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NULL SPACE PROPERTY FOR COMPRESSED SENSING WITH FRAMES VIA $l_q(0 < q \leq 1)$ SYNTHESIS*

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Abstract: This paper studies the reconstruction of the signals with sparse or nearly sparse representations in a given dictionary D via $l_q(0 < q \leq 1)$ synthesis method. A new null space property based on the given dictionary (D-NSP$_q$) is proposed. It is proved that sensing matrices satisfying the D-NSP$_q$ is not just a sufficient and necessary condition for l_q-synthesis to exactly recover signals sparse in D, but also a sufficient and necessary condition for l_q-synthesis to stably recover signals which are compressible in D. To the best of our knowledge, this new property is the first sufficient and necessary condition for successful signal recovery via l_q-synthesis. In addition, we also characterize the theoretical performance of reconstructed signals via l_q-synthesis in the case of noise.

Key words: *compressed sensing, frame sparse, l_q-synthesis, null space property*

Mathematics Subject Classification: *94A12, 94A15*

1 Introduction

Compared with the traditional Nyquist-Shannon sampling theory (see e.g., [4, 6, 16]), compressed sensing is a revolutionary innovation. The sparsity assumption plays an important role in signal reconstruction. A vector is called k-sparse if the number of its nonzero entries is no more than k. The fundamental idea of compressed sensing is to recover a sparse signal $x \in \mathbb{R}^d$ from its undersampled linear measurements $y = Ax + e$, where $y \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times d}(m \ll d)$, and $e \in \mathbb{R}^m$ is a vector of measurement errors with $\|e\|_2 \leq \epsilon$. The classical compressed sensing theory points out that, the sparse or nearly sparse signal x_0, can be successfully reconstructed through the following l_1-minimization model under certain conditions of measurement matrix A.

$$\hat{x} = \arg\min_{x\in\mathbb{R}^d} \|x\|_1, \quad \text{subject to} \quad \|Ax - y\|_2 \leq \epsilon, \tag{1.1}$$

where $\|\cdot\|_2$ is the Euclidean norm of vectors, $\|x\|_1 = \sum_{i=1}^{d} |x_i|$ denotes the l_1-norm. When $\epsilon = 0$, we call it the noiseless case, and $\epsilon > 0$, the noisy case.

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Another feasible way of recovering the unknown sparse signal x is the $l_q(0 < q < 1)$-minimization model

$$\hat{x} = \arg\min_{x \in \mathbb{R}^d} \|x\|_q^q \quad \text{subject to} \quad \|y - Ax\|_2 \le \epsilon, \tag{1.2}$$

where $\|x\|_q = (\sum_{i=1}^{d} |x_i|^q)^{1/q}$ is called l_q-norm. It is a quasi norm and does not satisfy the triangle inequality. It only satisfies the following q-triangle inequality: $\|x + y\|_q^q \le \|x\|_q^q + \|y\|_q^q$, for $x, y \in \mathbb{R}^d$.

One of the key research works of compressed sensing is designing an appropriate sensing matrix to ensure good reconstruction performance of minimization problem (1.1) and (1.2). The Restricted Isometry Property (RIP) introduced by Candès and Tao in [7], is shown to provide stable recovery of signals nearly sparse via (1.1). A matrix A satisfies RIP of order k, if there exists a constant $\delta_k \in [0, 1)$, such that

$$(1 - \delta_k)\|x\|_2^2 \le \|Ax\|_2^2 \le (1 + \delta_k)\|x\|_2^2,$$

holds for all k-sprase vectors $x \in \mathbb{R}^d$, and the smallest constant satisfying the above inequality is defined as the Restricted Isometry Constant (RIC). Various sufficient conditions based on the RIC for sparse signal recovery, exactly or stably, can be found in [5, 7, 19, 9, 28, 8, 11, 10, 12]. Null space property (NSP) is another well-known property used to characterize the sensing matrix. A matrix A satisfies the NSP of order k, which means for any $v \in \ker A \setminus \{0\}$, and any index set $|T| \le k$, it holds that

$$\|v_T\|_1 < \|v_{T^c}\|_1.$$

The NSP is a necessary and sufficient condition which guarantees the exact reconstruction of the sparse signal using the l_1-minimization model (1.1). Many works are based on NSP (see [17, 21, 20, 23, 24, 36]), especially [20], which proposed the stable NSP, the robust NSP, and used them to characterize the solutions of (1.1). Moreover, it was shown that NSP matrices can reach a similar stability result as RIP matrices, except that the constants may be larger[1, 31].

