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TWO METHODS FOR FINDING A SPARSE SOLUTION OF THE LINEAR COMPLEMENTARITY PROBLEM WITH Z-MATRIX*

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Abstract: In this paper, we propose two numerical methods to find a sparse solution of the linear complementarity problem (LCP) with a Z-matrix. The first one is an iterative method based on solving the lower-dimensional linear equations by using Gaussian elimination, which terminates at a sparsest solution of the LCP within a finite number of iterations, and the computational complexity of the method is $\mathcal{O}(\mu^3)$ where μ is the number of non-zero elements in the sparsest solution of the LCP. The second one is a fixed point iterative method starting from a feasible point of the LCP, which converges monotonically downward to a solution of the LCP, and specially, it can be used to find a sparse solution of the LCP if the starting point is sparse. Compared with several existing methods, the numerical results show the advantage and the effectiveness of the proposed methods.

Key words: *linear complementarity problem, Z-matrix, sparse solution, iterative method*

Mathematics Subject Classification: *90C33, 65K15*

1 Introduction

Given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $\mathbf{q} \in \mathbb{R}^n$, the *linear complementarity problem*, abbreviated as LCP$(A, \mathbf{q})$, is to find a vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathbf{x} \geq \mathbf{0}, \quad A\mathbf{x} + \mathbf{q} \geq \mathbf{0}, \quad \mathbf{x}^\top(A\mathbf{x} + \mathbf{q}) = 0,$$

which has been extensively studied due to its wide applications in engineering and economics, such as bimatrix game, market equilibrium, and so on [9, 11]. Moreover, as the optimality conditions for quadratic programs, it plays a vital role in optimization research [6]. In this paper, we denote the solution set of LCP$(A, \mathbf{q})$ by SOL$(A, \mathbf{q})$ and its feasible set by

$$\text{FEA}(A, \mathbf{q}) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0},\ A\mathbf{x} + \mathbf{q} \geq \mathbf{0}\}.$$

It is well known that sparsity often exists in many practical problems, such as compressed sensing, signal processing, machine learning, sensor location, and so on [8]. Sparsity related problems have attracted lots of attentions and obtained rapid developments in recent years

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[1, 7]. In the last decades, both the theory and numerical methods for finding a sparse solution of a system of linear equations have been studied extensively [10]. In contrast with the fast and great development in the sparse solution of linear equations, the study on the sparse solution of LCP$(A, \mathbf{q})$ is few, from the perspective of both theory and algorithms. Chen and Xiang [5] considered the characterization and computation of sparse solutions and least-p-norm $(0 < p < 1)$ solutions of LCP$(A, \mathbf{q})$, and provided conditions on A such that a sparse solution can be found by solving convex minimization. Shang *et al.* [19] proposed an l_p-norm $(0 < p < 1)$ regularized minimization to approximate the sparse solution of LCP$(A, \mathbf{q})$ by sequentially decreasing the regularization parameter, and proposed a half thresholding projection algorithm for $l_{1/2}$ regularization model in [18]. Recently, Zhou *et al.* [22] proposed a Newton hard thresholding pursuit to solve LCP$(A, \mathbf{q})$ via a new merit function.

When FEA$(A, \mathbf{q}) \neq \emptyset$ and A is a Z-matrix, FEA$(A, \mathbf{q})$ contains a least element, which is a sparsest solution of LCP$(A, \mathbf{q})$ [6]. For LCP$(A, \mathbf{q})$ with A being a Z-matrix, one of the classic methods for finding the least solution is to solve the following linear program [16]:

$$\begin{array}{ll} \min & \mathbf{p}^\top \mathbf{x} \\ \text{s.t.} & \mathbf{x} \geq \mathbf{0}, \quad A\mathbf{x} + \mathbf{q} \geq \mathbf{0} \end{array}$$

for any positive vector $\mathbf{p} \in \mathbb{R}^n$; and another classic method was proposed by Chandrasekaran [2], which can be viewed as a special principal pivoting method (also see Chapter 4 in [6]). Recently, Chen and Xiang [3] presented an implicit solution function for LCP$(A, \mathbf{q})$ with Z-matrix A, and apply it to find a sparsest solution of LCP$(A, \mathbf{q})$. Luo *et al.* [15] showed that a class of LCP$(A, \mathbf{q})$ can be exactly solved via a nonconvex l_p-norm minimization $(0 < p < 1)$. Moreover, the least element solution has been used in the time stepping scheme to find stable solutions of dynamic linear complementarity systems in [4, 20].

In this paper, we further study the numerical methods for finding a sparse solution of LCP$(A, \mathbf{q})$ with A being a Z-matrix. The contribution of this paper is as follows:

(i) Based on the discrimination of positive components in the sparsest solution of LCP$(A, \mathbf{q})$, we propose an iterative method by solving lower-dimensional linear equations to find a sparsest solution of LCP$(A, \mathbf{q})$. The obtained solution is the least solution of LCP$(A, \mathbf{q})$. In such a method, actually only a lower-dimensional linear system of no exceeding μ equations and μ variables need to be solved, where μ is the number of non-zero elements in the sparsest solution to LCP$(A, \mathbf{q})$.

(ii) Based on a new fixed point equation reconstruction for LCP$(A, \mathbf{q})$, we propose a fixed point iterative method for solving LCP$(A, \mathbf{q})$ starting from a feasible point, and prove that the resulting sequence of iterations descends monotonically to a solution of the problem. Combined with a strategy to find a sparse feasible solution, this method can be used to find a sparse solution for LCP$(A, \mathbf{q})$.

(iii) The two methods are simple and easy to implement. Numerical simulation results show that these two methods are superior to some classical methods [2, 16].

The rest of this paper is organized as follows. In Section 2, we give some basic symbols, definitions and results. An iterative method based on lower-dimensional linear equations for finding a sparsest solution of LCP$(A, \mathbf{q})$ is proposed in Section 3. In Section 4, we propose a monotonically decreasing fixed point method to solve LCP$(A, \mathbf{q})$, and give a preconditioning technique to find a feasible point for it. The convergence result is also established in this section. In Section 5, we show the numerical performances of the two proposed methods compared to the classical ones. Conclusions are given in Section 6.

2 Preliminaries

For a positive integer n, we use $[n]$ to denote $\{1, 2, \ldots, n\}$. Let index sets $T, S \subseteq [n]$. We use $|S|$ to denote the cardinality of the set S. For any $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x}_S$ denotes the subvector of $\mathbf{x}$ containing components corresponding to the indices in S. For a matrix $A \in \mathbb{R}^{n\times n}$, $\mathbf{a}_{\cdot j}$ denotes the j-th column of A, $A_{\cdot S}$ denotes the submatrix of A comprising columns corresponding to the indices in S, and A_{TS} denotes a matrix obtained by deleting from A all rows except those of the indices in T and all columns except those of the indices in S.

Let $\mathbb{R}^n_+ := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}\}$ and $\mathbb{R}^n_{++} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} > \mathbf{0}\}$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we use $\mathbf{x}^\top \mathbf{y}$ to denote the Euclidean inner product, $\mathbf{x} \circ \mathbf{y}$ to denote the Hadamard product, and $\|\mathbf{x}\|$ to denote the 2-norm of $\mathbf{x}$.

Definition 2.1 ([6])**.** A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is said to be a Z-matrix if its off-diagonal entries are non-positive.

Definition 2.2 ([6])**.** A subset $S \subseteq \mathbb{R}^n$ is bounded below if there exists a vector $\mathbf{u} \in \mathbb{R}^n$ such that $\mathbf{x} \geq \mathbf{u}$ for all vectors $\mathbf{x} \in S$. If such a vector $\mathbf{u}$ happens to belong to S, then $\mathbf{u}$ is called a least element of S.

Theorem 2.3 ([6])**.** *Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix and $\mathbf{q} \in \mathbb{R}^n$. If LCP$(A, \mathbf{q})$ is feasible, then FEA$(A, \mathbf{q})$ contains a least element $\mathbf{u}$. Moreover, $\mathbf{u}$ solves LCP$(A, \mathbf{q})$.*

Throughout the paper, we assume that FEA$(A, \mathbf{q}) \neq \emptyset$. Moreover, if $\mathbf{u}$ is the least element of FEA$(A, \mathbf{q})$, then $\mathbf{u}$ is said to be the least solution of LCP$(A, \mathbf{q})$.

3 An Iterative Method Based on Lower-Dimensional Equations

In this section, we propose an iterative method to find a sparsest solution of LCP$(A, \mathbf{q})$ with A being a Z-matrix by sequentially solving lower-dimensional linear equations, where the lower-dimensional linear equations are solved by Gaussian elimination method.

We have the following two simple observations. First, if $\mathbf{q} \in \mathbb{R}^n_+$, then it is easy to see that $\mathbf{0} \in \mathbb{R}^n$ is the unique sparsest solution of LCP$(A, \mathbf{q})$, which is also the least element of FEA$(A, \mathbf{q})$. Second, if $\mathbf{q} \notin \mathbb{R}^n_+$, i.e., there exists at least an index $i \in [n]$ such that $q_i < 0$, then we have that $a_{ii} > 0$ for all $i \in [n]$ satisfying $q_i < 0$, since A is a Z-matrix and FEA$(A, \mathbf{q}) \neq \emptyset$.

In the following, without loss of generality, we assume that $\mathbf{q} \notin \mathbb{R}^n_+$. We denote

$$\Upsilon := \{i \in [n] : q_i < 0\} \quad \text{and} \quad \Omega := \{i \in [n] : a_{ii} > 0\},$$

and $\Upsilon^c := [n] \setminus \Upsilon$ and $\Omega^c := [n] \setminus \Omega$. Denote $s := |\Upsilon|$ and $l := |\Omega|$. Then, by $\emptyset \neq \Upsilon \subseteq \Omega$, we have $0 < s \leq l \leq n$.

Remark 3.1. It is obvious that $\Upsilon = \Omega \bigcap \Upsilon$ and $\Upsilon^c = (\Omega \setminus \Upsilon) \bigcup \Omega^c$.

Next, we construct a matrix $B \in \mathbb{R}^{n\times l}$ satisfying $B = A_{\cdot\Omega}$, and denote

$$\text{FEA}(B, \mathbf{q}) := \{\mathbf{y} \in \mathbb{R}^l : \mathbf{y} \geq \mathbf{0}, B\mathbf{y} + \mathbf{q} \geq \mathbf{0}\}.$$

It is easy to see that $B = A$ when $\Omega^c = \emptyset$. We have the following result.

