critical direction for the problem (MP-PVI) at $(\bar{x},\bar{y})$ if $\varphi^{'}((\bar{x},\bar{y});(u,v)) \preceq 0$ and $0 \in DM((\bar{x},\bar{y},0);(u,v))$.

By the definition, (u,v) is a critical direction of (MP-PVI) if and only if there exist sequences $t_k \to 0$, $(u_k,v_k) \to (u,v)$ satisfying

$$\lim_{k\to\infty} \frac{\varphi(\bar{x}+t_k u_k, \bar{y}+t_k v_k) - \varphi(\bar{x},\bar{y})}{t_k} \preceq 0, \qquad \lim_{k\to\infty} \frac{\mathbf{d}(0, M(\bar{x}+t_k u_k, \bar{y}+t_k v_k))}{t_k} = 0.$$

The following theorem reveals the solution relationship between (MP-PVI) and (SMP-PVI).

Lemma 4.3 ([19])**.** *Suppose that $(\bar{x},\bar{y}) \in B$ is the weakly Pareto efficient solution of (MP-PVI), then $(\bar{x},\bar{y})$ is the minimum solution of (SMP-PVI).*

Next we give the directional optimality conditions of the problem (MP-PVI).

Theorem 4.4. *Let $K = R^l_+$ and $(\bar{x},\bar{y})$ be the weakly Pareto solution to the problem (MP-PVI). Assume that (u,v) is the critical direction for the problem (MP-PVI) at $(\bar{x},\bar{y})$. The function φ is Lipschitz continuous at $(\bar{x},\bar{y})$ in direction (u,v). Suppose that λ is the (uniquely) solution of the equation $F(\bar{x},\bar{y}) + \nabla g(\bar{y})^T\lambda = 0, \lambda \in N_{R^q_-}(g(\bar{y}))$. For each direction (u,v), there is η such that*

$$\nabla F(\bar{x},\bar{y})(u,v) + \nabla(\nabla g(\bar{y})^T\lambda)v + \nabla g(\bar{y})^T\eta = 0, \eta \in N_{\mathcal{K}_{R^q_-}(g(\bar{y}),\lambda)}(\nabla g(\bar{y})v). \tag{4.1}$$

Assume that the mapping $\mathcal{H}$ given in (3.3) is metric subregular at $(\bar{y},\lambda,0)$ in direction (v,η) and the implication

$$\left.\begin{array}{l} \nabla_x F(\bar{x},\bar{y})^T v^* = 0 \\ \nabla_y F(\bar{x},\bar{y})^T v^* + \nabla(\nabla g(\bar{y})^T\lambda)v^* + \nabla g(\bar{y})^T w^* = 0 \\ w^* \in D^*N_{R^q_-}((g(\bar{y}),\lambda);(\nabla g(\bar{y})v,\eta))(\nabla g(\bar{y})v^*) \end{array}\right\} \Rightarrow v^* = 0 \tag{4.2}$$

is fulfilled. Then there are $z^* \in R^l_+$ and $v^* \in R^m$ satisfying

$$\begin{aligned} 0 \in\ & \partial\langle z^*, \varphi\rangle((\bar{x},\bar{y});(u,v)) + \nabla F(\bar{x},\bar{y})^T v^* + \nabla(\nabla g(\bar{y})^T\lambda)v^* \\ & + \nabla g(\bar{y})^T D^*N_{R^q_-}((g(\bar{y}),\lambda);(\nabla g(\bar{y})v,\eta))(\nabla g(\bar{y})v^*). \end{aligned} \tag{4.3}$$

Proof. Since (u,v) is the critical direction for the problem (MP-PVI) at $(\bar{x},\bar{y})$, (u,v) is also the critical direction of (SMP-PVI). In fact, by the definition of the critical directions, there is sequences $t_k \to 0, (u_k,v_k) \to (u,v), w_k \to 0, \vartheta_k \to 0$, $\mu \in R^l_+$ such that $\varphi(\bar{x}+t_k u_k, \bar{y}+t_k v_k) = \varphi(\bar{x},\bar{y}) + t_k w_k - \mu$ and $0 + t_k\vartheta_k \in M(\bar{x}+t_k u_k, \bar{y}+t_k v_k)$. Hence,

