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FRANK-WOLFE TYPE THEOREMS FOR POLYNOMIAL VECTOR OPTIMIZATION PROBLEMS*

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Abstract: In this paper, we study the solvability of a polynomial vector optimization problem under the weak section-boundedness from below condition. We give a characterization of the weak section-boundedness from below condition. Under the weak section-boundedness condition, we prove the existence of weakly Pareto efficient solutions for a convex polynomial vector optimization problem. For the non-convex case, we prove the existence of Pareto efficient solutions when the convenience, non-degeneracy, and weak section-boundedness conditions are satisfied.

Key words: *polynomial vector optimization problem, Newton polyhedron at infinity, convenience, non-degeneracy, Weak section-boundedness from below, Frank-Wolfe type theorems*

Mathematics Subject Classification: *90C29, 90C46*

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1 Introduction

Throughout, $\mathbf{R}^n$ denotes the n-dimensional Euclidean space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$, and $\mathbf{R}^n_+$ denotes the non-negative orthant of $\mathbf{R}^n$. Let

$$f_1, f_2, \ldots, f_s, g_1, g_2, \ldots, g_p : \mathbf{R}^n \to \mathbf{R}$$

be polynomial functions. Consider the following polynomial vector optimization problem:

$$\mathrm{PVOP}(K, f): \qquad \mathbf{R}^s_+ - \mathrm{Min}_{x\in K} f(x),$$

where $f = (f_1, f_2, \ldots, f_s) : \mathbf{R}^n \to \mathbf{R}^s$ is a polynomial vector-valued function and

$$K = \{x \in \mathbf{R}^n : g_1(x) \le 0, g_2(x) \le 0, \ldots, g_p(x) \le 0\}.$$

In what follows, we always suppose that the constraint set K is nonempty.

Recall that a point $x^* \in K$ is a *Pareto efficient solution* of $\mathrm{PVOP}(K, f)$ if

$$f(x) - f(x^*) \notin -\mathbf{R}^s_+\backslash\{0\}, \quad \forall x \in K$$

and $x^* \in K$ is a *weakly Pareto efficient solution* of $\mathrm{PVOP}(K, f)$ if

$$f(x) - f(x^*) \notin -\text{ int } \mathbf{R}^s_+, \quad \forall x \in K.$$

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The Pareto efficient solution set and the weakly Pareto efficient solution set of PVOP(K, f) are denoted by $SOL^s(K, f)$ and $SOL^w(K, f)$, respectively. Clearly,

$$SOL^s(K, f) \subseteq SOL^w(K, f).$$

When $s = 1$, PVOP(K, f) collapses to the polynomial scalar optimization problem:

$$\text{PSOP}(K, f): \qquad \text{Min}_{x \in K} f(x),$$

whose optimal solution set is denoted by $SOL(K, f)$.

In 1956, Frank and Wolfe [8] proved that a quadratic function attains its infimum on a polyhedron provided that it is bounded from below on this polyhedron. This result has been known as the Frank-Wolfe theorem. Since then, many authors have been focusing on extensions and generalizations of the Frank-Wolfe theorem. For instance, in 1980, Perold [25] proved a Frank-Wolfe type theorem for the minimization problem with a non-quadratic objective function and a nonempty polyhedral constraint set. In 1999, Luo and Zhang [22] established a Frank-Wolfe type theorem for the minimization problem where the objective function is quadratic and the constraint set consists of finitely many quadratic inequalities. In 2002, Belousov and Klatte [2] proved a Frank-Wolfe type Theorem for the minimization problem with a convex polynomial objective function and a constraint set defined by finitely many convex polynomial functions. In 2006, Obuchowska [24] obtained a Frank-Wolfe type theorem for the minimization problem with a faithfully convex or quasiconvex polynomial objective function and a constraint set defined by a system of faithfully convex inequalities and/or quasiconvex polynomial inequalities. Dinh et al. [4] proved a Frank-Wolfe type theorem for a non-convex polynomial optimization problem under convenience and non-degeneracy conditions. For more results on Frank-Wolfe type theorems for scalar optimization problems, we refer the reader to [18, 20, 23, 7, 26] and the reference therein.

Recently, some researchers focused on the study of Frank-Wolfe type theorems for vector optimization problems. Kim et al. [16] proved the nonemptiness of the Pareto efficient solution set of an unconstrained polynomial vector optimization problem when the Palais-Smale condition holds and the objective function has a section bounded from below. Lee et al. [19] proved that a constrained vector optimization problem with the constraint set being a closed convex semi-algebraic set and the objective function being a convex vector polynomial has a nonempty Pareto efficient solution set if and only if its objective function has a section bounded from below.

Motivated by the above works, in this paper, we investigate Frank-Wolfe type theorems for the polynomial vector optimization problem PVOP(K, f) under a weak section-boundedness from below condition. The outline of this paper is as follows: In Section 2, we give the definition and the property of weak section-boundedness from below and recall some notations and preliminary results. In Section 3, we are devoted to establishing Frank-Wolfe type theorems for PVOP(K, f) under the weak section-boundedness from below condition.

2 Preliminaries

In this section, we give some concepts and results that will be used in this paper.

