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A NEW STABILITY ANALYSIS ON A NEURAL NETWORK METHOD FOR LINEAR COMPLEMENTARITY PROBLEMS

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Abstract: Consider a neural network method for a linear complementarity problem which is assumed to have a non-empty set of solutions. A new proof of sufficient conditions which guarantee the (globally) asymptotic stability of its equilibrium point is given. Another purpose of this paper is to point out a fatal mistake in the paper by Huang et al. [*ICIC Express Lett.*, 11(4), pp. 853–861, 2017].

Key words: *neural network, linear complementarity problem, asymptotic stability, equilibrium point*

Mathematics Subject Classification: *90C33, 90C30, 65K10*

1 Introduction

A linear complementarity problem (LCP) is to find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad f(x) = Mx + q \geq 0, \quad x^\top f(x) = 0, \tag{1.1}$$

where $M \in \mathbb{R}^{n\times n}, q \in \mathbb{R}^n$ are given. LCP (1.1) has been found a wide range of applications in various fields, such as quadratic programming, bimatrix games, equilibrium problems, and it is a hot topic in the optimization community, see e.g. [4, 18, 19] and references quoted therein.

It can be proved that (see [4]) x^* is a solution of LCP (1.1) if and only if x^* satisfies the following projection equation

$$P_\Omega[x - (Mx + q)] = x, \tag{1.2}$$

where $\Omega = \{x \in \mathbb{R}^n : x \geq 0\}$ and $P_\Omega(x) = \arg\min\{\|x - y\| : y \in \Omega\}$. Throughout this paper, we will assume that the solution set of LCP (1.1) is nonempty. It is well known that LCP (1.1) has a unique solution for all vectors $q \in \mathbb{R}^n$ if and only if $M \in \mathbb{R}^{n\times n}$ is a P-matrix, and LCP (1.1) with a positive semi-definite matrix can have multiple solutions (if it is feasible) [4].

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Many methods for solving LCP (1.1) have been proposed over the years, see, e.g., [18, 4, 14, 1, 17, 23] and references therein. From a continuous perspective, neural network (NN) methods have been extensively investigated for solving many optimization problems, such as LCP [10, 13, 15, 16], variational inequality [8, 21], horizontal LCP [7], linear projection equations [25, 26, 24] and others; see e.g. [12, 2] and references therein.

Based on (1.2), Huang et al. [10] used the following NN method for solving LCP (1.1):

$$\frac{dx(t)}{dt} = \lambda\{P_\Omega[x(t) - (Mx(t) + q)] - x(t)\} \doteq g(x(t)), \tag{1.3}$$

where $\lambda > 0$ is a scale parameter. It is easy to find that x^* is a solution of LCP (1.1) if and only if it is an equilibrium point of NN (1.3), i.e. $g(x^*) = 0$. To our knowledge, NN (1.3) owes to Friesz et al. [5] and Xia et al. [25, 26, 24] have analyzed its stability in a more general background.

The goals of this paper are twofold: to reanalyze the asymptotic stability of the equilibrium point of NN (1.3) in the condition that M is symmetric positive semi-definite and to point out a fatal mistake in the paper [10].

The rest of this paper is arranged as follows. In section 2, we state a new look on NN (1.3). Section 3 is devoted to the stability analysis of NN (1.3). In section 4, we make some comments on the paper [10]. In section 5, we give some conclusions.

Notation. We use $\mathbb{R}^{n\times n}$ to represent the set of all $n \times n$ real matrices and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. I represents the identity matrix of order n. $\cdot^\top$ stands for the transpose of a matrix or vector. $|x| = [|x_1|, |x_2|, \ldots, |x_n|]^\top$ denotes the component-wise absolute value of a vector $x = [x_1, x_2, \ldots, x_n]^\top \in \mathbb{R}^n$. $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product, which is defined as $\langle x, y\rangle \doteq x^\top y = \sum_{i=1}^n x_i y_i$ and $\|x\| \doteq \sqrt{\langle x, x\rangle}$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^n$. $\|A\|$ denotes the spectral norm of A and is defined by the formula $\|A\| \doteq \max\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\}$. A matrix $A \in \mathbb{R}^{n\times n}$ is symmetric if $A^\top = A$. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be positive definite (positive semi-definite) if $\langle Ax, x\rangle > 0$ (respectively, $\langle Ax, x\rangle \geq 0$) for all $0 \neq x \in \mathbb{R}^n$. A matrix $A \in \mathbb{R}^{n\times n}$ is called a P-matrix if every principal minor of A is positive. Evidently, a positive definite matrix is necessary a P-matrix. A symmetric and positive semi-definite (symmetric and positive definite) matrix $X \in \mathbb{R}^{n\times n}$ is denoted by $X \succeq 0$ ($X \succ 0$). Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric, $A \succeq B (A \succ B)$ means $A - B \succeq 0 (A - B \succ 0)$.