$$\begin{aligned} & \lim_{k\to\infty} \frac{(\phi_e \circ \varphi)(\bar{x}+t_k u_k, \bar{y}+t_k v_k) - (\phi_e \circ \varphi)(\bar{x},\bar{y})}{t_k} \\ = & \lim_{k\to\infty} \frac{(\phi_e(\varphi(\bar{x}+t_k u_k, \bar{y}+t_k v_k)) - (\phi_e(\varphi(\bar{x},\bar{y}))}{t_k} \\ = & \lim_{k\to\infty} \frac{\phi_e(\varphi(\bar{x},\bar{y}) + t_k w_k - \mu) - (\phi_e(\varphi(\bar{x},\bar{y}))}{t_k}. \end{aligned}$$

Since ϕ_e is sublinear, we have $\phi_e(\varphi(\bar{x},\bar{y})+t_kw_k-\mu)\le\phi_e(\varphi(\bar{x},\bar{y}))+\phi_e(t_kw_k)+\phi_e(-\mu)$, then

$$\begin{aligned}&\lim_{k\to\infty}\frac{(\phi_e\circ\varphi)(\bar{x}+t_ku_k,\bar{y}+t_kv_k)-(\phi_e\circ\varphi)(\bar{x},\bar{y})}{t_k}\\ \le\ &\lim_{k\to\infty}\frac{\phi_e(\varphi(\bar{x},\bar{y}))+\phi_e(t_kw_k)+\phi_e(-\mu)-\phi_e(\varphi(\bar{x},\bar{y}))}{t_k}\\ =\ &\lim_{k\to\infty}\frac{\phi_e(t_kw_k)+\phi_e(-\mu)}{t_k}.\end{aligned}$$

Due to $w_k\to0,\phi_e(t_kw_k)\to0$, and $-\mu\in R^l_-,\phi_e(-\mu)\le0$, then

$$\lim_{k\to\infty}\frac{(\phi_e\circ\varphi)(\bar{x}+t_ku_k,\bar{y}+t_kv_k)-(\phi_e\circ\varphi)(\bar{x},\bar{y})}{t_k}\le0,$$

which means that (u,v) is the critical direction of (SMP-PVI).

Next we illustrate that, under the qualification condition (4.2), the mapping M given in (3.1) is metric regular at $(\bar{x},\bar{y})$ in direction (u,v), which means it is also metric subregular at $(\bar{x},\bar{y})$ in direction (u,v). Since $M(x,y)=F(x,y)+Q(y)$, by the directional codervative calculus in [7], we obtain the directional coderivative of M:

$$\begin{aligned}D^*M((\bar{x},\bar{y},0);(u,v,0))(v^*)&=D^*(F+Q)((\bar{x},\bar{y},0);(u,v,0))(v^*)\\&=\nabla F(\bar{x},\bar{y})^Tv^*\\&\quad+D^*Q((\bar{y},-F(\bar{x},\bar{y}));(v,-\nabla F(\bar{x},\bar{y})(u,v)))(v^*).\end{aligned}$$

Under the metric subregularity of $\mathcal{H}$ given in (3.3) at $(\bar{y},\lambda,0)$ in direction (v,η), from Theorem 3.5, we have

$$\begin{aligned}D^*Q((\bar{y},-F(\bar{x},\bar{y}));(v,-\nabla F(\bar{x},\bar{y})(u,v)))(v^*)\subset\nabla(\nabla g(\bar{y})^T\lambda)v^*\\+\nabla g(\bar{y})^TD^*N_{R^q_-}((g(\bar{y}),\lambda);(\nabla g(\bar{y})v,\eta))(\nabla g(\bar{y})v^*).\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}D^*M((\bar{x},\bar{y},0);(u,v,0))(v^*)\subset\nabla F(\bar{x},\bar{y})^Tv^*+\nabla(\nabla g(\bar{y})^T\lambda)v^*\\+\nabla g(\bar{y})^TD^*N_{R^q_-}((g(\bar{y}),\lambda);(\nabla g(\bar{y})v,\eta))(\nabla g(\bar{y})v^*).\end{aligned}\tag{4.4}$$