Sparse signal recovery via $l_q(0 < q < 1)$-minimization has been studied in a series of literatures [19, 15, 22, 25, 32]. It was pointed out in [20] that $l_q(0 < q < 1)$-minimization method requires significantly fewer measurements if the sensing matrix is Gaussian. Compared with the l_1-minimization, a sufficiently sparse signal can be recovered perfectly with the $l_q(0 < q < 1)$-minimization under less restrictive RIP requirements [33]. The empirical results show that l_q-minimization method would take much less time than the l_0-minimization [14]. These interesting phenomena inspire more and more research on the $l_q(0 < q < 1)$ modeling although the l_q-minimization problem for $0 < q < 1$ is also NP-hard in general [20].

The $l_q(0 < q \le 1)$-minimization method shows good reconstruction performance for signals which are sparse in the standard orthonormal basis or some other orthonormal basis. However, in many practical applications, the signal of interest is not sparse in an orthonormal basis. More often than not, sparsity is expressed in terms of an overcomplete dictionary. This kind of signal is called dictionary-sparse signal or frame-sparse signal, and is called D-sparse signal when the dictionary D is given, while the signals which are nearly sparse in D will be called D-compressible. The signal $x \in \mathbb{R}^d$ we consider in this paper is now expressed as $x = Dz$, where $D \in \mathbb{R}^{d\times n}$, $d \ll n$ is some overcomplete dictionary of $\mathbb{R}^d$ and the coefficient $z \in \mathbb{R}^n$ is sparse or nearly sparse. The linear measurement is $y_0 = Ax_0$. We refer to [25, 3, 34, 30, 2, 27, 26, 35] and the reference therein for details.

A natural idea of recovering x_0 from the measurements y is to solve the minimization problem

$$\hat{z} = \arg\min_{z \in \mathbb{R}^n} \|z\|_1, \quad \text{subject to} \quad y = ADz \tag{1.3}$$

for the sparse coefficients $\hat{z}$ at first, then synthesizing it to get $\hat{x} = D\hat{z}$. This method is called l_1-synthesis, it is related to the l_1-analysis, which recovers the signal directly from solving the problem

$$\hat{x} = \arg\min_{x \in \mathbb{R}^d} \|D^* x\|_1, \quad \text{subject to} \quad y = Ax, \tag{1.4}$$

where D^* is the transpose of D.

Numerical experiments show that the l_1-synthesis method can often perform good reconstruction results, however, it has a fundamental distinction with the l_1-analysis method [18]. It was shown that there was a large theory gap between these two techniques. In [13] the authors introduced the l_1-synthesis method and analyzed the essential differences and relations between the two problems. They pointed out that since optimal dual based l_1-analysis is equivalent to l_1 synthesis [29], l_1-analysis appears to be a subproblem of the l_1-synthesis, and l_1-synthesis would be more natural and more thorough way than l_1-analysis. The main contribution of [13] is to establish the first necessary and sufficient condition for reconstructing D-sparse signal based on the l_1-synthesis method by using a new null space property of the frame D. Inspired by the work of [13], we consider frame-sparse signals recovery based on a new l_q-norm null space property of the dictionary D, via the l_q-synthesis method, where $0 < q \le 1$. For frame-sparse signal recovery in the noiseless case, we consider

$$\hat{z} = \text{argmin}\|z\|_q^q \qquad \text{subject to} \qquad y = ADz, \tag{1.5}$$

while, for the recovery of D-compressible signals in the case of the measurements are perturbed, we naturally consider the following method:

$$\hat{z} = \text{argmin}\|z\|_q^q \qquad \text{subject to} \qquad \|ADz - y\|_2 \le \epsilon. \tag{1.6}$$

In this paper, we generalize the D-NSP proposed by [13] to l_q-norm D-NSP (D-NSP$_q$), and show that the D-NSP$_q$, is a sufficient and necessary condition for the l_q-synthesis to exactly recover all D-sparse signals of order k. Moreover, when the measurements are perturbed and the signals are D-compressible, we prove that D-NSP$_q$ is still a sufficient and necessary condition for stable recovery.

