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RELAXATION INERTIAL PROJECTION ALGORITHMS FOR SOLVING MONOTONE VARIATIONAL INEQUALITY PROBLEMS*

Yan Zhang, Denglian Yang and Yongle Zhang†

Abstract: In this paper, we propose relaxation inertial projection algorithms for solving monotone variational inequality problem in Hilbert space. Each iteration of this algorithm only needs to calculate the projection onto a half-space once. The weak convergence and convergence rate of the algorithm is proved under the mapping is Lipschitz continuous and monotone. Further, by introducing a contraction mapping, a strong convergence algorithm is given for solving the monotone variational inequality. Numerical experiments are also manifested to show the efficiency and advantages of the proposed algorithms.

Key words: *variational inequality problem, monotone mapping, Lipschitz continuous, weak convergence, strong convergence*

Mathematics Subject Classification: *65K15, 47J05, 47J20, 47J25*

1 Introduction

The variational inequality problem, denoted by $VI(C, F)$, is to find a point $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \ \forall x \in C, \tag{1.1}$$

where F is a mapping from a Hilbert space $\mathcal{H}$ into itself, C is a nonempty closed convex subset in $\mathcal{H}$. The solution set of $VI(C, F)$ is denoted by $SOL(C, F)$.

Variational inequality theory plays an important role in many fields, such as transportation, mathematical programming, economics, and others[2, 20, 21, 32]. As an effective numerical method for solving $VI(C, F)$, projection method has been widely concerned by many authors. When projections onto C can be efficiently computed, the classical gradient projection method [13, 8, 22, 33, 12] and its variants are usually the prominent choices of algorithms for solving (1.1), due to their ease of implementation.

Projections onto C, however, are not necessarily easy to compute. For example, C is an ellipsoid and the other concrete instances that arise in applications, see [10, 14]. In this case, we mainly consider $C := \{x \in \mathcal{H} | c(x) \leq 0\}$, where $c : \mathcal{H} \to \mathbb{R}$ is a convex function. Fukushima [11] proposed relaxed projection algorithm for solving variational inequality

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†Corresponding Author

problem, which projection to the feasible set C was replaced with projection to a half-space $C_k := \{x \in \mathcal{H} | c(x_k) + \langle c'(x_k), x - x_k\rangle \leq 0\}$. Under the mapping F is Lipschitz continuous and strong monotone, the convergence of the algorithm is proved. In order to relax the strong monotonicity of F, Censor [12] proposed relaxed subgradient extragradient algorithm, which one projection to half-space C_k was replaced with twice projection to half-space. However, the convergence of [12] is an open question. Combining the inertial technique with [12], Cao and Guo [5] proposed the inertial subgradient extragradient method, and proved the convergence of the algorithm. But, in [12], the step size λ is determined by a constant which is depends on the solution of VI. He and Wu [16] proposed another way to relax the strong monotonicity of F, which is called the subgradient extragradient method. The difference between [16] and [12] is that, in [12], the second projection onto C_k is replaced by a projection onto T_k, which is another half-space containing C. In order to speed up the convergence rate of algorithms, in recent years, some scholars have combined the inertial method with these subgradient extragradient projection algorithms, see [18, 7, 27, 28].

In fact, strong convergence results of algorithms are more valuable than weak ones in practice, it is necessary to study strong convergence of algorithms. By introducing a contraction mapping, some scholars proposed strong convergence algorithms, as shown in [30, 31, 9, 26]. As far as we know, most of the algorithms for strong convergence are based on computing the projection onto the set C. Recently, [6] proposed a strong convergence algorithm for solving the monotone variational inequality, which needs to computing the projection onto the half-space C_k twice in each iteration.

In this paper, we propose an relaxation inertial projection algorithm for solving the variational inequalities. In this algorithm, each iteration only needs to calculate the projection onto the half-space C_k once, thus reducing the number of projections of previous algorithms, and the selection of parameters is no longer dependent on the solution of the variational inequality. The weak convergence and convergence rate of the algorithm is proved under the mapping F is Lipschitz continuous and monotone. Further, by introducing a contraction mapping, a strong convergence algorithm is given for solving the monotone variational inequality. Finally, we give some numerical examples to show the efficiency of our algorithm over some other algorithms in the literature.

2 Preliminaries

Let $\mathcal{H}$ be a Hilbert space with inner product and norm denoted respectively by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, C be a nonempty closed and convex subset of $\mathcal{H}$. Let $\mathbb{N}^+$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. The weak convergence of $\{x_k\}_{k=1}^+\infty$ to x is denoted by $x_k \rightharpoonup x$ as $k \to +\infty$, while the strong convergence of $\{x_k\}_{k=1}^+\infty$ to x is denoted by $x_k \to x$ as $k \to +\infty$. In addition, we denote $[t]_+$ by $\max\{t, 0\}$.

Definition 2.1. Let C be a nonempty closed convex subset of $\mathcal{H}$ and $F : \mathcal{H} \to \mathcal{H}$ be a mapping. Then

(i) F is said to be L-Lipschitz continuous on C, if there exists $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \ \forall x \ , y \in C.$$

If $L \in [0, 1)$, F is said to be contraction mapping.

(ii) F is said to be monotone on C if

$$\langle F(x) - F(y), x - y\rangle \geq 0, \ \forall x \ , y \in C.$$

Definition 2.2. A function $c : \mathcal{H} \to \mathbb{R}$ is said to be weakly lower semicontinuous (w-lsc) at $x \in \mathcal{H}$, if $x_k \rightharpoonup x$ implies $c(x) \leq \liminf_{k \to +\infty} c(x_k)$. We say c is weakly lower semicontinuous on $\mathcal{H}$, if for each $x \in \mathcal{H}$, c is weakly lower semicontinuous at x.

Definition 2.3. A function $c : \mathcal{H} \to \mathbb{R}$ is said to be Gâteaux differentiable at $x \in \mathcal{H}$, if there exists an element, denoted by $c'(x) \in \mathcal{H}$, such that

$$\lim_{t \to 0} \frac{c(x + t\nu) - c(x)}{t} = \langle \nu, c'(x) \rangle, \ \forall \nu \in \mathcal{H},$$

where $c'(x)$ is called the Gâteaux differential of c at x. We say c is Gâteaux differentiable on $\mathcal{H}$, if for each $x \in \mathcal{H}$, c is Gâteaux differentiable at x.

If c is a convex function and Gâteaux differentiable at x, we have that $c(y) \geq c(x) + \langle c'(x), y - x \rangle$, for any $y \in \mathcal{H}$, see [19].

Definition 2.4. Let C be a nonempty closed convex subset of $\mathcal{H}$. The normal cone to C at $x \in C$ is a multi-valued mapping defined by

$$N_C(x) := \{\xi \in \mathcal{H} | \langle \xi, z - x \rangle \leq 0, \ \forall z \in C\}.$$

Definition 2.5. Let $T : \mathcal{H} \to 2^{\mathcal{H}}$ be a multi-valued mapping defined on $\mathcal{H}$ and the following two conditions hold:

(i) T is monotone, i.e.

$$\langle u - v, x - y \rangle \geq 0, \ \forall u \in T(x), v \in T(y).$$

(ii) the graph $G(T) := \{(x, u) | u \in T(x)\}$ of T is not properly contained in the graph of any other monotone operator, i.e. for every $(x, u) \in \mathcal{H} \times \mathcal{H}$,

$$(x, u) \in G(T) \Leftrightarrow (\forall (y, v) \in G(T)) \ \langle x - y, u - v \rangle \geq 0.$$

Then T is called a maximal monotone mapping.

Let C be a nonempty closed convex subset of $\mathcal{H}$. For each point $x \in \mathcal{H}$, there exists a unique element in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \ \forall y \in C.$$

The mapping $P_C : \mathcal{H} \to C$ is called the metric projection, which has the following properties:

Lemma 2.6 ([3])**.** *Let C be a nonempty closed convex subset of $\mathcal{H}$. Given $x \in \mathcal{H}$, then*

(i) $\langle P_C(x) - x, y - P_C(x) \rangle \geq 0, \ \forall y \in C.$

(ii) $\|P_C(x) - y\|^2 \leq \|x - y\|^2 - \|x - P_C(x)\|^2, \ \forall y \in C.$

The following lemmas are crucial for the proof of our convergence theorems.

Lemma 2.7 ([3])**.** *For all $x, \ y \in \mathcal{H}$, the following equality holds:*

$$\|\alpha x + (1 - \alpha) y\|^2 = \alpha \|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2, \ \forall \alpha \in \mathbb{R}.$$

Lemma 2.8 ([17])**.** *Assume that the solution set $SOL(C, F)$ of $VI(C, F)$ is nonempty. Given $x^* \in C$, then $x^* \in SOL(C, F)$ if and only if either*

(i) $F(x^*) = 0$, *or*

(ii) $x^* \in \partial C$ *and there exists a positive constant* η *such that* $F(x^*) = -\eta c'(x^*)$.

Lemma 2.9 ([1])**.** *Let* $\{\varphi_k\}, \{\alpha_k\}$ *and* $\{\delta_k\}$ *be sequences in* $[0, +\infty)$ *such that*

$$\varphi_{k+1} \le \varphi_k + \alpha_k(\varphi_k - \varphi_{k-1}) + \delta_k, \quad \sum_{k=1}^{+\infty} \delta_k < +\infty,$$

and there exists a real number α *with* $0 \le \alpha_k \le \alpha < 1$, *for all* $k \in \mathbb{N}^+$, *Then the following results hold:*

(i) $\sum\limits_{k=1}^{+\infty} [\varphi_k - \varphi_{k-1}]_+ < +\infty$;

(ii) *there exists* $\varphi^* \in [0, +\infty)$ *such that* $\lim\limits_{k\to+\infty} \varphi_k = \varphi^*$.

Lemma 2.10 ([24])**.** *Let* C *be a nonempty closed convex subset of* $\mathcal{H}$. *Let* $F : \mathcal{H} \to \mathcal{H}$ *be a monotone and Lipschitz continuous mapping. Define*

$$T(x) := \begin{cases} F(x) + N_C(x), & x \in C; \\ \emptyset, & x \notin C. \end{cases}$$

Then T *is maximal monotone and* $0 \in T(x)$ *if and only if* $x \in SOL(C, F)$.