2.1 Weak section-boundedness from below

Let C be a nonempty subset of $\mathbf{R}^n$ and $F : \mathbf{R}^n \to \mathbf{R}^s$ be a vector-valued function with

$$F(x) = (F_1(x), F_2(x), \ldots, F_s(x)).$$

Definition 2.1 (See, e.g., [3, 12, 13, 15, 16, 21])**.** Let A be a subset of $\mathbf{R}^s$ and $\bar{t} \in \mathbf{R}^s$. The set $A \bigcap (\bar{t} - \mathbf{R}^s_+)$ is called a section of A at $\bar{t}$ and denoted by $[A]_{\bar{t}}$. The section $[A]_{\bar{t}}$ is said to be bounded if there exists $a \in \mathbf{R}^s$ such that

$$[A]_{\bar{t}} \subseteq a + \mathbf{R}^s_+.$$

Definition 2.2. A vector-valued function $F : \mathbf{R}^n \to \mathbf{R}^s$ is said to be *section-bounded from below* on C if there exists $x' \in C$ such that the section $[F(C)]_{F(x')}$ is bounded.

Remark that, by definition, a vector-valued function F is *section-bounded from below* on C if and only if there exist $x' \in C$ and $a = (a_1, a_2, \dots, a_s) \in \mathbf{R}^s$ such that

$$F_i(x) \geq a_i$$

for any $x \in C$ satisfying $F(x) \leq F(x')$ and each $i \in \{1, 2, \dots, s\}$. In [16], the section-boundedness from below has been used to derive Frank-Wolfe type theorem for a polynomial vector optimization problem. In this paper, we consider the weak section-boundedness from below on C for a vector-valued function F.

Definition 2.3. A vector-valued function $F = (F_1, F_2, \dots, F_s) : \mathbf{R}^n \to \mathbf{R}^s$ is said to be *weakly section-bounded from below* on C if there exist $\bar{x} \in C$ and $\bar{a} \in \mathbf{R}^s$ such that

$$F(x) - \bar{a} \notin - \text{ int } \mathbf{R}^s_+, \quad \forall x \in C_{\bar{x}},$$

where $C_{\bar{x}} = \{x \in C : F_i(x) \leq F_i(\bar{x}), i = 1, 2, \dots, s\}$.

Remark 2.4. By definition, section-boundedness from below implies weak section-boundedness from below. The following example shows that the inverse is not true in general.

Example 2.5. Consider the vector-valued function $F = (F_1, F_2)$ defined by

$$F_1(x_1, x_2) = x_2, F_2(x_1, x_2) = -x_1$$

and

$$C = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \in \mathbf{R}, x_2 \geq 0\}.$$

Let $\bar{x} = (1, 2)$ and $\bar{a} = (0, 1)$. Then $C_{\bar{x}} = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \geq 1, 2 \geq x_2 \geq 0\}$. It is not difficult to see that $(x_2, -x_1) - (0, 1) \notin -$ int $\mathbf{R}^2_+$ for any $(x_1, x_2) \in C_{\bar{x}}$. Thus, F is weakly section-bounded from below on C. On the other hand, let $y = (y_1, y_2) \in C$. By computation, we have $C_y = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \geq y_1, y_2 \geq x_2 \geq 0\}$. It is easy to see that F_2 is not bounded from below on C_y. As a result, F is not section-bounded from below on C.

Next, we give a characterization for the weak section-boundedness from below on C of a vector-valued function F, which plays an important role in proving the existence of Pareto efficient solutions.

Proposition 2.6. *Let $F = (F_1, F_2, \dots, F_s) : \mathbf{R}^n \to \mathbf{R}^s$ be a vector-valued function. Then F is weakly section-bounded from below on C if and only if there exist $x^* \in C$ and $i_0 \in \{1, 2, \dots, s\}$ such that F_{i_0} is bounded from below on $C^{i_0}_{x^*}$, where*

$$C^{i_0}_{x^*} = \{x \in C : F_i(x) \leq F_i(x^*), i = 1, 2, \dots, i_0 - 1, i_0 + 1, \dots, s\}.$$

Proof. Suppose that there exist $x^* \in C$ and $i_0 \in \{1, 2, \ldots, s\}$ such that F_{i_0} is bounded from below on $C^{i_0}_{x^*}$. Since $C_{x^*} \subseteq C^{i_0}_{x^*}$, F_{i_0} is bounded from below on $C_{x^*} = \{x \in C : F_i(x) \leq F_i(x^*), i = 1, 2, \ldots, s\}$. Then there exists $a_{i_0} \in \mathbf{R}$ such that $F_{i_0}(x) \geq a_{i_0}$ for all $x \in C_{x^*}$. Let $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_s) \in \mathbf{R}^s$ with $\bar{a}_{i_0} = a_{i_0}$. It follows that $F(x) - \bar{a} \notin -\operatorname{int} \mathbf{R}^s_+$ for any $x \in C_{x^*}$. As a result, F is weakly section-bounded from below on C.