The remainder of this paper is organized as follows. Some notations, definitions and some useful lemmas are introduced in section 2. In section 3, we present the main theorems for recovering D-sparse signals in the noiseless case and D-compressible signals in noisy case.

2 Preliminaries

We provide the notations of this paper roughly as follows. For a vector $z = (z_1, z_2, ..., z_n)^T \in \mathbb{R}^n$, let $\text{supp}(z) \subseteq \{1, 2, ..., n\}$ denote the support of z, which is the set of indices of nonzero entries in z. The l_0-norm of z is defined as $\|z\|_0 = |\text{supp}(x)|$, and z is called k-sparse when $|\text{supp}(x)| \le k$. $\|z\|_q = (\sum_{i=1}^{n} |z_i|^q)^{\frac{1}{q}}$ is the $l_q(0 < q \le 1)$-norm of vector z. Denote $z_T \in \mathbb{R}^n$ be the vector of which the i-th coordinate $(z_T)_i = z_i$, if $i \in T$, and $(z_T)_i = 0$ for $i \notin T$, and $z_{T^c} = z - z_T$. For a given frame $D \in \mathbb{R}^{d \times n}$, we define $D\Sigma_k = \{x \in \mathbb{R}^d |$if there exists $z \in$

$\mathbb{R}^n, \|z\|_0 \leq k$ such that $x = Dz\}$. Denote $\sigma_k(z_0) = \inf_{\|z\|_0 \leq k} \|z - z_0\|_q$ to be the optimal k-term approximation of z_0 in l_q-norm.

The following two new null space properties are very important in characterizing the reconstruction performance of l_q-synthesis methods (1.5) and (1.6).

Definition 2.1. (k-order l_q-norm null space property of a dictionary D (k-D-NSP$_q$)). Given a dictionary $D \in \mathbb{R}^{d\times n}$. A matrix $A \in \mathbb{R}^{m\times d}$ satisfies the D-NSP$_q$ of order k (k-D-NSP$_q$) means that, for any index set T with $|T| \leq k$, there exists $u \in \ker D$, such that

$$\|v_T + u\|_q^q < \|v_{T^c}\|_q^q, \quad \forall v \in D^{-1}(\ker A \setminus \{0\}). \tag{2.1}$$

Definition 2.2. (k-order l_q-norm strong null space property of a dictionary D (k-D-SNSP$_q$)) A matrix A satisfying the l_q-norm strong null space property with respect to D of order k if there is a positive constant c such that for any index set T with $|T| \leq k$, there exists $u \in \ker D$, such that

$$\|v_{T^c}\|_q^q - \|v_T + u\|_q^q \geq c\|Dv\|_2^q, \quad \forall v \in \ker(AD). \tag{2.2}$$

The following lemmas will be useful in the next part of the paper.

Lemma 2.3. *Let $A \in R^{m\times d}$, for any $a \in \ker A, b \in \ker A^{\perp}$, and $0 < p \leq 1$, the following inequality holds.*

$$\|a + b\|_2^q \leq \|a\|_2^q + \|b\|_2^q.$$

Proof. Since $a^T b = 0$, so $\|a + b\|_2^q = (\|a\|_2^2 + \|b\|_2^2)^{\frac{q}{2}}$. It is sufficient to show

$$(x^2 + y^2)^{\frac{q}{2}} \leq x^q + y^q, \tag{2.3}$$

for all positive real numbers x, y. Define the real function $f(x) = (1 + x^q)^{\frac{2}{q}} - 1 + x^2$, and then it will be easy to check that $f(x)$ is monotonically increasing when $x > 0$, then $f(x) > f(0) = 0$, and we will get $(1 + x^q)^{\frac{2}{q}} \geq 1 + x^2$, which implies (2.3) holds. □

Lemma 2.4. *Let a, b, c be positive numbers, $0 < p \leq 1$, then the inequality*

$$(a^q + b^q + c^q)^{\frac{1}{q}} \leq 3^{\frac{1}{q}-1}(a + b + c)$$

holds.