Theorem 3.2. *Suppose that $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a Z-matrix and $\mathbf{q} \in \mathbb{R}^n$. Let $B = A_{\cdot\Omega} \in \mathbb{R}^{n\times l}$ be the submatrix of A.*

(i) *If $\hat{\mathbf{x}} \in \mathrm{FEA}(A, \mathbf{q})$, then $\hat{\mathbf{y}} = \hat{\mathbf{x}}_\Omega \in \mathrm{FEA}(B, \mathbf{q})$.*

(ii) *If $\bar{\mathbf{y}} \in \mathrm{FEA}(B, \mathbf{q})$, then $\bar{\mathbf{x}} \in \mathrm{FEA}(A, \mathbf{q})$ with $\bar{\mathbf{x}}_\Omega = \bar{\mathbf{y}}$ and $\bar{\mathbf{x}}_{\Omega^c} = \mathbf{0}$.*

Proof. (i) Since $\hat{\mathbf{x}} \in \mathrm{FEA}(A, \mathbf{q})$, we have that $\hat{\mathbf{x}} \geq \mathbf{0}$ and $A\hat{\mathbf{x}} + \mathbf{q} \geq \mathbf{0}$. Then,

$$\hat{\mathbf{y}} = \hat{\mathbf{x}}_\Omega \geq \mathbf{0} \quad \text{and} \quad B\hat{\mathbf{y}} + \mathbf{q} = A_{\cdot\Omega}\hat{\mathbf{x}}_\Omega + \mathbf{q} \geq -A_{\cdot\Omega^c}\hat{\mathbf{x}}_{\Omega^c} \geq \mathbf{0},$$

where the last inequality holds since $A_{\cdot\Omega^c} \leq 0$ by the fact that A is a Z-matrix and $a_{ii} \leq 0$ for any $i \in \Omega^c$. Thus, $\hat{\mathbf{y}} \in \mathrm{FEA}(B, \mathbf{q})$.

(ii) Since $\bar{\mathbf{y}} \in \mathrm{FEA}(B, \mathbf{q})$, we have that $\bar{\mathbf{y}} \geq \mathbf{0}$ and $B\bar{\mathbf{y}} + \mathbf{q} \geq \mathbf{0}$. Then, we have

$$\bar{\mathbf{x}} \geq \mathbf{0} \quad \text{and} \quad A\bar{\mathbf{x}} + \mathbf{q} = A_{\cdot\Omega}\bar{\mathbf{x}}_\Omega + A_{\cdot\Omega^c}\bar{\mathbf{x}}_{\Omega^c} + \mathbf{q} = B\bar{\mathbf{y}} + \mathbf{q} \geq \mathbf{0}.$$

Thus, $\bar{\mathbf{x}} \in \mathrm{FEA}(A, \mathbf{q})$. The proof is complete.

Theorem 3.2 demonstrates that in order to find a sparsest solution of $\mathrm{LCP}(A, \mathbf{q})$, we only need to find the least element of $\mathrm{FEA}(B, \mathbf{q})$. This may reduce the number of variables to be examined when $\Omega^c \neq \emptyset$. In the following, instead of $\mathrm{LCP}(A, \mathbf{q})$, we consider $\mathrm{FEA}(B, \mathbf{q})$.

For convenience, we denote

$$\Omega \equiv \{i_1, i_2, \ldots, i_l\} \text{ with } i_1 < i_2 < \cdots < i_l, \tag{3.1}$$

and let the j-th column of $B \in \mathbb{R}^{n\times l}$ be the i_j-th column of A, i.e.,

$$\mathbf{b}_{\cdot j} := \mathbf{a}_{\cdot i_j} \text{ for all } j \in [l] \text{ satisfying } i_j \in \Omega. \tag{3.2}$$

Then, we can obtain the following result.

Theorem 3.3. *Suppose that $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a Z-matrix and $\mathbf{q} \in \mathbb{R}^n$. Let $B = A_{\cdot\Omega} \in \mathbb{R}^{n\times l}$ and $\bar{\mathbf{y}} \in \mathrm{FEA}(B, \mathbf{q})$. Then*

$$\bar{y}_j > 0, \quad \forall j \in [l] \text{ satisfying } i_j \in \Upsilon.$$

Proof. Suppose that $\bar{y}_{j_0} = 0$ for some $j_0 \in [l]$ with $i_{j_0} \in \Upsilon$, then there is a contradiction to $\bar{\mathbf{y}} \in \mathrm{FEA}(B, \mathbf{q})$, as we can show that for any $\bar{\mathbf{y}} \geq \mathbf{0}$,

$$(B\bar{\mathbf{y}} + \mathbf{q})_{i_{j_0}} = b_{i_{j_0}j_0}\bar{y}_{j_0} + \sum_{j\in[l]\setminus\{j_0\}} b_{i_{j_0}j}\bar{y}_j + q_{i_{j_0}} < 0. \tag{3.3}$$

In fact, (3.3) holds since $q_{i_{j_0}} < 0$ and $b_{i_{j_0}j} \leq 0$ for any $j \in [l] \setminus \{j_0\}$ by (3.2) and the assumption that A is a Z-matrix. The proof is complete.

Next, we design an iterative method to find a sparsest solution of $\mathrm{LCP}(A, \mathbf{q})$ based on Theorems 3.2 and 3.3. For convenience, we denote

$$\Upsilon \equiv \{h_1, h_2, \ldots, h_s\} \text{ with } h_1 < h_2 < \cdots < h_s,$$

and $\langle h_k \rangle := \{h_1, h_2, \ldots, h_k\}$ for any $k \in [s]$. Then, for any $h_p \in \Upsilon \subseteq \Omega$ with $p \in [s]$, there exists some $j_p \in [l]$ such that $h_p = i_{j_p}$. We consider the following system of linear equalities and inequalities with $\mathbf{y} \in \mathbb{R}^l_+$:

$$\begin{cases} b_{h_pj_p}y_{j_p} + \sum\limits_{j\in[l]\setminus\{j_p\}} b_{h_pj}y_j = -q_{h_p}, & \forall h_p \in \Upsilon, \\ \sum\limits_{j\in[l]} (-b_{ij})y_j \leq q_i, & \forall i \in \Upsilon^c. \end{cases} \tag{3.4}$$

For convenience, we will use $\langle j_k\rangle := \{j_1, \dots, j_k\}$ for any $k \in [s]$.

Procedure 1: We use Gaussian elimination method to solve the equations in (3.4), and update the inequalities in (3.4) meantime. Specifically, for any $k \in [s]$, (3.4) turns to

$$\begin{cases} y_{j_p} + \sum\limits_{j\in[l]\setminus\langle j_k\rangle} b^{(k)}_{h_p j} y_j = -q^{(k)}_{h_p}, & \forall h_p \in \langle h_k\rangle\ (p \in [k]),\\ \sum\limits_{j\in[l]\setminus\langle j_k\rangle} b^{(k)}_{ij} y_j = -q^{(k)}_{i}, & \forall i \in \Upsilon \setminus \langle h_k\rangle,\\ \sum\limits_{j\in[l]\setminus\langle j_k\rangle} (-b^{(k)}_{ij}) y_j \le q^{(k)}_{i}, & \forall i \in \Upsilon^c, \end{cases} \tag{3.5}$$

where for $k = 1$:

$$\begin{cases} q^{(1)}_{h_1} := \frac{q_{h_1}}{b_{h_1 j_1}}, & b^{(1)}_{h_1 j} := \frac{b_{h_1 j}}{b_{h_1 j_1}}\ (\forall j \in [l]\setminus\{j_1\}),\\ q^{(1)}_{i} := q_i - q^{(1)}_{h_1} b_{i j_1}, & b^{(1)}_{ij} := b_{ij} - b^{(1)}_{h_1 j} b_{i j_1}\ (\forall j \in [l]\setminus\{j_1\}), \quad \forall i \ne h_1, \end{cases}$$

and for any $k \in \{2, \dots, s\}$:

$$\begin{cases} q^{(k)}_{h_k} := \frac{q^{(k-1)}_{h_k}}{b^{(k-1)}_{h_k j_k}}, & b^{(k)}_{h_k j} := \frac{b^{(k-1)}_{h_k j}}{b^{(k-1)}_{h_k j_k}}\ (\forall j \in [l]\setminus\langle j_k\rangle),\\ q^{(k)}_{i} := q^{(k-1)}_{i} - q^{(k)}_{h_k} b^{(k-1)}_{i j_k}, & b^{(k)}_{ij} := b^{(k-1)}_{ij} - b^{(k)}_{h_k j} b^{(k-1)}_{i j_k}\ (\forall j \in [l]\setminus\langle j_k\rangle), \quad \forall i \ne h_k. \end{cases}$$

By **Procedure 1**, we finally obtain equivalently the following system of equations and inequalities with nonnegative variables y_j for all $j \in [l]$:

$$\begin{cases} y_{j_p} + \sum\limits_{j\in[l]\setminus\langle j_s\rangle} b^{(s)}_{h_p j} y_j = -q^{(s)}_{h_p}, & \forall h_p \in \Upsilon\ (p \in [s]),\\ \sum\limits_{j\in[l]\setminus\langle j_s\rangle} (-b^{(s)}_{ij}) y_j \le q^{(s)}_{i}, & \forall i \in \Upsilon^c. \end{cases} \tag{3.6}$$

Lemma 3.4. *In* **Procedure 1***, for any $h_p \in \Upsilon$ with $p \in [s]$, we have the following results.*

(i) *For any $k \in [s]$, it holds that $q^{(k)}_{h_p} < 0$ and*

$$b^{(k)}_{h_p j} \le 0, \quad \forall j \in [l]\setminus\langle j_k\rangle,\ j \ne j_p. \tag{3.7}$$

Specially, we have that $q^{(s)}_{h_p} < 0$ and

$$b^{(p)}_{h_p j} \le 0, \quad \forall j \in [l]\setminus\langle j_p\rangle. \tag{3.8}$$

(ii) *It holds that*

$$b^{(p-1)}_{h_p j_p} > 0, \quad \forall h_p \ne h_1,\ j_p \ne j_1. \tag{3.9}$$

Proof. We first show that (i) and (ii) hold when $k = 1$. Firstly, we show that (i) holds. After the first elimination, we have that

$$\begin{cases} q^{(1)}_{h_1} := \frac{q_{h_1}}{b_{h_1 j_1}} < 0,\\ q^{(1)}_{h_p} := q_{h_p} - q^{(1)}_{h_1} b_{h_p j_1} < 0, \quad \forall h_p \in \Upsilon \setminus \{h_1\}, \end{cases} \tag{3.10}$$

since $q_{h_1}, q_{h_p} < 0$ by $h_1 \in \Upsilon$, $b_{h_1 j_1} > 0$ by $h_1 \in \Omega$, and $b_{h_p j_1} \leq 0$ by (3.2) and the fact that A is a Z-matrix. Moreover, for any $j \in [l] \setminus \{j_1\}$, we have that

$$\begin{cases} b_{h_1 j}^{(1)} := \frac{b_{h_1 j}}{b_{h_1 j_1}} \leq 0, \\ b_{h_p j}^{(1)} := b_{h_p j} - b_{h_1 j}^{(1)} b_{h_p j_1} \leq 0, \quad \forall h_p \in \Upsilon \setminus \{h_1\}, \quad j \neq j_p, \end{cases} \tag{3.11}$$

since $b_{h_1 j_1} > 0$ and $b_{h_1 j}, b_{h_p j}, b_{h_p j_1} \leq 0$ by (3.2) and the assumption that A is a Z-matrix. Secondly, based on (i), we show that (ii) holds for $p = 2$, i.e.,

$$b_{h_2 j_2}^{(1)} > 0.$$

Suppose by contradiction that $b_{h_2 j_2}^{(1)} \leq 0$. We consider (3.5):

$$b_{h_2 j_2}^{(1)} y_{j_2} + \sum_{j \in [l] \setminus \langle j_2 \rangle} b_{h_2 j}^{(1)} y_j = -q_{h_2}^{(1)}. \tag{3.12}$$

By (3.11), we have that $b_{h_2 j}^{(1)} \leq 0$ for any $j \in [l] \setminus \langle j_2 \rangle$. This, together with $\mathbf{y} \in \mathbb{R}_+^l$, implies that the left-hand side of (3.12) is non-positive; while the right-hand side of (3.12) is positive by (3.10). This is a contradiction.