From (2.3), we know that the metric regularity of M at $(\bar{x},\bar{y})$ in direction (u,v) is equivalent to

$$0\in D^*M((\bar{x},\bar{y},0);(u,v,0))(v^*)\Rightarrow v^*=0.\tag{4.5}$$

Under the metric regularity of M at $(\bar{x},\bar{y})$ in direction (u,v), moreover, the functions ϕ_e and φ are Lipschitz continuous at $(\bar{x},\bar{y})$ in direction (u,v), then $\phi_e\circ\varphi$ is also Lipschitz continuous at $(\bar{x},\bar{y})$ in direction (u,v). Take a similar proof as [9, Theorem 7 (ii)], we obtain

$$0\in\partial(\phi_e\circ\varphi)((\bar{x},\bar{y});(u,v))+D^*M((\bar{x},\bar{y},0);(u,v,0))(v^*).\tag{4.6}$$

Apply the directional subdifferential chain rule in Theorem 2.5 to $\phi_e\circ\varphi$, taking $w=s=0$, we obtain that

$$\partial(\phi_e\circ\varphi)((\bar{x},\bar{y});(u,v))\subset D^*\varphi((\bar{x},\bar{y});(u,v))\partial\phi_e(\varphi(\bar{x},\bar{y})).\tag{4.7}$$

From Lemma 4.3 and [2, Proposition 5.1], there is $z^* \in \partial\phi_e(\varphi(\bar{x},\bar{y})) \subset R^l_+$ satisfying

$$D^*\varphi((\bar{x},\bar{y});(u,v))(z^*) = \partial\langle z^*,\varphi\rangle((\bar{x},\bar{y});(u,v)). \tag{4.8}$$

Combing (4.4) and (4.6-4.8), we obtain

$$\begin{aligned} 0 \in \quad & \partial\langle z^*,\varphi\rangle((\bar{x},\bar{y});(u,v)) + \nabla F(\bar{x},\bar{y})^T v^* + \nabla(\nabla g(\bar{y})^T\lambda)v^* \\ & + \nabla g(\bar{y})^T D^* N_{R^q_-}((g(\bar{y}),\lambda);(\nabla g(\bar{y})v,\eta))(\nabla g(\bar{y})v^*). \end{aligned} \tag{4.9}$$

The proof of the theorem is completed. □

Remark 4.5. In the paper [21], Theorem 4.2 provides the optimality condition of the problem (MP-PVI) under the linear independent constraints qualification, which is the matrix

$$\begin{pmatrix} J_{x,y}L(\bar{x},\bar{y},\lambda) \\ Jg_I(\bar{y}) \end{pmatrix} \tag{4.10}$$

is of full row rank. The qualification condition (4.10) is much stronger than the qualification condition (4.2). Under the qualification condition (4.2), the metric regularity of M at $(\bar{x},\bar{y})$ in direction (u,v) holds, which is (4.5). In fact, the constraint qualification (4.5) is a directional extension of the Mordukhovich criterion. In [21], under the linear independent constraints qualification (4.10), we investigate the Mordukhovich criterion of the inquired set-valued mappings, and obtain the optimality conditions. In this paper, by virtue of (4.2), we study the directional Mordukhovich criterion of the inquired set-valued mappings, and the directional optimality conditions are derived. The difference of them is that, before computing the directional limiting coderivatives of set-valued mappings, it needs to compute their graphic derivatives, which is also very complicated.

At last we give an example to illustrate the validity of the directional necessary optimality condition.

Example 4.6. Consider the problem (MP-PVI), where

$$\varphi(x,y) = \begin{pmatrix} x_1^2 - 2x_2 + y_1^2 \\ -x_2^2 + 2x_2 + y_2^2 - \frac{4}{3}y_2 \end{pmatrix}, \quad F(x,y) = \begin{pmatrix} y_1 - x_1 + x_2 - 0.5 \\ y_2 - 2x_1 + 2x_2 - 0.5 \end{pmatrix},$$

and

$$g(y) = \begin{pmatrix} -(y_1-1)^2 - 2y_2 + \frac{5}{4} \\ -(y_2-1)^2 - y_1 + \frac{3}{4} \end{pmatrix}.$$