Remark 2.5. The results of lemma 2.3 and lemma 2.4 can be easily derived from Hölder Inequality.

Given a index set T, and a vector $v \in D^{-1}(\ker A \setminus \{0\})$, for any $u \in \ker D$ and $t > 0$, defined the real functions

$$\phi_v(u, t) = \sup_{\tilde{u}\in\ker D} (\|(tv + u)_{T^c}\|_q^q - \|(tv + u)_T + \tilde{u}\|_q^q)$$

and

$$f_v(u, t) = \frac{\phi_v(u, t)}{t^p}. \tag{2.4}$$

Lemma 2.6. *Suppose that A satisfies the k-D-NSP_q, then for any given index set T and $v \in D^{-1}(\ker A \setminus \{0\})$, the function defined in (2.4) satisfies*

$$\inf_{u\in\ker D, t>0} f_v(u, t) > 0.$$

Proof. Since A satisfies the k-D-NSP$_q$, it is easy to see that $f_v(u,t) > 0$, for any $v \in D^{-1}(\ker A \setminus \{0\})$, and it is sufficient to show that there is no $v_0 \in D^{-1}(\ker A \setminus \{0\})$ such that $\inf_{u \in \ker D, t>0} f_{v_0}(u,t) = 0$. If this is not ture, then for any $\eta > 0$, there is $u_0 \in \ker D, t_0 > 0$ such that $f_{v_0}(u_0,t_0) < \eta$. By the definition of $f_{v_0}(u_0,t_0)$, that is

$$\frac{1}{t_0^q} \sup_{\tilde{u} \in \ker D} (\|(t_0 v_0 + u_0)_{T^c}\|_q^q - \|(t_0 v_0 + u_0)_T + \tilde{u}\|_q^q) < \eta.$$

This will lead to $\|(t_0 v_0 + u_0)_{T^c}\|_q^q - \|(t_0 v_0 + u_0)_T + \tilde{u}\|_q^q \le 0$, for any $\tilde{u} \in \ker D$, and it is contradicts with the assumption that A satisfies the k-D-NSP$_q$. □

3 Main Results

Theorem 3.1. *D-NSP$_q$ is a necessary and sufficient condition for l_q-synthesis (1.5) to successfully recover all signals in the set $D\Sigma_k$.*

Proof. (Sufficient part) Suppose that the sensing matrix A satisfies the D-NSP$_q$ of order k, then the lq-synthesis (1.5) can successfully recover all D-sparse signals $x \in D\Sigma_k$ from measurements $y = Ax$. Otherwise, there is a vector $x_0 \in D\Sigma_k$, the reconstruction of which is $\hat{x} = D\hat{z} \neq x_0$. Denote $x_0 = Dz_0$, where $|z_0| \le k$. Let $v = z_0 - \hat{z}$, it is easy to check that $v \in D^{-1}(\ker A \setminus \{0\})$. Denote T to be the support set of z_0, by the definition of D-NSP$_q$, there must exist a $u \in \ker D$, such that $\|v_T + u\|_q^q < \|v_{T^c}\|_q^q$, which implies $\|(z_0 - \hat{z})_T + u\|_q^q = \|z_0 - \hat{z}_T + u\|_q^q < \|\hat{z}_{T^c}\|_q^q$, and

$$\begin{aligned} \|z_0 + u\|_q^q &\le \|z_0 - \hat{z}_T + u\|_q^q + \|\hat{z}_T\|_q^q \\ &< \|\hat{z}_{T^c}\|_q^q + \|\hat{z}_T\|_q^q \\ &= \|\hat{z}\|_q^q. \end{aligned}$$

This leads to the contradiction of the assumption that $\hat{z}$ is a minimizer of the problem (1.5).