Next, we apply the induction, and assume that (i) and (ii) holds for case of $k-1$ with $k \in \{2, \ldots, s\}$, i.e., after the $(k-1)$-th elimination, the following results hold:

$$q_{h_p}^{(k-1)} < 0, \quad \forall h_p \in \Upsilon; \tag{3.13}$$

$$b_{h_p j}^{(k-1)} \leq 0, \quad \forall h_p \in \Upsilon \setminus \{h_{k-1}\}, \ \forall j \in [l] \setminus \langle j_{k-1} \rangle, \ j \neq j_p; \tag{3.14}$$

$$b_{h_k j_k}^{(k-1)} > 0. \tag{3.15}$$

In the following, we show that (i) and (ii) hold for case of k with $k \in \{2, \ldots, s\}$. Specifically, after the k-th elimination, from (3.13)-(3.15), we can obtain that (i) holds, i.e.,

$$\begin{cases} q_{h_k}^{(k)} := \frac{q_{h_k}^{(k-1)}}{b_{h_k j_k}^{(k-1)}} < 0, \\ q_{h_p}^{(k)} := q_{h_p}^{(k-1)} - q_{h_k}^{(k)} b_{h_p j_k}^{(k-1)} < 0, \quad \forall h_p \in \Upsilon \setminus \{h_k\}, \end{cases} \tag{3.16}$$

and also for any $j \in [l] \setminus \langle j_k \rangle$, it holds that

$$\begin{cases} b_{h_k j}^{(k)} := \frac{b_{h_k j}^{(k-1)}}{b_{h_k j_k}^{(k-1)}} \leq 0, \\ b_{h_p j}^{(k)} := b_{h_p j}^{(k-1)} - b_{h_k j}^{(k)} b_{h_p j_k}^{(k-1)} \leq 0, \quad \forall h_p \in \Upsilon \setminus \{h_k\}, \quad j \neq j_p. \end{cases} \tag{3.17}$$

Based on (i), we show that (ii) holds for $p \in \{3, \ldots, s\}$. It is worth noting that after $k = s$, **Procedure 1** terminates. After the k-th elimination for $k \in \{2, \ldots, s-1\}$, it holds that

$$b_{h_{k+1} j_{k+1}}^{(k)} > 0.$$

If not, from $b_{h_{k+1} j}^{(k)} \leq 0$ by (3.17) and $y_j \geq 0$ for any $j \in [l] \setminus \langle j_{k+1} \rangle$, it follows that the left-hand side of the h_{k+1}-th equation $b_{h_{k+1} j_{k+1}}^{(k)} y_{j_{k+1}} + \sum_{j \in [l] \setminus \langle j_k \rangle} b_{h_{k+1} j}^{(k)} y_j = -q_{h_{k+1}}^{(k)}$ in system (3.5) is non-positive, which contradicts to $-q_{h_{k+1}}^{(k)} > 0$ by (3.16).

The proof is complete.

Lemma 3.4 characterizes the property of the s equations in system (3.5) in **Procedure 1**. The next lemma discusses the property of inequalities in system (3.5) corresponding to index Υ^c after **Procedure 1**.

Lemma 3.5. *In* **Procedure 1***, for any $k \in [s]$, the following results hold.*

(i) *For any $i \in \Omega \setminus \Upsilon$, we have*

$$b_{ij}^{(k)} \leq 0, \quad \forall j \in [l] \setminus \langle j_k \rangle \text{ with } i_j \neq i. \tag{3.18}$$

(ii) *For any $i \in \Omega^c$, we have*

$$b_{ij}^{(k)} \leq 0, \quad \forall j \in [l] \setminus \langle j_k \rangle, \tag{3.19}$$

and hence, $q_i^{(s)} \geq 0$.

Proof. (i) We first show (3.18) when $k = 1$. For any $i \in \Omega \setminus \Upsilon$, it holds that

$$b_{ij}^{(1)} = b_{ij} - b_{h_1 j}^{(1)} b_{ij_1} \leq 0, \quad \forall j \in [l] \setminus \{j_1\} \text{ with } i_j \neq i,$$

since $b_{h_1 j}^{(1)} \leq 0$ by (3.8) and $b_{ij}, b_{ij_1} \leq 0$ by (3.2) and the assumption that A is a Z-matrix.

Let us apply the induction. Suppose that for any $k \in \{2, \ldots, s\}$, it holds that

$$b_{ij}^{(k-1)} \leq 0, \quad \forall j \in [l] \setminus \langle j_{k-1} \rangle \text{ with } i_j \neq i.$$

Then, by using $b_{h_k j}^{(k)} \leq 0$ from (3.8), we can derive that

$$b_{ij}^{(k)} = b_{ij}^{(k-1)} - b_{h_k j}^{(k)} b_{ij_k}^{(k-1)} \leq 0, \quad \forall j \in [l] \setminus \langle j_k \rangle \text{ with } i_j \neq i.$$

(ii) To show (3.19), for any $i \in \Omega^c$ when $k = 1$, we have

$$b_{ij}^{(1)} := b_{ij} - b_{h_1 j}^{(1)} b_{ij_1} \leq 0, \quad \forall j \in [l] \setminus \{j_1\},$$

since $b_{h_1 j}^{(1)} \leq 0$ by (3.8), and $b_{ij}, b_{ij_1} \leq 0$ by (3.2) and the assumption that A is a Z-matrix. Then, for $k \in \{2, \ldots, s\}$, by using $b_{h_k j}^{(k)} \leq 0$ from (3.8) and applying recursion, we can further obtain that

$$b_{ij}^{(k)} := b_{ij}^{(k-1)} - b_{h_k j}^{(k)} b_{ij_k}^{(k-1)} \leq 0, \quad \forall j \in [l] \setminus \langle j_k \rangle.$$

Next, we show that $q_i^{(s)} \geq 0$ for any $i \in \Omega^c$. By (3.19) and $y_j \geq 0$, together with the inequalities corresponding to $i \in \Omega^c$ in system (3.6), i.e., $\sum_{j \in [l] \setminus \langle j_s \rangle} (-b_{ij}^{(s)}) y_j \leq q_i^{(s)}$, the result holds obviously. The proof is complete.

Recall Remark 3.1, we have $\Upsilon^c = (\Omega \setminus \Upsilon) \cup \Omega^c$. From Lemma 3.5 (ii), we can obtain the following corollary.

Corollary 3.6. *After* **Procedure 1***, if there exists an index (or indices) $i \in \Upsilon^c$ satisfying $q_i^{(s)} < 0$, then it must belong to the set $\Upsilon^c \setminus \Omega^c = \Omega \setminus \Upsilon$.*

Next, we continue the procedure in two cases.

Case 1: $q_i^{(s)} \geq 0$ for all $i \in \Omega \setminus \Upsilon$.

In this case, we have $q_i^{(s)} \geq 0$ for all $i \in \Upsilon^c$ by Lemma 3.5, and can obtain the following theorem.

Theorem 3.7. *Suppose that $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a Z-matrix and $\mathbf{q} \in \mathbb{R}^n$. Let* $\mathrm{FEA}(A, \mathbf{q}) \neq \emptyset$. *After* **Procedure 1**, *if $q_i^{(s)} \geq 0$ for all $i \in \Omega \setminus \Upsilon$, then $\bar{\mathbf{x}} \in \mathbb{R}^n_+$ satisfying $\bar{\mathbf{x}}_\Upsilon = -\mathbf{q}_\Upsilon^{(s)}$ and $\bar{\mathbf{x}}_{\Upsilon^c} = \mathbf{0}$ is a sparsest solution of LCP$(A, \mathbf{q})$, which is also the least solution of LCP$(A, \mathbf{q})$.*

Proof. It is easy to verify that $\bar{\mathbf{x}} \in \mathrm{FEA}(A, \mathbf{q})$. On one hand, $\bar{\mathbf{x}} \in \mathbb{R}^n_+$ since $\bar{\mathbf{x}}_\Upsilon = -\mathbf{q}_\Upsilon^{(s)} \in \mathbb{R}^s_{++}$ by Lemma 3.4 (i). On the other hand, let $\bar{\mathbf{y}} = \bar{\mathbf{x}}_\Omega$, then $A\bar{\mathbf{x}} + \mathbf{q} = B\bar{\mathbf{y}} + \mathbf{q} \geq \mathbf{0}$ since $(B\bar{\mathbf{y}} + \mathbf{q})_\Upsilon = \mathbf{0}$ and $(B\bar{\mathbf{y}} + \mathbf{q})_{\Upsilon^c} \geq \mathbf{0}$ by (3.6). Obviously, $\bar{\mathbf{x}} \in \mathrm{SOL}(A, \mathbf{q})$. Furthermore, $\bar{\mathbf{x}}$ is a sparsest solution of LCP$(A, \mathbf{q})$ by Theorems 3.2 and 3.3.