Now assume that F is weakly section-bounded from below on C. Suppose on the contrary that for any $x^* \in C$ and any $j \in \{1, 2, \ldots, s\}$, we have that F_j is unbounded from below on $C^j_{x^*}$ where

$$C^j_{x^*} = \{x \in C : F_i(x) \leq F_i(x^*), i = 1, 2, \ldots, j-1, j+1, \ldots, s\}.$$

Then there exists a sequence $\{y_m\} \subseteq C^j_{x^*}$ such that $F_j(y_m) \leq -m \leq F_j(x^*)$ for all sufficiently large m. As a consequence, $y_m \in C_{x^*}$ and F_j is unbounded from below on C_{x^*} for each j. Notice that F_j is unbounded from below on $C^j_{x^*}$ if and only if it is unbounded from below on C_{x^*}.

For $j = 1$, there exists a sequence $\{x^1_k\} \subseteq C_{x^*}$ such that $F_1(x^1_k) \leq -k$ for all k. For each k, consider the following nonempty set

$$C_{x^1_k} = \{x \in C : F_i(x) \leq F_i(x^1_k), i = 1, 2, \ldots, s\}.$$

Then

$$C_{x^1_k} \subseteq C_{x^*} \subseteq C.$$

Since $x^1_k \in C$, by assumption, F_2 is unbounded from below on $C_{x^1_k}$. Then there exists $\{x^2_k\} \subset C_{x^1_k}$ such that

$$F_2(x^2_k) \leq -k \text{ and } F_1(x^2_k) \leq F_1(x^1_k) \leq -k, \quad \forall k.$$

Similarly, consider the following nonempty set

$$C_{x^2_k} = \{x \in C : F_i(x) \leq F_i(x^2_k), i = 1, 2, \ldots, s\}.$$

Then, we have

$$C_{x^2_k} \subseteq C_{x^1_k} \subseteq C_{x^*} \subseteq C,$$

and there exists $\{x^3_k\} \subset C_{x^2_k}$ such that

$$F_3(x^3_k) \leq -k, F_2(x^3_k) \leq F_2(x^2_k) \leq -k \text{ and } F_1(x^3_k) \leq F_1(x^2_k) \leq F_1(x^1_k) \leq -k, \quad \forall k.$$

Repeating this process, we can obtain that for any $x^* \in C$, there exists a sequence $\{x^s_k\}_k$ such that for all k,

$$x^s_k \in C_{x^{s-1}_k} \subseteq C_{x^{s-2}_k} \subseteq \cdots \subseteq C_{x^2_k} \subseteq C_{x^1_k} \subseteq C_{x^*} \subseteq C$$

and

$$\begin{aligned}
F_1(x^s_k) \leq F_1(x^{s-1}_k) \leq \cdots \leq F_1(x^1_k) &\leq -k, \\
F_2(x^s_k) \leq F_2(x^{s-1}_k) \leq \cdots \leq F_2(x^2_k) &\leq -k, \\
&\vdots \\
F_{s-1}(x^s_k) \leq F_{s-1}(x^{s-1}_k) &\leq -k, \\
F_s(x^s_k) &\leq -k.
\end{aligned}$$

As a result, for any $x^* \in C$ and any $a' = (a'_1, a'_2, \ldots, a'_s) \in \mathbf{R}^s$, there exists a sequence $\{x^s_k\}_k \subseteq C_{x^*}$ such that $F(x^s_k) < a'$ for all sufficiently large k. By Definition 2.3, let $x^* = \bar{x}$ and $a' = \bar{a}$. Then there exists a sequence $\{\bar{x}^s_k\}_k \subseteq C_{\bar{x}}$ such that $F(\bar{x}^s_k) - \bar{a} \in -\operatorname{int} \mathbf{R}^s_+$ for all sufficiently large k. This contradicts to the weak section-boundedness from below on C of F. The proof is completed. □

Remark 2.7. If $x^* \in SOL^w(C, F)$, then $F(x) - F(x^*) \notin -\operatorname{int} \mathbf{R}^s_+$ for any $x \in C$. Let $\bar{x} \in C$. Since $C_{\bar{x}} \subseteq C$, we have $F(x) - F(x^*) \notin -\operatorname{int} \mathbf{R}^s_+$ for any $x \in C_{\bar{x}}$. So, F is weakly section-bounded from below on C. Hence, the weak section-boundedness from below condition of F is necessary for the existence of (weakly) Pareto efficient solutions.

2.2 Newton polyhedra at infinity, convenience and non-degeneracy

Let $\mathbf{N}$ be the set of all natural numbers. Given $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{N}^n$, the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ is denoted by x^α. For any polynomial $h : \mathbf{R}^n \to \mathbf{R}$, we can write $h(x) = \sum_\alpha h_\alpha x^\alpha$ with $h_\alpha \in \mathbf{R}$. Now, we recall the definition of Newton polyhedra following Kouchnirenko and Khovanskii (see [14, 17]).

Definition 2.8 (See, e.g., [4, 5, 6, 10])**.** A set $\mathcal{N} \subseteq \mathbf{R}^n_+$ is said to be a *Newton polyhedron at infinity* if there is some finite subset P of $\mathbf{N}^n$ such that $\mathcal{N}$ is equal to the convex hull of the set $P \bigcup \{0\}$. And the Newton polyhedron at infinity $\mathcal{N}$ is said to be *convenient* if it intersects each coordinate axis in a point different from the origin.

