Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 18, No. 4, pp. 841–865, 2022

A SECOND ORDER INEXACT LOG-EXPONENTIAL REGULARIZATION METHOD FOR MATHEMATICAL PROGRAMS WITH COMPLEMENTARITY CONSTRAINTS*

Na Xu, Li-Ping Pang†, Jian Lv and Menglong Xue

Abstract: In recent years, many methods have been presented to overcome the difficulties in solving mathematical programs with complementarity constraints (MPCC). The regularization methods are one prominent class of solution methods. This paper concerns the convergence properties and practical implementation of a log-exponential regularization method. We prove that, under MPCC-LICQ assumption, a limit point of a sequence generated by the log-exponential regularization method is an M-stationary point, if approximate second order stationary conditions are satisfied. A second order primal-dual stabilized SQP method is presented to solve the regularized problems. The proposed method can generate the approximate second order stationary points of the regularized problems. Numerical results, demonstrating the effectiveness of our approach, are also presented.

Key words: *mathematical programs with complementarity constraints, inexact regularization method, second order methods, stabilized SQP method*

Mathematics Subject Classification: *90C30, 90C33, 90C46*

1 Introduction

In this paper, we consider the mathematical programs with complementarity constraints (MPCC)

$$\begin{aligned} \min_{z} \quad & f(z) \\ \text{s.t.} \quad & g(z) \le 0, h(z) = 0, \\ & \min\{G(z), H(z)\} = 0, \end{aligned} \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$, $h : \mathbb{R}^n \to \mathbb{R}^q$, $G : \mathbb{R}^n \to \mathbb{R}^m$ and $H : \mathbb{R}^n \to \mathbb{R}^m$ are twice continuously differentiable functions. MPCC is widely used in many fields such as engineering design, economics, and bilevel optimization [3, 15, 16, 18]. Note that the complementarity constraints in (1.1) can be equivalently rewritten as

$$G_i(z) \ge 0, \; H_i(z) \ge 0, \; G_i(z)H_i(z) = 0, \quad i = 1, 2, \ldots, m. \tag{1.2}$$

*This research is supported by the National Natural Science Foundation of China (No.11801503, No.12171219), Scientific Research Fund of Liaoning Provincial Education Department (No.LQ2019019, No.LJKZ0961), Zhejiang Provincial Natural Science Foundation of China (No.LY20A010025) and the Key Research and Development Projects of Shandong Province (No.2019GGX104089).

†Corresponding author.

Due to the special structure of complementarity constraints, the standard Mangasarian-Fromovitz constraint qualification is violated at any feasible point. As a consequence, some theoretical properties and numerical methods for nonlinear programming (NLP) are invalid to deal with MPCC.

In recent years, there have been proposed a number of approaches for MPCC such as interior point methods, penalty methods, relaxation methods, and regularization methods. See, for instance, [1, 2, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 19, 20, 22]. Among these methods, regularization methods have become very popular to solve MPCC. The basic idea of the regularization method is to replace MPCC by a sequence of the parametrized NLP. Therefore, the stationary point of MPCC can be obtained by solving a sequence of the regularized problems. The first regularization method for MPCC is introduced in Scholtes [20]. Under suitable assumptions, it is shown that any accumulation point of the exact KKT sequence is a C-stationary point. Some of these more recent methods have better convergence properties. The regularization methods by Kadrani et al. [9] and Kanzow & Schwartz [10] converge to M-stationary points as the limit of exact KKT points. Yin & Zhang [22] and Li et al.[14] present a regularization method by using the log-exponential function. They show that the sequence of exact KKT points converges to a C-stationary point. However, the regularized subproblems are not solved exactly from a numerical point of view. In [11], the authors replace exact KKT points by approximate stationary points, which are computationally feasible. It is shown that without any additional assumptions, the regularization methods by Kadrani et al. [9] and Kanzow & Schwartz [10] converge to weakly stationary points only, whereas the regularization method by Scholtes converges to a C-stationary point. In [19], the author presents a sequential optimality condition to improve the convergence results of the regularization methods from [9, 10].

The approximate stationary conditions are realistic from a numerical point of view and coincide with the usual termination criteria for many practical algorithms. However, the weaker convergence results hold when approximate stationary points are considered. Hence, we expect to find an inexact regularization method that may get better convergence results for MPCC. We focus on the log-exponential regularization method. The aim of this paper is to provide an explicit numerical method for MPCC which can deal with the gap between the theoretical results and practical implementation of the log-exponential regularization method. This paper shows that without any additional assumptions, the sequence generated by the log-exponential regularization method converges to an M-stationary point, if the sequence satisfies approximate second order stationary conditions. Motivated by this result, we propose a modification of the second order primal-dual stabilized SQP method in [5], where the regularized subproblems are solved inexactly from a numerical point of view. It is shown that with appropriate choices of the approximate stationary conditions, the algorithm is well defined. If all termination criteria are omitted, the iterates converge to an infeasible stationary point of the norm of the constraint violations or the sequence satisfies the approximate second order stationary conditions of the regularized subproblems. Finally, we provide a numerical implementation of the proposed method based on some test problems from MacMPEC database. Numerical results of the test problems certify the applicability of the approach.

The paper is organized as follows. In section 2, we review some concepts and the log-exponential regularization method for MPCC. In section 3, we show the convergence results by considering approximate second order stationary points. In section 4, we present a feasible strategy to generate approximate second order stationary points of the log-exponential regularized problems under suitable assumptions. The numerical results are presented in section 5. It implies that our method is efficient to solve MPCC from a practical point of view. We conclude the paper in section 6.

We give some notations needed in this paper. We denote that ν_i is the i-th element of vector $\nu^T = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n$. For a given index set $\mathcal{I}$, $[\nu]_\mathcal{I}$ or $\nu_\mathcal{I}$ denotes the subvector of components ν_i such that $i \in \mathcal{I} \cap \{1, \dots, n\}$. Similarly, given a symmetric matrix $M \in \mathbb{R}^{n\times n}$, $[M]_\mathcal{I}$ denotes the symmetric matrix with elements m_{ij} for $i, j \in \mathcal{I}\cap\{1, \dots, n\}$. Given vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$, the vector with i-th component $a_i b_i$ is denoted by $a \cdot b$. $\min\{a, b\}$ is the vector with components $\min\{a_i, b_i\}$. e_i denotes the i-th column of identity matrix I and $e = \sum e_i$. The support of $z \in \mathbb{R}^n$ is defined as $\text{supp}(z) = \{i \mid z_i \neq 0\}$.

2 An Inexact Log-exponential Regularization for MPCC

In this section, we review some concepts for MPCC. Let $\mathcal{F}$ be the feasible region of (1.1) and $z^* \in \mathcal{F}$, we set some index sets as follows:

$$\mathcal{I}_g(z^*) = \{i \mid g_i(z^*) = 0\}, \qquad \mathcal{I}_{+0} = \{i \mid G_i(z^*) > H_i(z^*) = 0\},$$
$$\mathcal{I}_{00} = \{i \mid G_i(z^*) = H_i(z^*) = 0\}, \qquad \mathcal{I}_{0+} = \{i \mid H_i(z^*) > G_i(z^*) = 0\}.$$

It is clear that $\left\{\mathcal{I}_{0+}, \mathcal{I}_{00}, \mathcal{I}_{+0}\right\}$ is a partition of $\{1, 2, \cdots, m\}$.

Definition 2.1 ([15, 21])**.** (1)We say that $z^* \in \mathcal{F}$ is a weakly stationary point of (1.1), if there exist multipliers $(\lambda, \mu, u, v) \in \mathbb{R}^p_+ \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ such that

$$\nabla f(z^*) + \nabla g(z^*)\lambda + \nabla h(z^*)\mu - \nabla G(z^*)u - \nabla H(z^*)v = 0, \tag{2.1}$$
$$\lambda_{\{1,\dots,p\}\setminus\mathcal{I}_g(z^*)} = 0, \quad v_{\mathcal{I}_{0+}} = 0, \quad u_{\mathcal{I}_{+0}} = 0. \tag{2.2}$$

(2)We say that $z^* \in \mathcal{F}$ is Clarke stationary (C-stationary) of (1.1), if there exist multipliers $(\lambda, \mu, u, v) \in \mathbb{R}^p_+ \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ such that (2.1)-(2.2) and $u_i v_i \geq 0, \quad i \in \mathcal{I}_{00}$.
(3)We say that $z^* \in \mathcal{F}$ is Mordukhovich stationary (M-stationary) of (1.1), if there exist multipliers $(\lambda, \mu, u, v) \in \mathbb{R}^p_+ \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ such that (2.1)-(2.2) and $u_i v_i = 0$ or $u_i > 0, \ v_i > 0$ for all $i \in \mathcal{I}_{00}$.
(4)We say that $z^* \in \mathcal{F}$ is strongly stationary (S-stationary) of (1.1), if there exist multipliers $(\lambda, \mu, u, v) \in \mathbb{R}^p_+ \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ such that (2.1)-(2.2) and $u_i \geq 0, \ v_i \geq 0$ for all $i \in \mathcal{I}_{00}$.

The relations among these stationarity concepts can be stated as follows:
S-stationarity ⇒ M-stationarity ⇒ C-stationarity ⇒ weak stationarity.

Definition 2.2 ([6, 21])**.** Let $z^* \in \mathcal{F}$ and $\mathcal{I}_h = \{1, 2, \dots, q\}$.
(1)*MPCC-LICQ* holds at z^* if the gradients

$$\{\nabla g_i(z^*)|i \in I_g(z^*)\} \cup \{\nabla h_i(z^*)|i \in I_h\} \cup \{\nabla G_i(z^*)|i \in \mathcal{I}_{0+} \cup \mathcal{I}_{00}\}$$
$$\cup \{\nabla H_i(z^*)|i \in \mathcal{I}_{+0} \cup \mathcal{I}_{00}\}$$

are linearly independent.
(2) *MPCC-MFCQ* holds at z^* if the gradients

$$\{\nabla g_i(z^*)|i \in I_g(z^*)\} \cup \{\{\nabla h_i(z^*)|i \in I_h\} \cup \{\nabla G_i(z^*)|i \in \mathcal{I}_{0+} \cup \mathcal{I}_{00}\}$$
$$\cup \{\nabla H_i(z^*)|i \in \mathcal{I}_{+0} \cup \mathcal{I}_{00}\}\}$$

are positive-linearly independent.

Definition 2.3 ([21])**.** The upper level strict complementarity (ULSC) holds at z^*, if $z^* \in \mathcal{F}$ is a weakly stationary point and $u_i v_i \neq 0$ for all $i \in \mathcal{I}_{00}$.

Now, we focus on the log-exponential regularized problem $NLP(t)$ [22, 14]:

$$\begin{array}{ll} \min\limits_{z} & f(z) \\ \text{s.t.} & g(z) \leq 0,\ h(z) = 0,\ \Phi(z;t) = 0, \end{array} \tag{2.3}$$

where $t > 0$, $\Phi(z;t) = (\phi_1(z;t), \ldots, \phi_m(z;t))^T$ and

$$\phi_i(z;t) = \begin{cases} -t\ln(\exp(-G_i(z)/t) + \exp(-H_i(z)/t)), & t > 0, \\ \min\{G_i(z), H_i(z)\}, & t = 0 \end{cases} \tag{2.4}$$

for $i = 1, 2, \ldots, m$. The above approximation function is used in [17]. From [17], we have

$$\lim_{t \searrow 0} \phi_i(z;t) = \min\{G_i(z), H_i(z)\}, \quad \forall i = 1, \ldots, m$$

and $\phi_i(z;t)$ is continuously differentiable for all $t > 0$. The gradient and Hessian with respect to z are given by

$$\nabla\phi_i(z;t) = \nu_{i1}(z;t)\nabla G_i(z) + \nu_{i2}(z;t)\nabla H_i(z), \tag{2.5}$$

$$\begin{aligned} \nabla^2\phi_i(z;t) = {} & \nu_{i1}(z;t)\nabla^2 G_i(z) + \nu_{i2}(z;t)\nabla^2 H_i(z) \\ & -\frac{1}{t}\nu_{i1}(z;t)\nu_{i2}(z;t)\left[\nabla G_i(z)\nabla G_i(z)^T + \nabla H_i(z)\nabla H_i(z)^T\right] \\ & +\frac{1}{t}\nu_{i1}(z;t)\nu_{i2}(z;t)\left[\nabla G_i(z)\nabla H_i(z)^T + \nabla H_i(z)\nabla G_i(z)^T\right], \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} \nu_{i1}(z;t) &= \frac{\exp(-G_i(z)/t)}{\exp(-G_i(z)/t) + \exp(-H_i(z)/t)}, \\ \nu_{i2}(z;t) &= \frac{\exp(-H_i(z)/t)}{\exp(-G_i(z)/t) + \exp(-H_i(z)/t)} \end{aligned} \tag{2.7}$$

and

$$\nu_{i1}(z;t),\ \nu_{i2}(z;t) \in (0,1), \quad \nu_{i1}(z;t) + \nu_{i2}(z;t) = 1, \quad \forall i = 1, \ldots, m. \tag{2.8}$$

In particular,

$$\begin{aligned} \lim_{z^k \to z^*, t_k \to 0} \nabla\phi_i(z^k;t_k) &= \nabla G_i(z^*) \text{for } i \in \mathcal{I}_{0+}, \\ \lim_{z^k \to z^*, t_k \to 0} \nabla\phi_i(z^k;t_k) &= \nabla H_i(z^*) \text{for } i \in \mathcal{I}_{+0}. \end{aligned} \tag{2.9}$$

We consider the inexact first and second order conditions for the regularized problem (2.3), because the numerical algorithms rarely terminate in the exact conditions.