Suppose that $\hat{\mathbf{x}} \in \mathrm{FEA}(A, \mathbf{q})$, let $\hat{\mathbf{y}} = \hat{\mathbf{x}}_\Omega$, then $\hat{\mathbf{y}} \in \mathrm{FEA}(B, \mathbf{q})$. By utilizing the feasibility condition and applying Gaussian elimination, when the procedure terminates, similar to (3.6), we have that

$$\hat{y}_{j_p} + \sum_{j\in[l]\setminus\langle j_s\rangle} b_{h_p j}^{(s)} \hat{y}_j \geq -q_{h_p}^{(s)}, \quad \forall h_p \in \Upsilon \text{ with } p \in [s],$$

and then, from $\hat{\mathbf{y}} \in \mathbb{R}^l_+$ and $b_{h_p j}^{(s)} \leq 0$ by (3.7), we can further deduce that $\hat{y}_{j_p} \geq -q_{h_p}^{(s)}$. Thus, it holds that

$$\hat{\mathbf{x}}_\Upsilon \geq -\mathbf{q}_\Upsilon^{(s)} = \bar{\mathbf{x}}_\Upsilon,$$

which together with the fact that $\hat{\mathbf{x}}_{\Upsilon^c} \geq \mathbf{0} = \bar{\mathbf{x}}_{\Upsilon^c}$, implies that $\bar{\mathbf{x}}$ is the least solution of LCP$(A, \mathbf{q})$. The proof is complete.

Case 2: There exists some index $i \in \Omega \setminus \Upsilon$ such that $q_i^{(s)} < 0$.

For this case, by using Lemma 3.5 (i), similar to the proof of Theorem 3.3, we have the following result.

Theorem 3.8. *Suppose that $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a Z-matrix and $\mathbf{q} \in \mathbb{R}^n$. Let $B = A_{\cdot\Omega} \in \mathbb{R}^{n\times l}$ and $\bar{\mathbf{y}} \in \mathrm{FEA}(B, \mathbf{q})$. Then for any $i \in \Omega \setminus \Upsilon$ satisfying $q_i^{(s)} < 0$, we have*

$$\bar{y}_j > 0, \quad \forall j \in [l] \setminus \langle j_s\rangle \text{ with } i_j = i.$$

In the following, we denote

$$\Omega \setminus \Upsilon \equiv \{g_{s+1}, \ldots, g_l\},$$

for convenience, then for any $g_{s+t} \in \Omega\setminus\Upsilon$ with $t \in [l-s]$, there must exist some $j_{s+t} \in [l]\setminus\langle j_s\rangle$ such that $g_{s+t} = i_{j_{s+t}}$ by (3.1). Without loss of generality, we assume $q_{g_{s+1}}^{(s)} < 0$, and consider the following linear system:

$$\begin{cases} y_{j_p} + \sum_{j\in[l]\setminus\langle j_s\rangle} b_{h_p j}^{(s)} y_j = -q_{h_p}^{(s)}, & \forall h_p \in \Upsilon \ (p \in [s]), \\ b_{g_{s+1} j_{s+1}}^{(s)} y_{j_{s+1}} + \sum_{j\in[l]\setminus\langle j_{s+1}\rangle} b_{g_{s+1} j}^{(s)} y_j = -q_{g_{s+1}}^{(s)}, & g_{s+1} \in \Omega \setminus \Upsilon. \\ \sum_{j\in[l]\setminus\langle j_s\rangle} (-b_{ij}^{(s)}) y_j \leq q_i^{(s)}, & \forall i \in \Upsilon^c \setminus \{g_{s+1}\}. \end{cases} \tag{3.20}$$

From $q_{g_{s+1}}^{(s)} < 0$, we have that $y_{j_{s+1}} > 0$ by Theorem 3.8, here $j_{s+1} \in [l] \setminus \langle j_s\rangle$ satisfies $i_{j_{s+1}} = g_{s+1}$. Then, we can derive that

$$b_{g_{s+1} j_{s+1}}^{(s)} > 0. \tag{3.21}$$

If not, by $b^{(s)}_{g_{s+1}j} \le 0$ for any $j \in [l] \setminus \langle j_{s+1} \rangle$ from (3.18), together with the fact that $y_j \ge 0$, the left-hand side of inequality *w.r.t.* index g_{s+1} in system (3.6) is nonnegative, which contradicts to $q^{(s)}_{g_{s+1}} < 0$.

Based on (3.21), we can use Gaussian elimination on the equations in (3.20) while updating the inequalities meantime, and let $\mathbf{q}^{(s,1)}_{\Upsilon^c}$ be the variant of $\mathbf{q}^{(s)}_{\Upsilon^c}$ after the elimination. For $g_{s+1} \in \Omega \setminus \Upsilon$, from $q^{(s)}_{g_{s+1}} < 0$ and $b^{(s)}_{g_{s+1}j_{s+1}} > 0$ by (3.21), we have that

$$q^{(s,1)}_{g_{s+1}} := \frac{q^{(s)}_{g_{s+1}}}{b^{(s)}_{g_{s+1}j_{s+1}}} < 0. \tag{3.22}$$

Furthermore, we examine that if there exists index $i \in \Upsilon^c \setminus \{g_{s+1}\}$ such that $q^{(s,1)}_i < 0$. If $q^{(s,1)}_i \ge 0$ for all $i \in \Upsilon^c \setminus \{g_{s+1}\}$, we terminate the procedure. Otherwise, suppose that $q^{(s,1)}_{g_{s+2}} < 0$ without loss of generality, then we continue to apply Gaussian elimination on the equation corresponding to index g_{s+2} as well as the previous obtained ones, while updating the left inequalities of the variant system of (3.20) at the same time. This process is repeated until the desired condition is met. Such a process is called **Procedure 2**.

Procedure 2: We assume that **Procedure 2** terminates after r-step elimination ($r \ge 1$). For any $k \in [r]$, we denote $\langle g_{s+k} \rangle = \{g_{s+1}, \dots, g_{s+k}\}$ for convenience. Then, system (3.20) turns to the following form in turn:

$$\begin{cases} y_{j_p} + \sum_{j \in [l] \setminus \langle j_{s+k} \rangle} b^{(s,k)}_{h_p j} y_j = -q^{(s,k)}_{h_p}, & \forall h_p \in \Upsilon \ (p \in [s]), \\ y_{j_{s+t}} + \sum_{j \in [l] \setminus \langle j_{s+k} \rangle} b^{(s,k)}_{g_{s+t} j} y_j = -q^{(s,k)}_{g_{s+t}}, & \forall g_{s+t} \in \langle g_{s+k} \rangle \subseteq (\Omega \setminus \Upsilon), \\ & \quad (t \in [k]) \\ \sum_{j \in [l] \setminus \langle j_{s+k} \rangle} (-b^{(s,k)}_{ij}) y_j \le q^{(s,k)}_i, & \forall i \in \Upsilon^c \setminus \langle g_{s+k} \rangle, \end{cases} \tag{3.23}$$

where

$$q^{(s,k)}_{g_{s+k}} = \frac{q^{(s,k-1)}_{g_{s+k}}}{b^{(s,k-1)}_{g_{s+k}j_{s+k}}}, \quad b^{(s,k)}_{g_{s+k}j} = \frac{b^{(s,k-1)}_{g_{s+k}j}}{b^{(s,k-1)}_{g_{s+k}j_{s+k}}} \ (\forall j \in [l] \setminus \langle j_{s+k} \rangle), \quad g_{s+k} \in \Omega \setminus \Upsilon,$$
$$\begin{cases} q^{(s,k)}_{h_p} := q^{(s,k-1)}_{h_p} - q^{(s,k)}_{g_{s+k}} b^{(s,k-1)}_{h_p j_{s+k}}, \\ b^{(s,k)}_{h_p j} := b^{(s,k-1)}_{h_p j} - b^{(s,k)}_{g_{s+k} j} b^{(s,k-1)}_{h_p j_{s+k}} \ (\forall j \in [l] \setminus \langle j_{s+k} \rangle), \end{cases} \quad \forall h_p \in \Upsilon \ (p \in [s]),$$
$$\begin{cases} q^{(s,k)}_i = q^{(s,k-1)}_i - q^{(s,k)}_{g_{s+k}} b^{(s,k-1)}_{i j_{s+k}}, \\ b^{(s,k)}_{ij} := b^{(s,k-1)}_{ij} - b^{(s,k)}_{g_{s+k} j} b^{(s,k-1)}_{i j_{s+k}} \ (\forall j \in [l] \setminus \langle j_{s+k} \rangle), \end{cases} \quad \forall i \in \Upsilon^c \setminus \{g_{s+k}\}.$$

Lemma 3.9. *After the k-th elimination in* **Procedure 2** *with $k \in [r]$, we have the following results.*

(i) $q^{(s,k)}_{g_{s+t}} < 0$ *for any* $g_{s+t} \in \Omega \setminus \Upsilon$ *with* $t \in [k]$.

(ii) *For any $j \in [l] \setminus \langle j_{s+k} \rangle$, it holds that*

$$\begin{cases} b^{(s,k)}_{g_{s+t}j} \le 0, & \forall g_{s+t} \in \Omega \setminus \Upsilon \text{ with } t \in [k], \\ b^{(s,k)}_{g_{s+t}j} \le 0, & \forall g_{s+t} \in \Omega \setminus \Upsilon \text{ with } t \in \{k+1, \dots, l-s\}, \ i_j \ne g_{s+t} \\ b^{(s,k)}_{ij} \le 0, & \forall i \in \Omega^c. \end{cases}$$

(iii) $q_i^{(s,k)} \geq 0$ *for any* $i \in \Omega^c$. *Or to say, if there exists index* $g_{s+k+1} \in \Upsilon^c \setminus \langle g_{s+k} \rangle$ *satisfying* $q_{g_{s+k+1}}^{(s,k)} < 0$*, then* g_{s+k+1} *must belong to* $(\Omega \setminus \Upsilon) \setminus \langle g_{s+k} \rangle$.

(iv) *For* $j_{s+k+1} \in [l] \setminus \langle j_{s+k} \rangle$ *satisfying* $i_{j_{s+k+1}} = g_{s+k+1}$ *with* $q_{g_{s+k+1}}^{(s,k)} < 0$*, it holds that* $b_{g_{s+k+1} j_{s+k+1}}^{(s,k)} > 0$ *and* $\bar{y}_{j_{s+k+1}} > 0$ *for any* $\bar{\mathbf{y}} \in \mathrm{FEA}(B, \mathbf{q})$.

