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NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS AND A NEW APPROACH FOR SOLVING THE SMOOTH MULTIOBJECTIVE FRACTIONAL CONTINUOUS-TIME PROGRAMMING PROBLEM*

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Abstract: In this paper, the multiobjective fractional continuous-time programming problem with inequality constraints is considered. We investigate the optimality conditions for this problem under weaker assumptions than in [34], Necessary optimality conditions are obtained under a suitable constraint qualification and a certain regularity condition without convexity/concavity assumptions. It is important to highlight that the assumptions of convexity/concavity on objective and constraint functions in [34] are stronger than the assumptions in this paper. Here, there are no assumptions of convexity/concavity for deriving necessary optimality conditions. Also, the constraint qualifications and a certain regularity condition presented in this paper are much less restrictive and easier to verify than the constraint qualifications given in [34]. This means that the necessary optimality conditions, set in this paper, are obtained under the weakest possible assumptions that are known to date. The already achieved results in the area of multiobjective fractional continuous-time programming are improved and more generalized in this paper. Also, we provide several examples to illustrate our results.

Key words: *continuous-time programming, multiobjective fractional continuous-time programming, optimality conditions*

Mathematics Subject Classification: *90C29, 90C30, 90C46*

1 Introduction

It is well known that multiobjective programming has significant applications in various fields, such as economics, operational research, machine learning, mechanical engineering, electric power systems and chemical engineering. In the past several decades, multiobjective programming problems have been subjected to numerous investigations. The main reason being their heavy usage in the aforementioned fields. For more multiobjective optimization applications, the reader is referred to [26, 24, 27].

Continuous-time programming problems have been introduced in [5]. This work was extended and built upon by Tyndall [29], who gave rigorous treatment to the scalar linear continuous-time programming problems. Since then the theory of scalar and multiobjective

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continuous-time programming has been intensively investigated and a large number of optimality conditions and dual models have been established. Unfortunately, it should be noted that the validity of some results in the field of scalar and multiobjective continuous-time programming was questioned in [2]. In the aforementioned paper, a transposition theorem in the infinite-dimensional spaces is developed under a suitable regularity condition.

One of the earliest papers in the field of multiobjective continuous-time programming is the paper by Singh [25] . In that paper, the author presented dual models for smooth continuous-time programming problems, under generalized convexity/concavity assumptions. In [11], using the concept of Karush-Kuhn-Tucker invexity, the authors studied the relationship of the multiobjective continuous-time programming problems with some related scalar problems. Also, they showed that Karush-Kuhn-Tucker pseudoinvexity is a necessary and sufficient condition for a vector Karush-Kuhn-Tucker solution to be a weakly efficient solution. In [22, 21], the authors introduced a nonsmooth multiobjective continuous-time programming problem and established the optimality conditions and duality theorems under generalized convexity assumptions on the functions involved. In [10, 19], the authors have established optimality conditions and duality theory for multiobjecitve continuous-time programming problem with inequality constraints where the objective and inequality constraint functions are preinvex in their second argument. In [23], the authors studied the relationships between vector variational inequalities and multiobjective continuous-time programming problems under generalized invexity assumptions. In that paper, the authors also obtained optimality conditions for multiobjective continuous-time programming problems and variational-like inequalities problems. Unfortunately, the results in the aforementioned papers are not valid. See [2, 13, 14].

Finite-dimensional multiobjective fractional problems have been studied in [7, 6, 4, 15, 17, 28, 31, 30, 16]. Multiobjective continuous-time programming has been the subject of numerous research endeavours, as stated in the previous paragraphs. Unfortunately, the same can not be stated for multiobjective fractional continuous-time programming problems, which is why the main focus of this paper is presenting a new approach for solving the smooth multiobjective fractional continuous-time programming problem. Also, in the present paper, necessary and sufficient optimality conditions are derived.

A brief overview of the previous work, in the field of study of the multiobjective fractional continuous-time programming problems, will be provided, nevertheless, for more information on the subject the reader is referred to [34, 35, 36]. In [35], necessary and sufficient saddle-point and stationary-point-type proper efficiency conditions are established for a class of continuous-time multiobjective fractional programming problems, defined in Sobolev space, with convex operator inequality, affine operator equality constraints and nonnegativity constraints. In [36], both semiparametric and parametric saddle-point-type and stationary-point-type necessary and sufficient proper efficiency conditions are established for a class of nonsmooth continuous-time multiobjective fractional programming problems, defined in the Banach space, with Volterra-type integral inequality and nonnegativity constraints.

In [34], the author has considered a smooth case, defined in the Banach space, with the assumptions of convexity/concavity and nonnegativity constraints. Based on the concept of properly efficient solutions, the author has also established optimality conditions and parametric duality models.

Here, a more general multiobjective fractional continuous-time programming problem with inequality constraints under weaker assumptions comparing to the assumptions in [34] is considered. Necessary optimality conditions are obtained under a suitable constraint qualification and a certain regularity condition without convexity/concavity assumptions. The fundamental tools for deriving these conditions are a transposition theorem, given in [2] and results from [20]. It is important to emphasize that the hypotheses of convexity/concavity on objective and constraint functions, for example in [34] or [36], are stronger than the assumptions in this paper. Here, there are no assumptions of convexity/concavity for obtaining necessary optimality conditions. Also, it should be highlighted that in the aforementioned papers, the nonegativity of the function $x(\cdot)$ on the interval $[0, T]$ is required, while this is not required in our paper. Also, the constraint qualifications and a suitable regularity condition presented in this paper are much less restrictive and easier to verify than the constraint qualifications given in [34] or [36]. This means that the necessary optimality conditions, set in this paper, are obtained under the weakest possible assumptions that are known to date.

The layout of this paper is as follows. Some definitions are presented in Sect. 2. In Sect. 3, the necessary optimality conditions are obtained. Also, an illustrative example is provided to indicate the usefulness of these conditions. In Sect. 4, the sufficient optimality criteria is established under concavity and generalized concavity assumptions.

$\boxed{2}$ Preliminaries and Statement of Problem

Let us consider the following multiobjective fractional continuous-time problem :

$$\begin{aligned} &\max \frac{\int_\Delta f(\tau, v(\tau))d\tau}{\int_\Delta g(\tau, v(\tau))d\tau} = \left(\frac{\int_\Delta f_1(\tau, v(\tau))d\tau}{\int_\Delta g_1(\tau, v(\tau))d\tau}, \ldots, \frac{\int_\Delta f_k(\tau, v(\tau))d\tau}{\int_\Delta g_k(\tau, v(\tau))d\tau}\right) \\ &\text{s.t.} \quad h_i(\tau, v(\tau)) \geqq 0, \quad i \in I = \{1, \ldots, m\} \ \text{ a.e. in } \Delta, \\ &\qquad v(\cdot) \in L_\infty(\Delta; \mathbb{R}^n). \end{aligned} \tag{MFCTP}$$

Here, by $L_\infty(\Delta; \mathbb{R}^n)$ we denote the Banach space of all n-dimensional essentially bounded Lebesgue measurable functions defined on $\Delta = [0, T] \subset \mathbb{R}$ with the corresponding norm $\|\cdot\|_\infty$ defined by

$$\|v(\cdot)\|_\infty = \max_{1 \leqq i \leqq n} \operatorname*{ess\,sup}_{\tau \in \Delta} |v_i(\tau)|.$$

Further, $f_j, g_j, h_i : \Delta \times \mathbb{R}^n \to \mathbb{R}$, $j \in J = \{1, \ldots, k\}$, $i \in I$, are given functions. If $\nu, \mu \in \mathbb{R}^k$, the following convention will be used:

(i) $\nu = \mu \iff \nu_j = \mu_j, \ j = 1, \ldots, k,$

(ii) $\nu \leqq \mu \iff \nu_j \leqq \mu_j, \ j = 1, \ldots, k,$

(iii) $\nu \leq \mu \iff \nu \leqq \mu$ and $\nu \neq \mu,$

(iv) $\nu < \mu \iff \nu_j < \mu_j, \ j = 1, \ldots, k,$

(v) $\nu \nleq \mu$ is the negation of $\nu \leq \mu$.

All integrals are given in the sense of Lebesgue and B denotes the open unit ball centered at the origin in $\mathbb{R}^n$. Also, all vectors in this paper are column vectors. Here, $\mathbb{R}^k_+$ denotes the positive orthant of $\mathbb{R}^k$. Let

$$\Phi = \{v(\cdot) \in L_\infty(\Delta; \mathbb{R}^n) : h(\tau, v(\tau)) \geqq 0 \ \text{ a.e. in } \Delta\}$$

be the set of feasible solutions of (MFCTP). For $v(\cdot) \in \Phi$, we also assume that

$$\int_{\Delta} f(\tau, v(\tau))d\tau \geqq 0 \quad \text{and} \quad \int_{\Delta} g(\tau, v(\tau))d\tau > 0. \tag{2.1}$$

Let $\varepsilon > 0$, $\hat{v}(\cdot) \in \Phi$ and $\hat{w} = \frac{\int_{\Delta} f(\tau,\hat{v}(\tau))d\tau}{\int_{\Delta} g(\tau,\hat{v}(\tau))d\tau} \in \mathbb{R}^k_+$. Suppose the following assumption (A) is valid:

(A)
- $f(\tau, \cdot)$, $g(\tau, \cdot)$ and $h(\tau, \cdot)$ are continuously differentiable on $\hat{v}(\tau) + \varepsilon \bar{B}$ for almost every $\tau \in \Delta$;
- $f(\cdot, v)$, $g(\cdot, v)$ and $h(\cdot, v)$ are Lebesgue measurable for each v, $h(\cdot, v(\cdot))$ is essentially bounded in Δ for all $v(\cdot) \in L_\infty(\Delta, \mathbb{R}^n)$;
- $\exists\ K > 0$, $\exists\ H > 0$ such that $\|\nabla f_j(\tau, \hat{v}(\tau)) - \hat{w}_j \nabla g_j(\tau, \hat{v}(\tau))\| \leqq K$, $\|\nabla h_i(\tau, \hat{v}(\tau))\| \leqq H$, a.e. in Δ, $j \in J$, $i \in I$.