Lemma 2.11 ([23])**.** *Assume that* C *is a nonempty subset of* $\mathcal{H}$ *and* $\{x_k\}$ *is a sequence in* $\mathcal{H}$ *such that the following two conditions hold:*

(i) $\forall x \in C, \ \lim\limits_{k\to+\infty} \|x_k - x\|$ *exists;*

(ii) *every sequential weak cluster point of* $\{x_k\}$ *belongs to* C.
Then $\{x_k\}$ *converges weakly to a point in* C.

Lemma 2.12 ([25])**.** *Let* $\{\Phi_k\}$ *be a sequence of nonnegative real numbers,* $\{s_k\}$ *be a sequence in* $(0, 1)$ *such that* $\sum\limits_{k=1}^{+\infty} s_k = +\infty$ *and* $\{\Omega_k\}$ *be a sequence of real numbers. Suppose that*

$$\Phi_{k+1} \le (1 - s_k)\Phi_k + s_k\Omega_k, \quad \forall k \ge 1.$$

If $\limsup\limits_{j\to+\infty} \Omega_{k_j} \le 0$ *for every subsequence* $\{\Phi_{k_j}\}$ *of* $\{\Phi_k\}$ *satisfying* $\liminf\limits_{j\to+\infty}(\Phi_{k_j+1} - \Phi_{k_j}) \ge 0$, *then* $\lim\limits_{k\to+\infty} \Phi_k = 0$.

3 Weak Convergence Algorithm

In this paper, the nonempty closed convex set C is defined as the

$$C := \{x \in \mathcal{H} | c(x) \le 0\}.$$

We always assume that the following conditions are satisfied:

Assumption 3.1. The solution set $SOL(C, F)$ is nonempty.

Assumption 3.2. The mapping $F : \mathcal{H} \to \mathcal{H}$ is monotone and L-Lipschitz continuous on $\mathcal{H}$.

Assumption 3.3. The function $c : \mathcal{H} \to \mathbb{R}$ satisfies the following conditions:

(i) $c(x)$ is a convex function;

(ii) $c(x)$ is weakly lower semicontinuous on $\mathcal{H}$;

(iii) $c(x)$is Gâteaux differentiable on $\mathcal{H}$ and $c'(x)$ is a L_1-Lipschitz continuous mapping on $\mathcal{H}$;

(iv) there exists a positive constant M' such that $\|F(x)\| \leq M'\|c'(x)\|$, for any $x \in \partial C$, where ∂C denotes the boundary of C.

3.1 Algorithm and Convergence Analysis

The following describes the framework structure of the algorithm:

Algorithm 3.1 (Weak convergence algorithm for (1.1))**. Initialization:** Choose $x_0,\ x_1 \in \mathcal{H}$ and $\sigma > 0,\ \gamma \in (0,1),\ M = M'L_1,\ \mu \in (0,1),\ \alpha \in [0,1)$.

Iterative Steps: Calculate x_{k+1} as follows:

Step 1. Given the iterates x_{k-1} and $x_k (k \geq 1)$, Set

$$\omega_k = x_k + \alpha_k(x_k - x_{k-1}),$$

where

$$\alpha_k = \begin{cases} \min\{\frac{1}{k^2\|x_k - x_{k-1}\|^2}, \alpha\}, & \text{if } x_k \neq x_{k-1}; \\ \alpha, & \text{otherwise.} \end{cases}$$

Step 2. Construct the half-space

$$C_k := \{x \in \mathcal{H} | c(\omega_k) + \langle c'(\omega_k), x - \omega_k\rangle \leq 0\}. \tag{3.1}$$

Compute

$$y_k = P_{C_k}(\omega_k - \lambda_k F(\omega_k)),$$

where $\lambda_k = \sigma\gamma^{l_k}$, with l_k is the smallest nonnegative integer l satisfying

$$(\sigma\gamma^l)^2\|F(\omega_k) - F(y_k)\|^2 + 2M\sigma\gamma^l\|\omega_k - y_k\|^2 \leq \mu^2\|\omega_k - y_k\|^2. \tag{3.2}$$

If $y_k = \omega_k$ then stop. Otherwise, go to **Step 3.**

Step 3. Calculate the next iterate,

$$x_{k+1} = y_k + \lambda_k(F(\omega_k) - F(y_k)).$$

Set $k \leftarrow k+1$ and go to **Step 1.**

Remark 3.2. One can see that for each $k \in \mathbb{N}^+,\ C \subseteq C_k$ in Algorithm 3.1. In fact, for each $x \in C$, $c(x_k) + \langle c'(x_k), x - x_k\rangle \leq c(x) \leq 0$.

Remark 3.3. In Algorithm 3.1, by the definition of $\{\alpha_k\}$, the following conclusions hold:

(i) $0 \leq \alpha_k \leq \alpha < 1,\ \forall k \in \mathbb{N}^+$;

(ii) $\exists M_1 > 0,\ \sum\limits_{k=1}^{+\infty} \alpha_k\|x_k - x_{k-1}\|^2 \leq \sum\limits_{k=1}^{+\infty} \frac{1}{k^2} < M_1;\ \lim\limits_{k\to+\infty} \alpha_k\|x_k - x_{k-1}\|^2 = 0.$

Remark 3.4. Suppose that Assumption 3.2 holds, then the line search (3.2) is well defined. In fact, from $\gamma \in (0,1)$, we have that there exists l such that $\sigma^2 L^2(\gamma^l)^2 + 2M\sigma\gamma^l \leq \mu^2$. This implies that

$$\begin{aligned}(\sigma\gamma^l)^2\|F(\omega_k) - F(P_{C_k}(\omega_k - \sigma\gamma^l F(\omega_k)))\|^2 + 2M\sigma\gamma^l\|\omega_k - P_{C_k}(\omega_k - \sigma\gamma^l F(\omega_k))\|^2 \\ \leq (\sigma^2 L^2(\gamma^l)^2 + 2M\sigma\gamma^l)\|\omega_k - P_{C_k}(\omega_k - \sigma\gamma^l F(\omega_k))\|^2 \\ \leq \mu^2\|\omega_k - P_{C_k}(\omega_k - \sigma\gamma^l F(\omega_k))\|^2.\end{aligned}$$

In the following analysis, we assume that Algorithm 3.1 always generates infinite sequences. In fact, if Algorithm 3.1 terminates with in finite steps, i.e., there exists k_0, such that $y_{k_0} = \omega_{k_0}$, then $y_{k_0} = P_{C_{k_0}}(y_{k_0} - \lambda_{k_0} F(y_{k_0}))$, which means y_{k_0} is a solution of $VI(C,F)$.

First, we give the following lemma, which plays a crucial role in the proof of the convergence of Algorithm 3.1.

Lemma 3.5. *Suppose that Assumption 3.1, 3.2 and 3.3 hold. Let $\{x_k\}, \{\omega_k\}$ and $\{y_k\}$ be sequences generated by Algorithm 3.1. Then for all $x^* \in SOL(C,F)$ and $k \in \mathbb{N}^+$, we have*

$$\|x_{k+1} - x^*\|^2 \leq \|\omega_k - x^*\|^2 - (1-\mu^2)\|\omega_k - y_k\|^2.$$

Proof. From the definition of x_{k+1}, we obtain

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &= \|y_k + \lambda_k(F(\omega_k) - F(y_k)) - x^*\|^2 \\ &= \|y_k - x^*\|^2 + 2\lambda_k\langle F(\omega_k) - F(y_k), y_k - x^*\rangle + \lambda_k^2\|F(\omega_k) - F(y_k)\|^2 \\ &= \|y_k - \omega_k\|^2 + 2\langle y_k - \omega_k, \omega_k - x^*\rangle + \|\omega_k - x^*\|^2 \\ &\quad + 2\lambda_k\langle F(\omega_k) - F(y_k), y_k - x^*\rangle + \lambda_k^2\|F(\omega_k) - F(y_k)\|^2 \\ &= \|y_k - \omega_k\|^2 + 2\langle y_k - \omega_k, \omega_k - y_k\rangle + 2\langle y_k - \omega_k, y_k - x^*\rangle + \|\omega_k - x^*\|^2 \\ &\quad + 2\lambda_k\langle F(\omega_k) - F(y_k), y_k - x^*\rangle + \lambda_k^2\|F(\omega_k) - F(y_k)\|^2 \qquad (3.3) \\ &= \|\omega_k - x^*\|^2 - \|\omega_k - y_k\|^2 + 2\langle \omega_k - \lambda_k F(\omega_k) - y_k, x^* - y_k\rangle \\ &\quad + 2\lambda_k\langle F(y_k), x^* - y_k\rangle + \lambda_k^2\|F(\omega_k) - F(y_k)\|^2 \\ &\overset{(a)}{\leq} \|\omega_k - x^*\|^2 - \|\omega_k - y_k\|^2 + 2\lambda_k\langle F(y_k), x^* - y_k\rangle + \lambda_k^2\|F(\omega_k) - F(y_k)\|^2 \\ &\overset{(b)}{\leq} \|\omega_k - x^*\|^2 - \|\omega_k - y_k\|^2 + 2\lambda_k\langle F(x^*), x^* - y_k\rangle + \lambda_k^2\|F(\omega_k) - F(y_k)\|^2,\end{aligned}$$

where (a) follows from the definition of y_k, Lemma 2.6(i) and $x^* \in SOL(C,F) \subseteq C \subseteq C_k$, and (b) holds because of the monotonicity of the mapping F.

If $F(x^*) = 0$, according to (3.3), we have

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &\leq \|\omega_k - x^*\|^2 - \|\omega_k - y_k\|^2 + \lambda_k^2\|F(\omega_k) - F(y_k)\|^2 \\ &\overset{(c)}{\leq} \|\omega_k - x^*\|^2 - \|\omega_k - y_k\|^2 + \mu^2\|\omega_k - y_k\|^2 \\ &= \|\omega_k - x^*\|^2 - (1-\mu^2)\|\omega_k - y_k\|^2,\end{aligned}$$

where (c) holds because of (3.2).