Definition 2.4. Let ε be a positive constant and $z \in \mathbb{R}^n$. If there exist multipliers $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^q$, $\delta \in \mathbb{R}^m$ such that

$$\begin{aligned} & \| \nabla f(z) + \sum_{i=1}^{p} \lambda_i \nabla g_i(z) + \sum_{i=1}^{q} \mu_i \nabla h_i(z) - \sum_{i=1}^{m} \delta_i \nabla\phi_i(z;t) \|_\infty \leq \varepsilon, \\ & g_i(z) \leq \varepsilon,\ \lambda_i \geq -\varepsilon,\ \ | \lambda_i g_i(z) | \leq \varepsilon, \quad i = 1, \ldots, p, \\ & \| h(z) \|_\infty \leq \varepsilon,\ \ | \phi_i(z;t) | \leq \varepsilon, \quad i = 1, \ldots, m, \end{aligned} \tag{2.10}$$

we say that z is an ε-KKT point of the regularized problem $NLP(t)$.

From [11], the authors prove the convergence results of the different regularization methods when approximate KKT points are considered. It is shown that without any additional assumptions, these regularization methods mentioned in [11] converge to weakly or C-stationary points only. In order to improve the convergence properties for MPCC, we have that z^k is an ε_k-KKT point of $NLP(t_k)$ and z^* is a limit point of $\{z^k\}$. z^k also satisfies the following condition:

$$d^T \nabla_z^2 L(z^k, \lambda^k, \mu^k, \delta^k) d \geq -\varepsilon_k \|d\|^2 \tag{2.11}$$

holds for every direction $d \in \bar{T}(z^k; t_k)$, where $L(z, \lambda, \mu, \delta) = f(z) + \sum_{i=1}^p \lambda_i g_i(z) + \sum_{i=1}^q \mu_i h_i(z) - \sum_{i=1}^m \delta_i \phi_i(z; t)$ and

$$\bar{T}(z^k; t_k) = \left\{ d \in \mathbb{R}^n | \begin{array}{ll} \nabla g_i(z^k)^T d = 0 \;\; i \in \mathcal{I}_g(z^*), & \nabla h_i(z^k)^T d = 0 \;\; i \in \mathcal{I}_h, \\ \nabla \phi_i(z^k; t_k)^T d = 0 \quad i = 1, \ldots, m. & \end{array} \right\}.$$

We say that z^k is an approximate second order stationary point of $NLP(t_k)$, if z^k is defined as an ε_k-KKT point satisfying (2.11).

3 Convergence Results

We focus on a sequence of the approximate second order stationary points generated by the log-exponential regularization method. We can obtain some convergence properties of (1.1) under the certain MPCC CQs. The proof is long and borrows some ideas from the proofs of Yin & Zhang [22] and Kanzow & Schwartz [11]. It becomes somewhat technical with some necessary modifications.

Theorem 3.1. *Let $\{t_k\} \searrow 0$, $\{\varepsilon_k\} \searrow 0$ and $\{z^k\}$ be a sequence of the approximate second order stationary points of $NLP(t_k)$. If $z^k \to z^*$ and MPCC-LICQ holds at z^*, then z^* is M-stationary. If in addition ULSC holds at z^*, then z^* is S-stationary.*

Proof. Let $(\lambda^k, \mu^k, \delta^k)$ be multipliers associated with $\{z^k\}$ satisfying the ε_k-KKT conditions (2.10) and condition (2.11) of $NLP(t_k)$. It is obvious that z^* is a feasible point of (1.1). Using (2.5) and (2.8), the first row of (2.10) can be written as follow:

$$\begin{aligned} \| \nabla f(z^k) &+ \sum_{i=1}^p \lambda_i^k \nabla g_i(z^k) + \sum_{i=1}^q \mu_i^k \nabla h_i(z^k) - \sum_{i=1}^m \delta_i^{1,k} \nabla G_i(z^k) - \sum_{i=1}^m \delta_i^{2,k} \nabla H_i(z^k) \\ &- \sum_{i \in \mathcal{I}_{+0}} \delta_i^k \nu_{i1}(z^k; t_k) \nabla G_i(z^k) - \sum_{i \in \mathcal{I}_{0+}} \delta_i^k \nu_{i2}(z^k; t_k) \nabla H_i(z^k) \|_\infty \leq \varepsilon_k, \end{aligned} \tag{3.1}$$

where

$$\delta_i^{1,k} = \begin{cases} \delta_i^k \nu_{i1}(z^k; t_k), & i \in \mathcal{I}_{0+} \cup \mathcal{I}_{00}, \\ 0, & \text{otherwise}, \end{cases} \qquad \delta_i^{2,k} = \begin{cases} \delta_i^k \nu_{i2}(z^k; t_k), & i \in \mathcal{I}_{+0} \cup \mathcal{I}_{00}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_{i1}(z^k; t_k),\ \nu_{i2}(z^k; t_k) \in (0, 1), \quad \nu_{i1}(z^k; t_k) + \nu_{i2}(z^k; t_k) = 1, \quad \forall i = 1, \ldots, m. \tag{3.2}$$

Then we claim that the multipliers $(\lambda^k,\ \mu^k,\ \delta^{1,k},\ \delta^{2,k},\ \delta^k_{\mathcal{I}_{0+} \cup \mathcal{I}_{+0}})$ are bounded. We derive a contradiction by assuming that they are unbounded. We can assume that there exists a subsequence $\mathcal{K}$ satisfying

$$\frac{(\lambda^k,\ \mu^k,\ \delta^{1,k},\ \delta^{2,k},\ \delta^k_{\mathcal{I}_{0+}\cup\mathcal{I}_{+0}})}{\|(\lambda^k,\ \mu^k,\ \delta^{1,k},\ \delta^{2,k},\ \delta^k_{\mathcal{I}_{0+}\cup\mathcal{I}_{+0}})\|} \xrightarrow{\mathcal{K}} (\tilde{\lambda},\ \tilde{\mu},\ \tilde{\delta^1},\ \tilde{\delta^2},\ \tilde{\delta}_{\mathcal{I}_{0+}\cup\mathcal{I}_{+0}}) \neq 0.$$

From (2.9) and (3.2), we obtain

$$\lim_{k\to+\infty}\nu_{i1}(z^k;t_k) = \begin{cases} 1, & i\in\mathcal{I}_{0+},\\ \nu^*_{i1}, & i\in\mathcal{I}_{00},\\ 0, & i\in\mathcal{I}_{+0},\end{cases} \qquad \lim_{k\to+\infty}\nu_{i2}(z^k;t_k) = \begin{cases} 1, & i\in\mathcal{I}_{+0},\\ \nu^*_{i2}, & i\in\mathcal{I}_{00},\\ 0, & i\in\mathcal{I}_{0+}\end{cases} \tag{3.3}$$

and $\nu^*_{i1}+\nu^*_{i2}=1$ for all $i\in\mathcal{I}_{00}$.

Then (3.1) divided by $\|(\lambda^k,\ \mu^k,\ \delta^{1,k},\ \delta^{2,k},\ \delta^k_{\mathcal{I}_{0+}\cup\mathcal{I}_{+0}})\|$. We obtain

$$\sum_{i=1}^{p}\tilde{\lambda}_i\nabla g_i(z^*)+\sum_{i=1}^{q}\tilde{\mu}_i\nabla h_i(z^*)-\sum_{i=1}^{m}\tilde{\delta}^1_i\nabla G_i(z^*)-\sum_{i=1}^{m}\tilde{\delta}^2_i\nabla H_i(z^*)=0 \ \text{ for } \ k\to+\infty, \tag{3.4}$$

where $\tilde{\lambda}\geq 0$. If $\tilde{\lambda}_i>0$, then $\lambda^k_i>c$ for some constant $c>0$ and k sufficiently large. We have $0\leq\mid g_i(z^k)\mid\leq\frac{\varepsilon_k}{|\lambda^k_i|}\leq\frac{\varepsilon_k}{c}\to 0$, which implies that $\text{supp}(\tilde{\lambda})\subseteq\mathcal{I}_g(z^*)$. Moreover, from the definition of $\delta^{1,k}$ and $\delta^{2,k}$, it is clear that $\text{supp}(\tilde{\delta^1})\subseteq\mathcal{I}_{0+}\cup\mathcal{I}_{00}$ and $\text{supp}(\tilde{\delta^2})\subseteq\mathcal{I}_{+0}\cup\mathcal{I}_{00}$. If $(\tilde{\lambda},\ \tilde{\mu},\ \tilde{\delta^1},\ \tilde{\delta^2})\neq 0$, (3.4) is a contradiction to MPCC-LICQ. If $(\tilde{\lambda},\ \tilde{\mu},\ \tilde{\delta^1},\ \tilde{\delta^2})=0$, there exists $i_0\in\mathcal{I}_{0+}\cup\mathcal{I}_{+0}$ such that $\tilde{\delta}_{i_0}\neq 0$. We consider the case $i_0\in\mathcal{I}_{0+}$ and get $\tilde{\delta}^1_{i_0}=\lim_{k\to+\infty}\frac{\delta^k_{i_0}\nu_{i_0 1}(z^k;t_k)}{\|(\lambda^k,\ \mu^k,\ \delta^{1,k},\ \delta^{2,k},\ \delta^k_{\mathcal{I}_{0+}\cup\mathcal{I}_{+0}})\|}=\tilde{\delta}_{i_0}\neq 0$. This contradicts the assumption $\tilde{\delta}^1_{i_0}=0$. The other case can be proved in the same way. Thus, the sequence $(\lambda^k,\ \mu^k,\ \delta^{1,k},\ \delta^{2,k},\ \delta^k_{\mathcal{I}_{0+}\cup\mathcal{I}_{+0}})$ is bounded and converges to $(\lambda^*,\ \mu^*,\ \delta^{1,*},\ \delta^{2,*},\ \delta^*_{\mathcal{I}_{0+}\cup\mathcal{I}_{+0}})$. Using (3.1) and

$$\delta^{1,*}_i\delta^{2,*}_i=\lim_{k\to+\infty}(\delta^k_i)^2\nu_{i1}(z^k;t_k)\nu_{i2}(z^k;t_k)\geq 0 \ \text{ for all } i\in\mathcal{I}_{00},$$

we conclude that z^* is a C-stationary point.

Now, we show by contradiction that z^* is M-stationary. We assume that there exists $i'\in\mathcal{I}_{00}$ such that $\delta^{1,*}_{i'}<0$ and $\delta^{2,*}_{i'}<0$. Thus, we have

$$\delta^{1,*}_{i'}=\lim_{k\to+\infty}\delta^k_{i'}\nu_{i'1}(z^k;t_k)=\delta^*_{i'}\nu^*_{i'1}<0 \ \text{ and } \ \delta^{2,*}_{i'}=\lim_{k\to+\infty}\delta^k_{i'}\nu_{i'2}(z^k;t_k)=\delta^*_{i'}\nu^*_{i'2}<0. \tag{3.5}$$

Consequently, $\nu^*_{i'1}\in(0,1)$, $\nu^*_{i'2}\in(0,1)$ and $\nu^*_{i'1}+\nu^*_{i'2}=1$. We assume that $\nu^*_{i'1}=\alpha\in(0,1)$. Since MPCC-LICQ holds at z^*, we may choose a bounded sequence d^k such that, for k sufficiently large,

$$\begin{array}{ll} \nabla g_i(z^k)^T d^k=0 & i\in\mathcal{I}_g(z^*),\\ \nabla h_i(z^k)^T d^k=0 & i=1,\ldots,q,\\ \nabla\phi(z^k;t_k)^T d^k=0 & i\in\mathcal{I}_{0+}\cup\mathcal{I}_{+0},\\ \nabla G_i(z^k)^T d^k=0,\ \ \nabla H_i(z^k)^T d^k=0 & i\in\mathcal{I}_{00}\backslash\{i'\},\\ \nabla G_{i'}(z^k)^T d^k=\nu_{i'2}(z^k;t_k),\ \ \nabla H_{i'}(z^k)^T d^k=-\nu_{i'1}(z^k;t_k). & \end{array} \tag{3.6}$$

Hence, we have $d^k\in\bar{T}(z^k;t_k)$ and

$$\begin{aligned} {d^k}^T\nabla^2_z L(z^k,\lambda^k,\mu^k,\delta^k)d^k &= {d^k}^T\Big[\nabla^2 f(z^k)+\sum_{i=1}^{p}\lambda^k_i\nabla^2 g_i(z^k)\\ &\quad+\sum_{i=1}^{q}\mu^k_i\nabla^2 h_i(z^k)-\sum_{i=1}^{m}\delta^k_i\nabla^2\phi_i(z^k;t_k)\Big]d^k, \end{aligned}$$

$$\begin{aligned}
&{d^k}^T \nabla^2 \phi_i(z^k;t_k) d^k \\
= \quad & \nu_{i1}(z^k;t_k){d^k}^T \nabla^2 G_i(z^k) d^k + \nu_{i2}(z^k;t_k){d^k}^T \nabla^2 H_i(z^k) d^k \\
& -\frac{1}{t_k}\nu_{i1}(z^k;t_k)\nu_{i2}(z^k;t_k){d^k}^T \Big[\nabla G_i(z^k)\nabla G_i(z^k)^T + \nabla H_i(z^k)\nabla H_i(z^k)^T\Big] d^k \\
& +\frac{1}{t_k}\nu_{i1}(z^k;t_k)\nu_{i2}(z^k;t_k){d^k}^T \Big[\nabla G_i(z^k)\nabla H_i(z^k)^T + \nabla H_i(z^k)\nabla G_i(z^k)^T\Big] d^k.
\end{aligned} \tag{3.7}$$

From (3.6), we obtain, for $i \in \mathcal{I}_{0+}$,

$$\begin{aligned}
{d^k}^T \nabla^2 \phi_i(z^k;t_k) d^k \quad = \quad & \nu_{i1}(z^k;t_k){d^k}^T \nabla^2 G_i(z^k) d^k + \nu_{i2}(z^k;t_k){d^k}^T \nabla^2 H_i(z^k) d^k \\
& -\frac{1}{t_k}\nu_{i1}(z^k;t_k)\nu_{i2}(z^k;t_k)\Big(\frac{\nu_{i2}(z^k;t_k)}{\nu_{i1}(z^k;t_k)}+1\Big)^2({d^k}^T\nabla H_i(z^k))^2
\end{aligned} \tag{3.8}$$

and $-\frac{1}{t_k}\nu_{i1}(z^k;t_k)\nu_{i2}(z^k;t_k)\Big(\frac{\nu_{i2}(z^k;t_k)}{\nu_{i1}(z^k;t_k)}+1\Big)^2({d^k}^T\nabla H_i(z^k))^2 \to 0 \quad \text{as } k\to+\infty$.