(Necessary part) Assuming l_q-synthesis (1.5) can successfully recover all signals in $D\Sigma_k$, we need to show that the sensing matrix A satisfies D-NSP$_q$. For any $v \in D^{-1}(\ker A \setminus \{0\})$ and any index set T with $|T| \le k$, denote $x_0 = Dv_T$, then $x_0 \in D\Sigma_k$, and let $y_0 = Ax_0$ be its measurements. Let $\hat{z}$ be the solution of (1.5), and $\hat{x} = D\hat{z}$ be the reconstructed signal By the assumption, we have $\hat{x} = x_0$, and there is a $u \in \ker D$, such that $\hat{z} = v_T + u$. Since $AD(v_T - v) = y$ and $v_T - v \neq v_T + u$ for any $u \in \ker D$, then $v_T - v$ cannot be a minimizer of (1.5), therefor we get $\|v_T + u\|_q^q < \|v_T - v\|_q^q = \|v_T^c\|_q^q$, which implies A is k-D-NSP$_q$. □

In classical compressed sensing theory, it is well-known that the null space property is a sufficient and necessary condition not just for the sparse signal recovery in noiseless case, but also for compressible signals with noisy measurements [1, 31]. We will show that this result can be generalized to D-NSP$_q$ when the reconstruction is carried on a signal which is sparse or nearly sparse in a given frame.

The D-SNSP$_q$ defined in definition 2.2 looks stronger than the D-NSP$_q$. We now show that, with this stronger property, D-compressible signals can be stably recovered via (1.6) as follows.

Theorem 3.2. *If the sensing matrix $A \in \mathbb{R}^{m \times d}$ satisfies k-D-SNSP$_q$, then any solution $\hat{z}$ of l_q-synthesis (1.6) satisfies*

$$\|D\hat{z} - x_0\|_2 \le C_1 \sigma_k(z_0) + C_2 \epsilon, \tag{3.1}$$

where z_0 is any representation of x_0 in D, $\sigma_k(z_0) = \inf_{\|z\|_0 \le k} \|z - z_0\|_q$,$C_1, C_2$ are constants.

Proof. Denote $x_0 = Dz_0$ be the unknown signal we want to recover and T is the index set with the k largest coefficients (in magnitude) of z_0. Denote $h = D(\hat{z} - z_0)$, and decompose it as $h = t + \eta$ where $t \in \ker A, \eta \in \ker A^{\perp}$. Let $w = D^T(DD^T)^{-1}t$, then $h = Dw + \eta$ with $Dw \in \ker A$. It is not difficult to know that

$$\|\eta\|_2 \leq \frac{1}{V_A}\|Ah\|_2 \leq \frac{2\epsilon}{V_A}, \tag{3.2}$$

where V_A is the smallest positive singular value of A. Let $\xi = D^T(DD^T)^{-1}\eta$, then $\eta = D\xi$,and it is easy to show

$$\|\xi\|_2 \leq \frac{1}{V_D}\|\eta\|_2 \leq \frac{2}{V_A V_D}\epsilon. \tag{3.3}$$

Since $h = D(w + \xi)$ and $h = D(\hat{z} - z_0)$,so $\hat{z} - z_0 = w + \xi + u_1$ with $u_1 \in \ker D$.