Definition 2.1. [8] A feasible solution $\hat{v}(\cdot)$ of (MFCTP) is said to be an efficient solution EMFCTP (a weak efficient solution WEMFCTP) of (MFCTP) if there is no other $v(\cdot) \in \Phi$ such that

$$\frac{\int_{\Delta} f(\tau, v(\tau))d\tau}{\int_{\Delta} g(\tau, v(\tau))d\tau} \geq \frac{\int_{\Delta} f(\tau, \hat{v}(\tau))d\tau}{\int_{\Delta} g(\tau, \hat{v}(\tau))d\tau}.$$

$$\left(\frac{\int_{\Delta} f(\tau, v(\tau))d\tau}{\int_{\Delta} g(\tau, v(\tau))d\tau} > \frac{\int_{\Delta} f(\tau, \hat{v}(\tau))d\tau}{\int_{\Delta} g(\tau, \hat{v}(\tau))d\tau}. \right)$$

The definition below can be seen as a continuous-time version of the definition in [18].

Definition 2.2. [18] Let $\varepsilon > 0$. A feasible solution $\hat{v}(\cdot)$ of (MFCTP) is said to be a locally efficient solution LEMFCTP (a locally weak efficient solution LWEMFCTP) of (MFCTP) if there exists a neighborhood

$$\mathcal{N}(\hat{v}(\cdot), \epsilon) = \left\{ v(\cdot) \in L_\infty(\Delta; \mathbb{R}^n) \ : \ v(\tau) \in \hat{v}(\tau) + \epsilon \bar{B}, \quad \text{a.e. in } \Delta \right\}$$

such that $\hat{v}(\cdot)$ is an EMFCTP (WEMFCTP) on $\Phi \cap N(\hat{v}(\cdot), \varepsilon)$.

Obviously, an EMFCTP is necessarily a LEMFCTP. Also, a WEMFCTP is a LWEMFCTP and a LEMFCTP is a LWEMFCTP. Let $b > 0$ and $\hat{v}(\cdot) \in \Phi$. Let $I_b(\tau) = \{ i \in I : 0 \leqq h_i(\tau, \hat{v}(\tau)) \leqq b \}$, for each $\tau \in \Delta$ and

$$\delta_i^b(\tau) = \begin{cases} 1, \ i \in I_b(\tau) \\ 0, \ \text{otherwise.} \end{cases}$$

In the sequel, we consider cones in $L_\infty(\Delta; \mathbb{R}^n)$. For more details about the cones and calculating dual cones in the theory of extremal problems, the reader is referred to [12]. Let

$\hat{v}(\cdot) \in \Phi$ be such that (A) is satisfied for some $\varepsilon > 0$. Let $b > 0$. Define:

$$\mathcal{T}_\Phi(\hat{v}(\cdot)) = \Bigg\{ \zeta(\cdot) \in L_\infty(\Delta; \mathbb{R}^n) : \exists \{v_n(\cdot)\} \subset \Phi, \, \{\alpha_n\} \subset \mathbb{R}_+ \downarrow 0, \lim_{n\to\infty} v_n(\cdot) = \hat{v}(\cdot), \, \zeta(\cdot) = \lim_{n\to\infty} \frac{v_n(\cdot) - \hat{v}(\cdot)}{\alpha_n} \Bigg\},$$

$$\mathcal{A}_j(\hat{v}(\cdot)) = \Bigg\{ \zeta(\cdot) \in L_\infty(\Delta; \mathbb{R}^n) : \int_\Delta \Big(\nabla f_j(\tau, \hat{v}(\tau))^T \zeta(\tau) - \frac{\int_\Delta f_j(\tau, \hat{v}(\tau)) d\tau}{\int_\Delta g_j(\tau, \hat{v}(\tau)) d\tau} \nabla g_j(\tau, \hat{v}(\tau))^T \zeta(\tau) \Big) d\tau > 0 \Bigg\}, \; j \in J,$$

$$\mathcal{F}_b(\hat{v}(\cdot)) = \Bigg\{ \zeta(\cdot) \in L_\infty(\Delta; \mathbb{R}^n) : h_i(\tau, \hat{v}(\tau)) + \delta_i^b(\tau) \nabla h_i(\tau, \hat{v}(\tau))^T \zeta(\tau) \geqq 0, \; i \in I \quad \text{a.e. in } \Delta \Bigg\},$$

where $\mathcal{T}_\Phi(\hat{v}(\cdot))$, $\mathcal{A}_j(\hat{v}(\cdot))$, $j \in J$ and $\mathcal{F}_b(\hat{v}(\cdot))$ denote Bouligand tangent cone (see [9]), the cone of ascent directions and the feasible direction cone at $\hat{v}(\cdot)$ in the continuous-time context, respectively. Also, we give continuous-time versions of Mangasarian-Fromovitz and Slater constraint qualification.

Definition 2.3. The constraint qualification (MFQ) is satisfied at $\hat{v}(\cdot) \in \Phi$, if there exist $\bar{\zeta}(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$ and $\hat{b} > 0$ such that

$$\nabla h_i(\tau, \hat{v}(\tau))^T \bar{\zeta}(\tau) \geqq \beta, \, i \in I_{\hat{b}}(\tau) \; \text{ a.e. in } \; \Delta \tag{MFQ}$$

for some $\beta > 0$.

Definition 2.4. Assume that $h_i(\tau, \cdot)$ is a concave almost everywhere in Δ, $i \in I$. We say that (SQ) is satisfied, if there exists $\bar{\eta}(\cdot) \in \Phi$ and $\hat{b} > 0$ such that

$$h_i(\tau, \bar{\eta}(\tau)) \geqq \beta, \, i \in I_{\hat{b}}(\tau) \; \text{ a.e. in } \; \Delta \tag{SQ}$$

for some $\beta > 0$.

Remark 2.5. In [20], the authors showed that the constraint qualification (SQ) is a sufficient condition for (MFQ) under an additional concavity assumption.

The following convention will be used:

$$\nabla f(\tau, \hat{v}(\tau)) = (\nabla f_1(\tau, \hat{v}(\tau)), \ldots, \nabla f_k(\tau, \hat{v}(\tau)))^T,$$
$$\nabla g(\tau, \hat{v}(\tau)) = (\nabla g_1(\tau, \hat{v}(\tau)), \ldots, \nabla g_k(\tau, \hat{v}(\tau)))^T,$$
$$\nabla h(\tau, \hat{v}(\tau)) = (\nabla h_1(\tau, \hat{v}(\tau)), \ldots, \nabla h_m(\tau, \hat{v}(\tau)))^T, \; e = (1, \ldots, 1)^T \in \mathbb{R}^k,$$
$$\Lambda = \left\{ \lambda \in \mathbb{R}^k \, : \, \lambda^T e = 1, \, \lambda \geq 0 \right\}, \; \Lambda^+ = \left\{ \lambda \in \mathbb{R}^k \, : \, \lambda^T e = 1, \, \lambda > 0 \right\}.$$

$\boxed{3}$ Necessary Conditions

We will be following the similar approach as in [20] to prove the following crucial lemma.

Lemma 3.1. *Let $\hat{v}(\cdot) \in \Phi$ be a LEMFCTP (LWEMFCTP). Assume that (A) and* (MFQ) *are satisfied at $\hat{v}(\cdot)$. Then,*

$$\mathcal{F}_b(\hat{v}(\cdot)) \cap \bigcap_{j=1}^{k} \mathcal{A}_j(\hat{v}(\cdot)) = \emptyset. \tag{3.1}$$

Proof. Auxiliary functionals $F_j : L_\infty(\Delta; \mathbb{R}^n) \to \mathbb{R}$, $j \in J$, are defined by

$$F_j(v(\cdot)) = \int_\Delta \left(f_j(\tau, v(\tau)) - \frac{\int_\Delta f_j(\tau, \hat{v}(\tau))d\tau}{\int_\Delta g_j(\tau, \hat{v}(\tau))d\tau} g_j(\tau, v(\tau)) \right) d\tau.$$

We will suppose that (3.1) is not true. Let $\mathcal{F}_b(\hat{v}(\cdot)) \cap \bigcap_{j=1}^{k} \mathcal{A}_j(\hat{v}(\cdot)) \neq \emptyset$. Then, Proposition 3.2. [20] implies that

$$\mathcal{T}_\Phi(\hat{v}(\cdot)) \cap \bigcap_{j=1}^{k} \mathcal{A}_j(\hat{v}(\cdot)) \neq \emptyset.$$