We denote by $\mathcal{N}_\infty$ the set of all the faces of $\mathcal{N}$ which do not contain the origin $\mathbf{0}$ in $\mathbf{R}^n$. Since the *Newton polyhedron at infinity* $\mathcal{N}$ is determined by the finite set $P \subseteq \mathbf{N}^n$, we can write $\mathcal{N} = \mathcal{N}(P)$. The *support* of the polynomial $h(x) = \sum_\alpha h_\alpha x^\alpha$, denoted by supp$(h)$, is a set of all $\alpha \in \mathbf{N}^n$ such that $h_\alpha \neq 0$. For simplicity the *Newton polyhedron at infinity* $\mathcal{N}(\text{supp}(h))$ is written by $\mathcal{N}(h)$. We say that $\mathcal{N}(h)$ is the *Newton polyhedron at infinity* of the polynomial h. The polynomial h is said to be *convenient* if $\mathcal{N}(h)$ is *convenient.* The set $\mathcal{N}_\infty(h)$ is defined as the set of all the faces of $\mathcal{N}(h)$ which do not contain the origin $\mathbf{0}$ in $\mathbf{R}^n$. Let Δ be a face in $\mathcal{N}(h)$. We write $h_\Delta(x) = \sum_{\alpha \in \Delta} h_\alpha x^\alpha$ with $h_\alpha \neq 0$.

Next, let $T = (T_1, T_2, \ldots, T_m) : \mathbf{R}^n \to \mathbf{R}^m$ be a vector polynomial. Let $\mathcal{N}(T)$ be the Minkowski sum $\mathcal{N}(T_1) + \mathcal{N}(T_2) + \cdots + \mathcal{N}(T_m)$. Then $\mathcal{N}(T)$ is also a Newton polyhedron at infinity. We say that $\mathcal{N}(T)$ is the *Newton polyhedron at infinity* of the vector polynomial T. The set $\mathcal{N}_\infty(T)$ is defined as the set of all the faces of $\mathcal{N}(T)$ which do not contain the origin $\mathbf{0}$ in $\mathbf{R}^n$. The vector polynomial T is said to be *convenient* if $\mathcal{N}(T_i)$ is *convenient* for each $i = 1, 2, \ldots, m$.

The following result follows from (ii1) and the proof of (ii2) of [4, Lemma 2.1].

Lemma 2.9. *Let $T = (T_1, T_2, \ldots, T_m) : \mathbf{R}^n \to \mathbf{R}^m$ be a vector polynomial. If T is convenient, then for any face $\Delta \in \mathcal{N}_\infty(T)$, there exists a unique collection of faces $\Delta_1 \in \mathcal{N}_\infty(T_1), \Delta_2 \in \mathcal{N}_\infty(T_2), \ldots, \Delta_m \in \mathcal{N}_\infty(T_m)$ such that*

$$\Delta = \Delta_1 + \Delta_2 + \cdots + \Delta_m.$$

Let Δ be a face of $\mathcal{N}(T)$. Again by (ii1) of [4, Lemma 2.1], we have the unique decomposition $\Delta = \Delta_1 + \Delta_2 + \cdots + \Delta_m$, where Δ_i is a face of $\mathcal{N}(T_i)$ for $i = 1, 2, \ldots, m$. The definition of the non-degeneracy at infinity of a vector polynomial T is given as follows:

Definition 2.10 ([4, Definition 2.3] (also see [5, 11]))**.** Let $T = (T_1, T_2, \ldots, T_m) : \mathbf{R}^n \to \mathbf{R}^m$ be a vector polynomial. T is said to be *non-degenerate at infinity* if for any face $\Delta \in \mathcal{N}_\infty(T)$

and any $x \in (\mathbf{R}\backslash\{0\})^n$, the rank of matrix H_Δ is equal to m, where

$$H_\Delta = \begin{pmatrix} x_1 \frac{\partial (T_1)_{\Delta_1}}{\partial x_1}(x) & \dots & x_n \frac{\partial (T_1)_{\Delta_1}}{\partial x_n}(x) & (T_1)_{\Delta_1}(x) & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ x_1 \frac{\partial (T_m)_{\Delta_m}}{\partial x_1}(x) & \dots & x_n \frac{\partial (T_m)_{\Delta_m}}{\partial x_n}(x) & 0 & \dots & (T_m)_{\Delta_m}(x) \end{pmatrix}.$$

3 Existence Results

In this section, we derive Frank-Wolfe type theorems for PVOP(K, f) under the weak section-boundedness from below condition.

3.1 Frank-Wolfe type theorem for the convex case

In this subsection, we investigate the existence of the weakly Pareto efficient solutions for PVOP(K, f) under convexity and weak section-boundedness from below conditions. To do so, we need the following result.

Lemma 3.1 (See [2, Theorem 3] and [1, Chapter II, Section 4, Theorem 13])**.** *Let $T_i : \mathbf{R}^n \to \mathbf{R}$ be a convex polynomial, $i = 0, 1, \dots, m$. Assume that*

$$C := \{x \in \mathbf{R}^n : T_1(x) \leq 0, T_2(x) \leq 0, \dots, T_m(x) \leq 0\}$$

and T_0 is bounded from below on C. Then T_0 attains its infimum on C.