For $i \in \mathcal{I}_{+0}$, we know that

$$\begin{aligned}
{d^k}^T \nabla^2 \phi_i(z^k;t_k) d^k \quad = \quad & \nu_{i1}(z^k;t_k){d^k}^T \nabla^2 G_i(z^k) d^k + \nu_{i2}(z^k;t_k){d^k}^T \nabla^2 H_i(z^k) d^k \\
& -\frac{1}{t_k}\nu_{i1}(z^k;t_k)\nu_{i2}(z^k;t_k)\Big(\frac{\nu_{i1}(z^k;t_k)}{\nu_{i2}(z^k;t_k)}+1\Big)^2({d^k}^T\nabla G_i(z^k))^2
\end{aligned} \tag{3.9}$$

and $-\frac{1}{t_k}\nu_{i1}(z^k;t_k)\nu_{i2}(z^k;t_k)\Big(\frac{\nu_{i1}(z^k;t_k)}{\nu_{i2}(z^k;t_k)}+1\Big)^2({d^k}^T\nabla G_i(z^k))^2 \to 0$ as $k\to+\infty$. Clearly, for $i \in \mathcal{I}_{0+}\cup\mathcal{I}_{+0}$, ${d^k}^T\nabla^2\phi_i(z^k;t_k)d^k$ is bounded.

For $i \in \mathcal{I}_{00}\backslash\{i'\}$,

$${d^k}^T \nabla^2 \phi_i(z^k;t_k) d^k = \nu_{i1}(z^k;t_k){d^k}^T \nabla^2 G_i(z^k) d^k + \nu_{i2}(z^k;t_k){d^k}^T \nabla^2 H_i(z^k) d^k. \tag{3.10}$$

From (3.6), we obtain, for $i = i'$,

$$\begin{aligned}
{d^k}^T \nabla^2 \phi_{i'}(z^k;t_k) d^k \quad = \quad & \nu_{i'1}(z^k;t_k){d^k}^T \nabla^2 G_{i'}(z^k) d^k + \nu_{i'2}(z^k;t_k){d^k}^T \nabla^2 H_{i'}(z^k) d^k \\
& -\frac{1}{t_k}\nu_{i'1}(z^k;t_k)\nu_{i'2}(z^k;t_k).
\end{aligned}$$

Using the boundedness of d^k, $\nu_{i1}(z^k;t_k)$, $\nu_{i2}(z^k;t_k)$, $(\lambda^k,\ \mu^k,\ \delta^{1,k},\ \delta^{2,k},\ \delta^k_{\mathcal{I}_{0+}\cup\mathcal{I}_{+0}})$,

$$\begin{aligned}
&{d^k}^T \nabla_z^2 L(z^k,\lambda^k,\mu^k,\delta^k) d^k = \\
&\qquad {d^k}^T[\nabla^2 f(z^k) + \sum_{i=1}^p \lambda_i^k \nabla^2 g_i(z^k) + \sum_{i=1}^q \mu_i^k \nabla^2 h_i(z^k) - \sum_{i\in\mathcal{I}_{0+}\cup\mathcal{I}_{+0}} \delta_i^k \nabla^2\phi(z^k;t_k)]d^k, \\
&\qquad -\sum_{i\in\mathcal{I}_{00}} {d^k}^T[\delta_i^{1,k}\nabla^2 G_i(z^k) + \delta_i^{2,k}\nabla^2 H_i(z^k)]d^k + \frac{1}{t_k}\delta_{i'}^k \nu_{i'1}(z^k;t_k)\nu_{i'2}(z^k;t_k),
\end{aligned}$$

we have that all terms in the above equation are bounded except the last one. From (3.5), we know that

$$\begin{aligned}
\lim_{k\to+\infty}\frac{1}{t_k}\delta_{i'}^k\nu_{i'1}(z^k;t_k)\nu_{i'2}(z^k;t_k) \quad = \quad & \lim_{k\to+\infty}\frac{1}{t_k}\delta_{i'}^k\nu_{i'1}(z^k;t_k)(1-\nu_{i'1}(z^k;t_k)) \\
& = (1-\alpha)\delta_{i'}^{1,*}\lim_{k\to+\infty}\frac{1}{t_k} = -\infty.
\end{aligned} \tag{3.11}$$

Hence, ${d^k}^T \nabla_z^2 L(z^k, \lambda^k, \mu^k, \delta^k) d^k \to -\infty$, which is a contradiction to the condition

$$ {d^k}^T \nabla_z^2 L(z^k, \lambda^k, \mu^k, \delta^k) d^k \geq -\varepsilon_k \|d^k\|^2. $$

Therefore, z^* is M-stationary of (1.1). Clearly, if in addition ULSC holds at z^*, then z^* is an S-stationary point. $\square$

The following example shows that we cannot obtain the conclusion of Theorem 3.1 under MPCC-MFCQ instead of MPCC-LICQ.

Example 3.2. Consider the three-dimensional MPCC

$$ \min_z -z_1 - z_3 \quad \text{s.t.} \; -z_2 \leq 0, \quad z_1^2 - z_2 \leq 0, \quad \min\{z_1, z_3\} = 0. $$

We know that MPCC-MFCQ holds at $z^* = (0,0,0)$ and MPCC-LICQ does not hold at z^*. Now we consider the log-exponential regularized problem

$$ \min_z -z_1 - z_3 \quad \text{s.t.} \; -z_2 \leq 0, \quad z_1^2 - z_2 \leq 0, \quad -t \ln(\exp(-z_1/t) + \exp(-z_3/t)) = 0 $$

with parameter $t > 0$. Let $t_k = k^{-\frac{1}{2}}$, $\varepsilon_k = 2k^{-\frac{1}{2}}$ and $z^k = (k^{-1}, k^{-2}, k^{-1}, k^{-1})$. It is clear that z^k togethers with the multipliers $(\lambda_1^k,\ \lambda_2^k,\ \delta^k) = (0, 0, -2)$ satisfying ε_k-KKT conditions and

$$ d^T \nabla_z^2 L(z^k, \lambda^k, \mu^k, \delta^k) d \geq -\varepsilon_k \|d\|^2 \quad \text{for all} \quad d \in \bar{T}(z^k; t_k). $$

Since, for all $k = 1, 2, \ldots$,

$$ \|\nabla L_z(z^k, \lambda^k, \delta^k)\|_\infty = \left\| \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} + \lambda_1^k \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \lambda_2^k \begin{pmatrix} 2z_1^k \\ -1 \\ 0 \end{pmatrix} - \delta^k \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \right\|_\infty \leq \varepsilon_k, $$

$$ \begin{array}{l} z_2^k \geq -\varepsilon_k, \quad \lambda_1^k \geq -\varepsilon_k, \quad | \lambda_1^k z_2^k | \leq \varepsilon_k, \quad (z_1^k)^2 - z_2^k \leq \varepsilon_k, \\ \lambda_2^k \geq -\varepsilon_k, \quad | \lambda_2^k((z_1^k)^2 - z_2^k) | \leq \varepsilon_k, \end{array} $$

$$ \begin{array}{rcl} | \phi(z^k; t_k) | & = & | -t_k \ln(\exp(-z_1^k/t_k) + \exp(-z_3^k/t_k)) | \\ & = & | -k^{-\frac{1}{2}} \ln(2 \exp(-k^{-\frac{1}{2}}) | \leq | -2k^{-\frac{1}{2}} | = \varepsilon_k \end{array} $$

and $\bar{T}(z^k; t_k) = \left\{ d \in \mathbb{R}^3 | \begin{array}{ll} (2k^{-1}, -1, 0)^T d = 0, & (0, -1, 0)^T d = 0, \\ (-\frac{1}{2}, 0, -\frac{1}{2})^T d = 0 & \end{array} \right\} = (0, 0, 0)^T.$

From the equation

$$ \nabla L(z^*, \lambda^*,\ u^*,\ v^*) = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} + \lambda_1^* \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \lambda_2^* \begin{pmatrix} 2z_1^* \\ -1 \\ 0 \end{pmatrix} - u^* \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - v^* \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0, $$

it means that $u^* = -1,\ v^* = -1$ and $\lambda_1^* = -\lambda_2^*$. There exist multipliers $u^* = -1,\ v^* = -1,\ \lambda_1^* = -\lambda_2^* = 0$ such that the conditions of C-stationarity. However, z^* is not an M-stationary point.

Furthermore, we focus on finding a sequence of the approximate second order stationary points. The existence of ε_k-KKT point of $NLP(t_k)$ satisfying (2.11) is discussed in the next section. We design an algorithm to generate $\{z^k\}$.

4 A Second Order Stabilized SQP Method

Our primary interest in this paper is to compute a solution of MPCC by solving a sequence of the regularized problems. Since the usual stopping criteria may fail to satisfy the KKT conditions, we use the approximate stationary conditions to define the stopping criteria for practical algorithm. In order to obtain M-stationarity, we consider the approximate second order stationary points of $NLP(t)$. First, we consider the basic strategy for MPCC as follows.

General Scheme of a Practical Algorithm for MPCC
Choose $t_k \searrow 0, \varepsilon_k \searrow 0$ and a stopping tolerance $\varepsilon' > 0$, Let z^0 be a feasible point of $NLP(t_0)$ and set $k = 0$.
 While $\varepsilon_k \geq \varepsilon'$ **do**
  Solve the regularization $NLP(t_k)$ and use z^k as starting vector.
  Find an approximate second order stationary point z^{k+1} of $NLP(t_k)$.
  Set $k \leftarrow k+1$.
 end while
Return the final iterate $z_{opt} = z^k$ and function value at z_{opt}.

In this section, we present a detail procedure to compute a sequence satisfying approximate first and second order conditions of $NLP(t)$. We recall the regularized problem (2.3), by adding the slack variables, it is equivalently written in the form

$$\min_{z,\hat{z}} f(z) \quad \text{s.t. } c(z,\hat{z};t) = 0, \quad \hat{z}_i \geq 0, \quad i = 1,2,\ldots,p, \tag{4.1}$$

where $c : \mathbb{R}^{n+p} \to \mathbb{R}^{q+m+p}$ and

$$c(z,\hat{z};t) = \begin{pmatrix} h(z) \\ \Phi(z;t) \\ g(z) + \hat{z} \end{pmatrix}. \tag{4.2}$$

We set $x = (z,\hat{z})^T$, $\widetilde{f}(x) = f(z)$ and rewrite (4.1) as follows

$$\min_{x} \widetilde{f}(x) \quad \text{s.t. } c(x;t) = 0, \quad x_i \geq 0, \quad i = n+1,\ldots,n+p. \tag{4.3}$$

We focus on the following approximate KKT conditions of (4.3), if there exists multiplier $y \in \mathbb{R}^{q+m+p}$ such that

$$r(x,y;t) = \left\| \begin{pmatrix} c(x;t) \\ [\nabla\widetilde{f}(x) - \nabla c(x;t)y]_{\mathcal{E}}, \\ [x]_{\mathcal{I}} \cdot [\nabla\widetilde{f}(x) - \nabla c(x;t)y]_{\mathcal{I}} \\ \min\{[x]_{\mathcal{I}}, [\nabla\widetilde{f}(x) - \nabla c(x;t)y]_{\mathcal{I}}\} \end{pmatrix} \right\| \leq \varepsilon, \tag{4.4}$$

where $\varepsilon > 0$, $\mathcal{E} = \{1,\ldots,n\}$ and $\mathcal{I} = \{n+1,\ldots,n+p\}$. The Hessian of the Lagrangian with respect to x is denoted by $H(x,y;t) = \nabla^2\widetilde{f}(x) - \sum_{i=1}^{q+m+p} y_i \nabla^2 c_i(x;t)$. We set

$$\mathcal{A}(x) = \{i \in \mathcal{I} : [x]_i = 0\}, \quad \mathcal{F}(x) = \{1,\ldots,n+p+q+m+p\} \setminus \mathcal{A}(x), \tag{4.5}$$

$$\mathcal{A}_+(x,y) = \{i \in \mathcal{A}(x) \mid [\nabla\widetilde{f}(x) - \nabla c(x;t)y]_i > 0\}, \tag{4.6}$$

$$\mathcal{A}_0(x,y) = \{i \in \mathcal{A}(x) \mid [\nabla\widetilde{f}(x) - \nabla c(x;t)y]_i = 0\}. \tag{4.7}$$

The idea of our method is to solve problem (4.3) inexactly. Hopefully, as t approaches to zero, the iterate approaches to a solution of MPCC. The proposed method is based on a stabilized sequential quadratic programming (SQP) method from Gill & Kungurtsev [5]. The problem format presented in [5] assumes that all variables are sign constrained, i.e., $x \geq 0$. However, the problem (4.3) is different from the problem considered in [5], since we only need the slack variables $x_{\mathcal{I}} \geq 0$. The proposed method can be adapted to the case where not all variables are sign constrained. Hence, we change the SQP method in [5] to solve (4.3) and consider the iteration of variable t. Since our approach is different from [5], the SQP merit function of (4.3) is given by

$$M(x, y; y^E, u, t) = \widetilde{f}(x) - c(x; t)^T y^E + \frac{1}{2u}\|c(x; t)\|^2 + \frac{1}{2u}\|c(x; t) + u(y - y^E)\|^2, \tag{4.8}$$

where y^E is a Lagrange multiplier estimate, u is a positive penalty parameter and t is a positive smoothing parameter. Note that $M(x, y; y^E, u, t)$ is at least twice continuously differentiable function with respect to x, y, and for any given y^E, u, t,

$$\nabla M(x, y; y^E, u, t) = \begin{pmatrix} \nabla \widetilde{f}(x) - \nabla c(x; t)(\pi(x; y^E, u, t) + (\pi(x; y^E, u, t) - y)) \\ u(y - \pi(x; y^E, u, t)) \end{pmatrix}, \tag{4.9}$$

$$\nabla^2 M(x, y; y^E, u, t) =$$

$$\begin{pmatrix} H(x, \pi(x; y^E, u, t) + (\pi(x; y^E, u, t) - y); t) + \frac{2}{u}\nabla c(x; t)\nabla c(x; t)^T & \nabla c(x; t) \\ \nabla c(x; t)^T & u\mathrm{I} \end{pmatrix}, \tag{4.10}$$

where $\pi(x; y^E, u, t) = y^E - c(x; t)/u$.