The weakly Pareto efficient solution of the problem is $(\bar{x},\bar{y}) = (0.5, 0.5, 0.5, 0.5)^T$. In this problem, we have

$$\nabla_x F(\bar{x},\bar{y}) = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}, \quad \nabla_y F(\bar{x},\bar{y}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \nabla g(\bar{y}) = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}.$$

$\nabla_x F(\bar{x},\bar{y})$ is not surjective and $\nabla_y F(\bar{x},\bar{y}), \nabla g(\bar{y})$ are surjective. The metric subregularity constraint qualification is fulfilled for the constraint $g(y) \in R^2_-$ at $\bar{y}$. $\lambda = 0$ is the uniquely solution of the equation $F(\bar{x},\bar{y}) + \nabla g(\bar{y})^T\lambda = 0, \lambda \in N_{R^2_-}(g(\bar{y}))$.

Next we verify the validity of the qualification condition (4.2). For the direction (u,v), there is $\eta \in N_{\mathcal{K}_{R^2_-}(g(\bar{y}),\lambda)}(\nabla g(\bar{y})v)$, $\mathcal{K}_{R^2_-}(g(\bar{y}),\lambda) = T_{R^2_-}(g(\bar{y})) \cap [\lambda]^\perp = R^2_-$ satisfying

$$\nabla F(\bar{x},\bar{y})(u,v) + \nabla(\nabla g(\bar{y})^T\lambda)v + \nabla g(\bar{y})^T\eta = 0.$$

Then

$$\begin{cases} v_1 = u_1 - u_2 - \eta_1 + \eta_2 \\ v_2 = 2u_1 - 2u_2 + 2\eta_1 - \eta_2, \end{cases}$$

and

$$\eta \in N_{\mathcal{K}_{R^2_-}(g(\bar{y}),\lambda)}(\nabla g(\bar{y})v) = N_{R^2_-}\begin{pmatrix} v_1 - 2v_2 \\ -v_1 + v_2 \end{pmatrix} = N_{R^2_-}\begin{pmatrix} -3u_1 + 3u_2 - 5\eta_1 + 3\eta_2 \\ u_1 - u_2 + 3\eta_1 - 2\eta_2 \end{pmatrix}. \tag{4.11}$$

Some calculations yield that for every $u \in R^2$ the set $T(u) := \{(v, \eta) \in R^2 \times R^2\,|(u, v, \eta)$ fulfills (4.11)$\}$ is not empty and

$$T(u) = \begin{cases} \{((\frac{3}{2}(u_1 - u_2), \frac{3}{2}(u_1 - u_2)), (0, \frac{1}{2}(u_1 - u_2)))\} & \text{if } u_1 > u_2, \\ \{((0,0), (-3(u_1 - u_2), -4(u_1 - u_2)))\} & \text{if } u_1 < u_2, \\ \{((0,0),(0,0))\} & \text{if } u_1 = u_2, \\ \emptyset & \text{otherwise.} \end{cases} \tag{4.12}$$

According to (4.12), we verify that (4.2) is satisfied. Since $g(\bar{y}) = \lambda = 0$, the left of (4.2) becomes

$$\begin{cases} v_1^* - v_2^* = 0 \\ v_1^* + \vartheta_1 - \vartheta_2 = 0 \\ v_2^* - 2\vartheta_1 + \vartheta_2 = 0 \\ (\vartheta, -\nabla g(\bar{y})v^*) \in N_{\mathrm{gph} N_{R^2_-}}(\nabla g(\bar{y})v, \eta), \end{cases} \tag{4.13}$$

and we will verify $v^* = 0$ from the following two cases.

(i) $u_1 > u_2$, $v = (\frac{3}{2}(u_1 - u_2), \frac{3}{2}(u_1 - u_2))$, $\eta = (0, \frac{1}{2}(u_1 - u_2))$. We obtain $\nabla g_1(\bar{y})v < 0, \eta_1 = 0$, $\nabla g_2(\bar{y})v = 0, \eta_2 = \frac{1}{2}(u_1 - u_2)$. It follows from [10] that $(\vartheta_1, \nabla g_1(\bar{y})v^*) \in \{0\} \times R$, $(\vartheta_2, \nabla g_2(\bar{y})v^*) \in R \times \{0\}$, so $\vartheta_1 = 0$, $v_1^* - v_2^* = 0$. Combing (4.13), we obtain that $v^* = 0$.