$\boxed{2}$ A New Look on NN (1.3)

In this section, we will give a new derivation of NN (1.3).

As is known, LCP (1.1) is equivalent to (see [4])

$$\min\{x, f(x)\} = 0.$$

Since, for $a, b \in \mathbb{R}^n$,

$$\min\{a, b\} = \frac{1}{2}(a + b - |a - b|),$$

LCP (1.1) can be transformed into

$$(I + M)x - |(I - M)x - q| = -q, \tag{2.1}$$

which is a special case of the following new generalized absolute value equations (NGAVE) [22]

$$Cx - |Dx - c| - d = 0, \tag{2.2}$$

where $C, D \in \mathbb{R}^{n\times n}$, $c, d \in \mathbb{R}^n$ are known, $x \in \mathbb{R}^n$ is to be determined and $|Dx - c|$ denotes the component-wise absolute value of the vector $Dx - c$.

We tentatively turn to study some properties of NGAVE (2.2). Throughout this paper, we also assume that the solution set of NGAVE (2.2) is nonempty.

The following theorem is extended from [22, Lemma 3.1].

Theorem 2.1. *NGAVE* (2.2) *is equivalent to the following extended vertical LCP (EVLCP)*

$$G(x) \doteq (C + D)x - c - d \geq 0, \quad H(x) \doteq (C - D)x + c - d \geq 0, \quad \langle G(x), H(x)\rangle = 0. \quad (2.3)$$

Proof. Actually, NGAVE (2.2) can be expressed as $Cx - d = |Dx - c|$. Let

$$s + t = Cx - d, \quad s - t = Dx - c,$$

then

$$s = \frac{(C + D)x - c - d}{2}, \quad t = \frac{(C - D)x + c - d}{2}.$$

In light of [22, Lemma 2.3], the assertion then follows immediately. □

By using Theorem 2.1, [22, Definition 2.3 and Lemma 2.1] and [20, Theorem 3], the following theorem is gained.

Theorem 2.2. *Let $C + D$ be nonsingular. Then for any $c, d \in \mathbb{R}^n$, NGAVE* (2.2) *has a unique solution if and only if $(C - D)(C + D)^{-1}$ is a P-matrix.*

EVLCP (2.3) is a special case of the general variational inequality (GVI) problem:

$$\text{find an } x^*, \text{ such that } G(x^*) \in \Omega', \ \langle v - G(x^*), H(x^*)\rangle \geq 0, \ \forall\, v \in \Omega', \quad (2.4)$$

where Ω' is a closed convex subset of $\mathbb{R}^n$. Indeed, EVLCP (2.3) is equivalent to GVI (2.4) with $\Omega' = \Omega = \{x \in \mathbb{R}^n : x \geq 0\}$ (The proof can be inspired by that of [6, 1.1.3 Proposition.]). If no otherwise specified, we remain $\Omega' = \Omega = \{x \in \mathbb{R}^n : x \geq 0\}$ in this paper.

For a nonempty closed convex subset Ω of $\mathbb{R}^n$, a fundamental property of the projection from $\mathbb{R}^n$ onto Ω is that

$$\langle u - P_\Omega[u], v - P_\Omega[u]\rangle \leq 0, \ \forall u \in \mathbb{R}^n, \ \forall\, v \in \Omega.$$

Using this property, we can observe that GVI (2.4) is equivalent to the following projection equation [9]

$$G(x^*) = P_\Omega\left[G(x^*) - H(x^*)\right]. \quad (2.5)$$

Let

$$r(x) \doteq G(x) - P_\Omega\left[G(x) - H(x)\right] \quad (2.6)$$

be the residual of (2.5). Similar to the proof of [3, Theorem 2.1], we can obtain the following basic theorem.