If $F(x^*) \neq 0$, from Lemma 2.8, we get $x^* \in \partial C$ and there exists a $\eta > 0$ such that $F(x^*) = -\eta c'(x^*)$. Using Assumption 3.3(iv), we deduce that $\eta \leq M'$. From the convexity of c, we obtain

$$c(y_k) \geq c(x^*) + \langle c'(x^*), y_k - x^*\rangle \overset{(d)}{=} \langle c'(x^*), y_k - x^*\rangle = \langle -\frac{1}{\eta}F(x^*), y_k - x^*\rangle,$$

where (d) follows from $x^* \in \partial C$. This implies that $\langle F(x^*), x^* - y_k \rangle \leq \eta c(y_k)$. Using the convexity of c again,

$$c(\omega_k) \geq c(y_k) + \langle c'(y_k), \omega_k - y_k \rangle.$$

Therefore,

$$\begin{aligned} c(y_k) &\leq c(\omega_k) - \langle c'(y_k), \omega_k - y_k \rangle \\ &\overset{(e)}{\leq} \langle c'(\omega_k), \omega_k - y_k \rangle - \langle c'(y_k), \omega_k - y_k \rangle \\ &= \langle c'(\omega_k) - c'(y_k), \omega_k - y_k \rangle \\ &\leq \|c'(\omega_k) - c'(y_k)\| \|\omega_k - y_k\| \\ &\leq L_1 \|\omega_k - y_k\|^2, \end{aligned}$$

where (e) holds because of $y_k \in C_k$. Thus, $\langle F(x^*), x^* - y_k \rangle \leq \eta c(y_k) \leq \eta L_1 \|\omega_k - y_k\|^2 \leq M' L_1 \|\omega_k - y_k\|^2 = M \|\omega_k - y_k\|^2$. Finally, from the relation (3.3), we get

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &\leq \|\omega_k - x^*\|^2 - \|\omega_k - y_k\|^2 + 2\lambda_k M \|\omega_k - y_k\|^2 + \lambda_k^2 \|F(\omega_k) - F(y_k)\|^2 \\ &\overset{(f)}{\leq} \|\omega_k - x^*\|^2 - \|\omega_k - y_k\|^2 + \mu^2 \|\omega_k - y_k\|^2 \\ &= \|\omega_k - x^*\|^2 - (1 - \mu^2) \|\omega_k - y_k\|^2, \end{aligned}$$

where (f) holds due to (3.2). This completes the proof. □

Lemma 3.6. *Suppose that Assumption 3.1, 3.2 and 3.3 hold. Let C_k be as shown (3.1). Assume that $\{\lambda_k\}$ is a positive real number sequence and $\{(\omega_k, y_k)\}$ is the sequence that satisfies $y_k = P_{C_k}(\omega_k - \lambda_k F(\omega_k))$. If there exists a subsequence $\{\omega_{k_j}\}$ of $\{\omega_k\}$ such that $\omega_{k_j} \rightharpoonup \hat{x} \in \mathcal{H}$ and $\lim_{j \to +\infty} \|\omega_{k_j} - y_{k_j}\| = 0$, then $\hat{x} \in SOL(C, F)$.*

Proof. Due to $y_{k_j} \in C_{k_j}$ and the definition of C_k, we have

$$c(\omega_{k_j}) + \langle c'(\omega_{k_j}), y_{k_j} - \omega_{k_j} \rangle \leq 0.$$

Then, using the Cauchy-Schwartz inequality,

$$c(\omega_{k_j}) \leq \|c'(\omega_{k_j})\| \|\omega_{k_j} - y_{k_j}\|.$$

Since $\omega_{k_j} \rightharpoonup \hat{x}$ and $\lim_{j \to +\infty} \|\omega_{k_j} - y_{k_j}\| = 0$, we get $\{\omega_{k_j}\}$ and $\{y_{k_j}\}$ are bounded and

$$y_{k_j} \rightharpoonup \hat{x}. \tag{3.4}$$

According to Assumption 3.3(iii) and the boundedness of $\{\omega_{k_j}\}$, we can deduce that $\{c'(\omega_{k_j})\}$ is bounded, so there exists $M_2 > 0$ such that $\|c'(\omega_{k_j})\| \leq M_2$ for all k_j, and then

$$c(\omega_{k_j}) \leq M_2 \|\omega_{k_j} - y_{k_j}\|.$$

From the weakly lower semicontinuous of c, we get

$$c(\hat{x}) \leq \liminf_{j \to +\infty} c(\omega_{k_j}) \leq \lim_{j \to +\infty} M_2 \|\omega_{k_j} - y_{k_j}\| = 0,$$

which means $\hat{x} \in C$. Now, we turn to showing $\hat{x} \in SOL(C, F)$. Define

$$T(x) := \begin{cases} F(x) + N_C(x), & x \in C, \\ \emptyset, & x \notin C, \end{cases}$$

where $N_C(x)$ is the normal cone of C at x. We know from Lemma 2.10 that T is a maximal monotone mapping. For arbitrary $(x,u) \in G(T)$, we have $u \in T(x) = F(x) + N_C(x)$. Equivalently,

$$\langle u - F(x), z - x\rangle \leq 0,\ \forall z \in C.$$

Setting $z = \hat{x}$, we get

$$\langle u - F(x), x - \hat{x}\rangle \geq 0.$$

and then

$$\begin{aligned}\langle u, x-\hat{x}\rangle &\geq \langle F(x), x-\hat{x}\rangle \\ &= \langle F(x), x-y_{k_j}\rangle + \langle F(x), y_{k_j}-\hat{x}\rangle \\ &= \langle F(x)-F(y_{k_j}), x-y_{k_j}\rangle + \langle F(y_{k_j})-F(\omega_{k_j}), x-y_{k_j}\rangle \\ &\quad + \langle F(\omega_{k_j}), x-y_{k_j}\rangle + \langle F(x), y_{k_j}-\hat{x}\rangle \\ &\overset{(g)}{\geq} \langle F(y_{k_j})-F(\omega_{k_j}), x-y_{k_j}\rangle + \langle F(\omega_{k_j}), x-y_{k_j}\rangle + \langle F(x), y_{k_j}-\hat{x}\rangle,\end{aligned} \tag{3.5}$$

where (g) holds due to the monotonicity of F. By the definition of y_{k_j} and Lemma 2.6(i), we have

$$\langle y_{k_j} - \omega_{k_j} + \lambda_{k_j}F(\omega_{k_j}), x - y_{k_j}\rangle \geq 0,\ \forall x \in C_{k_j}.$$

Since $C \subseteq C_{k_j}$, we can deduce

$$\langle F(\omega_{k_j}), x-y_{k_j}\rangle \geq \frac{1}{\lambda_{k_j}}\langle \omega_{k_j}-y_{k_j}, x-y_{k_j}\rangle,\ \forall x \in C. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain

$$\begin{aligned}\langle u, x-\hat{x}\rangle &\geq \langle F(y_{k_j})-F(\omega_{k_j}), x-y_{k_j}\rangle + \frac{1}{\lambda_{k_j}}\langle \omega_{k_j}-y_{k_j}, x-y_{k_j}\rangle + \langle F(x), y_{k_j}-\hat{x}\rangle \\ &\geq -\|F(y_{k_j})-F(\omega_{k_j})\|\|x-y_{k_j}\| - \frac{1}{\lambda_{k_j}}\|\omega_{k_j}-y_{k_j}\|\|x-y_{k_j}\| + \langle F(x), y_{k_j}-\hat{x}\rangle \\ &\geq -L\|\omega_{k_j}-y_{k_j}\|\|x-y_{k_j}\| - \frac{1}{\lambda_{k_j}}\|\omega_{k_j}-y_{k_j}\|\|x-y_{k_j}\| + \langle F(x), y_{k_j}-\hat{x}\rangle.\end{aligned} \tag{3.7}$$

By virtue of $\|\omega_{k_j} - y_{k_j}\| \to 0$, the boundedness of y_k and (3.4), taking $j \to +\infty$ in (3.7), we have

$$\langle u, x - \hat{x}\rangle \geq 0.$$

Then by the maximality of T, we know $0 \in T(\hat{x})$. Thanks to Lemma 2.10, we have $\hat{x} \in SOL(C,F)$. □

Theorem 3.7. *Under Assumption 3.1, 3.2 and 3.3, the sequence $\{x_k\}$ generated by the Algorithm 3.1 converges weakly to a point in $SOL(C,F)$.*

Proof. For any $x^* \in SOL(C,F)$, we have

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &\overset{(h)}{\leq} \|w_k - x^*\|^2 \\ &= \|x_k + \alpha_k(x_k - x_{k-1}) - x^*\|^2 \\ &= \|(1+\alpha_k)(x_k - x^*) - \alpha_k(x_{k-1}-x^*)\|^2 \\ &\overset{(i)}{=} (1+\alpha_k)\|x_k - x^*\|^2 - \alpha_k\|x_{k-1}-x^*\|^2 + (1+\alpha_k)\alpha_k\|x_k - x_{k-1}\|^2 \\ &\overset{(j)}{\leq} \|x_k - x^*\|^2 + \alpha_k(\|x_k - x^*\|^2 - \|x_{k-1}-x^*\|^2) + (1+\alpha)\alpha_k\|x_k - x_{k-1}\|^2,\end{aligned}$$

where (h) holds because of Lemma 3.5 and $\mu \in (0,1)$, (i) follows from Lemma 2.7, and (j) holds due to Remark 3.3(i). Letting $\varphi_k := \|x_k - x^*\|^2$ and $\delta_k := (1+\alpha)\alpha_k\|x_k - x_{k-1}\|^2$, by Remark 3.3(ii), we have $\sum_{k=1}^{+\infty} \delta_k < +\infty$. Applying Lemma 2.9, we obtain $\lim_{k\to+\infty} \|x_k - x^*\|^2$ exists and