Consequently, there exists $\zeta(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$ such that

$$\begin{aligned} &\int_\Delta \nabla \xi_1(\tau, \hat{v}(\tau))^T \zeta(\tau) d\tau > 0, \ldots, \int_\Delta \nabla \xi_k(\tau, \hat{v}(\tau))^T \zeta(\tau) d\tau > 0, \\ &\zeta(\cdot) = \lim_{n\to\infty} \frac{v_n(\cdot) - \hat{v}(\cdot)}{\alpha_n}, \end{aligned} \tag{3.2}$$

where

$$\xi_j(\tau, v(\tau)) = f_j(\tau, v(\tau)) - \frac{\int_\Delta f_j(\tau, \hat{v}(\tau))d\tau}{\int_\Delta g_j(\tau, \hat{v}(\tau))d\tau} g_j(\tau, v(\tau)), \ j \in J,$$

and $\{v_n(\cdot)\}$ is a sequence in the feasible set, such that $\lim_{n\to\infty} v_n(\cdot) = \hat{v}(\cdot)$. Also, $\{\alpha_n\}$ is a sequence of positive numbers converging to 0. Let $C_0 > 0$. Then, there is $N_0 > 0$ such that

$$\left\| \frac{v_n(\cdot) - \hat{v}(\cdot)}{\alpha_n} - \zeta(\cdot) \right\|_\infty \leqq C_0, \ \forall n \geqq N_0$$

and $\|\zeta(\cdot)\|_\infty = C_1$. If hypothesis (A) is satisfied, functionals

$$F_1(v(\cdot)) = \int_\Delta \xi_1(\tau, v(\tau))d\tau, \ldots, F_k(v(\cdot)) = \int_\Delta \xi_k(\tau, v(\tau))d\tau$$

are Fréchet differentiable. By construction of ξ_j and by the Mean Value Theorem (see [1, 3]), there is $\lambda \in (0, 1)$ such that

$$\begin{aligned} &\int_\Delta \left(\xi_j(\tau, v_n(\tau)) - \xi_j(\tau, \hat{v}(\tau)) \right) d\tau \\ &= \int_\Delta \xi_j(\tau, v_n(\tau)) d\tau \\ &= \int_\Delta \nabla \xi_j(\tau, (1-\lambda)v_n(\tau) + \lambda \hat{v}(\tau))^T \left(v_n(\tau) - \hat{v}(\tau) \right) d\tau, \ j \in J. \end{aligned} \tag{3.3}$$

Note that $\lim_{n\to\infty}((1-\lambda)v_n(\cdot)+\lambda\hat{v}(\cdot))=\hat{v}(\cdot)$. Further, there exists $N_1>0$, such that

$$\|\nabla\xi_j\left(\tau,(1-\lambda)v_n(\tau)+\lambda\hat{v}(\tau)\right)\|\leqq K,\quad \text{a.e. in } \Delta,\forall n\geqq N_1,\ j\in J.$$

Setting $N:=\max\{N_0,N_1\}$, we have for $j\in J$,

$$\|\nabla\xi_j\left(\tau,(1-\lambda)v_n(\tau)+\lambda\hat{v}(\tau)\right)\|\leqq K\quad \text{a.e. in } \Delta,\ \forall n\geqq N.$$

Also, we obtain

$$\left\|\frac{v_n(\tau)-\hat{v}(\tau)}{\alpha_n}\right\|\leqq\left\|\frac{v_n(\tau)-\hat{v}(\tau)}{\alpha_n}-\zeta(\tau)\right\|+\|\zeta(\tau)\|\leqq C,\ \forall n\geqq N,\quad \text{a.e. in } \Delta,$$

where $C=C_0+C_1$. Consider the sequences $\{\gamma_n^1(\tau)\}_{n=N}^{\infty},\ldots,\{\gamma_n^k(\tau)\}_{n=N}^{\infty}$, where $\{\gamma_n^j(\tau)\}_{n=N}^{\infty}\subset L_\infty(\Delta;\mathbb{R}^n)$ and

$$\gamma_n^j(\tau)=\nabla\xi_j\left(\tau,(1-\lambda)v_n(\tau)+\lambda\hat{v}(\tau)\right)^T\frac{v_n(\tau)-\hat{v}(\tau)}{\alpha_n},\ j\in J,\quad \text{a.e. in } \Delta.$$

For $B=C\cdot K$ we obtain

$$\begin{aligned}&\|\gamma_n^j(\tau)\|\leqq\left\|\nabla\xi_j\left(\tau,(1-\lambda)v_n(\tau)+\lambda\hat{v}(\tau)\right)\right\|\left\|\frac{v_n(\tau)-\hat{v}(\tau)}{\alpha_n}\right\|\leqq B,\\&\forall n\geqq N,\forall j\in J,\quad \text{a.e. in } \Delta\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\gamma_n^j(\tau)&=\lim_{n\to\infty}\nabla\xi_j\left(\tau,(1-\lambda)v_n(\tau)+\lambda\hat{v}(\tau)\right)^T\frac{v_n(\tau)-\hat{v}(\tau)}{\alpha_n}\\&=\nabla\xi_j(\tau,\hat{v}(\tau))^T\zeta(\tau),\ \ \forall j\in J.\end{aligned}$$

Lebesgue Dominated Convergence Theorem and (3.2) imply

$$\begin{aligned}&\lim_{n\to\infty}\int_\Delta\nabla\xi_j\left(\tau,(1-\lambda)v_n(\tau)+\lambda\hat{v}(\tau)\right)^T\frac{v_n(\tau)-\hat{v}(\tau)}{\alpha_n}\,d\tau\\&=\int_\Delta\lim_{n\to\infty}\nabla\xi_j\left(\tau,(1-\lambda)v_n(\tau)+\lambda\hat{v}(\tau)\right)^T\frac{v_n(\tau)-\hat{v}(\tau)}{\alpha_n}\,d\tau\\&=\int_\Delta\nabla\xi_j(\tau,\hat{v}(\tau))^T\zeta(\tau)d\tau>0,\ \forall j\in J.\end{aligned}$$

Therefore,

$$\int_\Delta\nabla\xi_j\left(\tau,(1-\lambda)v_n(\tau)+\lambda\hat{v}(\tau)\right)^T\frac{v_n(\tau)-\hat{v}(\tau)}{\alpha_n}\,d\tau>0,\ \forall j\in J,\tag{3.4}$$

for all sufficiently large n. Hence, for all $j\in J$, (3.3) and (3.4) imply

$$\begin{aligned}\int_\Delta\xi_j(\tau,v_n(\tau))d\tau&=\int_\Delta\nabla\xi_j(\tau,(1-\lambda)v_n(\tau)+\lambda\hat{v}(\tau))^T\left(v_n(\tau)-\hat{v}(\tau)\right)d\tau\\&=\alpha_n\int_\Delta\nabla\xi_j(\tau,(1-\lambda)v_n(\tau)+\lambda\hat{v}(\tau))^T\frac{v_n(\tau)-\hat{v}(\tau)}{\alpha_n}d\tau>0,\end{aligned}$$

so that

$$\int_\Delta \xi_j(\tau, v_n(\tau))d\tau > 0,\ \forall j \in J \tag{3.5}$$

for all sufficiently large n. By construction, inequality (3.5) can be rewritten as

$$\int_\Delta \left(f_j(\tau, v_n(\tau)) - \frac{\int_\Delta f_j(\tau, \hat{v}(\tau))d\tau}{\int_\Delta g_j(\tau, \hat{v}(\tau))d\tau} g_j(\tau, v_n(\tau)) \right) d\tau > 0\ \forall j \in J, \tag{3.6}$$

for all sufficiently large n.

Put

$$\hat{w} = \frac{\int_\Delta f_j(\tau, \hat{v}(\tau))d\tau}{\int_\Delta g_j(\tau, \hat{v}(\tau))d\tau} \geqq 0.$$

Then, inequality (3.6) can be rewritten as

$$\begin{aligned} &\int_\Delta \left(f_j(\tau, v_n(\tau)) - \hat{w} g_j(\tau, v_n(\tau))\right) d\tau \\ &= \int_\Delta f_j(\tau, v_n(\tau))d\tau - \hat{w} \int_0^T g_j(\tau, v_n(\tau))d\tau > 0\ \ \forall j \in J, \end{aligned}$$

for all sufficiently large n. Previous inequality implies

$$\frac{\int_\Delta f_j(\tau, v_n(\tau))d\tau}{\int_\Delta g_j(\tau, v_n(\tau))d\tau} > \frac{\int_\Delta f_j(\tau, \hat{v}(\tau))d\tau}{\int_\Delta g_j(\tau, \hat{v}(\tau))d\tau} = \hat{w}\ \forall j \in J,$$

for all sufficiently large n. Thus, $\hat{v}(\cdot)$ is not locally (weak) efficient solution of (MFCTP). Thus, the proof is complete. □

Let $\zeta \in \mathbb{R}^n$, $\hat{v}(\cdot)$ be a locally efficient solution for (MFCTP) and suppose that (A), (MFQ) are satisfied at $\hat{v}(\cdot)$ and

$$\begin{aligned} \phi_j(\tau, \zeta) &= -\int_\Delta \left(\nabla f_j(\tau, \hat{v}(\tau))^T \zeta - \hat{w}_j \nabla g_j(\tau, \hat{v}(\tau))^T \zeta\right) d\tau,\ \ j \in J, \\ \phi_i(\tau, \zeta) &= -\, h_i(\tau, \hat{v}(\tau)) - \delta_i^{\hat{b}}(\tau) \nabla h_i(\tau, \hat{v}(\tau))^T \zeta,\ \ i \in I, \end{aligned}$$

where

$$\hat{w} = \frac{\int_\Delta f(\tau, \hat{v}(\tau))d\tau}{\int_\Delta g(\tau, \hat{v}(\tau))d\tau} \in \mathbb{R}^k_+.$$

Let

$$\begin{cases} \phi_j(\tau, \zeta) < 0,\ j \in J, \\ \phi_i(\tau, \zeta) \leqq 0,\ i \in I, \\ \zeta \in \mathbb{R}^n, \end{cases} \tag{3.7}$$

be a system corresponding to the problem (MFCTP), $K = J \sqcup I$, and

$$\mathcal{I}(\tau, \zeta) = \left\{ i : \phi_i(\tau, \zeta) = \max_{r \in K} \phi_r(\tau, \zeta) \right\},\ \ \tau \in \Delta,\ \ \zeta \in \mathbb{R}^n.$$

Definition 3.2. (Regularity condition [2]) We say that the regularity condition (RC) is satisfied at $\hat{v}(\cdot)$, if there exist $\bar{v}(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$, reals $R \geqq 0$ and $\alpha > 0$ such that for almost every $\tau \in [0, 1]$,

$$\begin{gathered} \forall \zeta \in \mathbb{R}^n,\ \|\zeta - \bar{v}(\tau)\| \geqq R,\ \exists e = e(\tau, \zeta) \in \mathbb{R}^n,\ \|e\| = 1\ : \\ \langle \partial_\zeta \phi_i(\tau, \zeta), e \rangle \geqq \alpha \quad \forall i \in \mathcal{I}(\tau, \zeta). \end{gathered} \tag{RC}$$

Now, we give necessary optimality conditions for (MFCTP), using preceding Lemma and new tool presented in [2].