Theorem 2.3. *For $r(x)$ defined as in* (2.6)*, we have*

$$r(x) = Cx - |Dx - c| - d$$

and thus x^* is a solution of NGAVE (2.2) *if and only if $r(x^*) = 0$.*

On the basis of Theorem 2.3, we will construct the following NN method for solving NGAVE (2.2):

$$\frac{dx(t)}{dt} = \frac{1}{2}\gamma \left\{ P_\Omega \left[G(x(t)) - H(x(t)) \right] - G(x(t)) \right\}, \tag{2.7}$$

where $\gamma > 0$ is a scale parameter. Substituting (2.6) and (2.3) into (2.7), it is reduced to

$$\frac{dx(t)}{dt} = \frac{1}{2}\gamma[d + |Dx(t) - c| - Cx(t)] \doteq h(x(t)). \tag{2.8}$$

Now we turn back to LCP (1.1) or NGAVE (2.1). By setting $C = I + M$, $D = I - M$, $c = q$ and $d = -q$, (2.8) becomes

$$\frac{dx(t)}{dt} = \frac{1}{2}\gamma[|(I - M)x(t) - q| - (I + M)x(t) - q]. \tag{2.9}$$

On the other hand, since $\Omega = \{x \in \mathbb{R}^n : x \geq 0\}$,

$$P_\Omega[x] = \max\{x, 0\} = \frac{1}{2}(|x| + x). \tag{2.10}$$

Substituting (2.10) into (1.3), we have

$$\frac{dx(t)}{dt} = \frac{1}{2}\lambda[|(I - M)x(t) - q| - (I + M)x(t) - q]. \tag{2.11}$$

Evidently, (2.9) equals to (2.11) whenever $\lambda = \gamma$. In the following , we will always remain $\lambda = \gamma$.

3 Stability Analysis

In this section, we will analyze the stability of NN (2.8) and then apply the results to NN (1.3), or equivalently NN (2.11).

Firstly, we draw some concepts of dynamical systems. Consider the autonomous system

$$\frac{dz(t)}{dt} = F(z(t)), \tag{3.1}$$

where F: $\mathbb{R}^n \to \mathbb{R}^n$ is a mapping. Denoting $z(t; z(t_0))$ is the solution of (3.1) determined by the initial value condition $z(t_0) = z_0$. Recall that a point $z_e \in \mathbb{R}^n$ is called an equilibrium point of the dynamical system (3.1) if $F(z_e) = 0$. The following statements can be found in [11, Chapters 2 and 3].

Definition 3.1 (Lyapunov function)**.** Assume that $\mathcal{N} \subseteq \mathbb{R}^n$ is an open neighborhood of $\bar{z}$. A continuously differentiable function $V : \mathcal{N} \to \mathbb{R}$ is said to be a Lyapunov function at the state $\bar{z}$ (over the set $\mathcal{N}$) for (3.1) if

$$V(\bar{z}) = 0 \quad \text{and} \quad V(z) > 0 \quad \text{for} \quad \forall z \in \mathcal{N}, z \neq \bar{z},$$

$$\frac{dV(z(t))}{dt} = [\nabla V(z(t))]^\top F(z(t)) \leq 0,\ \forall z \in \mathcal{N}.$$

Theorem 3.2. (i) *An isolated equilibrium point z^* of (3.1) is Lyapunov stable if there exists a Lyapunov function V over some neighborhood $\mathcal{N}$ of z^*.*

(ii) *An isolated equilibrium point z^* of* (3.1) *is asymptotically stable if there exists a Lyapunov function V over some neighborhood $\mathcal{N}$ of z^* such that*

$$\frac{dV(z(t))}{dt} < 0,\ \forall z(t) \in \mathcal{N}, z(t) \neq z^*.$$

(iii) *An isolated equilibrium point z^* of* (3.1) *is globally asymptotically stable if there exists a Lyapunov function V in $\mathbb{R}^n$ satisfying*

$$\frac{dV(z(t))}{dt} < 0,\ \forall z(t) \in \mathbb{R}^n, z(t) \neq z^*,$$

$$\|z(t) - z^*\| \to \infty \Rightarrow V(z(t)) \to \infty.$$

Now we prove the existence and uniqueness of trajectories of NN (2.8).