$$\sum_{k=1}^{+\infty}[\|x_k - x^*\|^2 - \|x_{k-1} - x^*\|^2]_+ < +\infty.$$

Therefore, the sequence $\{x_k\}$ is bounded and

$$\lim_{k\to+\infty}[\|x_k - x^*\|^2 - \|x_{k-1} - x^*\|^2]_+ = 0. \tag{3.8}$$

By the definition of ω_k, we get

$$0 \le \|\omega_k - x_k\|^2 = \alpha_k^2\|x_k - x_{k-1}\|^2 \overset{(k)}{\le} \alpha \cdot \alpha_k\|x_k - x_{k-1}\|^2 \overset{(l)}{\to} 0 \ (k \to +\infty),$$

where (k) holds because of Remark 3.3(i) and (l) follows from Remark 3.3(ii). And then we obtain

$$\|\omega_k - x_k\| \to 0 \ (k \to +\infty). \tag{3.9}$$

This implies that the sequence $\{\omega_k\}$ is bounded thanks to the boundedness of $\{x_k\}$. We also have

$$\begin{aligned}
0 \le (1-\mu^2)\|\omega_k - y_k\|^2 &\overset{(m)}{\le} \|\omega_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 \\
&\overset{(n)}{=} (1+\alpha_k)\|x_k - x^*\|^2 - \alpha_k\|x_{k-1} - x^*\|^2 \\
&\quad + (1+\alpha_k)\alpha_k\|x_k - x_{k-1}\|^2 - \|x_{k+1} - x^*\|^2 \\
&\overset{(o)}{\le} \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 + (1+\alpha)\alpha_k\|x_k - x_{k-1}\|^2 \\
&\quad + \alpha_k(\|x_k - x^*\|^2 - \|x_{k-1} - x^*\|^2) \\
&\le \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 + (1+\alpha)\alpha_k\|x_k - x_{k-1}\|^2 \\
&\quad + \alpha_k[\|x_k - x^*\|^2 - \|x_{k-1} - x^*\|^2]_+ \overset{(p)}{\to} 0 \ (k \to +\infty),
\end{aligned}$$

where (m) follows from Lemma 3.5, (n) holds due to the definition of $\{\omega_k\}$ and Lemma 2.7, (o) follows from Remark 3.3(i) and (p) holds thanks to Remark 3.3(ii) and (3.8). Thus,

$$\|w_k - y_k\| \to 0 \ (k \to +\infty). \tag{3.10}$$

Due to the boundedness of $\{x_k\}$, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j} \rightharpoonup \bar{x} \in \mathcal{H}$. Since (3.9), we also have $\omega_{k_j} \rightharpoonup \bar{x}$ $(j \to +\infty)$. Combining (3.10) with Lemma 3.6, we get $\bar{x} \in SOL(C,F)$.

Therefore, we prove that:

(i) For any $x^* \in SOL(C,F)$, $\lim_{k\to+\infty} \|x_k - x^*\|$ exists;

(ii) Every sequential weak cluster point of $\{x_k\}$ belongs to $SOL(C,F)$.

By Lemma 2.11, the sequence $\{x_k\}$ converges weakly to an element of $SOL(C,F)$. This completes the proof. □

3.2 Convergence rate of Algorithm 3.1

In this section, we establish the convergence rate of Algorithm 3.1. Our result is based on $\omega_k \neq y_k$. Otherwise, y_k is a solution of $VI(C, F)$.

Theorem 3.8. *Suppose that Assumption 3.1, 3.2 and 3.3 hold. Let the sequences* $\{x_k\}$, $\{\omega_k\}$ *and* $\{y_k\}$ *be generated by Algorithm 3.1. Then for any* $x^* \in SOL(C, F)$*, there exist constants* M_3, $\epsilon > 0$ *such that the following estimate holds:*

$$\min_{1 \le i \le k} \|\omega_i - y_i\|^2 \le \frac{(1-\alpha)\|x_1 - x^*\|^2 + [\|x_1 - x^*\|^2 - \|x_0 - x^*\|^2]_+ + M_3}{\epsilon(1-\alpha)k}.$$

Proof. Letting ϵ satisfies $0 < \epsilon < 1 - \mu^2$ thanks to $0 < \mu < 1$. Then,

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &\overset{(q)}{\le} \|w_k - x^*\|^2 - (1-\mu^2)\|w_k - y_k\|^2 \\ &\le \|w_k - x^*\|^2 - \epsilon\|w_k - y_k\|^2, \end{aligned} \tag{3.11}$$

where (q) follows from Lemma 3.5. Therefore,

$$\begin{aligned} \epsilon\|\omega_k - y_k\|^2 &\le \|w_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 \\ &\overset{(r)}{=} (1+\alpha_k)\|x_k - x^*\|^2 - \alpha_k\|x_{k-1} - x^*\|^2 + (1+\alpha_k)\alpha_k\|x_k - x_{k-1}\|^2 \\ &\quad - \|x_{k+1} - x^*\|^2 \\ &\le \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 + \alpha_k[\|x_k - x^*\|^2 - \|x_{k-1} - x^*\|^2]_+ \\ &\quad + (1+\alpha_k)\alpha_k\|x_k - x_{k-1}\|^2 \\ &\overset{(s)}{\le} \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 + \alpha[\|x_k - x^*\|^2 - \|x_{k-1} - x^*\|^2]_+ \\ &\quad + (1+\alpha)\alpha_k\|x_k - x_{k-1}\|^2, \end{aligned}$$

where (r) holds because of the definition of ω_k and Lemma 2.7 and (s) follows from Remark 3.3(i). Letting $\Upsilon_k := \|x_k - x^*\|^2$ and $\Delta_k := (1+\alpha)\alpha_k\|x_k - x_{k-1}\|^2$, we obtain

$$\epsilon\|w_k - y_k\|^2 \le \Upsilon_k - \Upsilon_{k+1} + \alpha[\Upsilon_k - \Upsilon_{k-1}]_+ + \Delta_k.$$

Hence, we have

$$\begin{aligned} \epsilon\sum_{i=1}^{k}\|\omega_i - y_i\|^2 &\le \sum_{i=1}^{k}(\Upsilon_i - \Upsilon_{i+1}) + \alpha\sum_{i=1}^{k}[\Upsilon_i - \Upsilon_{i-1}]_+ + \sum_{i=1}^{k}\Delta_i \\ &\overset{(t)}{\le} \Upsilon_1 - \Upsilon_{k+1} + \alpha\sum_{i=1}^{k}[\Upsilon_i - \Upsilon_{i-1}]_+ + M_3 \\ &\overset{(u)}{\le} \Upsilon_1 + \alpha\sum_{i=1}^{k}[\Upsilon_i - \Upsilon_{i-1}]_+ + M_3 \\ &= \Upsilon_1 + \alpha\sum_{i=1}^{k}[\Upsilon_{i+1} - \Upsilon_i]_+ + \alpha[\Upsilon_1 - \Upsilon_0]_+ - \alpha[\Upsilon_{k+1} - \Upsilon_k]_+ + M_3 \\ &\le \Upsilon_1 + \alpha\sum_{i=1}^{k}[\Upsilon_{i+1} - \Upsilon_i]_+ + [\Upsilon_1 - \Upsilon_0]_+ + M_3, \end{aligned} \tag{3.12}$$

where (t) follows from Remark 3.3(ii) and $M_3 = (1+\alpha)M_1$ and (u) holds due to $\Upsilon_{k+1} \geq 0$. On the other hand, by (3.11), we get

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &\leq \|\omega_k - x^*\|^2\\ &= (1+\alpha_k)\|x_k - x^*\|^2 - \alpha_k\|x_{k-1} - x^*\|^2 + (1+\alpha_k)\alpha_k\|x_k - x_{k-1}\|^2\\ &\leq \|x_k - x^*\|^2 + \alpha_k(\|x_k - x^*\|^2 - \|x_{k-1} - x^*\|^2) + (1+\alpha_k)\alpha_k\|x_k - x_{k-1}\|^2\\ &\leq \|x_k - x^*\|^2 + \alpha[\|x_k - x^*\|^2 - \|x_{k-1} - x^*\|^2]_+ + (1+\alpha)\alpha_k\|x_k - x_{k-1}\|^2.\end{aligned}$$

or equivalently:

$$\Upsilon_{k+1} - \Upsilon_k \leq \alpha[\Upsilon_k - \Upsilon_{k-1}]_+ + \Delta_k.$$

Therefore,

$$[\Upsilon_{k+1} - \Upsilon_k]_+ \leq \alpha[\Upsilon_k - \Upsilon_{k-1}]_+ + \Delta_k \leq \alpha^k[\Upsilon_1 - \Upsilon_0]_+ + \sum_{i=1}^{k}\alpha^{i-1}\Delta_{k+1-i}.$$

Then, we have

$$\begin{aligned}\sum_{k=1}^{+\infty}[\Upsilon_{k+1} - \Upsilon_k]_+ &\leq \sum_{k=1}^{+\infty}\alpha^k[\Upsilon_1 - \Upsilon_0]_+ + \sum_{k=1}^{+\infty}\sum_{i=1}^{k}\alpha^{i-1}\Delta_{k+1-i}\\ &\leq \frac{\alpha}{1-\alpha}[\Upsilon_1 - \Upsilon_0]_+ + \frac{1}{1-\alpha}\sum_{k=1}^{+\infty}\Delta_k\\ &\leq \frac{\alpha}{1-\alpha}[\Upsilon_1 - \Upsilon_0]_+ + \frac{1}{1-\alpha}M_3.\end{aligned}$$

In light of (3.12) and the above inequality, we have

$$\begin{aligned}\epsilon\sum_{i=1}^{k}\|\omega_i - y_i\|^2 &\leq \Upsilon_1 + \alpha(\frac{\alpha}{1-\alpha}[\Upsilon_1 - \Upsilon_0]_+ + \frac{1}{1-\alpha}M_3) + [\Upsilon_1 - \Upsilon_0]_+ + M_3\\ &= \Upsilon_1 + \frac{\alpha^2 - \alpha + 1}{1-\alpha}[\Upsilon_1 - \Upsilon_0]_+ + \frac{1}{1-\alpha}M_3\\ &\leq \Upsilon_1 + \frac{1}{1-\alpha}[\Upsilon_1 - \Upsilon_0]_+ + \frac{1}{1-\alpha}M_3\\ &= \|x_1 - x^*\|^2 + \frac{1}{1-\alpha}[\|x_1 - x^*\|^2 - \|x_0 - x^*\|^2]_+ + \frac{1}{1-\alpha}M_3.\end{aligned}$$

which implies

$$\min_{1\leq i\leq k}\|\omega_i - y_i\|^2 \leq \frac{(1-\alpha)\|x_1 - x^*\|^2 + [\|x_1 - x^*\|^2 - \|x_0 - x^*\|^2]_+ + M_3}{\epsilon(1-\alpha)k}.$$