From [5, Theorem 1.3], the motivation of $M(x, y; y^E, u, t)$ as an SQP merit function is given by Gill et al. The result can be applied to problem (4.3) such as Theorem 4.1.

Theorem 4.1. *For each $t > 0$, if (x^t, y^t) is a solution of problem (4.3) that satisfies the second order sufficient optimality conditions, i.e., $r(x^t, y^t; t) = 0$,*

$$p^T H(x^t, y^t; t)p > 0 \quad \text{for all} \quad p \in \mathcal{C}(x^t, y^t; t)\backslash\{0\}, \tag{4.11}$$

where $\mathcal{C}(x^t, y^t; t) = \{p \mid \nabla c(x^t; t)^T p = 0,\ p_i = 0 \text{ for } i \in \mathcal{A}_+(x^t, y^t),\ p_i \geq 0 \text{ for } i \in \mathcal{A}_0(x^t, y^t)\}$, then for the choice $y^E = y^t$, there exists a positive $\bar{u}$ such that for all $0 < u < \bar{u}$, the point (x^t, y^t) satisfies the second order sufficient optimality conditions for the problem

$$\min_{x,y} \quad M(x, y; y^E, u, t) \quad \text{s.t. } x_i \geq 0,\ i = n+1, \ldots, n+p. \tag{4.12}$$

Let $v_k = (x_k, y_k)$ be the kth estimate of a primal-dual solution of (4.3) with $t > 0$ and u_k^R be a second penalty parameter such that $0 < u_k^R \leq u_k$. The quadratic model of the change in $M(x, y; y_k^E, u_k^R, t)$ is defined as

$$\mathcal{Q}_k(v; y_k^E, u_k^R, t) = \nabla M(v_k; y_k^E, u_k^R, t)^T(v - v_k) + \frac{1}{2}(v - v_k)^T \mathrm{B}(v_k; u_k^R, t)(v - v_k), \tag{4.13}$$

where $v = (x, y)$, y_k^E is an estimate of y_k, and

$$\mathrm{B}(v; u, t) = \begin{pmatrix} H(x, y; t) + \frac{2}{u}\nabla c(x; t)\nabla c(x; t)^T & \nabla c(x; t) \\ \nabla c(x; t)^T & u\mathrm{I} \end{pmatrix}. \tag{4.14}$$

Notice that $\mathrm{B}(v_k; u_k^R, t)$ is independent of y_k^E. According to the definition of $\mathcal{Q}_k(v; y_k^E, u_k^R, t)$, we consider the subproblem

$$\min_{v} \quad \mathcal{Q}_k(v; y_k^E, u_k^R, t) \quad \text{s.t. } v_i \geq 0, \; i = n+1, \ldots, n+p. \tag{4.15}$$

By introducing $\mathrm{B}(v; u_k^R, t)$, the subproblem (4.15) can be formally equivalent to the QP subproblem

$$\begin{aligned} \min_{x,y} \quad & \nabla \widetilde{f}(x_k)^T (x - x_k) + \frac{1}{2}(x - x_k)^T H(x_k, y_k; t)(x - x_k) + \frac{1}{2} u_k^R \|y\|^2 \\ \text{s.t.} \quad & c(x_k; t) + \nabla c(x_k; t)^T (x - x_k) + u_k^R (y - y_k^E) = 0, \\ & x_i \geq 0, \quad i = n+1, \ldots, n+p. \end{aligned} \tag{4.16}$$

At the start of the kth iteration, u_{k-1}^R, u_{k-1} and (x^k, y^k) are given. In order to compute y_k^E and u_k^R, it is necessary to define the ϵ-active set and its complement set, i.e.,

$$\mathcal{A}_\epsilon(x, y; u, t) = \{i \in \mathcal{I} : x_i \leq \epsilon, \text{ with } \epsilon \equiv \min\{\epsilon_1, \; \max\{u, r(x, y; t)^\gamma\}\}, \tag{4.17}$$

$$\mathcal{F}_\epsilon(x, y; u, t) = \{1, \ldots, n+p+q+m+p\} \setminus \mathcal{A}_\epsilon(x, y, u), \tag{4.18}$$

where $0 < \gamma < 1$, $0 < \epsilon_1 < 1$. Furthermore, the required non-negative scalar ξ_k is computed as part of the vector $(\xi_k, s_k^{(1)})$ which is given by [5, Algorithm 1]. From [5], we have that $s_k^{(1)} = (\mu_k^{(1)}, w_k^{(1)})$ and $s_k^{(1)T} \mathrm{B}(v_k; u_{k-1}^R, t) s_k^{(1)} = -\xi_k \|\mu_k^{(1)}\|^2$. The feasibility and optimality measures are given by

$$\eta(x_k; t) = \|c(x_k; t)\| = \left\| \begin{pmatrix} h(z) \\ \Phi(z; t) \\ g(z) + \hat{z} \end{pmatrix} \right\| \tag{4.19}$$

and

$$\omega(x_k, y_k, \xi_k; t) = \max \left\{ \left\| \begin{pmatrix} [\nabla \widetilde{f}(x_k) - \nabla c(x_k; t) y_k]_{\mathcal{E}}, \\ [x_k]_{\mathcal{I}} \cdot [\nabla \widetilde{f}(x_k) - \nabla c(x_k; t) y_k]_{\mathcal{I}} \\ \min\{[x_k]_{\mathcal{I}}, [\nabla \widetilde{f}(x_k) - \nabla c(x_k; t) y_k]_{\mathcal{I}}\} \end{pmatrix} \right\|, \; \xi_k \right\}. \tag{4.20}$$

Define

$$\psi_{\mathrm{v}}(x_k, y_k, \xi_k; t) = \eta(x_k; t) + \beta \omega(x_k, y_k, \xi_k; t), \quad \psi_{\mathrm{o}}(x_k, y_k, \xi_k; t) = \beta \eta(x_k; t) + \omega(x_k, y_k, \xi_k; t),$$

where β is fixed and $0 < \beta \ll 1$. These functions $\psi_{\mathrm{v}}(v_k, \xi_k; t)$ and $\psi_{\mathrm{o}}(v_k, \xi_k; t)$ are used to define the "V-iterate" and "O-iterate" in Algorithm 1. The new parameter is defined by

$$u_k^R = \begin{cases} \min\{u_{k-1}^R, \max\{r(x_k, y_k; t), \xi_k\}^\gamma\}, & \text{if } \max\{r(x_k, y_k; t), \xi_k\} > 0, \\ \frac{1}{2} u_{k-1}^R, & \text{otherwise,} \end{cases} \tag{4.21}$$

which is used for the local quadratic model. If v_k is not satisfied the conditions of "V-O-iterate", then v_k is checked to whether satisfy some conditions of the problem

$$\min_{x,y} \quad M(x, y; y_{k-1}^E, u_{k-1}^R, t) \quad \text{s.t. } x_i \geq 0, \; i = n+1, \ldots, n+p \tag{4.22}$$

with $t > 0$. These conditions are called the "M-iterate" in Step 3 of Algorithm 1. Otherwise, the y_k^E and u_k^R are fixed at the current values and (x_k, y_k) is called an "F-iterate".

Note that the problem (4.15) is not suitable to obtain a search direction because $\mathrm{B}(v_k; u_k^R, t)$ is not a positive definite matrix in general. In addition, the computation of a second order solution of a nonconvex QP is intractable in certain instances. Hence, it is necessary to consider the definition of a more appropriate quadratic model and line-search direction. The line-search direction is the sum of the descent direction d_k and the direction of negative curvature. The vector d_k is either the local descent direction or global descent direction.

An optimal point $\widehat{v}_k = d_k + v_k$ of the subproblem (4.15) satisfies the optimality conditions

$$\begin{gathered}[\nabla \mathcal{Q}_k(d_k + v_k; y_k^E, u_k^R, t)]_{\mathcal{F}(d_k+v_k)} = 0, \quad [\nabla \mathcal{Q}_k(d_k + v_k; y_k^E, u_k^R, t)]_{\mathcal{A}(d_k+v_k)} \geq 0 \ \text{ and} \\ [d_k + v_k]_i \geq 0, \quad i = n+1, \ldots, n+p, \end{gathered} \tag{4.23}$$

where the free and active set are evaluated at $d_k + v_k$. The local descent direction is computed by the QP subproblem [5]

$$\min_v \quad \mathcal{Q}_k(v; y_k^E, u_k^R, t) \quad \text{s.t.} \ [v]_{\mathcal{A}_\epsilon} = 0, \tag{4.24}$$

which is given by relaxing the optimality conditions (4.23).

The global descent step is computed by solving the convex QP subproblem [5]

$$\min_v \quad \widehat{\mathcal{Q}}_k(v; y_k^E, u_k^R, t) \quad \text{s.t.} \ v_i \geq 0, \ i = n+1, \ldots, n+p, \tag{4.25}$$

where $\widehat{\mathcal{Q}}_k(v; y_k^E, u_k^R, t) = \nabla M(v_k; y_k^E, u_k^R, t)^T (v - v_k) + \frac{1}{2}(v - v_k)^T \widehat{\mathrm{B}}(v_k; u_k^R, t)(v - v_k)$ and

$$\widehat{\mathrm{B}}(v_k; u_k^R, t) = \begin{pmatrix} \widehat{H}(x_k, y_k; t) + \frac{2}{u_k^R} \nabla c(x_k; t) \nabla c(x_k; t)^T & \nabla c(x_k; t) \\ \nabla c(x_k; t)^T & u_k^R \mathrm{I} \end{pmatrix}. \tag{4.26}$$

The $\widehat{H}(x, y; t)$ is defined so that $\widehat{\mathrm{B}}(v_k; u_k^R, t)$ is a positive definite matrix. The global descent direction is defined by $d_k = \widehat{v}_k - v_k$, where $\widehat{v}_k$ is the unique solution of the subproblem (4.25). The detail computations of d_k are summarized in Algorithm 2. The calculation of the direction of negative curvature s_k is given by Step 5 of Algorithm 1. The feasible second order strategy can be stated as follow.

Algorithm 1: Second order primal dual algorithm for MPCC

Step 0 (Initialization)
Given initial point $(x^1, \ y^1) \in \mathbb{R}^{n+p} \times \mathbb{R}^{q+m+p}$, parameters $t_1 > 0$, $\varepsilon_1 > 0$, $0 < \gamma < 1$, $0 < \epsilon_1 < 1$, $0 < \gamma_s < 0.5$, $0 < \zeta < 1$, $\theta > 0$ and the stopping tolerance $\varepsilon' > 0$. Choose $y_0^E \in \mathbb{R}^{q+m+p}$, $\tau_0 > 0$, $\psi_{\mathrm{v},0}^{max}$, $\psi_{\mathrm{o},0}^{max}$, $y_{\max} > 0$, $0 < u_0^R \leq u_1$. Let $k = 1, \ j = 1$.

Step 1 (Least curvature estimate of Q_k)
Compute the $\epsilon-$free set $\mathcal{F}_\epsilon(x_k, y_k; u_{k-1}^R, t_j)$ from (4.18) and $r(x_k, y_k; t_j)$ from (4.4);
$\nabla c_k = \nabla c(x_k; t_j)$, $H_k = H(x_k, y_k; t_j)$; $H_{\bar{\mathcal{F}}_\epsilon}$ and $\nabla c_{\bar{\mathcal{F}}_\epsilon}$ as submatrices of H_k and ∇c_k related to $\bar{\mathcal{F}}_\epsilon = \mathcal{F}_\epsilon(x_k, y_k; u_{k-1}^R, t_j) \cap \{1, \ldots, n+p\}$.

If $H_{\bar{\mathcal{F}}_\epsilon} + (1/u_{k-1}^R) \nabla c_{\bar{\mathcal{F}}_\epsilon} \nabla c_{\bar{\mathcal{F}}_\epsilon}^T$ is a positive semidefinite matrix **then** set

$\xi_k = 0$; $\mu_k^{(1)} = 0$; $w_k^{(1)} = 0$;

else

Compute a direction $\mu_{\bar{\mathcal{F}}_\epsilon} \neq 0$ satisfying

$\mu^T_{\bar{\mathcal{F}}_\epsilon}(H_{\bar{\mathcal{F}}_\epsilon}+(1/u^R_{k-1})\nabla c_{\bar{\mathcal{F}}_\epsilon}\nabla c^T_{\bar{\mathcal{F}}_\epsilon})\mu_{\bar{\mathcal{F}}_\epsilon}\le\theta\lambda_{\min}(H_{\bar{\mathcal{F}}_\epsilon}+(1/u^R_{k-1})\nabla c_{\bar{\mathcal{F}}_\epsilon}\nabla c^T_{\bar{\mathcal{F}}_\epsilon})\|\mu_{\bar{\mathcal{F}}_\epsilon}\|^2<0;$

$\xi_k = -\mu^T_{\bar{\mathcal{F}}_\epsilon}(H_{\bar{\mathcal{F}}_\epsilon} + (1/u^R_{k-1})\nabla c_{\bar{\mathcal{F}}_\epsilon}\nabla c^T_{\bar{\mathcal{F}}_\epsilon})\mu_{\bar{\mathcal{F}}_\epsilon}/\|\mu_{\bar{\mathcal{F}}_\epsilon}\|^2 > 0;$

$[\mu^{(1)}_k]_{\mathcal{A}_\epsilon} = 0;\ [\mu^{(1)}_k]_{\bar{\mathcal{F}}_\epsilon} = \mu_{\bar{\mathcal{F}}_\epsilon};\ w^{(1)}_k = -(1/u^R_{k-1})\nabla c^T_k \mu^{(1)}_k;$

end if

Set $s^{(1)}_k = (\mu^{(1)}_k, w^{(1)}_k)$.