Theorem 3.2. *Assume that $f_1, f_2, \dots, f_s, g_1, g_2, \dots, g_p : \mathbf{R}^n \to \mathbf{R}$ are convex polynomials. Then f is weakly section-bounded from below on K if and only if $SOL^w(K, f)$ is nonempty.*

Proof. "$\Leftarrow$": The result follows immediately from Remark 2.7.

"$\Rightarrow$": By Definition 2.3, there exist $\bar{x} \in K$ and $\bar{a} \in \mathbf{R}^s$ such that $f(x) - \bar{a} \notin -\operatorname{int} \mathbf{R}^s_+$ for any $x \in K_{\bar{x}}$, where $K_{\bar{x}} = \{x \in K : f_i(x) \leq f_i(\bar{x}), i = 1, 2, \dots, s\}$. Then $(f(K_{\bar{x}}) - \bar{a}) \bigcap (-\operatorname{int} \mathbf{R}^s_+) = \emptyset$. It follows that $\bar{a} \notin f(K_{\bar{x}}) + \operatorname{int} \mathbf{R}^s_+$. Since f is convex, we can easily check that the set $f(K_{\bar{x}}) + \operatorname{int} \mathbf{R}^s_+$ is convex. So, by the separation theorem of convex sets, there exists $\alpha \in \mathbf{R}^s\backslash\{0\}$ such that

$$\langle \alpha, \bar{a} \rangle \leq \langle \alpha, v \rangle$$

for any $v \in f(K_{\bar{x}}) + \operatorname{int} \mathbf{R}^s_+$. It is easy to verify that $\alpha \in \mathbf{R}^s_+\backslash\{0\}$ and so $\langle \alpha, \bar{a} \rangle \leq \langle \alpha, v' \rangle$ for any $v' \in f(K_{\bar{x}})$. Define $g_\alpha : \mathbf{R}^n \to \mathbf{R}$ by $g_\alpha(x) = \langle \alpha, f(x) \rangle$. Then g_α is a convex polynomial on $\mathbf{R}^n$ and is bounded from below on $K_{\bar{x}}$. By Lemma 3.1, g_α attains the infimum on $K_{\bar{x}}$ at some $x^* \in K_{\bar{x}}$. We claim that $x^* \in SOL^w(K, f)$. If not, then there exists $x_0 \in K$ such that $f_i(x_0) < f_i(x^*)$ for all $i \in \{1, 2, \dots, s\}$. Then

$$g_\alpha(x_0) = \langle \alpha, f(x_0) \rangle < \langle \alpha, f(x^*) \rangle = g_\alpha(x^*)$$

and $x_0 \in K_{\bar{x}}$ (since $x^* \in K_{\bar{x}}$). This contradicts to $x^* \in SOL(K_{\bar{x}}, g_\alpha)$. The proof is completed. □

Remark 3.3. Recently, Lee et al. derived in [19, Theorem 3.1] a Frank-Wolfe type theorem for a convex PVOP(K, f) by showing that $SOL^s(K, f)$ is nonempty if and only if there exists $z_0 \in \mathbf{R}^n$ such that $f(K) \bigcap (f(z_0) - \mathbf{R}^s_+)$ is nonempty and bounded. As a comparison, we has shown in Theorem 3.2 that a convex PVOP(K, f) admits a nonempty weak Pareto efficient solution set if and only if f is weakly section-bounded from below on K.

The following example illustrates Theorem 3.2.

Example 3.4. Consider the convex vector polynomial $f = (f_1, f_2)$ defined by

$$f_1(x_1, x_2) = x_1^2 + 2x_2, \quad f_2(x_1, x_2) = -x_1$$

and the constraint set $K = \{x \in \mathbf{R}^2 : x_1 \geq 0, x_2 \geq 0\}$. Let $\bar{x} = (1, 0)$. Then

$$K_{\bar{x}} = \{(x_1, x_2) \in K : x_1^2 + 2x_2 \leq 1, x_1 \geq 1\}.$$

It is easy to see that $f(x) - (0, 1) \notin -\operatorname{int} \mathbf{R}^s_+$ for any $x \in K_{\bar{x}}$. So, f is weakly section-bounded from below on K. By Theorem 3.2, $SOL^w(K, f)$ is nonempty. Indeed, it is easy to verify that $(0, 0) \in SOL^w(K, f)$.

The following example shows that Theorem 3.2 may not hold if f is non-convex.

Example 3.5. Consider the vector polynomial $f = (f_1, f_2)$ defined by

$$f_1(x_1, x_2) = (x_1x_2 - 1)^2 + 3x_1^2, \quad f_2(x_1, x_2) = (x_1x_2 - 1)^2 + 2x_1^2$$

and the constraint set $K = \mathbf{R}^2$. Then f is non-convex. For any $z = (z_1, z_2) \in \mathbf{R}^2$,

$$K_z = \{x \in \mathbf{R}^2 : f_1(x) \leq (z_1z_2 - 1)^2 + 3z_1^2, f_2(x) \leq (z_1z_2 - 1)^2 + 2z_1^2\}.$$

Clearly, $f(x) - (0, 0) \notin -\operatorname{int} \mathbf{R}^s_+$ for any $x \in K_z$. So f is weakly section-bounded from below on K. On the other hand, $SOL^w(K, f) = \emptyset$ because $f_1 > 0$, $f_2 > 0$, and $f(\frac{1}{n}, n) = (\frac{3}{n^2}, \frac{2}{n^2}) \to (0, 0)$ as $n \to +\infty$.