This completes the proof. □

4 Strong Convergence Algorithm

In this section, we give a strong convergence algorithm for (1.1). Let $f : \mathcal{H} \to \mathcal{H}$ be a contraction mapping with a coefficient $\rho \in [0, 1)$, then we introduce the following algorithm:

Algorithm 4.1 (Strong convergence algorithm for (1.1))**. Initialization:** Let $x_0,\ x_1 \in \mathcal{H}$ and $\sigma > 0,\ \gamma \in (0,1),\ M = M'L_1,\ \mu \in (0,1),\ \alpha \geq 0$. Choose three positive sequences $\{\varepsilon_k\} \subseteq [0,+\infty)$ and $\{\beta_k\} \subseteq (0,1)$ satisfying

$$\lim_{k\to+\infty} \beta_k = 0, \quad \sum_{k=1}^{+\infty} \beta_k = +\infty, \quad \sum_{k=1}^{+\infty} \varepsilon_k < +\infty, \quad \varepsilon_k = o(\beta_k).$$

Iterative Steps: Calculate x_{k+1} as follows:

Step 1. Assume that $x_{k-1},\ x_k \in \mathcal{H}$ for each $k \geq 1$, Set

$$\omega_k = x_k + \alpha_k(x_k - x_{k-1}),$$

where

$$\alpha_k = \begin{cases} \min\{\frac{\varepsilon_k}{\|x_k - x_{k-1}\|}, \alpha\}, & \text{if } x_k \neq x_{k-1}; \\ \alpha, & \text{otherwise.} \end{cases}$$

Step 2. Construct the half-space

$$C_k := \{x \in \mathcal{H} | c(\omega_k) + \langle c'(\omega_k), x - \omega_k\rangle \leq 0\}.$$

Compute

$$y_k = P_{C_k}(\omega_k - \lambda_k F(\omega_k)),$$

where $\lambda_k = \sigma\gamma^{l_k}$, with l_k is the smallest nonnegative integer l satisfying

$$(\sigma\gamma^l)^2\|F(\omega_k) - F(y_k)\|^2 + 2M\sigma\gamma^l\|\omega_k - y_k\|^2 \leq \mu^2\|\omega_k - y_k\|^2.$$

If $y_k = \omega_k$ then stop and $y_k \in SOL(C, F)$. Otherwise, go to **Step 3.**

Step 3. Compute

$$z_k = y_k + \lambda_k(F(\omega_k) - F(y_k)).$$

Step 4. Calculate the next iterate,

$$x_{k+1} = \beta_k f(z_k) + (1 - \beta_k)z_k.$$

Set $k \leftarrow k + 1$ and go to **Step 1.**

Remark 4.2. In Algorithm 4.1, the following conclusions hold:

(i) $\lim\limits_{k\to+\infty} \frac{\alpha_k}{\beta_k}\|x_k - x_{k-1}\| = 0$;

(ii) $\exists M_4 > 0,\ \frac{\alpha_k}{\beta_k}\|x_k - x_{k-1}\| \leq M_4$.

Indeed, by the definition of $\{\alpha_k\}$, we have $\alpha_k\|x_k - x_{k-1}\| \leq \varepsilon_k$ for all k, which implies that

$$\lim_{k\to+\infty} \frac{\alpha_k}{\beta_k}\|x_k - x_{k-1}\| \leq \lim_{k\to+\infty} \frac{\varepsilon_k}{\beta_k} = 0.$$

Thus (i) and (ii) hold.

Remark 4.3. By the similar argument of Remark 3.4, we have that the line search is well defined.

We still assumed that Algorithm 4.1 generates infinite sequences. First, we introduce the following lemma. The proof of this lemma is the same as that of Lemma 3.5 and Lemma 3.6. Here we omit it.

Lemma 4.4. *Suppose that Assumption 3.1, 3.2 and 3.3 hold. Let $\{z_k\}$, $\{\omega_k\}$ and $\{y_k\}$ be sequences generated by Algorithm 4.1. Then the following results hold:*

(i) $\|z_k - x^*\|^2 \le \|\omega_k - x^*\|^2 - (1-\mu^2)\|\omega_k - y_k\|^2,\ \forall x^* \in SOL(C,F),\ k \in \mathbb{N}^+$.

(ii) *If there exists a subsequence $\{\omega_{k_j}\}$ of $\{\omega_k\}$ such that $\omega_{k_j} \rightharpoonup \hat{x}$ and $\lim_{j\to+\infty}\|\omega_{k_j} - y_{k_j}\| = 0$, then $\hat{x} \in SOL(C,F)$.*

Next, we show that the sequences generated by the Algorithm 4.1 is bounded.

Lemma 4.5. *Suppose that Assumption 3.1, 3.2 and 3.3 hold. Let $f : \mathcal{H} \to \mathcal{H}$ be a contraction mapping with a coefficient $\rho \in [0,1)$. Then the sequences $\{x_k\}$, $\{\omega_k\}$, $\{z_k\}$ and $\{f(x_k)\}$ generated by Algorithm 4.1 are bounded.*

Proof. Let $x^* \in SOL(C,F)$, we have

$$\begin{aligned} \|z_k - x^*\| &\overset{(v)}{\le} \|\omega_k - x^*\| \\ &= \|x_k + \alpha_k(x_k - x_{k-1}) - x^*\| \\ &\le \|x_k - x^*\| + \beta_k \cdot \frac{\alpha_k}{\beta_k}\|x_k - x_{k-1}\| \\ &\overset{(w)}{\le} \|x_k - x^*\| + \beta_k M_4, \end{aligned} \tag{4.1}$$

where (v) follows from Lemma 4.4(i) and $\mu \in (0,1)$, and (w) holds due to Remark 4.2(ii). From the definition of x_{k+1}, we have

$$\begin{aligned} \|x_{k+1} - x^*\| &= \|\beta_k f(z_k) + (1-\beta_k)z_k - x^*\| \\ &\overset{(x)}{\le} \beta_k\|f(z_k) - x^*\| + (1-\beta_k)\|z_k - x^*\| \\ &\overset{(y)}{\le} \beta_k\|f(z_k) - f(x^*)\| + \beta_k\|f(x^*) - x^*\| + (1-\beta_k)\|z_k - x^*\| \\ &\le \beta_k\rho\|z_k - x^*\| + \beta_k\|f(x^*) - x^*\| + (1-\beta_k)\|z_k - x^*\| \\ &= (1-(1-\rho)\beta_k)\|z_k - x^*\| + \beta_k\|f(x^*) - x^*\| \\ &\overset{(z)}{\le} (1-(1-\rho)\beta_k)(\|x_k - x^*\| + \beta_k M_4) + \beta_k\|f(x^*) - x^*\| \\ &\overset{(a_1)}{\le} (1-(1-\rho)\beta_k)\|x_k - x^*\| + \beta_k(M_4 + \|f(x^*) - x^*\|) \\ &= (1-(1-\rho)\beta_k)\|x_k - x^*\| + (1-\rho)\beta_k\frac{M_4 + \|f(x^*) - x^*\|}{(1-\rho)} \\ &\le \max\{\|x_k - x^*\|, \frac{M_4 + \|f(x^*) - x^*\|}{(1-\rho)}\} \\ &\le \max\{\|x_0 - x^*\|, \frac{M_4 + \|f(x^*) - x^*\|}{(1-\rho)}\}, \end{aligned}$$

where (x) and (y) hold by the triangle inequality, (z) folllows from (4.1), and (a_1) holds due to $\rho \in [0,1)$ and $\beta_k \in (0,1)$. Therefore, $\{x_k\}$ is bounded.

Using (4.1), we also have $\{z_k\}$ and $\{\omega_k\}$ are bounded thanks to the boundedness of $\{x_k\}$ and $\{\beta_k\}$. Moreover, $\{f(x_k)\}$ is bounded by the fact that f is a contraction mapping. This completes the proof. □

Then, we give the following two lemmas which are important to the proof of strong convergence of Algorithm 4.1.

Lemma 4.6. *Suppose that Assumption 3.1, 3.2 and 3.3 hold. Let $f : \mathcal{H} \to \mathcal{H}$ be a contraction mapping with a coefficient $\rho \in [0,1)$ and $\{\beta_k\}$ be defined in Algorithm 4.1. Then for all $x^* \in SOL(C,F)$, there exists $M_5 > 0$ such that*

$$(1-\mu^2)\|\omega_k - y_k\|^2 \leq \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 + \beta_k M_5, \ \forall k \in \mathbb{N}^+.$$