Proof. We show that (i)-(iv) hold when $k = 1$. Firstly, (i) holds by (3.22) obviously. Secondly, we show that (ii) holds. For any $g_{s+t} \in \Omega \setminus \Upsilon$ with $t \in [l-s]$ and $j \in [l] \setminus \langle j_{s+1} \rangle$, from $b_{g_{s+t}j}^{(s)}, b_{g_{s+t}j_{s+1}}^{(s)} \leq 0$ by Lemma 3.5 (i) and $b_{g_{s+1}j_{s+1}}^{(s)} > 0$ by (3.21), we have

$$\begin{cases} b_{g_{s+1}j}^{(s,1)} := \dfrac{b_{g_{s+1}j}^{(s)}}{b_{g_{s+1}j_{s+1}}^{(s)}} \leq 0, \\ b_{g_{s+t}j}^{(s,1)} := b_{g_{s+t}j}^{(s)} - b_{g_{s+1}j}^{(s,1)} b_{g_{s+t}j_{s+1}}^{(s)} \leq 0, \quad \forall t \neq 1,\ j \neq j_{s+t}. \end{cases} \tag{3.24}$$

Moreover, for any $i \in \Omega^c$ and $j \in [l] \setminus \langle j_{s+1} \rangle$, from (3.24) and $b_{ij}^{(s)}, b_{ij_{s+1}}^{(s)} \leq 0$ by Lemma 3.5 (ii), we have

$$b_{ij}^{(s,1)} = b_{ij}^{(s)} - b_{g_{s+1}j}^{(s,1)} b_{ij_{s+1}}^{(s)} \leq 0, \tag{3.25}$$

Thirdly, based on (i) and (ii), we show that (iii) holds, i.e., if there exists $g_{s+2} \in \Upsilon^c \setminus \{g_{s+1}\}$ satisfying $q_{g_{s+2}}^{(s,1)} < 0$, we must have that

$$g_{s+2} \in (\Omega \setminus \Upsilon) \setminus \{g_{s+1}\}.$$

Suppose by contradiction that $g_{s+2} \in \Omega^c$, then by $y_j \geq 0$, combining with the fact that $-b_{ij}^{(s,1)} \geq 0$ for any $i \in \Omega^c$ and $j \in [l] \setminus \langle j_{s+1} \rangle$ by (3.25), the left-hand side of the inequality *w.r.t.* index g_{s+2} in system (3.23) is nonnegative, which contradicts to $q_{g_{s+2}}^{(s,1)} < 0$. Finally, we show that (iv) holds. Since $\bar{\mathbf{y}} \in \mathrm{FEA}(B, \mathbf{q})$, it follows that for $j_{s+2} \in [l] \setminus \langle j_{s+1} \rangle$ satisfying $i_{j_{s+2}} = g_{s+2}$ with $q_{g_{s+2}}^{(s,1)} < 0$,

$$b_{g_{s+2}j_{s+2}}^{(s,1)} \bar{y}_{j_{s+2}} + \sum_{j \in [l] \setminus \langle j_{s+2} \rangle} b_{g_{s+2}j}^{(s,1)} \bar{y}_j \geq -q_{g_{s+2}}^{(s,1)};$$

while, by (3.24), we have that $q_{g_{s+2}}^{(s,1)} < 0$ and $b_{g_{s+2}j}^{(s,1)} \leq 0$ for any $j \in [l] \setminus \langle j_{s+2} \rangle$. These as well as $\bar{\mathbf{y}} \geq \mathbf{0}$ imply that we must have $\bar{y}_{j_{s+2}} > 0$ and $b_{g_{s+2}j_{s+2}}^{(s,1)} > 0$.

Then, similar to the proof in the case of $k = 1$, we can obtain that (i)-(iv) hold after the k-th elimination with $k \in \{2, \ldots, r\}$ by applying the induction. The proof is complete.

After **Procedure 2**, (3.20) turns to the system as follows:

$$\begin{cases} y_{j_p} + \sum\limits_{j \in [l] \setminus \langle j_{s+r} \rangle} b_{h_p j}^{(s,r)} y_j = -q_{h_p}^{(s,r)}, & \forall h_p \in \Upsilon\ (p \in [s]), \\ y_{j_{s+t}} + \sum\limits_{j \in [l] \setminus \langle j_{s+r} \rangle} b_{g_{s+t}j}^{(s,r)} y_j = -q_{g_{s+t}}^{(s,r)}, & \forall g_{s+t} \in \Omega \setminus \Upsilon (t \in [r]), \\ \sum\limits_{j \in [l] \setminus \langle j_{s+r} \rangle} (-b_{ij}^{(s,r)}) y_j \leq q_i^{(s,r)}, & \forall i \in \Upsilon^c \setminus \langle g_{s+r} \rangle. \end{cases} \tag{3.26}$$

Next, we show that $\mathbf{q}_{\Upsilon}^{(s,k)} < 0$ for any $k \in [r]$ in **Procedure 2**.

Lemma 3.10. *In* **Procedure 2***, for any $h_p \in \Upsilon$ with $p \in [s]$, the following results hold for any $k \in [r]$.*

(i) $q_{h_p}^{(s,k)} < 0$,

(ii) $b_{h_p j}^{(s,k)} \leq 0$ *for any* $j \in [l] \setminus \langle j_{s+k} \rangle$.

Proof. After the first elimination in **Procedure 2**, for any $h_p \in \Upsilon$ with $p \in [s]$, we have

$$\begin{cases} q_{h_p}^{(s,1)} = q_{h_p}^{(s)} - q_{g_{s+1}}^{(s,1)} b_{h_p j_{s+1}}^{(s)} < 0, \\ b_{h_p j}^{(s,1)} = b_{h_p j}^{(s)} - b_{g_{s+1} j}^{(s,1)} b_{h_p j_{s+1}}^{(s)} \leq 0, \quad \forall j \in [l] \setminus \langle j_{s+1} \rangle, \end{cases}$$

where the first inequality holds since $q_{h_p}^{(s)} < 0$ by Lemma 3.4 (i), $q_{g_{s+1}}^{(s,1)} < 0$ by (3.22) and $b_{h_p j_{s+1}}^{(s)} \leq 0$ by (3.7), and the second inequality holds since $b_{h_p j}^{(s)}, b_{h_p j_{s+1}}^{(s)} \leq 0$ by (3.7) and $b_{g_{s+1} j}^{(s,1)} \leq 0$ by (3.24).

Now, let us apply the induction. Suppose that after the $(k-1)$-th elimination with $k \in \{2, \ldots, r\}$, we have that for any $h_p \in \Upsilon$ with $p \in [s]$,

$$\begin{cases} q_{h_p}^{(s,k-1)} < 0, \\ b_{h_p j}^{(s,k-1)} \leq 0, \quad \forall j \in [l] \setminus \langle j_{s+k-1} \rangle, \end{cases}$$

then, after the k-th elimination, from $q_{g_{s+k}}^{(s,k)} < 0$ by Lemma 3.4 (i) and $b_{g_{s+k} j}^{(s,k)} \leq 0$ by Lemma 3.4 (ii), it holds that for any $h_p \in \Upsilon$ with $p \in [s]$,

$$\begin{cases} q_{h_p}^{(s,k)} = q_{h_p}^{(s,k-1)} - q_{g_{s+k}}^{(s,k)} b_{h_p j_{s+k}}^{(s,k-1)} < 0, \\ b_{h_p j}^{(s,k)} = b_{h_p j}^{(s,k-1)} - b_{g_{s+k} j}^{(s,k)} b_{h_p j_{s+k}}^{(s,k-1)} \leq 0, \quad \forall j \in [l] \setminus \langle j_{s+k} \rangle. \end{cases}$$

The proof is complete.

After **Procedure 2**, we can obtain the following theorem.

Theorem 3.11. *Suppose that $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is a Z-matrix and $\mathbf{q} \in \mathbb{R}^n$. Let $\mathrm{FEA}(A, \mathbf{q}) \neq \emptyset$. When* **Procedure 2** *terminates, $\bar{\mathbf{x}} \in \mathbb{R}^n_+$ with*

$$\bar{\mathbf{x}}_{\Upsilon \cup \langle g_{s+r} \rangle} = -\mathbf{q}_{\Upsilon \cup \langle g_{s+r} \rangle}^{(s,r)} \quad \text{and} \quad \bar{\mathbf{x}}_{\Upsilon^c \setminus \langle g_{s+r} \rangle} = \mathbf{0}$$

is a sparsest solution of LCP$(A, \mathbf{q})$, which is also the least solution of LCP$(A, \mathbf{q})$.

Proof. First, we show that $\bar{\mathbf{x}} \in \mathrm{FEA}(A, \mathbf{q})$. On one hand, by Lemma 3.9 (i) and Lemma 3.10 (i), we have that $\bar{\mathbf{x}}_{\Upsilon \cup \langle g_{s+r} \rangle} = -\mathbf{q}_{\Upsilon \cup \langle g_{s+r} \rangle}^{(s)} > \mathbf{0}$, which implies that $\bar{\mathbf{x}} \in \mathbb{R}^n_+$. On the other hand, let $\bar{\mathbf{y}} = \bar{\mathbf{x}}_\Omega$, then $A\bar{\mathbf{x}} + \mathbf{q} = B\bar{\mathbf{y}} + \mathbf{q} \geq \mathbf{0}$ since $(B\bar{\mathbf{y}} + \mathbf{q})_{\Upsilon \cup \langle g_{s+r} \rangle} = \mathbf{0}$ and $(B\bar{\mathbf{y}} + \mathbf{q})_{\Upsilon^c \setminus \langle g_{s+r} \rangle} \geq \mathbf{0}$ by (3.26). Second, it is obvious that $\bar{\mathbf{x}} \in \mathrm{SOL}(A, \mathbf{q})$. Finally, by Theorems 3.2, 3.3 and 3.8 and Lemma 3.9 (iv), we obtain that $\bar{\mathbf{x}}$ is a sparsest solution of LCP$(A, \mathbf{q})$.

Suppose that $\hat{\mathbf{x}} \in \mathrm{FEA}(A, \mathbf{q})$, let $\hat{\mathbf{y}} = \hat{\mathbf{x}}_\Omega$, then $\hat{\mathbf{y}} \in \mathrm{FEA}(B, \mathbf{q})$. By utilizing the feasibility condition and applying Gaussian elimination, when the procedure terminates, similar to (3.26), for any $h_p \in \Upsilon$ with $p \in [s]$ and $g_{s+t} \in \Omega \setminus \Upsilon$ with $t \in [r]$, we have that

$$\begin{cases} \hat{y}_{j_p} + \sum\limits_{j \in [l] \setminus \langle j_{s+r} \rangle} b_{h_p j}^{(s,r)} \hat{y}_j \geq -q_{h_p}^{(s,r)}, \\ \hat{y}_{j_{s+t}} + \sum\limits_{j \in [l] \setminus \langle j_{s+r} \rangle} b_{g_{s+t} j}^{(s,r)} \hat{y}_j \geq -q_{g_{s+t}}^{(s,r)}, \end{cases}$$

and then, from $\hat{\mathbf{y}} \in \mathbb{R}^l_+$ and $b^{(s,r)}_{h_p j}, b^{(s,r)}_{g_{s+t} j} \leq 0$ by Lemma 3.10 (ii) and Lemma 3.9 (ii), we can further deduce that $\hat{y}_{j_p} \geq -q^{(s,r)}_{h_p}$ and $\hat{y}_{j_{s+t}} \geq -q^{(s,r)}_{g_{s+t}}$. Thus, it holds that

$$\hat{\mathbf{x}}_{\Upsilon \cup \langle g_{s+r} \rangle} \geq -\mathbf{q}^{(s,r)}_{\Upsilon \cup \langle g_{s+r} \rangle} = \bar{\mathbf{x}}_{\Upsilon \cup \langle g_{s+r} \rangle},$$

which together with the fact that $\hat{\mathbf{x}}_{\Upsilon^c \setminus \langle g_{s+r} \rangle} \geq \mathbf{0} = \bar{\mathbf{x}}_{\Upsilon^c \setminus \langle g_{s+r} \rangle}$, implies that $\bar{\mathbf{x}}$ is the least solution of LCP$(A, \mathbf{q})$. The proof is complete.