Theorem 3.3. *Let $\hat{v}(\cdot) \in \Phi$ be a LEMFCTP (LWEMFCTP). Suppose that (A),* (MFQ) *and* (RC) *are satisfied at $\hat{v}(\cdot)$. Then, there exists $(\hat{\lambda}, \hat{u}(\cdot)) \in \Lambda \times L_\infty(\Delta; \mathbb{R}^m)$ such that the following conditions are satisfied:*

$$\begin{aligned}\hat{\lambda}^T \left(\nabla f(\tau, \hat{v}(\tau)) - e \left(\frac{\int_\Delta f(\tau, \hat{v}(\tau)) d\tau}{\int_\Delta g(\tau, \hat{v}(\tau)) d\tau} \right)^T \nabla g(\tau, \hat{v}(\tau)) \right) \\ + \hat{u}(\tau)^T \nabla h(\tau, \hat{v}(\tau)) = 0 \quad \text{a.e. in } \Delta,\end{aligned} \tag{3.8}$$

$$\hat{u}(\tau)^T h(\tau, \hat{v}(\tau)) = 0, \ \hat{u}(\tau) \geqq 0 \quad \text{a.e. in } \Delta. \tag{3.9}$$

Proof. By Lemma 3.1, we have that there is no $\zeta(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$ such that system (3.7) is consistent. Then, by Theorem 1 [2], we have that there is a nonzero function $(\hat{\varphi}_1(\cdot), \ldots, \hat{\varphi}_k(\cdot), \hat{\mu}_1(\cdot), \ldots, \hat{\mu}_m(\cdot)) \in L_\infty(\Delta; \mathbb{R}_+^{k+m})$ with $\hat{\varphi}_j(\tau) \not\equiv 0$ for some $j \in J$ $(\tau \in \Delta)$, such that

$$\begin{aligned}&\sum_{j \in J} \hat{\varphi}_j(\tau) \int_\Delta \left(\nabla f_j(\tau, \hat{v}(\tau))^T \zeta - \hat{w}_j \nabla g_j(\tau, \hat{v}(\tau))^T \zeta \right) d\tau \\ &+ \sum_{i \in I} \hat{\mu}_i(\tau) \left(h_i(\tau, \hat{v}(\tau)) + \delta_i^{\hat{b}}(\tau) \nabla h_i(\tau, \hat{v}(\tau))^T \zeta \right) \leqq 0, \ \forall \zeta \in \mathbb{R}^n \quad \text{a.e. in } \Delta.\end{aligned}$$

Setting $\zeta \equiv 0$, we obtain $\sum_{i \in I} \hat{\mu}_i(\tau) h_i(\tau, \hat{v}(\tau)) \leqq 0$ a.e. in Δ. Since $\hat{v}(\cdot) \in \Phi$, we have that the opposite inequality is also satisfied. Therefore,

$$\hat{\mu}_i(\tau) h_i(\tau, \hat{v}(\tau)) = 0, \ i \in I, \quad \text{a.e. in } \Delta. \tag{3.10}$$

Hence

$$\begin{aligned}&\sum_{j \in J} \hat{\varphi}_j(\tau) \int_\Delta \left(\nabla f_j(\tau, \hat{v}(\tau))^T \zeta(\tau) - \hat{w}_j \nabla g_j(\tau, \hat{v}(\tau))^T \zeta(\tau) \right) d\tau \\ &+ \sum_{i \in I} \hat{\mu}_i(\tau) \delta_i^{\hat{b}}(\tau) \nabla h_i(\tau, \hat{v}(\tau))^T \zeta(\tau) \leqq 0 \quad \text{a.e. in } \Delta.\end{aligned} \tag{3.11}$$

Integrating (3.11) on Δ, we obtain

$$\begin{aligned}&\int_\Delta \left(\sum_{j \in J} \hat{\varphi}_j(\tau) \int_\Delta \left(\nabla f_j(r, \hat{v}(r))^T \zeta(r) - \hat{w}_j \nabla g_j(r, \hat{v}(r))^T \zeta(r) \right) dr \right) d\tau \\ &+ \int_\Delta \left(\sum_{i \in I} \hat{\mu}_i(\tau) \delta_i^{\hat{b}}(\tau) \nabla h_i(\tau, \hat{v}(\tau))^T \zeta(\tau) \right) d\tau \leqq 0,\end{aligned}$$

i.e.,

$$\begin{aligned}&\sum_{j \in J} \int_\Delta \hat{\varphi}_j(\tau) d\tau \int_\Delta \left(\nabla f_j(\tau, \hat{v}(\tau))^T \zeta(\tau) - \hat{w}_j \nabla g_j(\tau, \hat{v}(\tau))^T \zeta(\tau) \right) d\tau \\ &+ \int_\Delta \left(\sum_{i \in I} \hat{\mu}_i(\tau) \delta_i^{\hat{b}}(\tau) \nabla h_i(\tau, \hat{v}(\tau))^T \zeta(\tau) \right) d\tau \leqq 0.\end{aligned}$$

Setting

$$\hat{\psi}_j = \int_\Delta \hat{\varphi}_j(\tau)d\tau \geqq 0, \; j \in J, \tag{3.12}$$

with at least one strict inequality in (3.12) (according to the assumption $\hat{\varphi}_j(\tau) \not\equiv 0$ for some $j \in J$), it follows

$$\begin{aligned} &\int_\Delta \left(\sum_{j \in J} \hat{\psi}_j \; \left(\nabla f_j(\tau, \hat{v}(\tau))^T \zeta(\tau) - \hat{w}_j \nabla g_j(\tau, \hat{v}(\tau))^T \zeta(\tau) \right) \right) d\tau \\ &+ \int_\Delta \left(\sum_{i \in I} \hat{\mu}_i(\tau) \delta_i^{\hat{b}}(\tau) \nabla h_i(\tau, \hat{v}(\tau))^T \zeta(\tau) \right) d\tau \leqq 0. \end{aligned} \tag{3.13}$$

Now, dividing all terms in (3.13) by $\sum_{j \in J} \hat{\psi}_j > 0$ and defining

$$\hat{\lambda}_j = \frac{\hat{\psi}_j}{\sum_{j \in J} \hat{\psi}_j}, \quad \hat{u}_i(\tau) = \frac{\hat{\mu}_i(\tau)\delta_i^{\hat{b}}(\tau)}{\sum_{j \in J} \hat{\psi}_j}, \; j \in J, \; i \in I, \; \tau \in \Delta, \tag{3.14}$$

we have

$$\begin{aligned} &\int_\Delta \left(\sum_{j \in J} \hat{\lambda}_j \left(\nabla f_j(\tau, \hat{v}(\tau)) - \hat{w}_j \nabla g_j(\tau, \hat{v}(\tau)) \right) + \sum_{i \in I} \hat{u}_i(\tau) \nabla h_i(\tau, \hat{v}(\tau)) \right)^T \zeta(\tau) d\tau \\ &\quad \leqq 0 \;\; \forall \zeta(\cdot) \in L_\infty(\Delta; \mathbb{R}^n). \end{aligned} \tag{3.15}$$

Since (3.15) is valid for all $\zeta(\cdot) \in L_\infty(\Delta; \mathbb{R}^n)$, we obtain

$$\sum_{j \in J} \hat{\lambda}_j \left(\nabla f_j(\tau, \hat{v}(\tau)) - \hat{w}_j \nabla g_j(\tau, \hat{v}(\tau)) \right) + \sum_{i \in I} \hat{u}_i(\tau) \nabla h_i(\tau, \hat{v}(\tau)) = 0 \quad \text{a.e. in } \; \Delta,$$

i.e.,

$$\begin{aligned} &\sum_{j \in J} \hat{\lambda}_j \left(\nabla f_j(\tau, \hat{v}(\tau)) - \frac{\int_\Delta f_j(\tau, \hat{v}(\tau))d\tau}{\int_\Delta g_j(\tau, \hat{v}(\tau))d\tau} \nabla g_j(\tau, \hat{v}(\tau)) \right) \\ &+ \sum_{i \in I} \hat{u}_i(\tau) \nabla h_i(\tau, \hat{v}(\tau)) = 0 \quad \text{a.e. in } \; \Delta. \end{aligned} \tag{3.16}$$

Hence, (3.8) holds and (3.14) implies that $\lambda \in \Lambda$. Also, (3.10) and (3.14) imply (3.9). □

Here, there are no hypotheses of convexity/concavity for obtaining necessary optimality conditions. It is important to highlight that the hypotheses of convexity/concavity on objective and constraint functions in [34, 36] are stronger than the assumptions in this paper.

From Remark 2.5 we obtain the following necessary conditions for (MFCTP).