(ii) $u_1 < u_2 = 0$, $v = 0, \eta = (-3(u_1 - u_2), -4(u_1 - u_2))$, it has $\nabla g(\bar{y})v = 0$. By [10], it holds $(\vartheta_i, -\nabla g_i(\bar{y})v^*) \in R \times \{0\}, i = 1, 2$. Since $\nabla g(\bar{y})$ has full of row rank, it has $v^* = 0$.

(iii) $u_1 = u_2$, $v = \eta = 0$, from [10], it has $(\vartheta_i, -\nabla g_i(\bar{y})v^*) \in (R \times \{0\}) \cup (\{0\} \times R) \cup (R_+ \times R_-), i = 1, 2$. According the discussion in above (i) and (ii), when $(\vartheta_i, -\nabla g_i(\bar{y})v^*) \in (R \times \{0\}) \cup (\{0\} \times R)$, it also has $v^* = 0$. When $(\vartheta_i, -\nabla g_i(\bar{y})v^*) \in R_+ \times R_-, i = 1, 2$, from (4.13), by some calculation, we can also obtain that $v^* = 0$.

In a summary, we get that $v^* = 0$ from every possible directions and the qualification condition (4.2) holds.

Since the objective function φ is differentiable, it has $\partial\langle z^*, \varphi\rangle((\bar{x}, \bar{y}); (u, v)) = \nabla\langle z^*, \varphi\rangle(\bar{x}, \bar{y})$. From (4.3), we obtain that

$$\begin{cases} \nabla_x\langle z^*, \varphi\rangle(\bar{x}, \bar{y}) + \nabla_x F(\bar{x}, \bar{y})^T v^* = 0 \\ \nabla_y\langle z^*, \varphi\rangle(\bar{x}, \bar{y}) + \nabla_y F(\bar{x}, \bar{y})^T v^* + \nabla g(\bar{y})^T \vartheta = 0, \end{cases} \tag{4.14}$$

where $(\vartheta, -\nabla g(\bar{y})v^*) \in N_{\mathrm{gph} N_{R^2_-}}((g(\bar{y}), \lambda); (\nabla g(\bar{y})v, \eta))$ and η satisfies (4.1). Choose the direction $(u_1, u_2) = (-1, 1), v = (0, 0)$, there exist the multipliers $z^* = (1, 1)$ and $v^* =$

$(-\frac{1}{3}, -\frac{1}{3}), \vartheta = (0, \frac{2}{3})$ satisfying the optimality condition (4.14). In fact, from (4.14), we conclude that

$$\begin{cases} 1 + v_1^* + 2v_2^* = 0 \\ 1 + v_1^* + \vartheta_1 - \vartheta_2 = 0 \\ -\frac{1}{3} + v_2^* - 2\vartheta_1 + \vartheta_2 = 0. \end{cases}$$

Since $(u_1, u_2) = (-1, 1), v = (0, 0)$, we have $\eta = (6, 9)$, then $(\nabla g_1(\bar{y})v, \eta_1) = (0, 6)$, $(\nabla g_2(\bar{y})v, \eta_2) = (0, 9)$. Furthermore, it has one has $(\vartheta_1, -\nabla g_1(\bar{y})v^*) = (0, 0) \in (R \times \{0\})$ and $(\vartheta_2, -\nabla g_2(\bar{y})v^*) = (\frac{2}{3}, 0) \in R \times \{0\}$, which means that $(\vartheta, -\nabla g(\bar{y})v^*) \in N_{\mathrm{gph} N_{R^2_-}}(\nabla g(\bar{y})v, \eta)$. Thus for the given direction (u, v), there are nonzero multipliers z^* and v^* satisfying the optimality condition (4.14).

5 Conclusions

In this paper, we consider the directional optimality conditions for multiobjective program constrained by parameterized variational inequalities. When some information of the constraint system is not sufficient, the usual qualification conditions described by limiting coderivatives are not satisfied. The directional limiting coderivative, which uses the original data of the problem to study the local behavior of the set-valued mappings from relevant directions, can be utilized to redescribe these qualification conditions. Under the directional qualification conditions, the directional optimality conditions are obtained.

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