Remark 3.12. In the whole procedure, when Gaussian elimination is executed for the concerned equations, it can be seen from Corollary 3.6 and Lemma 3.9 (iii) that the concerned inequalities corresponding to index set Ω^c hold automatically. So, in the practical algorithm, we only deal with a lower-dimensional linear system with no more than l equations and l variables, and do not need to execute any updates for inequalities corresponding to index Ω^c. When $|\Omega^c|$ is large, this strategy could greatly decrease the computation cost in the iterations compared to Chandrasekaran's method [2].

In the following, we propose an iterative method based on lower-dimensional equations to find a sparsest solution of LCP$(A, \mathbf{q})$.

Algorithm 3.13. (An iterative method based on lower-dimensional equations (iLD))

(S0) Given $A \in \mathbb{R}^{n \times n}$ and $\mathbf{q} \in \mathbb{R}^n$. If $\mathbf{q} \geq \mathbf{0}$, then set $\mathbf{x} := \mathbf{0} \in \mathbb{R}^n$ and stop. Otherwise, let $\Upsilon := \{i \in [n] : q_i < 0\}$ and $\Omega := \{i \in [n] : a_{ii} > 0\}$. Set $\mathbf{x}_{\Omega^c} := \mathbf{0}$.

(S1) Gaussian elimination method is used to solve equations:

$$A_{\Upsilon\Upsilon} \mathbf{x}_\Upsilon = -\mathbf{q}_\Upsilon.$$

Compute $\mathbf{q}_{\Omega \setminus \Upsilon} := A_{(\Omega \setminus \Upsilon)\Upsilon} \mathbf{x}_\Upsilon + \mathbf{q}_{\Omega \setminus \Upsilon}$.

(S2) If $\mathbf{q}_{\Omega \setminus \Upsilon} \geq \mathbf{0}$, then set $\mathbf{x}_{\Omega \setminus \Upsilon} := \mathbf{0}$, and stop. Otherwise, choose $i_t \in \Omega \setminus \Upsilon$ satisfying $q_{i_t} < 0$, and set

$$\Upsilon := \Upsilon \bigcup \{i_t\},$$

and go to (S1).

Remark 3.14. We use μ to denote the number of non-zero elements in the sparsest solution to LCP$(A, \mathbf{q})$.

(i) In Algorithm 3.13, we first need to solve a system of linear equations with s equations and s variables; and then, we need to solve successively $\mu - s$ systems of linear equations where the numbers of variables and equations are gradually increasing. It should be emphasized that solving the first system of linear equations requires s-step elimination, while solving each subsequent system of linear equations requires only 1-step elimination, so the entire algorithm actually requires only μ-step elimination.

(ii) The computational complexity of Algorithm 3.13 is $\mathcal{O}(\mu^3)$. Thus, the smaller the number of non-zero elements in the sparsest solution to LCP$(A, \mathbf{q})$, the lower the computational cost of Algorithm 3.13.

4 A Fixed Point Iterative Method

In this section, we propose a fixed point method to solve LCP$(A,\mathbf{q})$, which monotonically decreasing converges to a solution of LCP$(A,\mathbf{q})$ from any given feasible point. Moreover, if the starting point is sparse, the proposed algorithm could quickly converge to a sparse solution of LCP$(A,\mathbf{q})$.

For any $A\in\mathbb{R}^{n\times n}$ and $\mathbf{q}\in\mathbb{R}^n$, we denote

$$F(\mathbf{x}) := \mathbf{x} - D(\mathbf{x}\circ(A\mathbf{x}+\mathbf{q})),\ \ \forall \mathbf{x}\in\mathbb{R}^n, \tag{4.1}$$

Then, we have the following result.

Proposition 4.1. *If* $\mathbf{x}\in \text{FEA}(A,\mathbf{q})$*, then* $\mathbf{x}\in\text{SOL}(A,\mathbf{q})$ *if and only if* $\mathbf{x}=F(\mathbf{x})$.

Inspired by Proposition 4.1, we design the following method to solve LCP$(A,\mathbf{q})$.

Algorithm 4.2. (A fixed point (FP) iterative method)

(S0) Given $A\in\mathbb{R}^{n\times n}$ and $\mathbf{q}\in\mathbb{R}^n$. If $\mathbf{q}\geq\mathbf{0}$, then set $\mathbf{x}:=\mathbf{0}\in\mathbb{R}^n$ and stop. Otherwise, if there exists $i\in[n]$ such that $q_i>0$, then set $q_{\max}:=\max\{q_i : q_i>0\}$; if not, set $q_{\max}:=1$. Choose $\gamma\in(0,1)$ such that $\gamma q_{\max}<1$, and $\mathbf{x}^0\in\text{FEA}(A,\mathbf{q})$. Set $k:=0$.

(S1) Let D^k be a diagonal matrix with its i-diagonal entry being

$$d_i^k := \begin{cases} \left(a_{ii}x_i^k+1\right)^{-1} & \text{if } a_{ii}>0, \\ 1 & \text{otherwise,}\end{cases} \quad \forall i\in[n]. \tag{4.2}$$

(S2) Set

$$\mathbf{x}^{k+1} := \mathbf{x}^k - \gamma D^k\left(\mathbf{x}^k\circ\left(A\mathbf{x}^k+\mathbf{q}\right)\right). \tag{4.3}$$

(S3) If $\min\{\mathbf{x}^{k+1}, A\mathbf{x}^{k+1}+\mathbf{q}\}=\mathbf{0}$, stop. Otherwise, set $k:=k+1$, and go to step (S1).

In the following, we assume $\mathbf{q}\notin\mathbb{R}^n_+$ without loss of generality, and denote $\mathcal{K}:=\{0,1,\ldots\}$.

Lemma 4.3. *Suppose that* $A=(a_{ij})\in\mathbb{R}^{n\times n}$ *is a* Z*-matrix and* $\mathbf{q}\in\mathbb{R}^n$*. Let the sequence* $\{\mathbf{x}^k\}$ *be generated by Algorithm 4.2. Then, we have the following results.*

(i) $\mathbf{x}^k\in\text{FEA}(A,\mathbf{q})$ *for all* $k\in\mathcal{K}$,

(ii) $\mathbf{x}^{k+1}\leq\mathbf{x}^k$ *for all* $k\in\mathcal{K}$.

Proof. (i) From the step (S0) of Algorithm 4.2, it follows that $\mathbf{x}^0\in\text{FEA}(A,\mathbf{q})$. In the following, we prove the conclusion in (i) by mathematical induction. That is, we assume that $\mathbf{x}^l\in\text{FEA}(A,\mathbf{q})$ for some $l\in\mathcal{K}$, and show that $\mathbf{x}^{l+1}\in\text{FEA}(A,\mathbf{q})$. Note that $\mathbf{x}^l\in\text{FEA}(A,\mathbf{q})$ means that

$$\mathbf{x}^l\geq\mathbf{0}\quad\text{and}\quad A\mathbf{x}^l+\mathbf{q}\geq\mathbf{0}. \tag{4.4}$$

To show $\mathbf{x}^{l+1}\in\text{FEA}(A,\mathbf{q})$, we divide the proof into the following two parts.

Part 1. We show that $\mathbf{x}^{l+1}\geq\mathbf{0}$. It follows from (4.3) that for any $i\in[n]$,

$$x_i^{l+1} = x_i^l - \gamma d_i^l x_i^l(A\mathbf{x}^l+\mathbf{q})_i = [1-\gamma d_i^l(A\mathbf{x}^l+\mathbf{q})_i]x_i^l. \tag{4.5}$$

For any $i \in [n]$, it is obvious that

$$(A\mathbf{x}^l + \mathbf{q})_i = a_{ii}x_i^l + \sum_{j\neq i} a_{ij}x_j^l + q_i, \tag{4.6}$$

Since A is a Z-matrix, it follows that for any $i \in [n]$, $a_{ij} \le 0$ for all $j \ne i$. We consider the following two cases.

Case 1. Suppose that $a_{ii} \le 0$ for some $i \in [n]$. In this case, combining the fact that $a_{ij} \le 0$ for all $j \ne i$ with (4.4) and (4.6), we can obtain that $(A\mathbf{x}^l)_i \le 0$ and $q_i \ge 0$. Moreover, by (4.2) we have $d_i^l = 1$. Thus, it follows from (4.5) that

$$x_i^{l+1} = [1 - \gamma d_i^l (A\mathbf{x}^l + \mathbf{q})_i]x_i^l \ge [1 - \gamma q_i]x_i^l \ge 0,$$

where the last inequality holds due to $\gamma q_i \le \gamma q_{\max} < 1$.

Case 2. Suppose that $a_{ii} > 0$ for some $i \in [n]$. In this case, we have that $d_i^l = \left(a_{ii}x_i^l + 1\right)^{-1}$ by (4.2).

By the fact that $a_{ij} \le 0$ for all $j \ne i$, combing with (4.4) and (4.6), we have that

$$\gamma d_i^l (A\mathbf{x}^l + \mathbf{q})_i \le \frac{\gamma a_{ii}x_i^l + \gamma q_i}{a_{ii}x_i^l + 1} < \frac{\gamma a_{ii}x_i^l + 1}{a_{ii}x_i^l + 1} < 1,$$

and hence, by (4.5), we have that

$$x_i^{l+1} = [1 - \gamma d_i^l (A\mathbf{x}^l + \mathbf{q})_i]x_i^l \ge \left[1 - \frac{\gamma a_{ii}x_i^l + 1}{a_{ii}x_i^l + 1}\right] x_i^l \ge 0.$$

Thus, we obtain that $x_i^{l+1} \ge 0$ for all $i \in [n]$. So $\mathbf{x}^{l+1} \ge \mathbf{0}$.