3.2 Frank-Wolfe type theorem for the non-convex case

In this subsection, we study the existence of the Pareto efficient solutions for a non-convex PVOP(K, f) when the convenience, the non-degeneracy at infinity, and the weak section-boundedness from below conditions are satisfied. We first give some lemmas.

Lemma 3.6 ([9, Proposition 13])**.** *Given $\lambda \in \operatorname{int} \mathbf{R}^s_+$ and $x' \in K$, define $g(x) = \sum_{j=1}^s \lambda_j f_j(x)$ and $K_{x'} = \{x \in K : f_i(x) \leq f_i(x'), i = 1, 2, \ldots, s\}$. If $x^* \in SOL(K_{x'}, g)$, then $x^* \in SOL^s(K, f)$.*

The following result has been established in the proof of [4, Theorem 1.1].

Lemma 3.7. *Let $T' = (T_0, T_1, \ldots, T_m) : \mathbf{R}^n \to \mathbf{R}^{m+1}$ be a vector polynomial. Assume that T' is convenient and non-degenerate at infinity and T_0 is bounded from below on the set*

$$C := \{x \in \mathbf{R}^n : T_1(x) \leq 0, \ldots, T_m(x) \leq 0\}.$$

Then T_0 is coercive on C in the sense that

$$\lim_{r \to \infty} \min_{x \in C, \|x\| = r} T_0(x) = +\infty.$$

The following result is simple but useful. It gives the translation invariance of the convenience and the non-degeneracy at infinity of a vector polynomial.

Lemma 3.8. *Let $T = (T_1, T_2, \ldots, T_m) : \mathbf{R}^n \to \mathbf{R}^m$ be a vector polynomial. If T is convenient and non-degenerate at infinity, then $T_b = (T_1 + b_1, T_2 + b_2, \ldots, T_m + b_m) : \mathbf{R}^n \to \mathbf{R}^m$ is convenient and non-degenerate at infinity for any $b = (b_1, b_2, \ldots, b_m) \in \mathbf{R}^m$.*

Proof. Let $b = (b_1, b_2, \ldots, b_m) \in \mathbf{R}^m$. By definition, we have $\mathcal{N}(T_i) = \mathcal{N}(T_i + b_i)$, $i = 1, 2, \ldots, m$. As a result, T_b is convenient (since T is convenient).

Next, we show that T_b is non-degenerate at infinity. Let $\Delta \in \mathcal{N}_\infty(T_b)$. Since $\mathcal{N}(T_b) = \mathcal{N}(T)$, we get $\Delta \in \mathcal{N}_\infty(T)$. Since T_b is convenient, by Lemma 2.9, we obtain $\Delta = \Delta_1 + \Delta_2 + \cdots + \Delta_m$, where $\Delta_i \in \mathcal{N}_\infty(T_i + b_i)$, $i = 1, 2, \ldots, m$. Since $\mathbf{0} \notin \cup_{i=1}^m \Delta_i$, we have $(T_i + b_i)_{\Delta_i} = (T_i)_{\Delta_i}$, $i = 1, 2, \ldots, m$. Since T is non-degenerate at infinity $\Delta \in \mathcal{N}_\infty(T)$, the rank of the matrix

$$\begin{pmatrix} x_1 \frac{\partial (T_1+b_1)_{\Delta_1}}{\partial x_1}(x) & \ldots & x_n \frac{\partial (T_1+b_1)_{\Delta_1}}{\partial x_n}(x) & (T_1+b_1)_{\Delta_1}(x) & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_1 \frac{\partial (T_m+b_m)_{\Delta_m}}{\partial x_1}(x) & \ldots & x_n \frac{\partial (T_m+b_m)_{\Delta_m}}{\partial x_n}(x) & 0 & \ldots & (T_m+b_m)_{\Delta_m}(x) \end{pmatrix}$$

is equal to m for any $x \in (\mathbf{R}\backslash\{0\})^n$. So, T_b is non-degenerate at infinity. □

As shown in the proof of Lemma 3.8, the translation invariance of the non-degeneracy at infinity of a vector polynomial depends on its convenience. The following example shows that if a vector polynomial is not convenience, then its translation invariance of the non-degeneracy at infinity may not hold.