Theorem 3.4. *Let $\hat{v}(\cdot) \in \Phi$ be a LEMFCTP (LWEMFCTP). Assume that (A),* (RC) *are satisfied at $\hat{v}(\cdot)$ and $h_i(\tau, \cdot)$ is a concave almost everywhere in Δ, $i \in I$. If* (SQ) *is satisfied, then there exists $(\hat{\lambda}, \hat{u}(\cdot)) \in \Lambda \times L_\infty(\Delta; \mathbb{R}^m)$ such that* (3.8)-(3.9) *are satisfied.*

Corollary 3.5. *The conditions* (3.8)-(3.9) *of the Theorem 3.3 (Theorem 3.4) are also necessary for $\hat{v}(\cdot)$ to be an efficient solution of* (MFCTP).

Let us consider the following example.

Example 3.6.

$$\text{(MFCTP)}\quad \begin{aligned} &\max \left(\frac{\int_0^1 e^{2\tau - v_1(\tau)} d\tau}{\int_0^1 e^{v_2(\tau)} d\tau}, \frac{\int_0^1 \left(2 - v_2(\tau)\right) d\tau}{\int_0^1 \left(2v_2(\tau) + \frac{1}{2}v_1^2(\tau) + \frac{2\tau}{3}\right) d\tau} \right) \\ &2\tau + 2v_2(\tau) - v_2^2(\tau) - v_1(\tau) \geqq 0 \quad \text{a.e. in } [0,1], \\ &-2\tau + 2v_2(\tau) - v_2^2(\tau) + v_1(\tau) \geqq 0 \quad \text{a.e. in } [0,1], \\ &v_1(\tau) - v_2(\tau) - 2\tau \geqq 0 \quad \text{a.e. in } [0,1], \\ &v(\cdot) \in L_\infty([0,1]; \mathbb{R}^2). \end{aligned}$$

It is obvious that for almost every $\tau \in [0,1]$, $\hat{v}(\tau) = (\hat{v}_1(\tau), \hat{v}_2(\tau)) = (2\tau, 0)$ is an EMFCTP, $\hat{w}_1 = 1$, $\hat{w}_2 = 2$ and $I_{\hat{b}}(\tau) = \{1, 2, 3\}$ for $\hat{b} = \frac{1}{2}$. It can be easily verified that for almost every $\tau \in [0,1]$,

$$\nabla f_1(\tau, \hat{v}(\tau)) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \nabla f_2(\tau, \hat{v}(\tau)) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \nabla g_1(\tau, \hat{v}(\tau)) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\nabla g_2(\tau, \hat{v}(\tau)) = \begin{pmatrix} 2\tau \\ 2 \end{pmatrix}, \nabla h_1(\tau, \hat{v}(\tau)) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \nabla h_2(\tau, \hat{v}(\tau)) = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

$$\nabla h_3(\tau, \hat{v}(\tau)) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Take $\bar{\zeta}(\tau) = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix}$ a.e. in $[0,1]$ and $\beta = \frac{1}{6}$. It follows $\nabla h_i(\tau, \hat{v}(\tau))^T \bar{\zeta}(\tau) \geqq \beta$, $i = 1, 2, 3$, a.e. in $[0,1]$. Thus, the constraint qualification (MFQ) is satisfied. Now, show that (RC) is satisfied. Let $(\zeta_1, \zeta_2) \in \mathbb{R}^2$ and for almost everywhere in $[0,1]$,

$$\begin{aligned} &\phi_1(\tau, \zeta) = \zeta_1 + \zeta_2,\ \phi_2(\tau, \zeta) = 2\zeta_1 + 5\zeta_2,\ \phi_3(\tau, \zeta) = \zeta_1 - 2\zeta_2, \\ &\phi_4(\tau, \zeta) = -\zeta_1 - 2\zeta_2,\ \phi_5(\tau, \zeta) = -\zeta_1 + \zeta_2. \end{aligned}$$

Define the following sets:

$$\begin{aligned} A_2 &= \{(\zeta_1, \zeta_2) \in \mathbb{R}^2 : \frac{1}{7}\zeta_1 + \zeta_2 \geqq 0,\ \frac{3}{4}\zeta_1 + \zeta_2 \geqq 0\}, \\ A_3 &= \{(\zeta_1, \zeta_2) \in \mathbb{R}^2 : \frac{1}{7}\zeta_1 + \zeta_2 \leqq 0,\ \zeta_1 \geqq 0\}, \\ A_4 &= \{(\zeta_1, \zeta_2) \in \mathbb{R}^2 : \zeta_1 \leqq 0,\ \zeta_2 \leqq 0\}, \\ A_5 &= \{(\zeta_1, \zeta_2) \in \mathbb{R}^2 : \frac{3}{4}\zeta_1 + \zeta_2 \leqq 0,\ \zeta_2 \geqq 0\}. \end{aligned}$$

It can be easily verified that for $(\zeta_1, \zeta_2) \in int\, A_i$, $\mathcal{I}(\tau, \zeta) = \{i\}$ a.e. in $[0,1]$, $i = 2, 3, 4, 5$ and $\bigcup_{i=2}^5 A_i = \mathbb{R}^2$. We have that $2 \notin \mathcal{I}(\tau, \zeta)$ for $(\zeta_1, \zeta_2) \in int\,(A_3 \cup A_4 \cup A_5)$. For $(\zeta_1, \zeta_2) \in int(A_2 \cup A_5)$, $\mathcal{I}(\tau, \zeta) = \{2, 5\}$ a.e. in $[0,1]$. For $(\zeta_1, \zeta_2) \in int(A_2 \cup A_3)$, $\mathcal{I}(\tau, \zeta) = \{2, 3\}$

a.e. in $[0,1]$. Further, we consider the system

$$\begin{aligned} \phi_1(\tau,\zeta) =& \zeta_1+\zeta_2<0, \\ \phi_2(\tau,\zeta) =& 2\zeta_1+5\zeta_2<0, \\ \phi_3(\tau,\zeta) =& \zeta_1-2\zeta_2 \leqq 0, \\ \phi_4(\tau,\zeta) =& -\zeta_1-2\zeta_2 \leqq 0, \\ \phi_5(\tau,\zeta) =& -\zeta_1+\zeta_2 \leqq 0, \\ & \zeta\in\mathbb{R}^2. \end{aligned} \tag{3.17}$$

Regularity condition (RC) is checked with $\bar{v}\equiv(0,0)$, $R\geqq 0$, $\alpha=\frac{1}{10}$ and for almost every $\tau\in[0,1]$,

(i) $e=e(\tau,\zeta)=(-\frac{4}{5},-\frac{3}{5})$ for $(\zeta_1,\zeta_2)\in int\,(A_3\cup A_4\cup A_5)$,

(ii) $e=e(\tau,\zeta)=(0,1)$ for $(\zeta_1,\zeta_2)\in int\,(A_2\cup A_5)$ or $(\zeta_1,\zeta_2)\in int\,A_2$, and

(iii) $e=e(\tau,\zeta)=(\frac{19}{20},\frac{\sqrt{39}}{20})$ for $(\zeta_1,\zeta_2)\in int\,(A_2\cup A_3)$.

We have that necessary optimality conditions are satisfied for $\hat{\lambda}_1=1$, $\hat{\lambda}_2=0$, $\hat{u}_1(\tau)=1$, $\hat{u}_2(\tau)=\frac{1}{3}$ and $\hat{u}_3(\tau)=\frac{5}{3}$ a.e. in $[0,1]$.

We conclude that the constraint qualifications (MFQ), (SQ) and regularity condition (RC) are much less restrictive and easier to verify than the constraint qualifications given in [34] or [36].

In the following example, we will show that the necessary conditions may not hold without constraint qualification (MFQ).

Example 3.7.

$$\text{(MFCTP)}\quad \begin{aligned} &\max\left(\frac{\int_0^1\left(1-v^2(\tau)\right)d\tau}{\int_0^1 e^{v(\tau)}d\tau},\ \frac{\int_0^1\left(\tau-v(\tau)\right)d\tau}{\int_0^1\left(1+v^2(\tau)\right)d\tau}\right) \\ &\quad -v^2(\tau)\geqq 0 \quad \text{a.e. in } [0,1], \\ &\quad v(\cdot)\in L_\infty([0,1];\mathbb{R}). \end{aligned}$$

It is obvious that for almost every $\tau\in[0,1]$, $\hat{v}(\tau)=0$ is an EMFCTP, $\hat{w}_1=1$, $\hat{w}_2=\frac{1}{2}$ and $I_{\hat{b}}(\tau)=\{1\}$ for $\hat{b}>0$, where $h_1(\tau,v(\tau)):=-v^2(\tau)$, $f_1(\tau,v(\tau)):=1-v^2(\tau)$, $g_1(\tau,v(\tau)):=e^{v(\tau)}$, $f_2(\tau,v(\tau)):=\tau-v(\tau)$ and $g_2(\tau,v(\tau)):=1+v^2(\tau)$. It can be easily verified that for almost every $\tau\in[0,1]$,

$$\nabla f_1(\tau,\hat{v}(\tau))=0,\ \nabla g_1(\tau,\hat{v}(\tau))=1,\ \nabla f_2(\tau,\hat{v}(\tau))=-1,\ \nabla g_2(\tau,\hat{v}(\tau))=0,\ \nabla h_1(\tau,\hat{v}(\tau))=0.$$

Therefore, we conclude that constraint qualification (MFQ) is not valid for all $\bar{\gamma}(\cdot)\in L_\infty([0,1];\mathbb{R}^n)$ and $\beta>0$. Further, the condition (3.8) in the function $\hat{v}(\tau)\equiv 0$ can be rewritten as

$$\hat{\lambda}_1+\hat{\lambda}_2=0.$$

But, on the other hand,

$$\hat{\lambda}_1+\hat{\lambda}_2=1,$$

and such a $\hat{\lambda}=\left(\hat{\lambda}_1,\hat{\lambda}_2\right)$ does not exist. Therefore, the necessary conditions (3.8)-(3.9) are not satisfied.