Step 2 (Check the approximate second order conditions)

If

$$r(x_k, y_k; t_j) \le \varepsilon_j, \quad \xi_k \le \varepsilon_j \quad \text{and} \quad u^R_{k-1} \le \varepsilon_j, \tag{4.27}$$

then set $t_{j+1} = \zeta t_j;\quad \varepsilon_{j+1} = \zeta\varepsilon_j;$

If $\varepsilon_j < \varepsilon'$, **stop**;

else set $j \leftarrow j + 1$ and go to Step 1;

else go to Step 3.

Step 3 (Compute the new multipliers and parameters)

If $\psi_{\rm v}(x_k, y_k, \xi_k; t_j) \le \frac{1}{2}\psi^{max}_{{\rm v},k-1}$, **then** [V-iterate]

set $\psi^{max}_{{\rm v},k} = \frac{1}{2}\psi^{max}_{{\rm v},k-1}$, $y^E_k = y_k$, $\tau_k = \tau_{k-1}$and u^R_k as in (4.21);

else if $\psi_{\rm o}(x_k, y_k, \xi_k; t_j) \le \frac{1}{2}\psi^{max}_{{\rm o},k-1}$, **then** [O-iterate]

set $\psi^{max}_{{\rm o},k} = \frac{1}{2}\psi^{max}_{{\rm o},k-1}$, $y^E_k = y_k$, $\tau_k = \tau_{k-1}$ and u^R_k as in (4.21);

else if [M-iterate]

$$\begin{aligned}
&\|[\nabla_x M(x_k, y_k; y^E_{k-1}, u^R_{k-1}, t_j)]_{\mathcal{E}}\| \le \tau_{k-1},\\
&\|[x_k]_{\mathcal{I}} \cdot [\nabla_x M(x_k, y_k; y^E_{k-1}, u^R_{k-1}, t_j)]_{\mathcal{I}}\| \le \tau_{k-1},\\
&\|\min\{[x_k]_{\mathcal{I}}, [\nabla_x M(x_k, y_k; y^E_{k-1}, u^R_{k-1}, t_j)]_{\mathcal{I}}\}\| \le \tau_{k-1},\\
&\|\nabla_y M(x_k, y_k; y^E_{k-1}, u^R_{k-1}, t_j)\| \le \tau_{k-1}u^R_{k-1},\\
&\xi_k \le \tau_{k-1}
\end{aligned} \tag{4.28}$$

then set $y^E_k = \max\{-y_{\max}e,\ \min\{y_k, y_{\max}e\}\}$, $\tau_k = \frac{1}{2}\tau_{k-1}$ and

$$u^R_k = \begin{cases} \min\{\frac{1}{2}u^R_{k-1}, \max\{r(x_k, y_k; t_j), \xi_k\}^\gamma\}, & \text{if } \max\{r(x_k, y_k; t_j), \xi_k\} > 0,\\ \frac{1}{2}u^R_{k-1}, & \text{otherwise,}\end{cases} \tag{4.29}$$

else set $\quad y^E_k = y^E_{k-1}, \quad \tau_k = \tau_{k-1}, \quad u^R_k = u^R_{k-1}.$ [F-iterate]

end if

Step 4 (Termination criterion)

If $\min\{\|c(x_k; t_j)\|, \varepsilon_j\} > u^R_k$ and

$$\left\| \begin{pmatrix} [\nabla c(x_k; t_j)c(x_k; t_j)]_{\mathcal{E}} \\ \min\{[x_k]_{\mathcal{I}}, [\nabla c(x_k; t_j)c(x_k; t_j)]_{\mathcal{I}}\} \end{pmatrix} \right\| \le \varepsilon_j \quad \text{with } k \text{ an M}-\text{iterate} \tag{4.30}$$

then set $t_{j+1} = \zeta t_j;\quad \varepsilon_{j+1} = \zeta\varepsilon_j;$

If $\varepsilon_j < \varepsilon'$, **stop**;

else set $j \leftarrow j + 1$, $\tau_{k-1} = \tau_k$, $u^R_{k-1} = u^R_k$ and go to Step 1;

else go to Step 5.

Step 5 (Compute the search direction)

Compute the set $\mathcal{F}_\epsilon(x_k, y_k; u^R_k, t_j)$ from (4.18); d_k from Algorithm 2;

If $\nabla M(v_k; y_k^E, u_k^R, t_j)^T s_k^{(1)} > 0$, **then** set $s_k^{(2)} = -s_k^{(1)}$;
else set $s_k^{(2)} = s_k^{(1)}$;
Let $\bar{d}_k$ and $\mu_k^{(2)}$ be the first $n+p$ components of d_k and $s_k^{(2)}$;
Compute $\sigma_k = \arg\max_{\sigma\geq 0}\{\sigma|\ [x_k + \bar{d}_k + \sigma\mu_k^{(2)}]_{\mathcal{I}} \geq 0,\ \|\sigma[\mu_k^{(2)}]_{\mathcal{I}}\| \leq \max\{\xi_k, \|[\bar{d}_k]_{\mathcal{I}}\|\}\}$ and $s_k = (\mu_k, w_k) = \sigma_k s_k^{(2)}$.
If $d_k \neq 0,\ s_k = 0$ and (x_k, y_k) is a V-O-iterate, **then** set $l_k = 1$;
else set $l_k = 2$.

Step 6 (Line search strategy)
Compute $\Delta v_k = d_k + s_k$, $u_k = \max\{u_k^R, u_k\}$ and $\nabla M = \nabla M(v_k; y_k^E, u_k^R, t_j)$. Define the univariate function $\Psi_k(\alpha; u, t_j) = M(v_k + \alpha\Delta v_k; y_k^E, u, t_j)$ with respect to α, the line-search model function
$\varphi_k(\alpha; u, l_k, t_j) = \Psi_k(0; u, t_j) + \alpha\Psi_k'(0; u, t_j) + \frac{1}{2}(l_k - 1)\alpha^2 \min\{0, \Delta v_k^T \mathrm{B}(v_k; u_{k-1}^R, t_j)\Delta v_k\}$ and
$\rho_k(\alpha; u, l_k, t_j) = (\Psi_k(0; u, t_j) - \Psi_k(\alpha; u, t_j))/(\varphi_k(0; u_k^R, l_k, t_j) - \varphi_k(\alpha; u_k^R, l_k, t_j))$.
If $d_k = 0,\ s_k = 0$, **then** set $\alpha_k = 1$;
else if $d_k \neq 0$ or $\nabla M^T s_k < 0$ or $u_k^R = u_{k-1}^R$, **then** set $\alpha_k = 1$;
while $\rho_k(\alpha_k; u_k^R, l_k, t_j) < \gamma_s$ and $\rho_k(\alpha_k; u_k, l_k, t_j) < \gamma_s$, **do** $\alpha_k = \frac{1}{2}\alpha_k$;
end while
else set $[d_k = 0,\ s_k \neq 0,\ \xi_k > 0]$

$$\xi_k^R = -s_k^T \nabla^2 M(v_k; y_k^E, u_k^R, t_j) s_k / \|\mu_k\|^2; \tag{4.31}$$

if $\xi_k^R > \gamma_s \xi_k$, **then** set $\alpha_k = 1$;
while $\rho_k(\alpha_k; u_k^R, l_k, t_j) < \gamma_s$ and $\rho_k(\alpha_k; u_k, l_k, t_j) < \gamma_s$, **do** $\alpha_k = \frac{1}{2}\alpha_k$;
end while
else set $\alpha_k = 0$.
end if
end if
Set

$$u_{k+1} = \begin{cases} u_k, & \text{if } \rho_k(\alpha_k; u_k, l_k, t_j) \geq \gamma_s \text{ or } d_k = s_k = 0 \text{ or } \alpha_k = 0, \\ \max\{\frac{1}{2}u_k, u_k^R\}, & \text{otherwise,} \end{cases} \tag{4.32}$$

$v_{k+1} = (x_{k+1}, y_{k+1}) = v_k + \alpha_k d_k + \alpha_k s_k$, $k \leftarrow k+1$ and go to Step 1.

Remark 4.1. In Step 1 of Algorithm 1, we introduce the new index set $\bar{\mathcal{F}}_\epsilon = \mathcal{F}_\epsilon(x_k, y_k; u_{k-1}^R, t_j) \cap \{1, \ldots, n+p\}$ instead of $\mathcal{F}_\epsilon(x_k, y_k; u_{k-1}^R, t_j)$ used in [5]. We suppose that Step 1 is always successful to determine whether or not $H_{\bar{\mathcal{F}}_\epsilon} + (1/u_{k-1}^R)\nabla c_{\bar{\mathcal{F}}_\epsilon} \nabla c_{\bar{\mathcal{F}}_\epsilon}^T$ is positive semidefinite and compute a $\mu_{\mathcal{F}_\epsilon} \neq 0$ if $H_{\bar{\mathcal{F}}_\epsilon} + (1/u_{k-1}^R)\nabla c_{\bar{\mathcal{F}}_\epsilon} \nabla c_{\bar{\mathcal{F}}_\epsilon}^T$ is not positive semidefinite. The similar line search strategy was proposed by [5].

Note that Algorithm 1 and Algorithm 2 are different from the Algorithm 5 and 2 in [5], since they are related to the iteration of variable t_j and adapted to the case that all variables are not sign constrained. In order to obtain an M-stationary point of MPCC, we consider the approximate second order conditions (see Step 2 of Algorithm 1), which establish by the new definition of $r(x, y; t)$. In addition, the new $r(x, y; t)$ have effect on the M-iterate conditions and ψ_{v}, ψ_{o} of V-O-iterate (see Step 3 of Algorithm 1). The special structure of

Algorithm 2: Computation of the descent direction d_k

Given v_k,y_k^E,u_k^R,∇c_k,H_k from Algorithm 1. Set $0 < \widetilde{\gamma} < \min\{\gamma, 1-\gamma\} < 1$;
$\mathrm{B} = \mathrm{B}(v_k; u_k^R, t_j)$;
$\nabla M_k = \nabla M(v_k; y_k^E, u_k^R, t_j)$; $\kappa_k = r(v_k; t_j)^{\widetilde{\gamma}}$; $[\widehat{v}_k^{(0)}]_{\mathcal{A}_\epsilon} = 0$; $[\widehat{v}_k^{(0)}]_{\mathcal{F}_\epsilon} = [v_k]_{\mathcal{F}_\epsilon}$;
$\mathcal{Q}_k(v) = \mathcal{Q}_k(v; y_k^E, u_k^R, t_j)$; $\widehat{\mathcal{Q}}_k(v) = \widehat{\mathcal{Q}}_k(v; y_k^E, u_k^R, t_j)$; Compute $\widehat{\mathrm{B}}$ from B.

Step 1 (Local descent direction)
  If $\mathrm{B}_{\mathcal{F}_\epsilon}$ is a positive definite matrix and v_k is a V-O-iterate, **then** set
    $[\Delta\widehat{v}_k^{(0)}]_{\mathcal{A}_\epsilon} = 0$; Solve $\mathrm{B}_{\mathcal{F}_\epsilon}[\Delta\widehat{v}_k^{(0)}]_{\mathcal{F}_\epsilon} = -[\nabla \mathcal{Q}_k(\widehat{v}_k^{(0)})]_{\mathcal{F}_\epsilon}$;
    $\widehat{v}_k = \widehat{v}_k^{(0)} + \Delta\widehat{v}_k^{(0)}$; $d_k = \widehat{v}_k - v_k$;
    If $v_k + d_k$ is feasible, $\nabla M_k^T d_k < 0$ and $[\nabla \mathcal{Q}_k(v_k + d_k)]_{\mathcal{A}_\epsilon} \geq -\kappa_k e$, **then**
    go to step 3;
    else go to step 2.
  else go to step 2.

Step 2 (Global descent direction)
  Set $[\widehat{v}_k^{(0)}]_{\mathcal{A}_\epsilon} = 0$; Solve $\widehat{\mathrm{B}}_{\mathcal{F}_\epsilon}[\Delta\widehat{v}_k^{(0)}]_{\mathcal{F}_\epsilon} = -[\nabla \widehat{\mathcal{Q}}_k(\widehat{v}_k^{(0)})]_{\mathcal{F}_\epsilon}$; Compute $\widehat{\alpha}_0 \geq 0$ and $\widehat{v}_k^{(1)}$ such that $\widehat{v}_k^{(1)} = \widehat{v}_k^{(0)} + \widehat{\alpha}_0 \Delta\widehat{v}_k^{(0)}$ is feasible;
  From the starting point $\widehat{v}_k^{(1)}$, use the active set method to solve
  the convex subproblem (4.25) to obtain $\widehat{v}_k$; $d_k = \widehat{v}_k - v_k$.

Step 3 Return to d_k.

$r(x, y; t)$ as shown in (4.4) is applied to Algorithm 1 to find a sequence of ε-KKT points satisfying (2.11).

In order to obtain the convergence results, two assumptions are given as follows.

Assumption 4.1. The sequence $\{x_k\}$ generated by Algorithm 1 is contained in a compact set.