Example 3.9. Consider the vector polynomial $T = (T_1, T_2)$ defined by

$$T_1(x_1, x_2) = x_1x_2 + 1, T_2(x_1, x_2) = x_2 + 1.$$

Then

$$\mathcal{N}(T_1) = \text{co}\{(0,0), (1,1)\}, \mathcal{N}(T_2) = \text{co}\{(0,0), (0,1)\}.$$

It is easy to see that neither T_1 nor T_2 are convenient. Next, we check that $T = (T_1, T_2)$ is non-degenerate at infinity. Note that

$$\mathcal{N}(T) = \mathcal{N}(T_1) + \mathcal{N}(T_2) = \text{co}\{(0,0), (0,1), (1,1), (1,2)\}.$$

Then $\mathcal{N}_\infty(T)$ has five faces as follows:

$$\Delta_1 = \text{co}\{(1,1), (1,2)\} = \Delta_1^1 + \Delta_1^2 = (1,1) + \text{co}\{(0,0), (0,1)\},$$

$$\Delta_2 = \text{co}\{(0,1), (1,2)\} = \Delta_2^1 + \Delta_2^2 = \text{co}\{(0,0), (1,1)\} + (0,1),$$

$$\Delta_3 = (0,1) = \Delta_3^1 + \Delta_3^2 = (0,0) + (0,1),$$

$$\Delta_4 = (1,2) = \Delta_4^1 + \Delta_4^2 = (1,1) + (0,1),$$

$$\Delta_5 = (1,1) = \Delta_5^1 + \Delta_5^2 = (1,1) + (0,0),$$

where the face $\Delta_i^j \in \mathcal{N}(T_j)$ for each $i = 1, 2, 3, 4, 5, j = 1, 2$. The matrix

$$H_{\Delta_1} = \begin{pmatrix} x_1 \frac{\partial (T_1)_{\Delta_1^1}}{\partial x_1}(x) & x_2 \frac{\partial (T_1)_{\Delta_1^1}}{\partial x_2}(x) & (T_1)_{\Delta_1^1}(x) & 0 \\ x_1 \frac{\partial (T_2)_{\Delta_1^2}}{\partial x_1}(x) & x_2 \frac{\partial (T_2)_{\Delta_1^2}}{\partial x_2}(x) & 0 & (T_2)_{\Delta_1^2}(x) \end{pmatrix} =$$

$$\begin{pmatrix} x_1x_2 & x_1x_2 & x_1x_2 & 0 \\ 0 & x_2 & 0 & x_2+1 \end{pmatrix}.$$

By similar calculations, we have

$$H_{\Delta_2} = \begin{pmatrix} x_1x_2 & x_1x_2 & x_1x_2+1 & 0 \\ 0 & x_2 & 0 & x_2 \end{pmatrix},$$

$$H_{\Delta_3} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & x_2 & 0 & x_2 \end{pmatrix},$$
$$H_{\Delta_4} = \begin{pmatrix} x_1x_2 & x_1x_2 & x_1x_2 & 0 \\ 0 & x_2 & 0 & x_2 \end{pmatrix},$$
$$H_{\Delta_5} = \begin{pmatrix} x_1x_2 & x_1x_2 & x_1x_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that the rank of matrix H_{Δ_j} is equal to 2 for any $x = (x_1, x_2) \in (\mathbf{R}\backslash\{0\})^2$ and each $j = 1, 2, \ldots, 5$. Let $b = (-1, -1)$, by similar calculations, we can easily check that T_b is not non-degenerate at infinity.

Theorem 3.10. *Assume that $G = (f_1, f_2, \ldots, f_s, g_1, g_2, \ldots, g_p) : \mathbf{R}^n \to \mathbf{R}^{s+p}$ is convenient and non-degenerate at infinity. Then f is weakly section-bounded from below on K if and only if $SOL^s(K, f)$ is nonempty.*

Proof. "$\Leftarrow$": Since $\emptyset \neq SOL^s(K, f) \subseteq SOL^w(K, f)$, the result follows immediately from Remark 2.7.

"$\Rightarrow$": The conclusion holds trivially when K is bounded. We now assume that K is unbounded. Since f is weakly section-bounded from below on K, by Proposition 2.6, there exist $x^* \in K$ and $i_0 \in \{1, 2, \ldots, s\}$ such that f_{i_0} is bounded from below on $K^{i_0}_{x^*}$, where

$$K^{i_0}_{x^*} = \{x \in K : f_i(x) \le f_i(x^*), i = 1, 2, \ldots, i_0 - 1, i_0 + 1, \ldots, s\}.$$

Recall that $K_{x^*} = \{x \in K : f_i(x) - f_i(x^*) \le 0, i = 1, 2, \ldots, s\}$. Since G is convenient and non-degenerate at infinity, by Lemma 3.8, we have that G_b is convenient and non-degenerate at infinity, where $G_b = G + b$ and

$$b = (-f_1(x^*), \ldots, -f_{i_0-1}(x^*), 0, -f_{i_0+1}(x^*), \ldots, -f_s(x^*), 0, \ldots, 0).$$

By Lemma 3.7, f_{i_0} is coercive on $K^{i_0}_{x^*}$. As a consequence,

$$K_{x^*} = \{x \in K^{i_0}_{x^*} : f_{i_0}(x) \le f_{i_0}(x^*)\}$$

is compact. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in$ int $\mathbf{R}^n_+$. Consider the following polynomial scalar optimization problem:

$$\inf_{x \in K_{x^*}} \sum_{i=1}^{s} \gamma_i f_i(x).$$

Since K_{x^*} is compact, we obtain that the above problem have an optimal solution x_0 on K_{x^*}. By Lemma 3.6, we get $x_0 \in SOL^s(K, f)$. The proof is completed. □

Remark 3.11.