Let $v = w + u_1$, then $\hat{z} - z_0 = v + \xi$ and $v \in \ker AD$. By the assumption, A satisfies the k-D-SNSP$_q$, then there is a $u \in \ker D$ such that

$$\|v_{T^c}\|_q^q - \|v_T + u\|_q^q \geq c\|Dv\|_2^q.$$

Therefor

$$\begin{aligned}
\|v + z_{0,T}\|_q^q - \| - u + z_{0,T}\|_q^q &= \|v_{T^c} + v_T + z_{0,T}\|_q^q - \| - u_T + z_{0,T} - u_{T^c}\|_q^q \\
&= \|v_{T^c}\|_q^q + \|v_T + z0, T\|_q^q - \| - u_T + z_{0,T}\|_q^q - \|u_{T^c}\|_q^q \\
&= \|v_{T^c}\|_q^q - (\|u_T - z_{0,T}\|_q^q - \|v_T + z_{0,T}\|_q^q) - \|u_{T^c}\|_q^q \\
&\geq \|v_{T^c}\|_q^q - \|u_T + v_T\|_q^q - \|u_{T^c}\|_q^q \\
&= \|v_{T^c}\|_q^q - \|u + v_T\|_q^q \geq c\|Dv\|_2^q.
\end{aligned} \tag{3.4}$$

On the other side, since $\hat{z}$ is a minimizer, we have

$$\begin{aligned}
\| - u + z_{0,T}\|_q^q + \|z_{0,T^c}\|_q^q &\geq \| - u + z_0\|_q^q \geq \|\hat{z}\|_q^q \\
&= \|z_0 + v + \xi\|_q^q \geq \|v + z_0\|_q^q - \|\xi\|_q^q \\
&\geq \|v + z_{0,T}\|_q^q - \|z_{0,T^c}\|_q^q - \|\xi\|_q^q.
\end{aligned}$$

By rearranging the above inequality, we will obtain

$$\|v + z_{0,T}\|_q^q - \| - u + z_{0,T}\|_q^q \leq 2\|z_{0,T^c}\|_q^q + \|\xi\|_q^q. \tag{3.5}$$

Combining (3.4) with (3.5), we get

$$c\|Dv\|_2^q \leq 2\|z_{0,T^c}\|_q^q + \|\xi\|_q^q.$$

Using the Hölder inequality with $\|\xi\|_q^q$, the above inequality will become

$$\|Dv\|_2^q \leq \frac{2}{c}\|z_{0,T^c}\|_q^q + \frac{n^{1-\frac{q}{2}}}{c}\|\xi\|_2^q. \tag{3.6}$$

Finally, using (3.2),(3.6), and lemma 2.3

$$\begin{aligned}
\|h\|_2^q &= \|Dv + D\xi\|_2^q = \|Dv + \eta\|_2^q \leq \|Dv\|_2^q + \|\eta\|_2^q, \\
&\leq \frac{2}{c}\|z_{0,T^c}\|_q^q + \frac{n^{1-\frac{q}{2}}}{c}\|\xi\|_2^q + \|\eta\|_2^q.
\end{aligned}$$

That is

$$\|\hat{x} - x_0\|_2 \le (\frac{2}{c}\|z_{0,T^c}\|_q^q + \frac{n^{1-\frac{q}{2}}}{c}\|\xi\|_2^q + \|\eta\|_2^q)^{\frac{1}{q}}.$$

By using lemma 2.4,(3.2) and (3.3), the above inequality can be modified such that

$$\|\hat{x} - x_0\|_2 \le C_1\|z_{0,T^c}\|_q + C_2\epsilon = C_1\sigma_k(z_0) + C_2\epsilon, \tag{3.7}$$

where $C_1 = 3^{\frac{1}{q}-1}(\frac{2}{c})^{\frac{1}{q}}$,$C_2 = 3^{\frac{1}{q}-1}\frac{2}{V_A}[(\frac{n^{1-\frac{q}{2}}}{c})^{\frac{1}{q}}\frac{1}{V_D} + 1]\epsilon$. □

Remark 3.3. (a) When $p = 1$, our result is consistent with Theorem 5.2 in [13].

(b) When z_0 is k-sparse and $\epsilon = 0$, it means that D-SNSP$_q$ sparse signals can be exactly recovery by (1.5).

By definition 2.2, it is obvious that D-SNSP$_q$ is not weaker than D-NSP$_q$. We want to find it out that how much stronger it is than D-NSP$_q$. The following theorem shows that these two conditions are actually the same.