Proof. According to $x^* \in SOL(C,F)$ and the definition of x_{k+1}, we have

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 &= \|\beta_k f(z_k) + (1-\beta_k)z_k - x^*\|^2 \\
&\overset{(b_1)}{\leq} \beta_k \|f(z_k) - x^*\|^2 + (1-\beta_k)\|z_k - x^*\|^2 \\
&\leq \beta_k (\|f(z_k) - f(x^*)\| + \|f(x^*) - x^*\|)^2 + (1-\beta_k)\|z_k - x^*\|^2 \\
&\leq \beta_k (\rho\|z_k - x^*\| + \|f(x^*) - x^*\|)^2 + (1-\beta_k)\|z_k - x^*\|^2 \\
&\leq \beta_k (\|z_k - x^*\| + \|f(x^*) - x^*\|)^2 + (1-\beta_k)\|z_k - x^*\|^2 \\
&= \beta_k \|z_k - x^*\|^2 + \beta_k(2\|z_k - x^*\|\|f(x^*) - x^*\| + \|f(x^*) - x^*\|^2) \\
&\quad + (1-\beta_k)\|z_k - x^*\|^2 \\
&\leq \|z_k - x^*\|^2 + \beta_k M_6 \\
&\overset{(c_1)}{\leq} \|\omega_k - x^*\|^2 - (1-\mu^2)\|\omega_k - y_k\|^2 + \beta_k M_6 \\
&\overset{(d_1)}{\leq} (\|x_k - x^*\| + \beta_k M_4)^2 - (1-\mu^2)\|\omega_k - y_k\|^2 + \beta_k M_6 \\
&\leq \|x_k - x^*\|^2 + \beta_k M_7 - (1-\mu^2)\|\omega_k - y_k\|^2 + \beta_k M_6,
\end{aligned}$$

where (b_1) follows from Lemma 2.7, (c_1) holds due to Lemma 4.4(i), (d_1) holds thanks to (4.1), $M_6 := \sup_{k\in\mathbb{N}^+}\{2\|z_k - x^*\|\|f(x^*) - x^*\| + \|f(x^*) - x^*\|^2\}$ and $M_7 := \sup_{k\in\mathbb{N}^+}\{2M_4\|x_k - x^*\| + \beta_k M_4^2\}$. By the boundedness of $\{x_k\}$ and $\{z_k\}$, we get $M_6 \in (0,+\infty)$, $M_7 \in (0,+\infty)$. Letting $M_5 := M_6 + M_7$, then

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - (1-\mu^2)\|\omega_k - y_k\|^2 + \beta_k M_5.$$

Therefore, we get

$$(1-\mu^2)\|\omega_k - y_k\|^2 \leq \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 + \beta_k M_5.$$

This completes the proof. □

Lemma 4.7. *Under the conditions of Lemma 4.6, there exists $M_8 > 0$ such that*

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 \leq{}& (1-(1-\rho)\beta_k)\|x_k - x^*\|^2 + (1-\rho)\beta_k \\
&\cdot \Big(\frac{M_8}{1-\rho}\cdot\frac{\alpha_k}{\beta_k}\|x_k - x_{k-1}\| + \frac{2}{1-\rho}\langle f(x^*) - x^*, x_{k+1} - x^*\rangle\Big).
\end{aligned}$$

Proof. For all $x^* \in SOL(C,F)$, we have

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 &= \|\beta_k f(z_k) + (1-\beta_k) z_k - x^*\|^2 \\
&= \|\beta_k (f(z_k) - f(x^*)) + (1-\beta_k)(z_k - x^*) + \beta_k (f(x^*) - x^*)\|^2 \\
&\overset{(e_1)}{\leq} \|\beta_k (f(z_k) - f(x^*)) + (1-\beta_k)(z_k - x^*)\|^2 + 2\beta_k \langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&\overset{(f_1)}{\leq} \beta_k \|f(z_k) - f(x^*)\|^2 + (1-\beta_k)\|z_k - x^*\|^2 + 2\beta_k \langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&\leq \beta_k \rho^2 \|z_k - x^*\|^2 + (1-\beta_k)\|z_k - x^*\|^2 + 2\beta_k \langle f(x^*) - x^*, x_{k+1} - x^*\rangle \qquad (4.2) \\
&\overset{(g_1)}{\leq} \beta_k \rho \|z_k - x^*\|^2 + (1-\beta_k)\|z_k - x^*\|^2 + 2\beta_k \langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&= (1 - (1-\rho)\beta_k)\|z_k - x^*\|^2 + 2\beta_k \langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&\overset{(h_1)}{\leq} (1 - (1-\rho)\beta_k)\|\omega_k - x^*\|^2 + 2\beta_k \langle f(x^*) - x^*, x_{k+1} - x^*\rangle,
\end{aligned}$$

where (e_1) holds according to $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$, (f_1) follows from Lemma 2.7, (g_1) holds due to $\rho \in [0,1)$, and (h_1) holds because (4.1). According to the definition of ω_k, we deduce

$$\begin{aligned}
\|\omega_k - x^*\|^2 &= \|x_k + \alpha_k (x_k - x_{k-1}) - x^*\|^2 \\
&= \|x_k - x^*\|^2 + 2\alpha_k \|x_k - x^*\|\|x_k - x_{k-1}\| + \alpha_k^2 \|x_k - x_{k-1}\|^2.
\end{aligned}$$

Combining (4.2) and the above equality, we obtain

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 &\leq (1 - (1-\rho)\beta_k)(\|x_k - x^*\|^2 + 2\alpha_k \|x_k - x^*\|\|x_k - x_{k-1}\| + \alpha_k^2 \|x_k - x_{k-1}\|^2) \\
&\quad + 2\beta_k \langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&\overset{(i_1)}{\leq} (1 - (1-\rho)\beta_k)\|x_k - x^*\|^2 + 2\alpha_k \|x_k - x^*\|\|x_k - x_{k-1}\| + \alpha_k^2 \|x_k - x_{k-1}\|^2 \\
&\quad + 2\beta_k \langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&= (1 - (1-\rho)\beta_k)\|x_k - x^*\|^2 + \alpha_k \|x_k - x_{k-1}\|(2\|x_k - x^*\| + \alpha_k \|x_k - x_{k-1}\|) \\
&\quad + 2\beta_k \langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&\overset{(j_1)}{\leq} (1 - (1-\rho)\beta_k)\|x_k - x^*\|^2 + \alpha_k \|x_k - x_{k-1}\|(2\|x_k - x^*\| + \alpha \|x_k - x_{k-1}\|) \\
&\quad + 2\beta_k \langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&\leq (1 - (1-\rho)\beta_k)\|x_k - x^*\|^2 + \alpha_k \|x_k - x_{k-1}\| M_8 + 2\beta_k \langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&= (1 - (1-\rho)\beta_k)\|x_k - x^*\|^2 + (1-\rho)\beta_k \\
&\quad \cdot \Big(\frac{M_8}{1-\rho} \cdot \frac{\alpha_k}{\beta_k}\|x_k - x_{k-1}\| + \frac{2}{1-\rho}\langle f(x^*) - x^*, x_{k+1} - x^*\rangle\Big),
\end{aligned}$$

where (i_1) holds because $\beta_k \in (0,1)$ and $\rho \in [0,1)$, (j_1) follows from $0 \leq \alpha_k \leq \alpha$ and $M_8 := \sup_{k \in \mathbb{N}^+} \{2\|x_k - x^*\| + \alpha \|x_k - x_{k-1}\|\}$. Since $\{x_k\}$ is bounded, we know $M_8 \in (0, +\infty)$. This completes the proof. □

Finally, we prove the strong convergence of Algorithm 4.1.

Theorem 4.8. *Suppose that Assumption 3.1, 3.2 and 3.3 hold. Let $f : \mathcal{H} \to \mathcal{H}$ be a contraction mapping with a coefficient $\rho \in [0,1)$. Then, the sequence $\{x_k\}$ generated by Algorithm 4.1 converges strongly to an element $x^* \in SOL(C,F)$, where $x^* = P_{SOL(C,F)}(f(x^*))$.*

Proof. Letting $\Phi_k := \|x_k - x^*\|^2$, $s_k := (1-\rho)\beta_k$ and $\Omega_k := \frac{M_8}{1-\rho}\cdot\frac{\alpha_k}{\beta_k}\|x_k - x_{k-1}\| + \frac{2}{1-\rho}\langle f(x^*) - x^*, x_{k+1} - x^*\rangle$, by Lemma 4.7, we obtain that

$$\Phi_{k+1} \leq (1 - s_k)\Phi_k + s_k\Omega_k,$$

where $s_k \in (0,1)$ and $\sum_{k=1}^{+\infty} s_k = +\infty$. In order to prove $\Phi_k = \|x_k - x^*\| \to 0$, by Lemma 2.12, we only need to show $\limsup_{j\to+\infty} \Omega_{k_j} \leq 0$ for every subsequence $\{\Phi_{k_j}\}$ of $\{\Phi_k\}$ satisfying $\liminf_{j\to+\infty}(\Phi_{k_j+1} - \Phi_{k_j}) \geq 0$. For this, suppose that there exists a subsequence $\{\|x_{k_j} - x^*\|\}$ of $\{\|x_k - x^*\|\}$ such that

$$\liminf_{j\to+\infty}(\|x_{k_j+1} - x^*\| - \|x_{k_j} - x^*\|) \geq 0. \tag{4.3}$$

Thus,

$$\begin{aligned}
0 &\leq \limsup_{j\to+\infty}(1-\mu^2)\|\omega_{k_j} - y_{k_j}\|^2 \\
&\overset{(k_1)}{\leq} \limsup_{j\to+\infty}[\|x_{k_j} - x^*\|^2 - \|x_{k_j+1} - x^*\|^2 + \beta_{k_j}M_5] \\
&\leq \limsup_{j\to+\infty}[\|x_{k_j} - x^*\|^2 - \|x_{k_j+1} - x^*\|^2] + \lim_{j\to+\infty}\beta_{k_j}M_5 \\
&= -\liminf_{j\to+\infty}[\|x_{k_j+1} - x^*\|^2 - \|x_{k_j} - x^*\|^2] \\
&= -\liminf_{j\to+\infty}[(\|x_{k_j+1} - x^*\| - \|x_{k_j} - x^*\|)(\|x_{k_j+1} - x^*\| + \|x_{k_j} - x^*\|)] \\
&\overset{(l_1)}{\leq} 0,
\end{aligned}$$

where (k_1) follows from Lemma 4.6, and (l_1) holds due to (4.3) and the boundedness of $\{x_k\}$. This implies that

$$\lim_{j\to+\infty}\|\omega_{k_j} - y_{k_j}\| = 0. \tag{4.4}$$

Therefore,

$$\begin{aligned}
\|z_{k_j} - \omega_{k_j}\| &= \|y_{k_j} + \lambda_{k_j}(F(\omega_{k_j}) - F(y_{k_j})) - \omega_{k_j}\| \\
&\leq \|y_{k_j} - \omega_{k_j}\| + \lambda_{k_j}\|F(\omega_{k_j}) - F(y_{k_j})\| \overset{(m_1)}{\to} 0 \ (j \to +\infty),
\end{aligned} \tag{4.5}$$

where (m_1) holds due to (4.4) and Lipschitz continuity of F. Furthermore, according to the definition of x_{k+1}, $\lim_{k\to+\infty}\beta_k = 0$ and the boundedness of $\{f(z_k)\}$ and $\{z_k\}$, we get

$$\|x_{k_j+1} - z_{k_j}\| = \|\beta_{k_j}f(z_{k_j}) + (1-\beta_{k_j})z_{k_j} - z_{k_j}\| = \beta_{k_j}\|f(z_{k_j}) - z_{k_j}\| \to 0 \ (j \to +\infty). \tag{4.6}$$