Assumption 4.2. The sequence $\{\widehat{H}(x_k, y_k; t_j)\}$ is chosen to satisfy

$$\|\widehat{H}(x_k, y_k; t_j)\| \leq \widehat{H}_{\max} \quad \text{and} \quad \lambda_{\min}(\widehat{H}(x_k, y_k; t_j) + (1/u_k^R)\nabla c_k \nabla c_k^T)) \geq \underline{\lambda}_{\min} \tag{4.33}$$

for some $\widehat{H}_{\max}$, $\underline{\lambda}_{\min} > 0$ and all $k \geq 0$.

Similarly, these assumptions are given in [5]. If steps 2 and 4 are omitted from Algorithm 1, the infinite sequence $v_k = (x_k, y_k)$ is generated by Algorithm 1. Based on these assumptions and the similar analysis from [5], we know that there exists an infinite set of V-O-iterates or M-iterates and $u_k^R \to 0$ for all fixed $j \geq 1$. If the set of V-O-iterates is finite, then the set of M-iterates is infinite. We denote the set $\mathcal{M} = \{k : \text{iterate } k \text{ is an M}-\text{iterate}\}$. For each fixed j, the limit point x^j of $\{x_k\}_{k\in\mathcal{M}}$ satisfies $c(x^j; t_j) \neq 0$ and it is a KKT point for the problem

$$\min_x \ \frac{1}{2}\|c(x; t_j)\|^2 \quad \text{s.t. } x_i \geq 0, \ i = n+1, \ldots, n+p. \tag{4.34}$$

In the following theorem we prove that, if Step 2 and Step 4 are present, the conditions (4.27) or (4.30) hold in a finite number of iterates k for all fixed j.

Theorem 4.2. *If Algorithm 1 is implemented with a positive value of ε_j in Step 2 and Step 4, then the conditions (4.27) or (4.30) hold in a finite number of iterates k.*

Proof. We suppose by contradiction that there is a j' such that (4.27) and (4.30) fail to hold for all $k \geq 0$. It implies that the infinite sequence $\{v_k\}$ is generated by Algorithm 1 for fixed j'. There are two case to be considered. Suppose that there are infinitely V-O-iterates given by the sequence $\mathcal{S}$. By the update used for V-O-iterates and $u_k^R \to 0$, we obtain that (4.27) will be satisfied for all $k \in \mathcal{S}$ sufficiently large. This implies that the conditions (4.27) hold after a finite number of iterates. Next, we consider the case that the set of V-O-iterates is finite. By Assumption 4.1 and the optimality conditions of problem (4.34), there exists a subsequence $\mathcal{M}_1 \subseteq \mathcal{M}$ of M-iterates such that

$$\lim_{k\in\mathcal{M}_1} x_k = x^{j'}, \quad \left\| \begin{pmatrix} [\nabla c(x^{j'};t_{j'})c(x^{j'};t_{j'})]_{\mathcal{E}} \\ \min\{[x^{j'}]_{\mathcal{I}}, [\nabla c(x^{j'};t_{j'})c(x_k;t_{j'})]_{\mathcal{I}}\} \end{pmatrix} \right\| = 0 \quad \text{and} \quad c(x^{j'};t_{j'}) \neq 0 \tag{4.35}$$

It also follows from $u_k^R \to 0$, (4.35) and (4.19) that

$$\min\{\eta(x_k;t_{j'}), \varepsilon_{j'}\} > u_k^R \quad \text{for all } k \in \mathcal{M} \text{ sufficiently large.} \tag{4.36}$$

The combination of (4.35) and the definition of $\mathcal{M}$ mean that (4.30) must hold for $k \in \mathcal{M}$ sufficiently large. Hence, the conditions (4.30) are satisfied in a finite number of iterates k. □

Hence, for all $j \geq 1$, there must exist a $k = k(j)$ such that (4.27) or (4.30) hold. The result plays an important role in the convergence analysis. It is clear that Algorithm 1 will terminate after a finite number of iterations if the termination conditions ($\varepsilon_j < \varepsilon'$) are present. In the subsequent analysis, we assume that termination conditions ($\varepsilon_j < \varepsilon'$) of steps 2 and 4 are omitted, and set $j \leftarrow j+1$. As a consequence, there are infinitely many iterations j in Algorithm 1. We turn to consider the properties of limit points x^* with iteration j.

The next result establishes that, in the case that the iterations satisfying (4.27) is finite, there exists an infeasible limit point, i.e., $c(x^*;0) \neq 0$.

Theorem 4.3. *Assume that x^* is a limit point of $\{x_{k(j)}\}$ and the sequence $\{x_{k(j)}\}$ satisfies (4.27) is finite. Then $c(x^*;0) \neq 0$ and x^* is a stationary point of the nonsmooth problem*

$$\min_x \quad \frac{1}{2}\|c(x;0)\|^2 \quad \text{s.t. } x_i \geq 0,\ i = n+1, \ldots, n+p. \tag{4.37}$$

Proof. By Theorem 4.2, we have that, for all j sufficiently large, there exists a $k(j)$ such that $\min\{\|c(x_{k(j)};t_j)\|, \varepsilon_j\} > u_{k(j)}^R$ and

$$\left\| \begin{pmatrix} [\nabla c(x_{k(j)};t_j)c(x_{k(j)};t_j)]_{\mathcal{E}} \\ \min\{[x_{k(j)}]_{\mathcal{I}}, [\nabla c(x_{k(j)};t_j)c(x_{k(j)};t_j)]_{\mathcal{I}}\} \end{pmatrix} \right\| \leq \varepsilon_j \quad \text{with } k(j) \text{ an M}-\text{iterate.} \tag{4.38}$$

We have

$$\lim_{j\to+\infty} [\nabla c(x_{k(j)};t_j)c(x_{k(j)};t_j)]_{\mathcal{E}} = 0, \tag{4.39}$$

$$\lim_{j\to+\infty} \min\{[x_{k(j)}]_{\mathcal{I}}, [\nabla c(x_{k(j)};t_j)c(x_{k(j)};t_j)]_{\mathcal{I}}\} = 0. \tag{4.40}$$

According to (4.39) and [22, Theorem 2.1], it is obvious that, for $i \in \mathcal{E}$,

$$\lim_{j\to+\infty} \operatorname{dist}\{[\nabla c(x_{k(j)};t_j)]_i,\ [\partial c(x^*;0)]_i\} = 0 \quad \text{and} \quad 0 \in [\partial c(x^*;0)c(x^*;0)]_i, \tag{4.41}$$

where $\partial c(x^*;0) = \text{conv}\{\lim_{v\to+\infty} \nabla c(x^v;0) \mid x^v \to x^*, x^v \notin \mathcal{F}_c\}$ and $\mathcal{F}_c$ denotes the set that $c(x;0)$ fails to be differentiable.

For $i \in \mathcal{I}$, there are two cases to consider. Firstly, the index $i \in \mathcal{I}$ such that $[x^*]_i = 0$. In this case, we have that $\lim_{j\to+\infty}[\nabla c(x_{k(j)};t_j)c(x_{k(j)};t_j)]_i$ is nonnegative. From [22, Theorem 2.1], there exist $V \in \partial c(x^*;0)$ such that $\min\{[x^*]_i, [Vc(x^*;0)]_i\}$=0. Secondly, taking into account that $[x^*]_i > 0,\ i \in \mathcal{I}$, we obtain that $\lim_{j\to+\infty}[\nabla c(x_{k(j)};t_j)c(x_{k(j)};t_j)]_i = 0$. Similarly, it is clear that $0 \in [\partial c(x^*;0)c(x^*;0)]_i$. Hence, x^* is a stationary point of problem (4.37).

Since $k(j)$ is an M-iterate, it means that $x_{k(j)}$ satisfies

$$\begin{aligned}
&\|[\nabla_x M(x_{k(j)}, y_{k(j)}; y^E_{k-1(j)}, u^R_{k-1(j)}, t_j)]_{\mathcal{E}}\| \le \tau_{k-1(j)},\\
&\|[x_{k(j)}]_{\mathcal{I}} \cdot [\nabla_x M(x_{k(j)}, y_{k(j)}; y^E_{k-1(j)}, u^R_{k-1(j)}, t_j)]_{\mathcal{I}}\| \le \tau_{k-1(j)},\\
&\|\min\{[x_{k(j)}]_{\mathcal{I}}, [\nabla_x M(x_{k(j)}, y_{k(j)}; y^E_{k-1(j)}, u^R_{k-1(j)}, t_j)]_{\mathcal{I}}\}\| \le \tau_{k-1(j)},\\
&\|\nabla_y M(x_{k(j)}, y_{k(j)}; y^E_{k-1(j)}, u^R_{k-1(j)}, t_j)\| \le \tau_{k-1(j)} u^R_{k-1(j)}, \quad \xi_{k(j)} \le \tau_{k-1(j)}.
\end{aligned}$$

We have

$$\begin{aligned}
&\lim_{j\to+\infty} \|\min\{[x_{k(j)}]_{\mathcal{I}}, [\nabla \widetilde{f}(x_{k(j)}) - \nabla c(x_{k(j)};t_j) y^E_{k-1(j)}\\
&\qquad\qquad + \frac{1}{u^R_{k-1(j)}} \nabla c(x_{k(j)};t_j) c(x_{k(j)};t_j)]_{\mathcal{I}}\}\| = 0,\\
&\lim_{j\to+\infty} \|[x_{k(j)}]_{\mathcal{I}} \cdot [\nabla \widetilde{f}(x_{k(j)}) - \nabla c(x_{k(j)};t_j) y^E_{k-1(j)}\\
&\qquad\qquad + \frac{1}{u^R_{k-1(j)}} \nabla c(x_{k(j)};t_j) c(x_{k(j)};t_j)]_{\mathcal{I}}\| = 0,\\
&\lim_{j\to+\infty} \|[\nabla \widetilde{f}(x_{k(j)}) - \nabla c(x_{k(j)};t_j) y^E_{k-1(j)}\\
&\qquad\qquad + \frac{1}{u^R_{k-1(j)}} \nabla c(x_{k(j)};t_j) c(x_{k(j)};t_j)]_{\mathcal{E}}\| = 0, \quad (4.42)\\
&\lim_{j\to+\infty} \|\pi(x_{k(j)}; y^E_{k-1(j)}, u^R_{k-1(j)}, t_j) - y_{k(j)}\| = 0, \qquad \lim_{j\to+\infty} \xi_{k(j)} = 0,
\end{aligned}$$

where $\pi(x_{k(j)}; y^E_{k-1(j)}, u^R_{k-1(j)}, t_j) = y^E_{k-1(j)} - c(x_{k(j)};t_j)/u^R_{k-1(j)}$. These limits follow from the definition of M-iterates and the fact $\lim_{j\to+\infty} \tau_{k-1(j)} = 0$ is enforced by Algorithm 1. It remains to show that $\|c(x^*;0)\| \ne 0$. This leads to a contradiction. If $\|c(x^*;0)\| = 0$, then it follows from (4.42) that $x_{k(j)}$ must satisfy (4.27) for j sufficiently large. This completes the proof. □

In Section 3, the constraint qualification is a crucial important requirement in the analysis of convergence. Combining with MPCC-LICQ, we consider the relationship between the approximate second order stationary points of (4.3) and (2.3). In what follows, we show that the sequence of multipliers $\{y_{k(j)}\}$ is uniformly bounded, if there are infinite many iterations satisfy (4.27) and MPCC-LICQ holds at the limit point. Then, we show that it is reasonable to get the approximate second order stationary points of (2.3) by Algorithm 1. Hence, our method is realistic from a numerical point of view.

Theorem 4.4. *Let $\mathcal{J}$ be the index set $\mathcal{J} = \{j : v_{k(j)}$ satisfies (4.27)$\}$. Consider the infinitely sequence $\{v_{k(j)}\} = \{(x_{k(j)}, y_{k(j)})\}$, where $\{v_{k(j)}\}$ is generated by Algorithm 1 and satisfies*

(4.27). Suppose that $\{x_{k(j)}\} \xrightarrow{\mathcal{J}} x^* = (z^*, \hat{z}^*)^T$ as $j \to +\infty$. If MPCC-LICQ holds at z^*, then the sequence $\{y_{k(j)}\}$ is uniformly bounded.