Theorem 3.4. *Let $A \in \mathbb{R}^{m\times d}$, $D \in \mathbb{R}^{d\times n}$, matrix A satisfying D-NSP$_q$ is equivalent to A satisfying D-SNSP$_q$ with the same order.*

Proof. Suppose A satisfies $k - D$-NSP$_q$. For any $w \in \ker AD$, take $u = 0$, when $w = 0$, and $u = -w$ for $w \neq 0$,$Dw = 0$, then $\|w_{T^c}\|_q^q - \|w_T + u\|_q^q = 0$, and (2.2) holds for any positive number C. To complete the proof, we just need to show the function

$$F(w) = \sup_{\tilde{u}\in\ker D}\frac{\|w_{T^c}\|_q^q - \|w_T + \tilde{u}\|_q^q}{\|Dw\|_2^q}$$

has a positive lower bound on $D^{-1}(\ker A \setminus \{0\})$ for every $|T| \le k$.

Decompose w into two parts as $w = tv + u$, where $u = \mathrm{P}_{\ker D}w$, $tv = \mathrm{P}_{(\ker D)^\perp}w$, with $\|v\|_2 = 1$, and $t > 0$. By the definition of infimum, we have

$$\inf_{w\in D^{-1}(\ker A\setminus\{0\})} F(w) = \inf_{v\in\ker D^\perp, \|v\|_2=1}\inf_{u\ \ker D, t>0} f_v(u,t)/\|Dv\|_2.$$

By Lemma 2.6, the function $\inf_{u\in\ker D, t>0} f_v(u,t)$ is always positive. Since $(\ker D)^\perp \cap \mathbb{S}^{n-1}$ is a compact set, it is sufficient to prove that the function $\inf_{u\in\ker D, t>0} f_v(u,t)$ is lower-semi continuous with respect to v.

Since, for any $v \in D^{-1}(\ker A \setminus \{0\})$ and any $\eta > 0$, there is a $\delta = n^{1-1/q}\eta^{1/q} > 0$, such that for any $\|e\|_1 < \delta$,

$$\begin{aligned} f_{v+e}(u,t) &= \sup_{\tilde{u}\in\ker D}\frac{\|(tv + te + u)_{T^c}\|_q^q - \|(tv + te + u)_T + \tilde{u}\|_q^q}{t^q} \\ &\ge \sup_{\tilde{u}\in\ker D}\frac{\|(tv + u)_{T^c}\|_q^q - \|(tv + u)_T + \tilde{u}\|_q^q}{t^q} - \|e\|_q^q \\ &\ge \sup_{\tilde{u}\in\ker D}\frac{\|(tv + u)_{T^c}\|_q^q - \|(tv + u)_T + \tilde{u}\|_q^q}{t^q} - n^{1-q}\|e\|_1^q. \end{aligned}$$

Taking the infimum over u in $\ker D$ and $t > 0$ of both sides, we get

$$\inf_{u\in\ker D, t>0} f_{v+e}(u,t) \ge \inf_{u\in\ker D, t>0} f_v(u,t) - \eta,$$

which shows that the function is a lower semi continuous, and the proof is completed. □

4 Conclusion

In this paper, we generalized the D-NSP proposed by [13] to D-NSP$_q$. We proved in theorem 3.1 that this new property is equivalent to the exact recovery of D-sparse signals via l_q-synthesis. In addition, a stable reconstruction result of D-compressible signals via l_q-synthesis in noise case was given in theorem 3.2. To the best of our knowledge, these studies provide the first characterization of signal recovery with dictionaries via l_q-synthesis approach.

By theorem 3.4, we proved that A satisfies D-SNSP$_q$ is equivalent to A satisfies D-NSP$_q$ with the same order. Combined with theorem 3.1 and theorem 3.2 it is clear that D-NSP$_q$ is not only a sufficient and necessary condition for the success of l_q-synthesis without measurement noise, but also sufficient and necessary condition for stability of l_q-synthesis in the noisy case.

These results are helpful to characterize the reconstruction performance of l_q-synthesis approach, and of great significance to study and design the measurement matrix A.

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