(i) Theorem 3.10 can be regarded as a vectorial version of [4, Theorem 1.1] (also see [11, Theorem 4.3]);

(ii) The assumption that G is convenient in Theorem 3.10 is essential. Indeed, consider Example 3.5. It is easy to check that the vector polynomial $f = (f_1, f_2)$ is non-degenerate at infinity, but not convenient. Furthermore, f is weakly sectioned-bounded from below. However, $SOL^s(K, f) = \emptyset$.

Next, we give an example to illustrate Theorem 3.10.

Example 3.12. Consider the vector polynomial $f=(f_1,f_2)$ defined by

$$f_1(x_1,x_2)=x_1^6+2x_2^6+M_1(x_1,x_2), f_2(x_1,x_2)=x_1^4-x_2^4+M_2(x_1,x_2),$$

where M_1 is a polynomial of degree at most 5 and M_2 is a polynomial of degree at most 3. The constraint set K is given by

$$K=\{x\in\mathbf{R}^2 : f_3(x_1,x_2)=1-x_1^2-2x_2^2\le 0\}.$$

Then

$$\mathcal{N}(f_1)=\mathrm{co}\{(0,0),(6,0),(0,6)\}, \mathcal{N}(f_2)=\mathrm{co}\{(0,0),(4,0),(0,4)\}$$

and

$$\mathcal{N}(f_3)=\mathrm{co}\{(0,0),(2,0),(0,2)\}.$$

So f_1, f_2 and f_3 are convenient. Next, we need to check that the vector polynomial $G=(f_1,f_2,f_3)$ is non-degenerate at infinity. It is worth noting that the Newton polyhedra at infinity of G is

$$\mathcal{N}(G)=\mathcal{N}(f_1)+\mathcal{N}(f_2)+\mathcal{N}(f_3)=\mathrm{co}\{(0,0),(12,0),(0,12)\}.$$

Then $\mathcal{N}_\infty(G)$ has three faces as follows:

$$\begin{aligned}\Delta_1&=\mathrm{co}\{(12,0),(0,12)\}=\Delta_1^1+\Delta_1^2+\Delta_1^3\\&=\mathrm{co}\{(6,0),(0,6)\}+\mathrm{co}\{(4,0),(0,4)\}+\mathrm{co}\{(2,0),(0,2)\},\end{aligned}$$

$$\Delta_2=(12,0)=\Delta_2^1+\Delta_2^2+\Delta_2^3=(6,0)+(4,0)+(2,0),$$
$$\Delta_3=(0,12)=\Delta_3^1+\Delta_3^2+\Delta_3^3=(0,6)+(0,4)+(0,2),$$

where the face $\Delta_i^j\in\mathcal{N}_\infty(f_j)$ for each $i=1,2,3, j=1,2,3$. The matrix

$$H_{\Delta_1}=\begin{pmatrix} x_1\frac{\partial(f_1)_{\Delta_1^1}}{\partial x_1}(x) & x_2\frac{\partial(f_1)_{\Delta_1^1}}{\partial x_2}(x) & (f_1)_{\Delta_1^1}(x) & 0 & 0\\ x_1\frac{\partial(f_2)_{\Delta_1^2}}{\partial x_1}(x) & x_2\frac{\partial(f_2)_{\Delta_1^2}}{\partial x_2}(x) & 0 & (f_2)_{\Delta_1^2}(x) & 0\\ x_1\frac{\partial(f_2)_{\Delta_1^3}}{\partial x_1}(x) & x_2\frac{\partial(f_2)_{\Delta_1^3}}{\partial x_2}(x) & 0 & 0 & (f_3)_{\Delta_1^3}(x)\end{pmatrix}$$

$$=\begin{pmatrix} 6x_1^6 & 12x_2^6 & x_1^6+2x_2^6 & 0 & 0\\ 4x_1^4 & -4x_2^4 & 0 & x_1^4-x_2^4 & 0\\ -2x_1^2 & -4x_2^2 & 0 & 0 & -x_1^2-2x_2^2\end{pmatrix}.$$

By similar calculations, we have

$$H_{\Delta_2}=\begin{pmatrix} 6x_1^6 & 0 & x_1^6 & 0 & 0\\ 4x_1^4 & 0 & 0 & x_1^4 & 0\\ -2x_1^2 & 0 & 0 & 0 & -x_1^2\end{pmatrix}$$

and

$$H_{\Delta_3}=\begin{pmatrix} 0 & 12x_2^6 & 2x_2^6 & 0 & 0\\ 0 & -4x_2^4 & 0 & -x_2^4 & 0\\ 0 & -4x_2^2 & 0 & 0 & -2x_2^2\end{pmatrix}.$$

It is easy to see that the rank of matrix H_{Δ_1}, H_{Δ_2} and H_{Δ_3} are all equal to 3 for any $x=(x_1,x_2)\in(\mathbf{R}\backslash\{0\})^2$. By definition, G is non-degenerate at infinity. On the other hand, f_1 is bounded from below on K. By Proposition 2.6, f is weakly section-bounded from below on K. Hence, $SOL^s(K,f)$ is nonempty by Theorem 3.10.

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