From the definition of ω_k, we obtain

$$\|\omega_{k_j} - x_{k_j}\| = \|x_{k_j} + \alpha_{k_j}(x_{k_j} - x_{k_j-1}) - x_{k_j}\| = \beta_{k_j}\cdot\frac{\alpha_{k_j}}{\beta_{k_j}}\|x_{k_j} - x_{k_j-1}\| \overset{(n_1)}{\to} 0 \ (j \to +\infty), \tag{4.7}$$

where (n_1) follows from the boundedness of $\{\beta_k\}$ and Remark 4.2(i). Therefore, we have

$$\|x_{k_j+1} - x_{k_j}\| \leq \|x_{k_j+1} - z_{k_j}\| + \|z_{k_j} - \omega_{k_j}\| + \|\omega_{k_j} - x_{k_j}\| \to 0 \ (j \to +\infty), \tag{4.8}$$

Since the sequence $\{x_{k_j}\}$ is bounded, it follows that there exists a subsequence $\{x_{k_{j_i}}\}$ of $\{x_{k_j}\}$ converging weakly to a point $\bar{x} \in \mathcal{H}$ such that

$$\limsup_{j\to+\infty}\langle f(x^*) - x^*, x_{k_j} - x^*\rangle = \lim_{i\to+\infty}\langle f(x^*) - x^*, x_{k_{j_i}} - x^*\rangle = \langle f(x^*) - x^*, \bar{x} - x^*\rangle. \quad (4.9)$$

From (4.7), we obtain $\omega_{k_{j_i}} \rightharpoonup \bar{x}$ $(i \to +\infty)$. Combining (4.4) and Lemma 4.4(ii), we get $\bar{x} \in SOL(C,F)$. Therefore,

$$\begin{aligned}
\limsup_{j\to+\infty}\Omega_{k_j} &= \limsup_{j\to+\infty}\Big(\frac{M_8}{1-\rho}\cdot\frac{\alpha_{k_j}}{\beta_{k_j}}\|x_{k_j} - x_{k_j-1}\| + \frac{2}{1-\rho}\langle f(x^*) - x^*, x_{k_j+1} - x^*\rangle\Big) \\
&\overset{(o_1)}{=} \frac{2}{1-\rho}\limsup_{j\to+\infty}\langle f(x^*) - x^*, x_{k_j+1} - x^*\rangle \\
&\leq \frac{2}{1-\rho}\limsup_{j\to+\infty}\langle f(x^*) - x^*, x_{k_j+1} - x_{k_j}\rangle + \frac{2}{1-\rho}\limsup_{j\to+\infty}\langle f(x^*) - x^*, x_{k_j} - x^*\rangle \\
&\overset{(p_1)}{=} \frac{2}{1-\rho}\limsup_{j\to+\infty}\langle f(x^*) - x^*, x_{k_j} - x^*\rangle \\
&\overset{(q_1)}{=} \frac{2}{1-\rho}\langle f(x^*) - x^*, \bar{x} - x^*\rangle \\
&\overset{(r_1)}{\leq} 0,
\end{aligned}$$

where (o_1) holds because of Remark 4.2(i), (p_1) follows from (4.8), (q_1) holds according to (4.9), and (r_1) is true in view of $x^* = P_{SOL(C,F)}f(x^*)$, $\bar{x} \in SOL(C,F)$ and Lemma 2.6(i). In view of Lemma 2.12 we have $\lim_{k\to+\infty}\|x_k - x^*\| = 0$. This completes the proof. □

5 Numerical Experiments

Numerical experiments will be presented in this section to illustrate the performance of our proposed methods. All programs are written in Matlab R2019a and performed on a PC Desktop Intel(R) Core(TM) i5-6200U CPU@2.30GHz 2.30GHz, RAM 4.00GB.

Example 5.1. Consider the linear mapping $F(x) = Kx$. The feasible set $C \subseteq \mathbb{R}^n$ is an ellipsoid in $\mathbb{R}^n$ defined as

$$C := \{x \in \mathbb{R}^n : (x-d)^T P(x-d) \leq r^2\},$$

where K and P are positive definite matrices, $d(\neq 0) \in \mathbb{R}^n$ and $r > 0$.

Define $c : \mathbb{R}^n \to \mathbb{R}$ by $c(x) = \frac{1}{2}[(x-d)^T P(x-d) - r^2]$, then C is a level set of c, i.e., $C = \{x \in \mathbb{R}^n : c(x) \leq 0\}$. It is easy to verify that $c'(x) = P(x-d)$. Obviously,

$$\|c'(x) - c'(y)\| = \|P(x-d) - P(y-d)\| = \|P(x-y)\| \leq \|P\|\|x-y\|,\ \forall x, y \in \mathbb{R}^n.$$

So, $c'(x)$ is a $\|P\|$-Lipschitz continuous mapping, i.e., $L_1 = \|P\|$. We use $\lambda_{\max}$ and $\lambda_{\min}$ to represent the maximum and minimum eigenvalues of P, respectively. Then,

$$\lambda_{\min}\|x-d\|^2 \leq (x-d)^T P(x-d) \leq \lambda_{\max}\|x-d\|^2,\ \forall x \in \mathbb{R}^n.$$

Note that $(x-d)^T P(x-d) - r^2 = 0$ holds for all $x \in \partial C$, we have

$$\frac{r}{\sqrt{\lambda_{\max}}} \leq \|x-d\| \leq \frac{r}{\sqrt{\lambda_{\min}}},\ \forall x \in \partial C.$$

Therefore, we get

$$\frac{\|F(x)\|}{\|c'(x)\|} = \frac{\|Kx\|}{\|P(x-d)\|} \le \frac{\|K\|(\|x-d\|+\|d\|)}{\lambda_{\min}\|x-d\|} \le \frac{\|K\|\sqrt{\lambda_{\max}}(r+\sqrt{\lambda_{\min}}\|d\|)}{\lambda_{\min}^{\frac{3}{2}} r}, \ \forall x \in \partial C.$$

This implies that

$$M' = \frac{\|K\|\sqrt{\lambda_{\max}}(r+\sqrt{\lambda_{\min}}\|d\|)}{\lambda_{\min}^{\frac{3}{2}} r}.$$

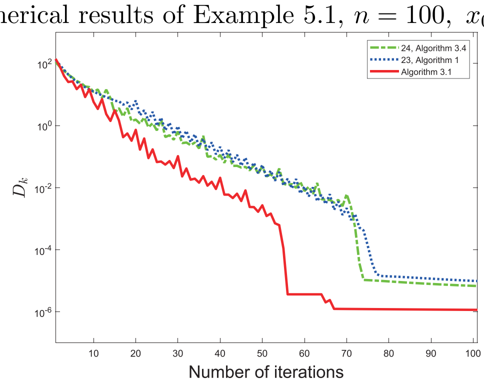
Hence, we obtain

$$M = M'L_1 = \frac{\|P\|\|K\|\sqrt{\lambda_{\max}}(r+\sqrt{\lambda_{\min}}\|d\|)}{\lambda_{\min}^{\frac{3}{2}} r}.$$

We compare [16, Algorithm 3.4],[15, Algorithm 1] with Algorithm 3.1, and use their corresponding parameters as following. The numerical experimental results are shown in Figure 1.

1. [16, Algorithm 3.4]: $\sigma = 1,\ \rho = 0.9,\ \nu = 0.91,\ D_k = \|x_k - y_k\|$.
2. [15, Algorithm 1]: $\xi = 1, \eta = 2,\ \theta = 0.99,\ \gamma = 1.5,\ D_k = \|x_k - y_k\|$.
3. Algorithm 3.1: $\sigma = 5,\ \gamma = 0.1,\ \mu = 0.9,\ \alpha = 0.6,\ D_k = \|\omega_k - y_k\|$.

Figure 1: Numerical results of Example 5.1, $n = 100,\ x_0 = x_1$ random.



Example 5.2. Consider $\mathcal{H} := L^2[0,1]$ with inner product

$$\langle x, y\rangle := \int_0^1 x(t)y(t)dt,$$

and induced norm

$$\|x\| := (\int_0^1 |x(t)|^2 dt)^{\frac{1}{2}}.$$

Define a mapping $F : \mathcal{H} \to \mathcal{H}$ by $F(x(t)) = \max\{0, x(t)\}, \forall x \in \mathcal{H}$. Denote by $C[0,1]$ the continuous function space defined on the interval [0,1] and choose an arbitrary fixed $\varphi \in C[0,1]$. Let $C := \{x \in \mathcal{H} : \|\varphi x\| \le 1\}$. It is very easy to verify that C is a nonempty closed convex subset of $\mathcal{H}$. In particular, if we choose $\varphi(t) = 1,\ \forall t \in [0,1]$, then C becomes the unit closed ball of $\mathcal{H}$, i.e., $C = \{x \in \mathcal{H} : \|x\| \le 1\}$.

Define $c : \mathcal{H} \to \mathbb{R}$ by $c(x) = \frac{1}{2}(\|\varphi x\|^2 - 1),\ \forall x \in \mathcal{H}$, then c is a convex function and C is a level set of c, i.e., $C = \{x \in \mathcal{H} : c(x) \le 0\}$. Also, it is easy to see that c is differentiable on $\mathcal{H}$ and $c'(x) = \varphi^2 x,\ \forall x \in \mathcal{H}$. In addition, for any $x, y \in \mathcal{H}$ we have

$$\|c'(x) - c'(y)\| = \|\varphi^2(x-y)\| \le (\max_{t\in[0,1]} |\varphi(t)|)^2\|x-y\|,$$

thus c is a $(\max_{t\in[0,1]} |\varphi(t)|)^2-$Lipschitz continuous mapping, i.e., $L_1 = (\max_{t\in[0,1]} |\varphi(t)|)^2$.