4 Sufficient Conditions

The next results establish sufficient optimality conditions for (MFCTP). The proofs of these results will be based on the concavity/ convexity and generalized concavity assumptions imposed on the functions involved. In the following, we will use the basic properties of concave/convex and quasiconcave functions.

Theorem 4.1. *Assume that there exist* $\hat{v}(\cdot) \in \Phi$ *for* (MFCTP) *and* $(\hat{\lambda}, \hat{u}(\cdot)) \in \Lambda^+ \times L_\infty(\Delta; \mathbb{R}^m)$ *such that the following conditions are satisfied:*

$$\begin{aligned}\hat{\lambda}^T \left(\nabla f(\tau, \hat{v}(\tau)) - e \left(\frac{\int_\Delta f(\tau, \hat{v}(\tau)) d\tau}{\int_\Delta g(\tau, \hat{v}(\tau)) d\tau} \right)^T \nabla g(\tau, \hat{v}(\tau)) \right) \\ + \hat{u}(\tau)^T \nabla h(\tau, \hat{v}(\tau)) = 0 \quad \text{a.e. in } \Delta,\end{aligned} \tag{4.1}$$

$$\hat{u}(\tau)^T h(\tau, \hat{v}(\tau)) = 0, \ \hat{u}(\tau) \geqq 0 \quad \text{a.e. in } \Delta. \tag{4.2}$$

If the function $f(\tau, \cdot)$ *is concave almost everywhere in* Δ*,* $g(\tau, \cdot)$ *is convex almost everywhere in* Δ *and* $\hat{u}(\tau)^T h(\tau, \cdot)$ *is quasiconcave almost everywhere in* Δ*, then* $\hat{v}(\cdot)$ *is an EMFCTP.*

Proof. Suppose that $\hat{v}(\cdot)$ is not EMFCTP. Then there exists $\bar{v}(\cdot) \in \Phi$ such that

$$\frac{\int_\Delta f_j(\tau, \bar{v}(\tau)) d\tau}{\int_\Delta g_j(\tau, \bar{v}(\tau)) d\tau} \geqq \frac{\int_\Delta f_j(\tau, \hat{v}(\tau)) d\tau}{\int_\Delta g_j(\tau, \hat{v}(\tau)) d\tau} = \hat{w}_j, \ \forall j \in J, \tag{4.3}$$

and

$$\frac{\int_\Delta f_i(\tau, \bar{v}(\tau)) d\tau}{\int_\Delta g_i(\tau, \bar{v}(\tau)) d\tau} > \frac{\int_\Delta f_i(\tau, \hat{v}(\tau)) d\tau}{\int_\Delta g_i(\tau, \hat{v}(\tau)) d\tau} = \hat{w}_i, \ \text{for some } i \in J. \tag{4.4}$$

So, (4.3) and (4.4) can be rewritten as

$$\int_\Delta \left(f_j(\tau, \bar{v}(\tau)) - \hat{w}_j g_j(\tau, \bar{v}(\tau)) \right) d\tau \geqq 0, \ \forall j \in J \tag{4.5}$$

and

$$\int_\Delta \left(f_i(\tau, \bar{v}(\tau)) - \hat{w}_i g_i(\tau, \bar{v}(\tau)) \right) d\tau > 0, \ \text{for some } i \in J. \tag{4.6}$$

Since $\hat{\lambda} \in \Lambda^+$, (4.5) and (4.6) imply

$$\hat{\lambda}_j \int_\Delta \left(f_j(\tau, \bar{v}(\tau)) - \hat{w}_j g_j(\tau, \bar{v}(\tau)) \right) d\tau \geqq 0, \ \forall j \in J \tag{4.7}$$

and

$$\hat{\lambda}_i \int_\Delta \left(f_i(\tau, \bar{v}(\tau)) - \hat{w}_i g_i(\tau, \bar{v}(\tau)) \right) d\tau > 0, \ \text{for some } i \in J. \tag{4.8}$$

From (4.7) and (4.8) we have

$$\sum_{j \in J} \hat{\lambda}_j \int_\Delta \left(f_j(\tau, \bar{v}(\tau)) - \hat{w}_j g_j(\tau, \bar{v}(\tau)) \right) d\tau > 0. \tag{4.9}$$

Since $\bar{v}(\cdot) \in \Phi$ and (4.2) holds, we have

$$\sum_{i \in I} \hat{u}_i(\tau) h_i(\tau, \bar{v}(\tau)) \geqq \sum_{i \in I} \hat{u}_i(\tau) h_i(\tau, \hat{v}(\tau)), \ \ \forall \bar{v}(\cdot) \in \Phi, \quad \text{a.e. in } \Delta. \tag{4.10}$$

Since $\hat{u}(\tau)^T h(\tau,\cdot)$ is quasiconcave almost everywhere in Δ, (4.10) yields

$$\sum_{i\in I}\hat{u}_i(\tau)\nabla h_i(\tau,\hat{v}(\tau))^T(\bar{v}(\tau)-\hat{v}(\tau)) \geqq 0 \ \forall \bar{v}(\cdot)\in\Phi, \quad \text{a.e. in } \Delta. \tag{4.11}$$

Then, (4.1) and (4.11) imply

$$\sum_{j\in J}\hat{\lambda}_j\left(\nabla f_j(\tau,\hat{v}(\tau))-\hat{w}_j\nabla g_j(\tau,\hat{v}(\tau))\right)^T\left(\bar{v}(\tau)-\hat{v}(\tau)\right)\leqq 0 \ \forall\bar{v}(\cdot)\in\Phi, \quad \text{a.e. in } \Delta.$$

Since, $\hat{w}\geqq 0$, $f_j(\tau,\cdot)$ and $-g_j(\tau,\cdot)$ are concave almost everywhere in Δ, it follows that

$$\sum_{j\in J}\hat{\lambda}_j\left(f_j(\tau,\cdot)-\hat{w}_j g_j(\tau,\cdot)\right)$$

is concave almost everywhere in Δ. Therefore,

$$\sum_{j\in J}\hat{\lambda}_j\left(f_j(\tau,\bar{v}(\tau))-\hat{w}_j g_j(\tau,\bar{v}(\tau))\right)-\sum_{j\in J}\hat{\lambda}_j\left(f_j(\tau,\hat{v}(\tau))-\hat{w}_j g_j(\tau,\hat{v}(\tau))\right)\leqq 0, \ \text{a.e. in } \Delta. \tag{4.12}$$

Integrating (4.12) on Δ, we have

$$\sum_{j\in J}\hat{\lambda}_j\int_\Delta\left(f_j(\tau,\bar{v}(\tau))-\hat{w}_j g_j(\tau,\bar{v}(\tau))\right)d\tau-\sum_{j\in J}\hat{\lambda}_j\int_\Delta\left(f_j(\tau,\hat{v}(\tau))-\hat{w}_j g_j(\tau,\hat{v}(\tau))\right)d\tau\leqq 0,$$

i.e.,

$$\sum_{j\in J}\hat{\lambda}_j\int_\Delta\left(f_j(\tau,\bar{v}(\tau))-\hat{w}_j g_j(\tau,\bar{v}(\tau))\right)d\tau\leqq 0.$$

This inequality contradicts (4.9). Therefore, $\hat{v}(\cdot)$ must be an EMFCTP. □

Example 4.2.

$$\text{(MFCTP)}\quad \begin{array}{c} \max\ \left(\dfrac{\int_0^1\left(2+2\tau-v^2(\tau)\right)d\tau}{\int_0^1 e^{v(\tau)}d\tau},\ \dfrac{\int_0^1\left(2+\tau-v(\tau)\right)d\tau}{\int_0^1\left(1+v^2(\tau)\right)d\tau}\right) \\ v(\tau)\geqq 0 \quad \text{a.e. in } [0,1], \\ \tau+1-v(\tau)\geqq 0 \quad \text{a.e. in } [0,1], \\ v(\cdot)\in L_\infty([0,1];\mathbb{R}). \end{array}$$

Let $f_1(\tau,v(\tau)):=2+2\tau-v^2(\tau)$, $g_1(\tau,v(\tau)):=e^{v(\tau)}$, $f_2(\tau,v(\tau)):=2+\tau-v(\tau)$, $g_2(\tau,v(\tau)):=1+v^2(\tau)$, $h_1(\tau,v(\tau)):=v(\tau)$ and $h_2(\tau,v(\tau)):=\tau+1-v(\tau)$. Note that $\hat{\lambda}=\left(\hat{\lambda}_1,\hat{\lambda}_2\right)=\left(\frac{1}{2},\frac{1}{2}\right)$, $\hat{v}(\tau)=0$, $\hat{u}(\tau)=(\hat{u}_1(\tau),\hat{u}_2(\tau))=(2,0)$ a.e. $\tau\in[0,1]$, satisfy sufficient conditions (4.1)-(4.2). Also, $f_1(\tau,\cdot)$, $f_2(\tau,\cdot)$, $g_1(\tau,\cdot)$, $g_2(\tau,\cdot)$, $h_1(\tau,\cdot)$ and $h_2(\tau,\cdot)$ satisfy all the assumptions of Theorem 4.1. Hence, we conclude that $\hat{v}(\tau)$ is an EMFCTP.