Proof. Since infinitely many $\{v_{k(j)}\} = \{(x_{k(j)}, y_{k(j)})\}$ satisfy the conditions (4.27), then we have

$$r(x_{k(j)}, y_{k(j)}; t_j) \le \varepsilon_j, \quad \xi_{k(j)} \le \varepsilon_j \quad \text{and} \quad u^R_{k-1(j)} \le \varepsilon_j. \tag{4.43}$$

It is clear that

$$\lim_{j\to+\infty} r(x_{k(j)}, y_{k(j)}; t_j) = 0 \quad \text{and} \quad \mathcal{I}_g(z^*) = \mathcal{I}(\hat{z}^*), \tag{4.44}$$

where $x^* = (z^*, \hat{z}^*)$ and $\mathcal{I}(\hat{z}^*) = \{i \in \mathcal{I} \mid [\hat{z}^*]_i = 0\}$. If the sequence $\{y_{k(j)}\}$ is unbounded, then there exists a subsequence $\mathcal{J}_1 \subseteq \mathcal{J}$ such that

$$\frac{y_{k(j)}}{\|y_{k(j)}\|} \xrightarrow{\mathcal{J}_1} y^* \ne 0. \tag{4.45}$$

Let $\varrho^j \in \mathbb{R}^p$ denote a sequence such that $\varrho^j = \max\{0, [\nabla \widetilde{f}(x_{k(j)}) - \nabla c(x_{k(j)}; t_j) y_{k(j)}]_{\mathcal{I}}\} \ge 0$. According to the definition of ϱ^j, (4.44) and (4.45) imply that

$$\lim_{j\to+\infty} \left(\nabla \widetilde{f}(x_{k(j)}) - \nabla c(x_{k(j)}; t_j) y_{k(j)} - \begin{pmatrix} 0 \\ \varrho^j \end{pmatrix} \right) = 0 \tag{4.46}$$

and $\{\varrho^j / \|y_{k(j)}\|\}_{j\in\mathcal{J}_1}$ is bounded. Without loss of generality, we suppose that $\frac{\varrho^j}{\|y_{k(j)}\|} \xrightarrow{\mathcal{J}_1} \widetilde{y}^*$ and $\widetilde{y}^* \ge 0$. It is not hard to get from (4.46), (4.2), (3.3) and $\{x_{k(j)}\} \to x^*$ that

$$\begin{aligned} 0 &= \lim_{j\in\mathcal{J}_1} \frac{1}{\|y_{k(j)}\|} \left(\nabla \widetilde{f}(x_{k(j)}) - \nabla c(x_{k(j)}; t_j) y_{k(j)} - \begin{pmatrix} 0 \\ \varrho^j \end{pmatrix} \right) \\ &= \lim_{j\in\mathcal{J}_1} \frac{1}{\|y_{k(j)}\|} \\ &\quad \begin{pmatrix} \nabla f(z_{k(j)}) - \sum_{i=1}^{q} [y^h_{k(j)}]_i \nabla h_i(z_{k(j)}) - \sum_{i=1}^{m} [y^\phi_{k(j)}]_i \nabla \phi_i(z_{k(j)}; t_j) - \sum_{i=1}^{p} [y^g_{k(j)}]_i \nabla g_i(z_{k(j)}) \\ -y^g_{k(j)} - \varrho^j \end{pmatrix}, \end{aligned} \tag{4.47}$$

where $y_{k(j)} = (y^h_{k(j)}, y^\phi_{k(j)}, y^g_{k(j)})^T$ and $y^* = (y^{h,*}, y^{\phi,*}, y^{g,*})^T$. It must hold that $\widetilde{y}^* = -y^{g,*}$ and

$$\begin{aligned} &\sum_{i=1}^{q} [y^{h,*}]_i \nabla h_i(z^*) + \sum_{i\in\mathcal{I}_{0+}\cup\mathcal{I}_{00}} [y^{\phi,*}]_i \nu^*_{i1} \nabla G_i(z^*) \\ &\qquad + \sum_{i\in\mathcal{I}_{+0}\cup\mathcal{I}_{00}} [y^{\phi,*}]_i \nu^*_{i2} \nabla H_i(z^*) + \sum_{i=1}^{p} [y^{g,*}]_i \nabla g_i(z^*) = 0, \end{aligned}$$

where

$$\nu^*_{i1} \in [0,1], \nu^*_{i2} \in [0,1] \quad \text{and} \quad \nu^*_{i1} + \nu^*_{i2} = 1. \tag{4.48}$$

According to (4.44) and (4.47), we have $\mathrm{supp}(y^{g,*}) = \mathrm{supp}(\widetilde{y}^*) \subseteq \mathcal{I}(\hat{z}^*) = \mathcal{I}_g(z^*)$. Since MPCC-LICQ holds at z^*, it implies that $y^{h,*} = 0$, $[y^{\phi,*}]_i \nu^*_{i1} = 0$ for $i \in \mathcal{I}_{0+} \cup \mathcal{I}_{00}$, $[y^{\phi,*}]_i \nu^*_{i2} = 0$ for $i \in \mathcal{I}_{+0} \cup \mathcal{I}_{00}$ and $y^{g,*} = 0$. From (4.48), we have $y^{\phi,*} = 0$. Therefore, it follows that $y^* = 0$, which contradicts that $y^* \ne 0$. □

Theorem 4.5. *Assume that the infinite sequence $\{(x_{k(j)}, y_{k(j)})\}$ satisfies (4.27) generated by Algorithm 1 and $x^* = (z^*, \hat{z}^*)$ is a limit point of $\{x_{k(j)}\}$. If MPCC-LICQ holds at z^*, then we get a sequence of the approximate second order stationary points of $NLP(t_j)$ for j sufficiently large. Hence, z^* is M-stationary of MPCC. In addition, if ULSC is satisfied at z^*, then z^* is S-stationary.*

Proof. Let $\{(x_{k(j)}, y_{k(j)})\}$ be a sequence generated by Algorithm 1 and satisfy the conditions of (4.27). It follows from (4.43) that

$$\|c(x_{k(j)}; t_j)\| = \left\| \begin{pmatrix} h(z_{k(j)}) \\ \Phi(z_{k(j)}; t_j) \\ g(z_{k(j)}) + \hat{z}_{k(j)} \end{pmatrix} \right\| \le \varepsilon_j, \tag{4.49}$$

which implies that

$$\|h(z_{k(j)})\|_\infty \le \varepsilon_j, \quad |\phi_i(z_{k(j)}; t_j)| \le \varepsilon_j, \ i = 1, \dots, m \ \text{ and} \tag{4.50}$$

$$|g_i(z_{k(j)}) + [\hat{z}_{k(j)}]_i| \le \varepsilon_j, \ i = 1, \dots, p. \tag{4.51}$$

We can rewrite (4.51) as

$$|g_i(z_{k(j)})| \le \varepsilon_j + |[\hat{z}_{k(j)}]_i|, \ i = 1, \dots, p. \tag{4.52}$$

From (4.43), we have $\|[\nabla \widetilde{f}(x_{k(j)}) - \nabla c(x_{k(j)}; t_j) y_{k(j)}]_{\mathcal{E}}\| \le \varepsilon_j$ and

$$\nabla \widetilde{f}(x_{k(j)}) - \nabla c(x_{k(j)}; t_j) y_{k(j)} = \begin{pmatrix} \nabla f(z_{k(j)}) - \sum_{i=1}^{q} [y^h_{k(j)}]_i \nabla h_i(z_{k(j)}) - \sum_{i=1}^{m} [y^\phi_{k(j)}]_i \nabla \phi_i(z_{k(j)}; t_j) - \sum_{i=1}^{p} [y^g_{k(j)}]_i \nabla g_i(z_{k(j)}) \\ -y^g_{k(j)} \end{pmatrix}, \tag{4.53}$$

where $y_{k(j)} = (y^h_{k(j)}, y^\phi_{k(j)}, y^g_{k(j)})^T$. It is obvious that

$$\begin{aligned} \| \nabla f(z_{k(j)}) &- \sum_{i=1}^{q} [y^h_{k(j)}]_i \nabla h_i(z_{k(j)}) \\ &- \sum_{i=1}^{m} [y^\phi_{k(j)}]_i \nabla \phi_i(z_{k(j)}; t_j) - \sum_{i=1}^{p} [y^g_{k(j)}]_i \nabla g_i(z_{k(j)}) \|_\infty \le \varepsilon_j. \end{aligned} \tag{4.54}$$

From (4.43) and (4.53), we obtain

$$\left\| \begin{pmatrix} -\hat{z}_{k(j)} y^g_{k(j)} \\ \min\{\hat{z}_{k(j)}, -y^g_{k(j)}\} \end{pmatrix} \right\| \le \varepsilon_j. \tag{4.55}$$

Since MPCC-LICQ holds at z^*, in view of Theorem 4.4, we know that the sequence $\{y_{k(j)}\}$ is uniformly bounded. There exists a positive constant C_1 such that $\|y_{k(j)}\|_\infty \le C_1$ for j sufficiently large. If $[-y^g_{k(j)}]_i > [\hat{z}_{k(j)}]_i$, then $|[\hat{z}_{k(j)}]_i| \le \varepsilon_j$. It follows from (4.52) that

$$|g_i(z_{k(j)})| \le 2\varepsilon_j, \quad [-y^g_{k(j)}]_i \ge -\varepsilon_j \ \text{ and } \ |[-y^g_{k(j)}]_i g_i(z_{k(j)})| \le 2C_1 \varepsilon_j. \tag{4.56}$$

If $[-y^g_{k(j)}]_i \le [\hat{z}_{k(j)}]_i$, then $|[-y^g_{k(j)}]_i| \le \varepsilon_j$ and $[\hat{z}_{k(j)}]_i \ge -\varepsilon_j$. According to Assumption 4.1, there exists a positive constant C_2 such that $\|\hat{z}_{k(j)}\|_\infty \le C_2$ for j sufficiently large. From (4.51) and (4.52), we have $g_i(z_{k(j)}) \le 2\varepsilon_j$ and $|g_i(z_{k(j)})| \le \varepsilon_j + C_2$. It is clear that

$$g_i(z_{k(j)}) \le 2\varepsilon_j, \quad [-y^g_{k(j)}]_i \ge -\varepsilon_j \quad \text{and} \quad |[-y^g_{k(j)}]_i g_i(z_{k(j)})| \le (\varepsilon_j + C_2)\varepsilon_j. \tag{4.57}$$

From (4.50), (4.54), (4.56) and (4.57), we obtain that $z_{k(j)}$ is an $\tilde{\varepsilon}_j$-KKT point of $NLP(t_j)$, where $\tilde{\varepsilon}_j = \max\{\ 2\varepsilon_j,\ (\varepsilon_j + C_2)\varepsilon_j,\ 2C_1\varepsilon_j\ \}$.

Furthermore, we focus on the condition (2.11). From (4.49), we have $u^R_{k-1(j)} \to 0$ as $j \to +\infty$ and $\mathcal{I}(\hat{z}^*) = \mathcal{I}_g(z^*)$. Consider the set

$$\widehat{\mathcal{C}}(x_{k(j)}, y_{k(j)}; t_j) = \left\{ p \in \mathbb{R}^{n+p} | \begin{array}{c} \nabla c(x_{k(j)}; t_j)^T p = 0, \\ \\ p_i = 0 \quad i \in \mathcal{A}_\epsilon(x_{k(j)}, y_{k(j)}; u^R_{k-1(j)}, t_j) \end{array} \right\} \tag{4.58}$$

and the definition of ϵ-active set (4.17). It implies that, for j sufficiently large,

$$\mathcal{A}_\epsilon(x_{k(j)}, y_{k(j)}; u^R_{k-1(j)}, t_j) \subseteq \mathcal{A}(x^*). \tag{4.59}$$

Hence, we have $\widetilde{\mathcal{C}}(x_{k(j)}; t_j) \subseteq \widehat{\mathcal{C}}(x_{k(j)}, y_{k(j)}; t_j)$ for j sufficiently large, where

$$\begin{aligned} & \widetilde{\mathcal{C}}(x_{k(j)}; t_j) \\ = \ & \{p \in \mathbb{R}^{n+p} |\ \nabla c(x_{k(j)}; t_j)^T p = 0,\ p_i = 0 \quad i \in \mathcal{A}(x^*)\} \\ = \ & \left\{ \begin{pmatrix} p_{\mathcal{E}} \\ p_{\mathcal{I}} \end{pmatrix} \mid \begin{array}{l} \nabla h_i(z_{k(j)})^T p_{\mathcal{E}} = 0 \quad i = 1, \ldots, q,\ \nabla \phi_i(z_{k(j)}; t_j)^T p_{\mathcal{E}} = 0 \quad i = 1, \ldots, m, \\ \nabla g_i(z_{k(j)})^T p_{\mathcal{E}} + [p_{\mathcal{I}}]_i = 0 \quad i = 1, \ldots, p, \quad [p_{\mathcal{I}}]_i = 0 \quad i \in \mathcal{I}_g(z^*) \end{array} \right\} \end{aligned}$$

and $p_{\mathcal{E}} = (p_1, \ldots, p_n)^T$, $p_{\mathcal{I}} = (p_{n+1}, \ldots, p_{n+p})^T$. Let d denote any vector such that $d \in \bar{T}(z_{k(j)}; t_j)$, where

$$\bar{T}(z_{k(j)}; t_j) = \left\{ d \in \mathbb{R}^n | \begin{array}{c} \nabla g_i(z_{k(j)})^T d = 0 \quad i \in \mathcal{I}_g(z^*), \qquad \nabla h_i(z_{k(j)})^T d = 0 \quad i \in \mathcal{I}_h, \\ \\ \nabla \phi_i(z_{k(j)}; t_j)^T d = 0 \quad i = 1, \ldots, m \end{array} \right\}.$$