In practical calculation, we choose $\varphi(t) := e^{-t},\ \forall t \in [0,1]$. Then we get $L_1 = 1$. Define

$$x_+(t) = \begin{cases} x(t), & \text{if } x(t) \geq 0; \\ 0, & \text{if } x(t) < 0. \end{cases}$$

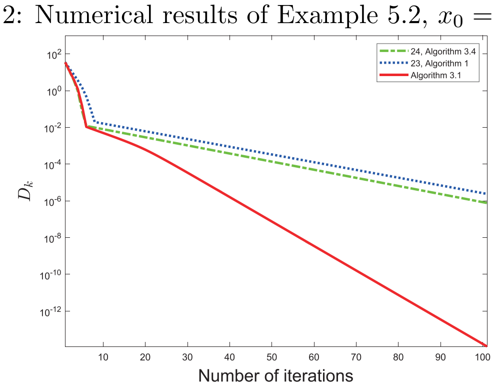
We deduce that

$$\frac{\|Fx\|}{\|c'(x)\|} = \frac{\|x_+(t)\|}{\|\varphi^2 x\|} \leq \frac{\|x\|}{(\min_{t\in[0,1]} |\varphi(t)|)^2 \|x\|} = e^2.$$

So we have $M' = e^2$. Hence, we obtain $M = M'L_1 = e^2$.

For weak convergence algorithms, we compare [16, Algorithm 3.4], [15, Algorithm 1] with Algorithm 3.1. The respective parameters are consistent with those in Example 5.1. The numerical experimental results are shown in Figure 2.

Figure 2: Numerical results of Example 5.2, $x_0 = x_1 = 20e^{-t}$.



For strong convergence algorithms, we compare [29, Algorithm 3.2], [4, Algorithm 5.1] with Algorithm 4.1. Note that [29, Algorithm 3.2] and [4, Algorithm 5.1] are difficult to implement because it seems not easy to find the explicit expression of projection operator P_C. So, here we use the technique proposed in this paper, that is, to replace C with C_k in $k-th$ step iteration. We call [29, Algorithm 3.2] and [4, Algorithm 5.1] with C_k the **relaxed-Algorithm 3.2** and **relaxed-Algorithm 5.1**, respectively. The parameters are shown below and the numerical experimental results are shown in Figure 3.

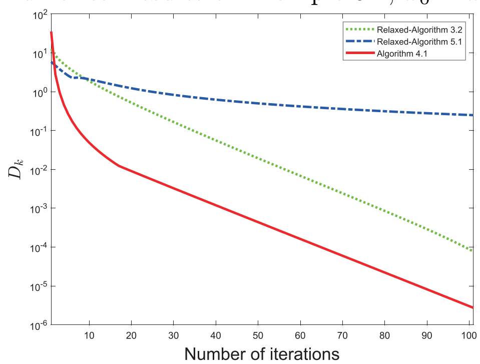
1. **relaxed-Algorithm 3.2**: $\tau = \alpha = 1,\ \mu = 0.9,\ \beta_k = \frac{1}{k+1},\ \varepsilon_k = \beta_k^2,\ f(x) = \frac{1}{2}x,\ D_k = \|\omega_k - y_k\|$.

2. Algorithm 4.1: $\sigma = 5,\ \gamma = 0.1,\ \mu = 0.9,\ \alpha = 0.6,\ \beta_k = \frac{1}{2(k+1)},\ \varepsilon_k = \beta_k^2,\ f(x) = \frac{9}{10}x,\ D_k = \|\omega_k - y_k\|$.

6 Conclusion

In this paper, we present relaxation inertial projection algorithms for solving monotone variational inequality problem in Hilbert space. In these algorithms, each iteration only needs to calculate the projection onto the half-space C_k once, thus reducing the number of projections of previous algorithms, and the selection of parameters is no longer dependent on the solution of the variational inequality. Besides, the step size can be selected in some

Figure 3: Numerical results of Example 5.2, $x_0 = x_1 = 20e^{-t}$.



adaptive ways, which means that we have no need to know or to estimate the Lipschitz constant of the mapping. Hence, we improve and extend some recent results in the literature. In order to ensure the convergence of algorithms, Assumption 3.3(iv) is needed. Sometimes, this condition is not easy to verify, so how to weaken or remove this condition will be an interesting question.

References

[1] F. Alvarez and H. Attouch, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping, *Set-Valued Anal.* 9 (2001) 3–11.

[2] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities: Applications to Free Boundary Problems*, Wiley, New York, 1984.

[3] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Space*, CMS Books in Mathematics, Springer, New York, 2011.

[4] G. Cai, A. Gibali, O.S. Iyiola and Y. Shehu, A new double-projection method for solving variational inequalities in Banach spaces, *J. Optim. Theory Appl.* 178 (2018) 219–239.

[5] Y. Cao and K. Guo, On the convergence of inertial two-subgradient extragradient method for variational inequality problems, *Optimization* 69 (2020) 1237–1253.

[6] J.X. Chen and M.L. Ye, A new inertial two-subgradient extragradient algorithm for variational inequality problems, *Adv. Math. (China)* 51 (2022) 165–182.

[7] Q.L. Dong, Y.J. Cho, L.L. Zhong and T.M. Rassias, Inertial projection and contraction algorithms for variational inequalities, *J. Global Optim.* 70 (2018) 687–704.

[8] F. Facchinei and J.S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer Series in Operations Research, Vols I And II. Springer, New York, 2003.

[9] J. Fan and X. Qin, Weak and strong convergence of inertial Tseng's extragradient algorithms for solving variational inequality problems, *Optimization* 70 (2021) 1195-1216.

[10] R.M. Freund, P. Grigas and R. Mazumder, An extended frank-wolfe method with in-face directions, and its application to low-rank matrix completion, *SIAM J. Optim.* 27 (2015) 319–346.

[11] M. Fukushima, A relaxed projection method for variational inequalities, *Math. Program.* 35 (1986) 58–70.

[12] Y. Gensor, A. Gibali and S. Reich, Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space, *Optimization* 61 (2010) 1119–1132.

[13] A.A. Goldstein, Convex programming in Hilbert space, *Bull. Amer. Math. Soc.* 70 (1964) 709–710.

[14] Z. Harchaoui, A. Juditsky and A. Nemirovski, Conditional gradient algorithms for norm-regularized smooth convex optimization, *Math. Program.* 152 (2015) 75–112.

[15] S.N. He, Q.L. Dong and H.L. Tian, Relaxed projection and contraction methods for solving Lipschitz continuous monotone variational inequalities, *RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* 113 (2019) 2773–2791.

[16] S.N. He and T. Wu, A modified subgradient extragradient method for solving monotone variational inequalities, *J. Inequal. Appl.* 2017 (2017) 1–14.

[17] S. He and H.K. Xu, Uniqueness of supporting hyperplanes and an alternative to solutions of variational inequalities, *J. Global Optim.* 57 (2013) 1375–1384.

[18] D.V. Hieu, Y.J. Cho and Y. Xiao, Modified accelerated algorithms for solving variational inequalities, *Int. J. Comput. Math.* 97 (2020) 2233–2258.

[19] J.B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of Convex Analysis*, Springer Science & Business Media, 2004.

[20] E.N. Khobotov, Modification of the extra-gradient method for solving variational inequalities and certain optimization problems, *USSR Comput. Math. Math. Phys.* 27 (1987) 120–127.

[21] I.V. Konnov, *Combined Relaxation Methods for Variational Inequalities*, Springer, Berlin, 2001.

[22] G.M. Korpelevich, The extragradient method for finding saddle points and for other problems, *Ekon. Mat. Metody.* 12 (1976) 747–756.

[23] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* 73 (1967) 591–597.

[24] R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Trans. Amer. Math. Soc.* 149 (1970) 75–88.

[25] S. Saejung and P. Yotkaew, Approximation of zeros of inverse strongly monotone operators in Banach spaces, *Nonlinear Anal.* 75 (2012) 742–750.

[26] Y. Shehu and O.S. Iyiola, Strong convergence result for monotone variational inequalities, *Numer. Algorithms* 76 (2017) 259–282.

[27] Y. Shehu and O.S. Iyiola, Projection methods with alternating inertial steps for variational inequalities: weak and linear convergence, *Appl Numer. Math.* 157 (2020) 315–337.

[28] Y. Shehu, O.S. Iyiola, X.H. Li and Q.L. Dong, Convergence analysis of projection method for variational inequalities, *Comput. Appl Math.* 38 (2019) 1–21.

[29] D.V. Thong, D.V. Hieu and T.M. Rassias, Self adaptive inertial subgradient extragradient algorithms for solving pseudomonotone variational inequality problems, *Optim. Lett.* 14 (2020) 115–144.

[30] D.V. Thong and Y.J. Cho, New strong convergence theorem of the inertial projection and contraction method for variational inequality problems, *Numer. Algorithms* 84 (2020) 285–305.

[31] D.V. Thong, Y. Shehu and O.S. Iyiola, Weak and strong convergence theorems for solving pseudo-monotone variational inequalities with non-Lipschitz mappings, *Numer. Algorithms* 84 (2020) 795–823.

[32] R. Trémolières, J.L. Lions and R. Glowinski, *Numerical Analysis of Variational Inequalities*, Elsevier, 2011.

[33] P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, *SIAM J. Control Optim.* 38 (2000) 431-446.

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Yan Zhang
School of Mathematics
Sichuan Normal University
Chengdu 610068, China
E-mail address: yanzhang_9@163.com

Denglian Yang
School of Mathematics
Sichuan Normal University
Chengdu 610068, China
E-mail address: denglian_yang@163.com

Yongle Zhang
School of Mathematics
Visual Computing and Virtual
Reality Key Laboratory of Sichuan Province
Sichuan Normal University
Chengdu 610068, China
E-mail address: yongle-zhang@163.com