Theorem 4.3. *Assume that there exist $\hat{v}(\cdot)\in\Phi$ for* (MFCTP) *and $(\hat{\lambda},\hat{u}(\cdot))\in\Lambda^+\times L_\infty(\Delta;\mathbb{R}^m)$ such that the following conditions are satisfied:*

$$\begin{aligned}\hat{\lambda}^T\left(\nabla f(\tau,\hat{v}(\tau))-e\left(\frac{\int_\Delta f(\tau,\hat{v}(\tau))d\tau}{\int_\Delta g(\tau,\hat{v}(\tau))d\tau}\right)^T\nabla g(\tau,\hat{v}(\tau))\right)&\\ +\hat{u}(\tau)^T\nabla h(\tau,\hat{v}(\tau))=0 \quad \text{a.e. in } \Delta,&\end{aligned}\tag{4.13}$$

$$\hat{u}(\tau)^T h(\tau, \hat{v}(\tau)) = 0,\ \hat{u}(\tau) \geqq 0 \quad \text{a.e. in } \Delta. \tag{4.14}$$

If the function $\hat{\lambda}^T \left(f(\tau,\cdot) - e\hat{w}^T g(\tau,\cdot)\right)$ *is pseudoconcave almost everywhere in* Δ *and* $\hat{u}(\tau)^T h(\tau,\cdot)$ *is quasiconcave almost everywhere in* Δ*, then* $\hat{v}(\cdot)$ *is an EMFCTP.*

Proof. Suppose that $\hat{v}(\cdot)$ is not EMFCTP. Then there exists $\bar{v}(\cdot) \in \Phi$ such that

$$\frac{\int_\Delta f_j(\tau,\bar{v}(\tau))d\tau}{\int_\Delta g_j(\tau,\bar{v}(\tau))d\tau} \geqq \frac{\int_\Delta f_j(\tau,\hat{v}(\tau))d\tau}{\int_\Delta g_j(\tau,\hat{v}(\tau))d\tau} = \hat{w}_j,\ \forall j \in J, \tag{4.15}$$

and

$$\frac{\int_\Delta f_i(\tau,\bar{v}(\tau))d\tau}{\int_\Delta g_i(\tau,\bar{v}(\tau))d\tau} > \frac{\int_\Delta f_i(\tau,\hat{v}(\tau))d\tau}{\int_\Delta g_i(\tau,\hat{v}(\tau))d\tau} = \hat{w}_i,\ \text{for some } i \in J. \tag{4.16}$$

So, (4.15) and (4.16) can be rewritten as

$$\int_\Delta \left(f_j(\tau,\bar{v}(\tau)) - \hat{w}_j g_j(\tau,\bar{v}(\tau))\right) d\tau \geqq 0,\ \forall j \in J \tag{4.17}$$

and

$$\int_\Delta \left(f_i(\tau,\bar{v}(\tau)) - \hat{w}_i g_i(\tau,\bar{v}(\tau))\right) d\tau > 0,\ \text{for some } i \in J. \tag{4.18}$$

Since $\hat{\lambda} \in \Lambda^+$, (4.17) and (4.18) imply

$$\sum_{j\in J} \hat{\lambda}_j \int_\Delta \left(f_j(\tau,\bar{v}(\tau)) - \hat{w}_j g_j(\tau,\bar{v}(\tau))\right) d\tau > 0. \tag{4.19}$$

Since $\bar{v}(\cdot) \in \Phi$ and (4.14) holds, we have

$$\sum_{i\in I} \hat{u}_i(\tau) h_i(\tau,\bar{v}(\tau)) \geqq \sum_{i\in I} \hat{u}_i(\tau) h_i(\tau,\hat{v}(\tau)),\quad \forall \bar{v}(\cdot) \in \Phi, \quad \text{a.e. in } \Delta. \tag{4.20}$$

Since $\hat{u}(\tau)^T h(\tau,\cdot)$ is quasiconcave almost everywhere in Δ, (4.20) yields

$$\sum_{i\in I} \hat{u}_i(\tau) \nabla h_i(\tau,\hat{v}(\tau))^T (\bar{v}(\tau) - \hat{v}(\tau)) \geqq 0\ \forall \bar{v}(\cdot) \in \Phi, \quad \text{a.e. in } \Delta. \tag{4.21}$$

Then, (4.13) and (4.21) imply

$$\sum_{j\in J} \hat{\lambda}_j \left(\nabla f_j(\tau,\hat{v}(\tau)) - \hat{w}_j \nabla g_j(\tau,\hat{v}(\tau))\right)^T \left(\bar{v}(\tau) - \hat{v}(\tau)\right) \leqq 0\ \forall \bar{v}(\cdot) \in \Phi, \quad \text{a.e. in } \Delta.$$

Since, $\hat{\lambda}^T \left(f(\tau,\cdot) - e\hat{w}^T g(\tau,\cdot)\right)$ is pseudoconcave almost everywhere in Δ, we have

$$\sum_{j\in J} \hat{\lambda}_j \left(f_j(\tau,\bar{v}(\tau)) - \hat{w}_j g_j(\tau,\bar{v}(\tau))\right) - \sum_{j\in J} \hat{\lambda}_j \left(f_j(\tau,\hat{v}(\tau)) - \hat{w}_j g_j(\tau,\hat{v}(\tau))\right) \leqq 0,\ \text{a.e. in } \Delta. \tag{4.22}$$

Integrating (4.22) on Δ, we have

$$\sum_{j\in J} \hat{\lambda}_j \int_\Delta \left(f_j(\tau,\bar{v}(\tau)) - \hat{w}_j g_j(\tau,\bar{v}(\tau))\right) d\tau - \sum_{j\in J} \hat{\lambda}_j \int_\Delta \left(f_j(\tau,\hat{v}(\tau)) - \hat{w}_j g_j(\tau,\hat{v}(\tau))\right) d\tau \leqq 0,$$

i.e.,

$$\sum_{j\in J}\hat{\lambda}_j\int_{\Delta}\left(f_j(\tau,\bar{v}(\tau))-\hat{w}_j g_j(\tau,\bar{v}(\tau))\right)d\tau \leqq 0.$$

This inequality contradicts (4.19). Therefore, $\hat{v}(\cdot)$ must be an EMFCTP. □

We also derive sufficient conditions for (MFCTP), where $\hat{\lambda}\in\Lambda$.

Theorem 4.4. *Assume that there exist $\hat{v}(\cdot)\in\Phi$ for* (MFCTP) *and $(\hat{\lambda},\hat{u}(\cdot))\in\Lambda\times L_\infty(\Delta;\mathbb{R}^m)$ such that the following conditions are satisfied:*

$$\begin{gathered}\hat{\lambda}^T\left(\nabla f(\tau,\hat{v}(\tau))-e\left(\frac{\int_\Delta f(\tau,\hat{v}(\tau))d\tau}{\int_\Delta g(\tau,\hat{v}(\tau))d\tau}\right)^T\nabla g(\tau,\hat{v}(\tau))\right)\\ +\hat{u}(\tau)^T\nabla h(\tau,\hat{v}(\tau))=0 \quad \text{a.e. in } \Delta,\end{gathered}\tag{4.23}$$

$$\hat{u}(\tau)^T h(\tau,\hat{v}(\tau))=0,\ \hat{u}(\tau)\geqq 0 \quad \text{a.e. in } \Delta. \tag{4.24}$$

If the function $f(\tau,\cdot)$ is concave almost everywhere in Δ, $g(\tau,\cdot)$ is convex almost everywhere in Δ, $\hat{u}(\tau)^T h(\tau,\cdot)$ is quasiconcave almost everywhere in Δ and $\hat{\lambda}^T(f(\tau,\cdot)-e\hat{w}^T g(\tau,\cdot))$ is strictly concave almost everywhere in Δ, then $\hat{v}(\cdot)$ is an EMFCTP.

Proof. Suppose that $\hat{v}(\cdot)$ is not EMFCTP. Then there exists $\bar{v}(\cdot)\in\Phi$ such that

$$\int_\Delta\left(f_j(\tau,\bar{v}(\tau))-\hat{w}_j g_j(\tau,\bar{v}(\tau))\right)d\tau\geqq 0,\ \forall j\in J \tag{4.25}$$

and

$$\int_\Delta\left(f_i(\tau,\bar{v}(\tau))-\hat{w}_i g_i(\tau,\bar{v}(\tau))\right)d\tau>0,\ \text{for some } i\in J. \tag{4.26}$$

Since $\hat{\lambda}\in\Lambda$, (4.25) and (4.26) imply

$$\sum_{j\in J}\hat{\lambda}_j\int_{\Delta}\left(f_j(\tau,\bar{v}(\tau))-\hat{w}_j g_j(\tau,\bar{v}(\tau))\right)d\tau \geqq 0. \tag{4.27}$$

From $\bar{v}(\cdot)\in\Phi$ and (4.24), we obtain

$$\sum_{i\in I}\hat{u}_i(\tau)h_i(\tau,\bar{v}(\tau))\geqq\sum_{i\in I}\hat{u}_i(\tau)h_i(\tau,\hat{v}(\tau)),\ \ \forall\bar{v}(\cdot)\in\Phi,\quad \text{a.e. in } \Delta. \tag{4.28}$$

Since $\hat{u}(\tau)^T h(\tau,\cdot)$ is quasiconcave almost everywhere in Δ, (4.28) yields

$$\sum_{i\in I}\hat{u}_i(\tau)\nabla h_i(\tau,\hat{v}(\tau))^T(\bar{v}(\tau)-\hat{v}(\tau))\geqq 0,\ \forall\bar{v}(\cdot)\in\Phi,\quad \text{a.e. in } \Delta. \tag{4.29}$$

Then, (4.23) and (4.29) imply

$$\sum_{j\in J}\hat{\lambda}_j\left(\nabla f_j(\tau,\hat{v}(\tau))-\hat{w}_j\nabla g_j(\tau,\hat{v}(\tau))\right)^T\left(\bar{v}(\tau)-\hat{v}(\tau)\right)\leqq 0\ \forall\bar{v}(\cdot)\in\Phi,\quad \text{a.e. in } \Delta.$$