There exists a $\begin{pmatrix} p_{\mathcal{E}} \\ p_{\mathcal{I}} \end{pmatrix}$ such that $d = p_{\mathcal{E}}$ and $\begin{pmatrix} p_{\mathcal{E}} \\ p_{\mathcal{I}} \end{pmatrix} \in \widetilde{\mathcal{C}}(x_{k(j)}; t_j)$ with $\|p\| = \|\begin{pmatrix} p_{\mathcal{E}} \\ p_{\mathcal{I}} \end{pmatrix}\| = 1$. If $p_{\bar{\mathcal{F}}_\epsilon}$ is the vector of components of p associated with the set $\bar{\mathcal{F}}_\epsilon = \mathcal{F}_\epsilon(x_{k(j)}, y_{k(j)}; u^R_{k-1(j)}, t_j) \cap \{1, \ldots, n+p\}$, then the definition of p as a vector of unit norm in the set $\widetilde{\mathcal{C}}(x_{k(j)}; t_j) \subseteq \widehat{\mathcal{C}}(x_{k(j)}, y_{k(j)}; t_j)$ implies that $\|p_{\bar{\mathcal{F}}_\epsilon}\| = 1$. Furthermore, $\nabla c(x_{k(j)}; t_j)p = 0$ and $\xi_{k(j)} \le \varepsilon_j$ imply that

$$\begin{aligned} & \frac{d^T \nabla^2_z L(z_{k(j)}, -y^g_{k(j)}, -y^h_{k(j)}, y^\phi_{k(j)}; t_j) d}{\|d\|^2} = p^T H(x_{k(j)}, y_{k(j)}; t_j) p \\ & = p^T (H(x_{k(j)}, y_{k(j)}; t_j) + (1/u^R_{k-1}) \nabla c(x_{k(j)}; t_j) \nabla c(x_{k(j)}; t_j)^T) p \\ & = p^T_{\bar{\mathcal{F}}_\epsilon} (H_{\bar{\mathcal{F}}_\epsilon, k(j)} + (1/u^R_{k-1}) \nabla c_{\bar{\mathcal{F}}_\epsilon, k(j)} \nabla c^T_{\bar{\mathcal{F}}_\epsilon, k(j)}) p_{\bar{\mathcal{F}}_\epsilon} \\ & \ge \lambda_{\min}(H_{\bar{\mathcal{F}}_\epsilon, k(j)} + (1/u^R_{k-1}) \nabla c_{\bar{\mathcal{F}}_\epsilon, k(j)} \nabla c^T_{\bar{\mathcal{F}}_\epsilon, k(j)}) \|p_{\bar{\mathcal{F}}_\epsilon}\|^2 \\ & \ge \frac{\mu^T_{\bar{\mathcal{F}}_\epsilon, k(j)} (H_{\bar{\mathcal{F}}_\epsilon, k(j)} + (1/u^R_{k-1(j)}) \nabla c_{\bar{\mathcal{F}}_\epsilon, k(j)} \nabla c^T_{\bar{\mathcal{F}}_\epsilon, k(j)}) \mu_{\bar{\mathcal{F}}_\epsilon, k(j)}}{\theta \|\mu_{\bar{\mathcal{F}}_\epsilon, k(j)}\|^2} \\ & = -\frac{1}{\theta} \xi_{k(j)} \ge -\frac{1}{\theta} \varepsilon_j, \end{aligned}$$

where

$$H_{\bar{\mathcal{F}}_\epsilon,k(j)} + (1/u^R_{k-1})\nabla c_{\bar{\mathcal{F}}_\epsilon,k(j)}\nabla c^T_{\bar{\mathcal{F}}_\epsilon,k(j)} = [H(x_{k(j)}, y_{k(j)}; t_j) + (1/u^R_{k-1})\nabla c(x_{k(j)}; t_j)\nabla c(x_{k(j)}; t_j)^T]_{\bar{\mathcal{F}}_\epsilon}.$$

Denote $\bar{\varepsilon}_j = \max\{ \frac{1}{\theta}\varepsilon_j, \tilde{\varepsilon}_j \}$, we obtain a sequence of the approximate second order stationary points of $NLP(t_j)$ for j sufficiently large. Therefore, the desired results can be obtained from Theorem 3.1. □

According to Theorem 4.5, we note that the sequence generated by Algorithm 1 satisfies the approximate second order stationary conditions for j sufficiently large under suitable conditions. Our feasible strategy shows the advantages of approximate stationary conditions as termination criteria from a numerical point of view.

5 Numerical Results

In this section, we concern a numerical implementation of Algorithm 1 described in the previous section. We apply Algorithm 1 to compute an approximate optimal solution of (1.1). To investigate the numerical behavior of Algorithm 1 for MPCC, we write an experimental code and test some problems in MacMPEC database available at https://wiki.mcs.anl.gov/leyffer/index.php/MacMPEC. These problems are transformed in the form (4.3) and coded in Matlab 2014b. In our experiments, we chose the values for the parameters as: $\gamma = 0.5, \epsilon_1 = 10^{-5}, \gamma_s = 10^{-3}, \theta = 10^{-5}, \varepsilon' = 10^{-5}$, $\tau_0 = 1$, $\widetilde{\gamma} = 0.2$, $\beta = 10^{-5}$, $\psi^{max}_{\mathrm{v},0} = 10^3, \psi^{max}_{\mathrm{o},0} = 10^3, y_{\max} = 10^6, u^R_0 = 10^{-4}, u_1 = 1$. Table 5.1 summarizes the results, where f_{gen} is the recommended function value in MacMPEC database, f^* denotes the final value and z^* is the final solution. The maximal degree of constraint violation from [6] is usually used to measure the feasibility of the final iterate z^*, which is defined by

$$\mathrm{maxvio}(z^*) = \max\{\|\max\{g(z^*), 0\}\|, \|h(z^*)\|, \|\min\{G(z^*), H(z^*)\}\|\}.$$

The objective function value and maxvio(z^*) can measure the accuracy of the solutions for MPCC.

From Table 5.1, we notice that Algorithm 1 does well on almost test problems of relatively small and medium sizes. Algorithm 1 can obtain the same optimal value for these test problems as the recommended function value in MacMPEC database except for ex 9.1.2 and ex 9.2.5. Actually, the final objective values of ex 9.1.2 and ex 9.2.5 are similar to the numerical results in [8]. We discuss the numerical results of ex 9.1.2 and ex 9.2.5 in more detail. By our algorithm, we obtain that the final iterates z^* of ex 9.1.2 and ex 9.2.5 tend to (3, 0, 6, 9, 0, 0, 0, 0, 1, 0) and (3, 5, 2, 9, 1, 0, 0, 0) respectively. By verification, these limit points are M-stationary points of MPCC and satisfy MPCC-LICQ. Hence, these numerical results confirm our theoretical convergence properties, which implies Algorithm 1 is reasonable. Furthermore, the maximum violation of all constraints is less than 10^{-4} for all test instances. Numerical results of these test problems indicate the applicability of our algorithm.

6 Conclusion

In this paper, we focus on an inexact log-exponential regularization method for MPCC. Due to the fact that the KKT points are not useful from a practical standpoint. We consider

Table 5.1: Numerical results on some problems from MacMPEC.

	Problem	f_{gen}	f^*	maxvio(z^*)
1	bard1	17.000	17.000	6.750e-14
2	bard1m	17.000	17.000	5.695e-05
3	dempe	28.250	28.250	1.870e-09
4	desilva	-1.000	-1.000	5.598e-05
5	df1	0.000	2.000e-08	2.082e-06
6	ex 9.1.1	-13.000	13.000	5.112e-05
7	ex 9.1.2	-6.250	-3.004	2.138e-05
8	ex 9.1.4	-37.000	-37.000	3.803e-05
9	ex 9.1.5	-1.000	-1.000	7.919e-05
10	ex 9.1.6	-49.000	-49.000	5.798e-06
11	ex 9.1.8	-3.250	-3.248	1.428e-05
12	ex 9.2.1	17.000	17.000	9.318e-05
13	ex 9.2.2	100.000	100.000	7.145e-05
14	ex 9.2.4	0.500	0.500	5.758e-08
15	ex 9.2.5	6.000	9.000	1.320e-05
16	ex 9.2.6	-1.000	-1.000	2.403e-06
17	ex 9.2.9	2.000	2.001	2.549e-05
18	gauvin	20.000	20.000	4.169e-06
19	flp2	0.000	2.603e-12	8.721e-09
20	gnash10	-230.823	-230.823	8.215e-08
21	gnash11	-129.912	-129.912	2.261e-06
22	gnash12	-36.933	-36.933	1.053e-07
23	gnash13	-7.062	-7.062	8.609e-08
24	gnash14	-0.179	-0.179	4.406e-05
25	gnash15	-354.699	-354.699	8.000e-05
26	gnash16	-241.442	-241.442	3.886e-07
27	gnash17	-90.749	-90.749	1.137e-06
28	gnash18	-25.698	-25.698	1.736e-07
29	gnash19	-6.117	-6.117	1.303e-05
30	jr1	0.500	0.500	0
31	jr2	0.500	0.500	5.194e-10
32	kth1	0.000	2.787e-08	1.999e-05
33	kth2	0.000	0.000	0
34	kth3	0.500	0.500	2.912e-07
35	nash1a	0.000	1.652e-13	1.098e-05
36	nash1b	0.000	3.638e-15	2.323e-05
37	nash1c	0.000	5.902e-13	1.769e-05
38	nash1d	0.000	1.093e-09	7.784e-06
39	nash1e	0.000	1.977e-13	1.125e-05
40	outrata31	3.208	3.208	8.966e-05
41	outrata32	3.449	3.449	5.704e-06
42	outrata33	4.604	4.604	5.573e-07
43	outrata34	6.593	6.593	7.463e-06
44	qpec1	80.000	80.000	2.167e-08
45	qpec2	45.000	45.000	8.860e-07
46	ralph2	0.000	-2.243e-13	3.484e-08
47	scholtes1	2.000	2.000	2.694e-08
48	scholtes3	0.500	0.500	1.253e-12
49	scholtes4	0.000	-6.286e-09	3.248e-09
50	scholtes5	1.000	1.000	2.186e-09
51	scale1	1.000	1.000	2.802e-11
52	scale2	1.000	1.000	2.737e-08
53	scale3	1.000	1.000	3.320e-07
54	scale4	1.000	1.000	1.629e-08
55	scale5	100.000	100.000	4.971e-09
56	stackelberg1	-3266.670	-3266.670	5.555e-09

a sequence of the approximate second order stationary points of the regularized problems and the limit point is M-stationary of MPCC. In addition, we propose a second order primal-dual stabilized SQP method to solve MPCC, where the regularized subproblems are solved inexactly. It is shown that the approximate conditions are appropriate choices of the termination criteria. Furthermore, the experiments illustrate that the approach is effective to solve MPCC from a practical point of view.

Acknowledgements

The authors would like to thank the editor and anonymous referees whose careful comments and suggestions have greatly improved the quality of this work.

References

[1] R. Andreani, G. Haeser, L.D. Secchin and P.J.S. Silva, New sequential optimality conditions for mathematical programs with complementarity constraints and algorithmic consequences, *SIAM J. Optim.* 29 (2019) 3201–3230.

[2] R. Andreani, L.D. Secchin and P.J.S. Silva, Convergence properties of a second order augmented Lagrangian method for mathematical programs with complementarity constraints, *SIAM J. Optim.* 28 (2018) 2574–2600.

[3] S. Dempe, Annotated bibliography on bilevel programming and mathematical programs with equilibrium constraints, *Optimization* 52 (2003) 333–359.

[4] R. Fletcher, S. Leyffer, D. Ralph and S. Scholtes, Local convergence of SQP methods for mathematical programs with equilibrium constraints, *SIAM J. Optim.* 17 (2006) 259–286.

[5] P. E. Gill and V. Kungurtsev, A stabilized SQP method: global convergence, *IMA J. Numer. Anal.* 37 (2017) 407–443.

[6] T. Hoheisel, C. Kanzow and A. Schwartz, Theoretical and numerical comparison of relaxation methods for mathematical programs with complementarity constraints, *Math. Program.* 137 (2011) 257–288.

[7] S. He, T. Shen and J. Zhang, A smoothing method for mathematical programs with complementarity constraints, *Pacific J. Optim.* 16 (2020) 525–545.

[8] S. Jiang, Z. Jin, C. Chen and G. Lin, Smoothing partial exact penalty splitting method for mathematical programs with equilibrium constraints, *J. Optim. Theory Appl.* 70 (2018) 223–236.

[9] A. Kadrani, J. P. Dussault and A. Benchakroun, A new regularization scheme for mathematical programs with complementarity constraints, *SIAM J. Optim.* 20 (2009) 78–103.

[10] C. Kanzow and A. Schwartz, A new regularization method for mathematical programs with complementarity constraints with strong convergence properties, *SIAM J. Optim.* 23 (2013) 770–798.

[11] C. Kanzow and A. Schwartz, The price of inexactness: convergence properties of relaxation methods for mathematical programs with complementarity constraints revisited, *Math. Oper. Res.* 40 (2015) 253–275.

[12] G.H. Lin and M. Fukushima, A modified relaxation scheme for mathematical programs with complementarity constraints, *Ann. Oper. Res.* 133 (2005) 63–84.

[13] S. Leyfer, G. López-Calva and J. Nocedal, Interior methods for mathematical programs with complementarity constraints, *SIAM J. Optim.* 17 (2006) 52–77.

[14] Y. Li, T. Tan and X. Li, A log-exponential smoothing method for mathematical programs with complementarity constraints, *Appl. Math. Comput.* 218 (2012) 5900–5909.

[15] Z.Q. Luo, J.S. Pang and D. Ralph, *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, Cambridge, 1996.

[16] J. Liu, T. Zhang, Y.X. Fan, B. Han and Y. Zheng, An objective penalty method for optimistic bilevel programming problems, *J. Oper. Res. Soc. China.* 8 (2020) 177–187.

[17] H.D. Qi and L.Z. Liao, A smoothing Newton method for extended vertical linear complementarity problems, *SIAM J. Matrix Anal. Appl.* 21 (1999) 45–66.

[18] Z. Wan and J. Chen, On bilevel variational inequalities, *J. Oper. Res. Soc. China* 1 (2013) 483–510.

[19] A. Ramos, Mathematical programs with equilibrium constraints: a sequential optimality condition, new constraint qualifications and algorithmic consequences, *Optim. Methods Softw.* 36 (2021) 45–81.

[20] S. Scholtes, Convergence properties of a regularization scheme for mathematical programs with complementarity constraints, *SIAM J. Optim.* 11 (2001) 918–936.

[21] H. Scheel and S. Scholtes, Mathematical programs with complementarity constraints: Stationarity, optimality, and sensitivity, *Math. Oper. Res.* 25 (2000) 1–22.

[22] H. Yin and J. Zhang, Global convergence of a smooth approximation method for mathematical programs with complementarity constraints, *Math. Methods Oper. Res.* 64 (2006) 255–269.

Manuscript received 17 September 2021
revised 23 December 2021
accepted for publication 24 January 2022

Na Xu
School of Mathematics
Liaoning Normal University, Dalian 116029, China
E-mail address: xuna19890224@163.com

Li-Ping Pang
School of Mathematical Sciences
Dalian University of Technology, Dalian 116024, China
E-mail address: lppang@dlut.edu.cn

Jian Lv
School of Finance, Zhejiang University of
Finance and Economics, Hangzhou 310018, China
E-mail address: lvjian328@163.com

Menglong Xue
School of Mathematical Sciences
Dalian University of Technology, Dalian 116024, China
E-mail address: mlx0819@mail.dlut.edu.cn