Part 2. We show that $A\mathbf{x}^{l+1} + \mathbf{q} \ge \mathbf{0}$. For any $i \in [n]$, it follows from (4.3) that

$$(A\mathbf{x}^{l+1} + \mathbf{q})_i = (A\mathbf{x}^l + \mathbf{q})_i - [A(\gamma D^l(\mathbf{x}^l \circ (A\mathbf{x}^l + \mathbf{q})))]_i. \tag{4.7}$$

If $a_{ii} \le 0$, by the step (S1) of Algorithm 4.2, we have that D^k is a diagonal matrix with positive diagonal entries, and then $\gamma D^k(\mathbf{x}^l \circ (A\mathbf{x}^l + \mathbf{q})) \in \mathbb{R}^n_+$. Thus, from the fact that $a_{ij} \le 0$ for all $j \ne i$, it follows that $[A(\gamma D^l(\mathbf{x}^l \circ (A\mathbf{x}^l + \mathbf{q})))^{m-1}]_i \le 0$, and hence, by (4.7), it holds that

$$(A\mathbf{x}^{l+1} + \mathbf{q})_i \ge (A\mathbf{x}^l + \mathbf{q})_i \ge 0.$$

In the following, we assume that $a_{ii} > 0$. In this case, by using the the fact that $a_{ij} \le 0$ for all $j \ne i$ and $\gamma D^l(\mathbf{x}^l \circ (A\mathbf{x}^l + \mathbf{q})) \in \mathbb{R}^n_+$, we have that

$$\begin{aligned}(A\mathbf{x}^{l+1} + \mathbf{q})_i &\ge (A\mathbf{x}^l + \mathbf{q})_i - a_{ii}[\gamma d_i^l x_i^l (A\mathbf{x}^l + \mathbf{q})_i] \\ &= [1 - \gamma a_{ii} d_i^l x_i^l](A\mathbf{x}^l + \mathbf{q})_i.\end{aligned}$$

Recall that $d_i^l = \left(a_{ii}x_i^l + 1\right)^{-1}$ by (4.2) in this case, we have that

$$\gamma a_{ii} d_i^l x_i^l = \gamma d_i^l (a_{ii} x_i^l) = \gamma \frac{a_{ii}x_i^l}{a_{ii}x_i^l + 1} < \gamma,$$

and hence,

$$1 - \gamma a_{ii} d_i^l x_i^l > 1 - \gamma.$$

So,

$$(A\mathbf{x}^{l+1} + \mathbf{q})_i \ge (1 - \gamma)(A\mathbf{x}^l + \mathbf{q})_i \ge 0.$$

Thus, we can obtain that $A\mathbf{x}^{l+1} + \mathbf{q} \geq \mathbf{0}$.

Combining Part 1 with Part 2, $\mathbf{x}^{l+1} \in \mathrm{FEA}(A, \mathbf{q})$. Then, it holds that $\mathbf{x}^k \in \mathrm{FEA}(A, \mathbf{q})$ for all $k \in \mathcal{K}$ by induction.

(ii) From the proof of (i), as well as (4.2) and (4.3), it holds that $\mathbf{x}^{k+1} \leq \mathbf{x}^k$ for all $k \in \mathcal{K}$. The proof is complete.

Theorem 4.4. *Suppose that $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a Z-matrix and $\mathbf{q} \in \mathbb{R}^n$. Let the sequence $\{\mathbf{x}^k\}$ be generated by Algorithm 4.2. Then, the sequence $\{\mathbf{x}^k\}$ descends monotonically, and it converges to a solution of LCP$(A, \mathbf{q})$.*

Proof. By Lemma 4.3, we obtain that the sequence $\{\mathbf{x}^k\}$ descends monotonically and has a lower bound of zero. In the following, we show convergence of $\{\mathbf{x}^k\}$. Without loss of generality, we assume that $\{\mathbf{x}^k\}$ is an infinite sequence. Then, by the monotone bounded convergence theorem, we have that $\{\mathbf{x}^k\}$ must converge. We use $\mathbf{x}^*$ to denote its limit point. Then,

$$\begin{cases} \lim\limits_{k\to\infty} A\mathbf{x}^k + \mathbf{q} = A\mathbf{x}^* + \mathbf{q}, \\ \lim\limits_{k\to\infty} \mathbf{x}^k \circ (A\mathbf{x}^k + \mathbf{q}) = \mathbf{x}^* \circ (A\mathbf{x}^* + \mathbf{q}). \end{cases}$$

For any $i \in [n]$, we denote the limit point of sequence $\{d_i^k\}$ by d_i^*, then

$$d_i^* := \begin{cases} (a_{ii}x_i^* + 1)^{-1} & \text{if } a_{ii} > 0, \\ 1 & \text{otherwise,} \end{cases} \qquad \forall i \in [n].$$

Denote $D^* := \mathrm{diag}\{d_i^* : i \in [n]\}$. Since

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \gamma D^k(\mathbf{x}^k \circ (A\mathbf{x}^k + \mathbf{q})), \quad \forall k \in \mathcal{K},$$

it follows that $D^*(\mathbf{x}^* \circ (A\mathbf{x}^* + \mathbf{q})) = \mathbf{0}$, i.e., $\mathbf{x}^* \circ (A\mathbf{x}^* + \mathbf{q}) = \mathbf{0}$. Moreover, by $\mathbf{x}^k \geq \mathbf{0}$ and $A\mathbf{x}^k + \mathbf{q} \geq \mathbf{0}$ for all $k \in \mathcal{K}$, we have that $\mathbf{x}^* \geq \mathbf{0}$ and $A\mathbf{x}^* + \mathbf{q} \geq \mathbf{0}$. Therefore, $\mathbf{x}^*$ is a solution to LCP$(A, \mathbf{q})$. The proof is complete.

Note that Algorithm 4.2 converges monotonically downward to a solution of LCP$(A, \mathbf{q})$ from a given feasible point. Obviously, if the starting point is sparse, then we can obtain a sparse solution of LCP$(A, \mathbf{q})$ by Algorithm 4.2. In general, it is difficult to obtain a sparse feasible solution for LCP$(A, \mathbf{q})$. Here, we have the following remark.

Remark 4.5. (i) Since the matrix in LCP$(A, \mathbf{q})$ is required to be a Z-matrix, the sparse feasible solutions of some of these problems can be directly observed by using the properties of Z-matrices. For example, we can use the following result:

(R) Suppose that $\bar{\mathbf{x}} \in \mathrm{FEA}(A, \mathbf{q})$ is a sparest solution of LCP$(A, \mathbf{q})$. If $q_i = 0$ and $a_{ii} \leq 0$ for some $i \in [n]$, then $\bar{x}_j = 0$ for any $j \in [n]$ satisfying $a_{ij} \neq 0$.

To illustrate this point, let's consider LCP$(A, \mathbf{q})$, where

$$A = \begin{pmatrix} -2 & 0 & -1 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & -2 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{q} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix}.$$

It is clear that A is a Z-matrix. Let $\bar{\mathbf{x}} \in \mathrm{FEA}(A, \mathbf{q})$ be a sparest solution of LCP$(A, \mathbf{q})$.

(a) Since $q_1 = 0$ and $a_{11} < 0$, it follows from the above result (R) that $\bar{x}_1 = \bar{x}_3 = 0$ since $a_{11}, a_{13} \neq 0$.

(b) Since $q_3 = 0$ and $\bar{x}_3 = 0$, it follows from $a_{32} < 0$ that $\bar{x}_2 = 0$.

Thus, $\bar{\mathbf{x}} = (0, 0, 0, \alpha)^\top$ with $\alpha \geq 2$ is a sparse feasible solution of LCP$(A, \mathbf{q})$.

(ii) For large-scale or complex problems, however, it is difficult to directly observe their sparse feasible solutions. Therefore, how to systematically give a sparse feasible solution of such a class of problems is important. Next, we consider how to find a spare feasible solution of LCP$(A, \mathbf{q})$. By Theorem 3.2, instead of FEA$(A, \mathbf{q})$, we may consider FEA$(B, \mathbf{q})$ to reduce computing cost. For system of inequalities in FEA$(B, \mathbf{q})$, by introducing slack variables and/or artificial variables, we can obtain the following system of equalities and inequalities:

$$\begin{cases} \sum_{j\in[l]} b_{ij} y_j - y_{l+i} \quad +w_i \quad = \quad -q_i, \qquad \forall i \in \Upsilon, \\ \sum_{j\in[l]} b_{ij} y_j - y_{l+i} \qquad\qquad = \quad -q_i, \qquad \forall i \in \Upsilon^c, \\ \mathbf{y} \geq 0,\ u_i \geq 0,\ w_i \geq 0\ \forall i \in [s],\ v_i \geq 0\ \forall i \in [n-s], \end{cases} \tag{4.8}$$

where each y_{l+i} $(i \in [n])$ is a slack variable, and each w_i $(i \in \Upsilon)$ is an artificial variable. It is not difficult to see that FEA$(B, \mathbf{q}) \neq \emptyset$ if and only if (4.8) has at least one solution such that $w_i = 0$ for all $i \in \Upsilon$. We denote the set of all points satisfying (4.8) by Θ, and consider the following linear programming with $\mathbf{w} = (w_i)_{i\in\Upsilon}$ and $\mathbf{y}_l := (y_{l+i})_{i\in[n]}$:

$$\begin{array}{ll} \min & \sum_{i\in\Upsilon} w_i \\ \text{s.t.} & \mathbf{u}^\top := \left(\mathbf{y}^\top\ \mathbf{y}_l^\top\ \mathbf{w}^\top\right)^\top \in \Theta. \end{array} \tag{4.9}$$

Suppose that $\mathbf{u}^*$ satisfying $(\mathbf{u}^*)^\top := \left((\mathbf{y}^*)^\top\ (\mathbf{y}_l^*)^\top\ (\mathbf{w}^*)^\top\right)^\top$ is an optimal solution to the linear programming (4.9). Then, if FEA$(B, \mathbf{q}) \neq \emptyset$, we have $w_i^* = 0$ for all $i \in \Upsilon$, and $\mathbf{y}^* \in$ FEA$(B, \mathbf{q})$.

In equalities in (4.8), all the numbers on the right-hand sides are nonnegative, the matrix of the coefficients on the left-hand sides contains a unit matrix, and all variables are nonnegative, then simplex algorithm can be directly applied to solve the linear programming (4.9).

(iii) It should be noted that the proposed method in (ii) for finding sparse feasible solutions is somewhat expensive. In practice, we can make use of the properties of Z-matrix and inequality to reduce the size of the problem, and then use this method to find the sparse feasible solutions to the problem.

5 Numerical Experiments

In this section, we implement the two proposed iterative methods to solve LCP$(A, \mathbf{q})$, including Algorithm 3.13 (iLD) and Algorithm 4.2 (FP). We compare the two proposed methods with Chandrasekaran's method (Pivot) [2] and a single linear program (LP) method by Mangasarian [16] for finding a sparse solution of LCP$(A, \mathbf{q})$ with given different Z-matrices A and vectors $\mathbf{q}$.

5.1 Experiment settings

All the experiments were performed on a Macbook Pro (2.3 GHz, 8 GB of RAM) using MATLAB 2021a.