Since $\hat{\lambda}\in\Lambda$, $\hat{w}\geqq 0$ and $\sum_{j\in J}\hat{\lambda}_j(f_j(\tau,\cdot)-\hat{w}_j g_j(\tau,\cdot))$ is strictly concave almost everywhere in Δ, it follows

$$\begin{gathered}\sum_{j\in J}\hat{\lambda}_j\left(f_j(\tau,\bar{v}(\tau))-\hat{w}_j g_j(\tau,\bar{v}(\tau))\right)-\sum_{j\in J}\hat{\lambda}_j\left(f_j(\tau,\hat{v}(\tau))-\hat{w}_j g_j(\tau,\hat{v}(\tau))\right)<0\\ \text{a.e. in } \Delta.\end{gathered}\tag{4.30}$$

Integrating (4.30) on Δ, we obtain

$$\sum_{j\in J}\hat{\lambda}_j\int_{\Delta}\left(f_j(\tau,\bar{v}(\tau))-\hat{w}_j g_j(\tau,\bar{v}(\tau))\right)d\tau<0.$$

This inequality contradicts (4.27). Therefore, $\hat{v}(\cdot)$ must be an EMFCTP. $\square$

Example 4.5.

$$\text{(MFCTP)}\quad \begin{aligned} &\max\ \left(\frac{\int_0^1\left(2-v^2(\tau)\right)d\tau}{\int_0^1 e^{v(\tau)}d\tau},\ \frac{\int_0^1\left(1-v(\tau)\right)d\tau}{\int_0^1\left(1+v^2(\tau)\right)d\tau}\right)\\ &v(\tau)\geqq 0\quad \text{a.e. in } [0,1],\\ &\frac{1}{2}-v(\tau)\geqq 0\quad \text{a.e. in } [0,1],\\ &v(\cdot)\in L_\infty([0,1];\mathbb{R}).\end{aligned}$$

Let $f_1(\tau,v(\tau)):=2-v^2(\tau)$, $g_1(\tau,v(\tau)):=e^{v(\tau)}$, $f_2(\tau,v(\tau)):=1-v(\tau)$, $g_2(\tau,v(\tau)):=1+v^2(\tau)$, $h_1(\tau,v(\tau)):=v(\tau)$ and $h_2(\tau,v(\tau)):=\frac{1}{2}-v(\tau)$. Note that $\hat{\lambda}=\left(\hat{\lambda}_1,\hat{\lambda}_2\right)=(0,1)$, $\hat{v}(\tau)=0$, $\hat{u}(\tau)=(\hat{u}_1(\tau),\hat{u}_2(\tau))=(1,0)$ a.e. $\tau\in[0,1]$, satisfy sufficient conditions (4.23)-(4.24). Also, $f_1(\tau,\cdot)$, $f_2(\tau,\cdot)$, $g_1(\tau,\cdot)$, $g_2(\tau,\cdot)$, $h_1(\tau,\cdot)$ and $h_2(\tau,\cdot)$ satisfy all the assumptions of Theorem 4.4. Hence, we conclude that $\hat{v}(\tau)$ is an EMFCTP.

Theorem 4.6. *Assume that there exist $\hat{v}(\cdot)\in\Phi$ for* (MFCTP) *and $(\hat{\lambda},\hat{u}(\cdot))\in\Lambda\times L_\infty(\Delta;\mathbb{R}^m)$ such that the following conditions are satisfied:*

$$\begin{aligned}\hat{\lambda}^T\left(\nabla f(\tau,\hat{v}(\tau))-e\left(\frac{\int_\Delta f(\tau,\hat{v}(\tau))d\tau}{\int_\Delta g(\tau,\hat{v}(\tau))d\tau}\right)^T\nabla g(\tau,\hat{v}(\tau))\right)&\\ +\hat{u}(\tau)^T\nabla h(\tau,\hat{v}(\tau))=0\quad\text{a.e. in }\Delta,&\end{aligned}\tag{4.31}$$

$$\hat{u}(\tau)^T h(\tau,\hat{v}(\tau))=0,\ \hat{u}(\tau)\geqq 0\quad\text{a.e. in }\Delta.\tag{4.32}$$

If the function $\hat{\lambda}^T\left(f(\tau,\cdot)-e\hat{w}^T g(\tau,\cdot)\right)$ is quasiconcave almost everywhere in Δ and $\hat{u}(\tau)^T h(\tau,\cdot)$ is strictly quasiconcave almost everywhere in Δ, then $\hat{v}(\cdot)$ is an EMFCTP.

Proof. Suppose that $\hat{v}(\cdot)$ is not EMFCTP. Then there exists $\bar{v}(\cdot)\in\Phi$ such that

$$\frac{\int_\Delta f_j(\tau,\bar{v}(\tau))d\tau}{\int_\Delta g_j(\tau,\bar{v}(\tau))d\tau}\geqq\frac{\int_\Delta f_j(\tau,\hat{v}(\tau))d\tau}{\int_\Delta g_j(\tau,\hat{v}(\tau))d\tau}=\hat{w}_j,\ \forall j\in J,\tag{4.33}$$

and

$$\frac{\int_\Delta f_i(\tau,\bar{v}(\tau))d\tau}{\int_\Delta g_i(\tau,\bar{v}(\tau))d\tau}>\frac{\int_\Delta f_i(\tau,\hat{v}(\tau))d\tau}{\int_\Delta g_i(\tau,\hat{v}(\tau))d\tau}=\hat{w}_i,\ \text{for some } i\in J.\tag{4.34}$$

So, (4.33) and (4.34) can be rewritten as

$$\int_\Delta\left(f_j(\tau,\bar{v}(\tau))-\hat{w}_j g_j(\tau,\bar{v}(\tau))\right)d\tau\geqq 0,\ \forall j\in J\tag{4.35}$$

and

$$\int_\Delta\left(f_i(\tau,\bar{v}(\tau))-\hat{w}_i g_i(\tau,\bar{v}(\tau))\right)d\tau>0,\ \text{for some } i\in J.\tag{4.36}$$

Since $\hat{\lambda} \in \Lambda$, (4.35) and (4.36) imply

$$\sum_{j\in J} \hat{\lambda}_j \int_\Delta \left(f_j(\tau, \bar{v}(\tau)) - \hat{w}_j g_j(\tau, \bar{v}(\tau))\right) d\tau \geqq 0. \tag{4.37}$$

Since $\bar{v}(\cdot) \in \Phi$ and (4.32) holds, we have

$$\sum_{i\in I} \hat{u}_i(\tau) h_i(\tau, \bar{v}(\tau)) \geqq \sum_{i\in I} \hat{u}_i(\tau) h_i(\tau, \hat{v}(\tau)), \quad \forall \bar{v}(\cdot) \in \Phi, \quad \text{a.e. in } \Delta. \tag{4.38}$$

Since $\hat{u}(\tau)^T h(\tau, \cdot)$ is strictly quasiconcave almost everywhere in Δ, (4.38) yields

$$\sum_{i\in I} \hat{u}_i(\tau) \nabla h_i(\tau, \hat{v}(\tau))^T (\bar{v}(\tau) - \hat{v}(\tau)) > 0 \ \forall \bar{v}(\cdot) \in \Phi, \quad \text{a.e. in } \Delta, \tag{4.39}$$

for all $\bar{v}(\cdot) \in \Phi$ such that $\bar{v}(\tau) \neq \hat{v}(\tau)$ a.e. in $[0, T]$. Then, (4.31) and (4.39) imply

$$\sum_{j\in J} \hat{\lambda}_j \left(\nabla f_j(\tau, \hat{v}(\tau)) - \hat{w}_j \nabla g_j(\tau, \hat{v}(\tau))\right)^T (\bar{v}(\tau) - \hat{v}(\tau)) < 0 \ \ \forall \bar{v}(\cdot) \in \Phi, \quad \text{a.e. in } \Delta.$$

Since, $\hat{\lambda}^T \left(f(\tau, \cdot) - e\hat{w}^T g(\tau, \cdot)\right)$ is quasiconcave almost everywhere in Δ, we have

$$\sum_{j\in J} \hat{\lambda}_j \left(f_j(\tau, \bar{v}(\tau)) - \hat{w}_j g_j(\tau, \bar{v}(\tau))\right) - \sum_{j\in J} \hat{\lambda}_j \left(f_j(\tau, \hat{v}(\tau)) - \hat{w}_j g_j(\tau, \hat{v}(\tau))\right) < 0, \ \text{a.e. in } \Delta. \tag{4.40}$$

Integrating (4.40) on Δ, we have

$$\sum_{j\in J} \hat{\lambda}_j \int_\Delta \left(f_j(\tau, \bar{v}(\tau)) - \hat{w}_j g_j(\tau, \bar{v}(\tau))\right) d\tau - \sum_{j\in J} \hat{\lambda}_j \int_\Delta \left(f_j(\tau, \hat{v}(\tau)) - \hat{w}_j g_j(\tau, \hat{v}(\tau))\right) d\tau < 0,$$

i.e.,

$$\sum_{j\in J} \hat{\lambda}_j \int_\Delta \left(f_j(\tau, \bar{v}(\tau)) - \hat{w}_j g_j(\tau, \bar{v}(\tau))\right) d\tau < 0.$$

This inequality contradicts (4.37). Therefore, $\hat{v}(\cdot)$ must be an EMFCTP. □

5 Conclusion

This paper has presented the optimality conditions and a new approach to smooth multiobjective fractional continuous-time programming. The necessary conditions have been obtained without convexity assumptions. What was not under the paper's intention are duality results of the initial nonsmooth problem and developing its optimality conditions without convexity assumptions. To this end, the necessary and sufficient conditions in this paper are good starting points.

Numerical methods for linear fractional continuous-time problem have been proposed in [33, 32]. Also, there does not exist a numerical algorithm to solve the multiobjective fractional continuous-time programming problem. This could possibly be a significant path for some future work.

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