Lemma 3.3. *The function h defined as in* (2.8) *is Lipschitz continuous in $\mathbb{R}^n$.*

Proof. For any $x_1, x_2 \in \mathbb{R}^n$, using certain properties of vector norm and $\||x| - |y|\| \leq \|x-y\|$, we have

$$\begin{aligned}\|h(x_1) - h(x_2)\| &= \left\|\frac{1}{2}\gamma[d + |Dx_1 - c| - Cx_1] - \frac{1}{2}\gamma[d + |Dx_2 - c| - Cx_2]\right\| \\ &= \frac{1}{2}\gamma\|C(x_2 - x_1) + (|Dx_1 - c| - |Dx_2 - c|)\| \\ &\leq \frac{1}{2}\gamma(\|C\| + \|D\|)\|x_1 - x_2\|,\end{aligned}$$

which indicates that $h(x)$ is a Lipschitz continuous function in $\mathbb{R}^n$ with Lipschitz constant $\frac{1}{2}\gamma\|(\|C\| + \|D\|)$. $\square$

The following theorem then follows from Lemma 3.3 and [11, Theorem 2.3].

Theorem 3.4. *There exists a unique solution $x(t; x(t_0))$ with $t \in [0, \infty)$ for NN* (2.8) *with any initial value $x(t_0) = x_0$.*

The following theorem lays the foundation of the stability analysis of NN (2.8) and its proof is inspired by that of [9, Theorem 2].

Theorem 3.5. *If x^* is a solution of NGAVE* (2.2) *and $C^\top C \succeq D^\top D$, then*

$$(x - x^*)^\top C^\top r(x) \geq \frac{1}{2}\|r(x)\|^2, \quad \forall\, x \in \mathbb{R}^n. \tag{3.2}$$

Proof. As mentioned previously, x^* is also a solution of EVLCP (2.3) or GVI (2.4) with $\Omega = \{x \in \mathbb{R}^n : x \geq 0\}$. Since $\Omega = \{x \in \mathbb{R}^n : x \geq 0\} \subseteq \mathbb{R}^n$ is a closed convex set and $G(x^*) \in \Omega$, it follows from (2) that

$$[v - P_\Omega(v)]^\top [P_\Omega(v) - G(x^*)] \geq 0, \quad \forall\, v \in \mathbb{R}^n.$$

On one hand, taking $v \doteq G(x) - H(x)$, we deduce that

$$[r(x) - H(x)]^\top \{P_\Omega[G(x) - H(x)] - G(x^*)\} \geq 0. \tag{3.3}$$

On the other hand, due to x^* is a solution of GVI (2.4), we have

$$G(x^*) \in \Omega, \quad H(x^*)^\top (v - G(x^*)) \geq 0, \quad \forall\, v \in \Omega,$$

where $\Omega = \{x \in \mathbb{R}^n : x \geq 0\}$. Since $P_\Omega(\cdot) \in \Omega$, then we obtain

$$H(x^*)^\top \{P_\Omega[G(x) - H(x)] - G(x^*)\} \geq 0. \tag{3.4}$$

Adding (3.3) to (3.4) and utilizing

$$P_\Omega[G(x) - H(x)] - G(x^*) = [G(x) - G(x^*)] - r(x),$$

then it can be concluded that

$$\begin{aligned} &\{[G(x) - G(x^*)] + [H(x) - H(x^*)]\}^\top r(x) \\ &\quad \geq \|r(x)\|^2 + [G(x) - G(x^*)]^\top [H(x) - H(x^*)]. \end{aligned} \tag{3.5}$$

From the definitions of G and H in (2.3), we immediately obtain

$$G(x) + H(x) = 2(Cx - d) \tag{3.6}$$

and

$$[G(x) - G(x^*)]^\top [H(x) - H(x^*)] = (x - x^*)^\top (C^\top C - D^\top D)(x - x^*). \tag{3.7}$$

Combining (3.6) and (3.7) with the inequality (3.5) leads to

$$2(x - x^*)^\top C^\top r(x) \geq \|r(x)\|^2 + (x - x^*)^\top (C^\top C - D^\top D)(x - x^*), \quad \forall\, x \in \mathbb{R}^n,$$

which together with $C^\top C \succeq D^\top D$ yields the desired result. □

Now we are in position to give the stability theorem of NN (2.8).