For Algorithm 4.2, we take $\gamma = \min\{1, 1/q_{\max}\} - (2e-2)$. Moreover, we use the stopping criteria $\|\min\{\mathbf{x}^{k+1}, A\mathbf{x}^{k+1}+\mathbf{q}\}\| \leq 1e{-}14$ in (S3), and set the maximum number of iterations to be 200.

For all the compared algorithms, we show the numerical results in the perspective of the CPU time (CPU) and recovery error (Rel.Err $:= \|\langle \mathbf{x}^{k+1}, A\mathbf{x}^{k+1} + \mathbf{q}\rangle\|$), as well as the sparsity of output $\|\mathbf{x}^k\|_0 := |\{i \in [n] : x_i^k \neq 0\}|$. Here, we think $x_i^k \neq 0$ if $x_i^k > 1e-4$.

5.2 Numerical results

In this subsection, we illustrate the numerical results of different methods for solving LCP$(A, \mathbf{q})$ with different matrices $A \in \mathbb{R}^{n\times n}$ and vectors $\mathbf{q} \in \mathbb{R}^n$.

Example 5.1 ([22])**.** Let us consider LCP$(A, \mathbf{q})$, where

$$A = I_n - \frac{1}{n}\mathbf{e}\mathbf{e}^\top = \begin{pmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & 1-\frac{1}{n} \end{pmatrix} \quad \text{and} \quad \mathbf{q} = \begin{pmatrix} \frac{1}{n} - 1 \\ \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{pmatrix}.$$

Here I_n is the identity matrix of order n, and $\mathbf{e} = (1, 1, \ldots, 1)^\top \in \mathbb{R}^n$. Clearly A is a positive semidefinite Z-matrix, which is widely used in statistics. It is easy to verify that for any scalar $\alpha \geq 0$, $\mathbf{x} = (\alpha + 1, \alpha, \ldots, \alpha)^\top \in \mathbb{R}^n$ are solutions to LCP$(A, \mathbf{q})$. Among all the solutions, $\hat{\mathbf{x}} = (1, 0, \ldots, 0)^\top \in \mathbb{R}^n$ is the unique sparsest solution.

Table 5.1: Numerical performances for Example 5.1.

n	Algo.	CPU (s)	Sparsity	Output
1000	Pivot	0.17	1	$(1, 0, \ldots, 0)^\top$
1000	**iLD**	**0.05**	1	$(1, 0, \ldots, 0)^\top$
1000	LP	2.15	1	$(1, 0, \ldots, 0)^\top$
1000	**FP**	1.07	1	$(1, 0, \ldots, 0)^\top$

Example 5.2. Let us consider LCP$(A, \mathbf{q})$, where

$$A = \begin{pmatrix} 1 & q & r & s & q & \cdots & \\ s & -1 & q & r & s & \ddots & \\ q & s & 1 & \ddots & \ddots\ddots & \ddots & q \\ r & \ddots & \ddots & \ddots & q & r & s \\ & \ddots & q & s & 1 & q & r \\ & \ddots & r & q & s & -1 & q \\ & \cdots & s & r & q & s & 1 \end{pmatrix} \in \mathbb{R}^{n\times n} \quad \text{and} \quad \mathbf{q} = \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{3} \\ 0 \\ 1 \\ -\frac{1}{6} \\ \vdots \end{pmatrix} \in \mathbb{R}^n$$

with $q = -\frac{p}{n}$, $r = -\frac{p}{n+1}$ and $s = -\frac{p}{n+2}$ for any $p > 0$.

Table 5.2: Numerical performances for Example 5.2 with $p = 1$.

n	Algo.	CPU (s)	Sparsity	Rel.Err	Output
10	Pivot	0.11	3	2.02e-17	$(0, 0, 0.3648, 0, 0, 0.2134, 0, 0, 0.1637, 0)^\top$
10	**iLD**	**0.05**	3	2.02e-17	$(0, 0, 0.3648, 0, 0, 0.2134, 0, 0, 0.1637, 0)^\top$
10	LP	0.65	3	1.05e-17	$(0, 0, 0.3648, 0, 0, 0.2134, 0, 0, 0.1637, 0)^\top$
10	**FP**	0.17	3	**1.38e-18**	$(0, 0, 0.3648, 0, 0, 0.2134, 0, 0, 0.1637, 0)^\top$
100	Pivot	0.67	33	**1.20e-17**	$(0, 0, 0.3494, \ldots, 0.0298, 0, 0, 0.0295, 0)^\top$
100	**iLD**	**0.1**	33	**1.20e-17**	$(0, 0, 0.3494, \ldots, 0.0298, 0, 0, 0.0295, 0)^\top$
100	LP	0.64	33	1.70e-17	$(0, 0, 0.3494, \ldots, 0.0298, 0, 0, 0.0295, 0)^\top$
100	**FP**	0.18	33	1.94e-17	$(0, 0, 0.3494, \ldots, 0.0298, 0, 0, 0.0295, 0)^\top$
500	Pivot	5.16	166	**5.68e-18**	$(0, 0, 0.3383, \ldots, 0.0077, 0, 0, 0.0076, 0, 0)^\top$
500	**iLD**	**0.33**	166	**5.68e-18**	$(0, 0, 0.3383, \ldots, 0.0077, 0, 0, 0.0076, 0, 0)^\top$
500	LP	0.87	166	4.38e-17	$(0, 0, 0.3383, \ldots, 0.0077, 0, 0, 0.0076, 0, 0)^\top$
500	**FP**	0.35	166	3.72e-17	$(0, 0, 0.3383, \ldots, 0.0077, 0, 0, 0.0076, 0, 0)^\top$
1000	Pivot	24.19	333	**3.02e-17**	$(0, 0, 0.3362, \ldots, 0.0042, 0, 0, 0.0042, 0)^\top$
1000	**iLD**	**1.35**	333	**3.02e-17**	$(0, 0, 0.3362, \ldots, 0.0042, 0, 0, 0.0042, 0)^\top$
1000	LP	3.66	333	1.28e-16	$(0, 0, 0.3362, \ldots, 0.0042, 0, 0, 0.0042, 0)^\top$
1000	**FP**	1.44	333	9.95e-17	$(0, 0, 0.3362, \ldots, 0.0042, 0, 0, 0.0042, 0)^\top$

Example 5.3. Consider LCP$(A, \mathbf{q})$, where

$$\mathbf{q} = (\mathbf{q}_{n_1}^\top, \ldots, \mathbf{q}_{n_1}^\top)^\top \in \mathbb{R}^n$$

with $\mathbf{q}_{n_1} = (-1, 1, \ldots, 1)^\top \in \mathbb{R}^{n_1}$, and

$$A = \begin{pmatrix} C & -I & & & \\ -I & C & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & C & -I \\ & & & -I & C \end{pmatrix} \in \mathbb{R}^{n \times n}$$

with $I \in \mathbb{R}^{n_1 \times n_1}$ to be an identity matrix and

$$C = \begin{pmatrix} 4 & -1 & & & \\ -1 & -4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & -4 \end{pmatrix} \in \mathbb{R}^{n_1 \times n_1}.$$

Here, $n = mn_1$.

Table 5.3: Numerical performances for Example 5.3.

(m, n_1)	Algo.	CPU (s)	Sparsity	Rel.Err	Output
(2,50)	Pivot	0.79	50	3.22e-15	$(0.3660, 0, 0.4641, 0, \ldots, 0.3660, 0)^\top$
(2,50)	**iLD**	**0.08**	50	3.22e-15	$(0.3660, 0, 0.4641, 0, \ldots, 0.3660, 0)^\top$
(2,50)	LP	0.58	50	9.96e-16	$(0.3660, 0, 0.4641, 0, \ldots, 0.3660, 0)^\top$
(2,50)	**FP**	0.17	50	**8.29e-16**	$(0.3660, 0, 0.4641, 0, \ldots, 0.3660, 0)^\top$
(20,50)	Pivot	4.84	50	3.22e-15	$(0.3660, 0, \ldots, 0, 0.3660, 0, \ldots, 0)^\top$
(20,50)	**iLD**	**0.2**	50	3.22e-15	$(0.3660, 0, \ldots, 0, 0.3660, 0, \ldots, 0)^\top$
(20,50)	LP	0.49	50	9.96e-16	$(0.3660, 0, \ldots, 0, 0.3660, 0, \ldots, 0)^\top$
(20,50)	**FP**	**0.2**	50	**8.29e-16**	$(0.3660, 0, \ldots, 0, 0.3660, 0, \ldots, 0)^\top$
(50,100)	Pivot	206.33	100	1.33e-14	$(0.3660, 0, \ldots, 0, 0.3660, 0, \ldots, 0)^\top$
(50,100)	**iLD**	14.84	100	1.33e-14	$(0.3660, 0, \ldots, 0, 0.3660, 0, \ldots, 0)^\top$
(50,100)	LP	1.16	100	**7.18e-16**	$(0.3660, 0, \ldots, 0, 0.3660, 0, \ldots, 0)^\top$
(50,100)	**FP**	**0.54**	100	1.10e-15	$(0.3660, 0, \ldots, 0, 0.3660, 0, \ldots, 0)^\top$

From Tables 5.1-5.3, we can observe the following facts. Firstly, the proposed iLD method always cost fewer times compared with Pivot method, and FP method always cost fewer times compared with LP method. The gap is more pronounced for the large scale problems. Furthermore, the two proposed methods almost always cost fewer time compared to the others. Secondly, the FP method could always converge to the least solution of LCP$(A, \mathbf{q})$, which further shows the convergence property of the method numerically. Thirdly, the iLD method seems to have more superiority compared to the others for LCP$(A, \mathbf{q})$ with smaller dimensionality, and the FP method seems to have a stronger performance compared to others for LCP$(A, \mathbf{q})$ with larger dimensionality.

In conclusion, the experimental results further demonstrate the advantages of the proposed methods in numerical calculation.

6 Conclusions

In this paper, we proposed two numerical iterative methods for finding a sparse solution of LCP$(A, \mathbf{q})$ with A being a Z-matrix, and they are an iterative method based on lower-dimensional linear equations and a fixed point iterative method. The first iterative method terminates at the unique least solution of LCP$(A, \mathbf{q})$. The computational cost of this method depends not on the size of the problem but on the sparsity of the solution. Therefore, when the sparsity of the solution is smaller, the calculation cost of the method is lower. The advantage of the second method is that it has the property of monotone descending convergence, while the disadvantage is that a feasible starting point is required. Obviously, the sparser the starting points, the sparser the solutions obtained by this method.

The tensor complementarity problem, which is generalized as a linear complementarity problem, has been actively studied in recent years [12, 13, 17]. In particular, the problem of finding sparse solutions of the tensor complementarity problem has been studied [14, 21]. We believe that the analytical method in this paper can be extended to find sparse solutions for tensor complementarity problems and improve the existing methods.

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