Theorem 3.6. *If the coefficient matrix C is symmetric positive definite, $C + D$ is nonsingular and $C^\top C \succeq D^\top D$, then the equilibrium point of NN (2.8) is asymptotically stable. Furthermore, if $C^\top C \succ D^\top D$, then the equilibrium point of NN (2.8) is unique and globally asymptotically stable.*

Proof. Let $x(t) = x(t; x(t_0))$ be the solution of (2.8) with initial value $x(t_0) = x_0$ and x^* is the equilibrium point nearby x_0. Consider the following Lyapunov function

$$V(x) = e^{(x - x^*)^\top C (x - x^*)} - 1, \quad x \in \mathbb{R}^n.$$

It is easy to see that $V(x^*) = 0$ and C is positive definite implies that $V(x) > 0$ for all $x \neq x^*$. In addition, it follows from (2.3), (2.7), (3.2) and the symmetry of C that

$$\begin{aligned} \frac{dV(x(t))}{dt} &= \nabla V(x(t))^\top \frac{dx(t)}{dt} \\ &= -\frac{1}{2}\gamma e^{(x(t) - x^*)^\top C (x(t) - x^*)} \left[C(x(t) - x^*) + C^\top (x(t) - x^*)\right]^\top r(x(t)) \\ &\leq -\frac{1}{2}\gamma e^{(x(t) - x^*)^\top C (x(t) - x^*)} \|r(x(t))\|^2 < 0, \quad \forall x(t) \neq x^*. \end{aligned}$$

According to Theorem 3.2 (ii), the equilibrium point x^* of NN (2.8) is asymptotically stable.

In addition, one can prove that the nonsingularity of $C + D$ and $C^\top C \succ D^\top D$ imply that NGAVE (2.2) has a unique solution (according to Theorem 2.2) and thus NN (2.8) has a unique equilibrium point. Since $V(x) \to \infty$ as $\|x - x^*\| \to \infty$, it follows from Theorem 3.2 (iii) that the unique equilibrium point x^* of NN (2.8) is globally asymptotically stable. □

Before ending this section, we turn to NN (2.11). Since $C = I + M$ and $D = I - M$, $C + D = 2I$ is nonsingular. In addition, C is symmetric positive definite and $C^\top C - D^\top D = 2(M + M^\top)$ is symmetric positive semi-definite if M is symmetric semi-positive definite. Moreover, if M is symmetric positive definite, then both C and $C^\top C - D^\top D$ are symmetric positive definite. Thus, we have the following corollary.

Corollary 3.7. *If M is symmetric positive semi-definite, then the equilibrium point of NN* (2.11) *is asymptotically stable. Furthermore, if M is symmetric positive definite, then the equilibrium point of NN* (2.11) *is unique and globally asymptotically stable.*

4 Comments on the Paper [10]

In [10], Huang et al. tried to discuss the global asymptotic stability of the equilibrium point of NN (1.3), under the condition that M is positive definite (unsymmetry is possible). However, their proof of [10, Theorem 3.4] may contain at least one fatal error, as pointed out below. For the sake of completeness, part of [10, Theorem 3.4] is included here.

Theorem 4.1 ([10, Theorem 3.4])**.** *If the matrix M is positive definite, then NN* (1.3) *is stable in the sense of Lyapunov and is globally convergent to the unique equilibrium point of NN* (1.3)*.*

Theorem 4.1 attempts to get rid of the symmetry of M. However, there exists at least one fatal error in the proof of [10, Theorem 3.4].

During the proof of [10, Theorem 3.4], a Lyapunov function candidate

$$V_0(x) = \frac{1}{2}\|P_\Omega(x - Mx - q) - x\|^2 - (Mx + q)^\top [P_\Omega(x - Mx - q) - x] \tag{4.1}$$

was used, where $\Omega = \{x \in \mathbb{R}^n : x \geq 0\}$. From which they got

$$\frac{dV_0(x)}{dx} = -[P_\Omega(x - Mx - q) - x] - M^\top [P_\Omega(x - Mx - q) - x] + Mx + q. \tag{4.2}$$

However, we assert that (4.2) is generally incorrect. The reason may be that the authors overlooked the fact that $P_\Omega(x - Mx - q)$ is depend on x. Here, we give a counterexample to point out this fatal mistake. Concretely, let $x - Mx - q \geq 0$, then $P_\Omega(x - Mx - q) = x - Mx - q$ and (4.1) becomes

$$V_0(x) = \frac{1}{2}\|Mx + q\|^2 + (Mx + q)^\top (Mx + q) = \frac{3}{2}\|Mx + q\|^2,$$

from which we obtain

$$\frac{dV_0(x)}{dx} = 3M^\top (Mx + q). \tag{4.3}$$

On the other hand, (4.2) reduces to

$$\frac{dV_0(x)}{dx} = (2I + M^\top)(Mx + q). \tag{4.4}$$

Obviously, in general, (4.3) is not equal to (4.4). Specially, (4.3) equals to (4.4) if $M = I$, in which case $P_\Omega[q]$ has nothing to do with x.

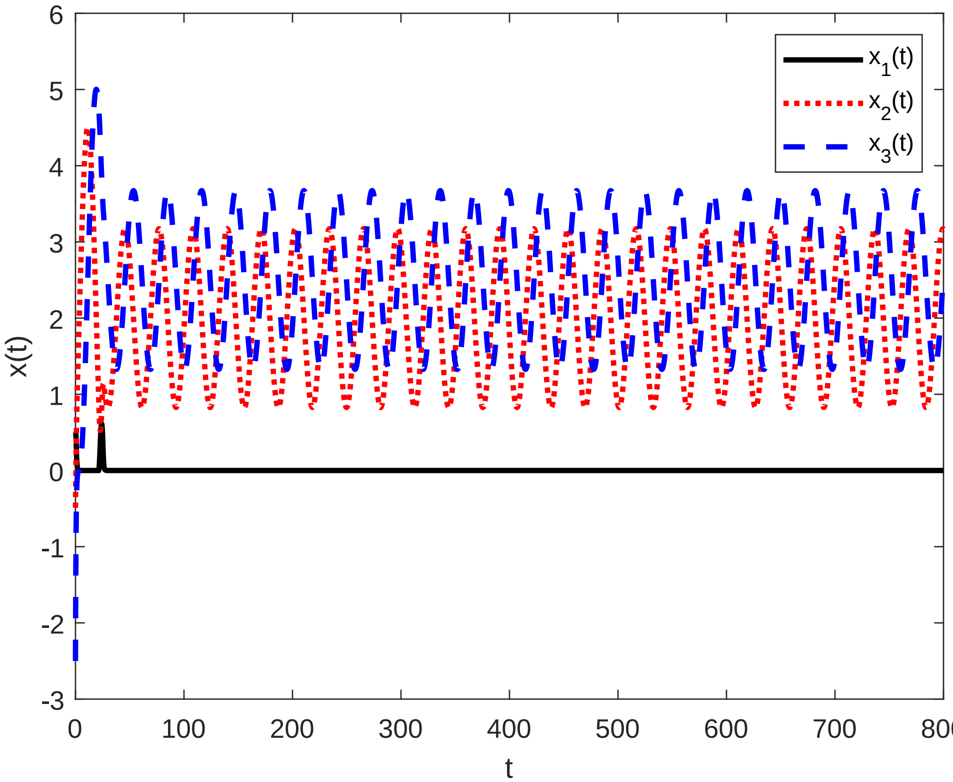

Figure 1: Phase diagrams of (1.3) with $\lambda = 2$. "ode45" is used to solve the ordinary differential equation.

The symmetric positive semi-definiteness of M is also used in [25, 26, 24], however, our proof here differs from theirs. In addition, symmetry seems necessary for the result. For example, consider the following LCP (1.1) [7] with

$$M = \begin{bmatrix} 0 & 0.5 & -0.1 \\ -0.5 & 0 & 0.1 \\ 0.1 & -0.1 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ -0.25 \\ 0.2 \end{bmatrix}.$$

One can easy to see that this M is asymmetric positive semi-definite and the LCP has unique solution $[0, 2, 2.5]^\top$. NN (1.3) is used to solve this LCP and the initial point is $[0.51, -0.49, -2.5]^\top$. Numerical simulation is shown in Figure 1, which indicates that NN (1.3) is not stable.

5 Brief Conclusion

Based on a new look on the neural network method proposed in [5] for solving linear complementarity problems, a new proof of a sufficient condition for its stability is given from the perspective of generalized variational inequalities. A counterexample is found to demonstrate the necessary of symmetry in the sufficient condition.

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