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STABLE QUATERNION PRINCIPAL COMPONENT PURSUIT*

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Abstract: The relaxed quaternion principal component pursuit is studied to recover low-rank quaternion matrix and sparse quaternion matrix with small entry-wise noise. Stable estimates of the low-rank quaternion matrix and the sparse quaternion matrix are provided by solving a convex minimization problem. The result in this paper generalizes the relaxed principal component pursuit from the case of real matrices to the case of quaternion matrices.

Key words: *quaternion principal component pursuit, low-rank quaternion matrix, sparse quaternion matrix, matrix recovery*

Mathematics Subject Classification: *15A83, 90C25, 15B33*

1 Introduction

In many practical problems of interest, it is desired to recover a matrix from samples of its entries. When the matrix is real, in order to reduce the dimensionality of the recovery problem out of big data, it is often assumed that the given data are located near a low-dimensional subspace. Hence, principal component analysis (PCA) was proposed in [8, 10]. Suppose that a real matrix of large data can be decomposed into the sum of a low-rank matrix and a sparse matrix. Candès et al. in [1] proposed the principal component pursuit (PCP) model. They proved that the low-rank and sparse matrices were recovered precisely by solving a convex optimization problem under relatively weak assumptions. However, in practice, the original data often contain noise, which must be considered in the matrix decomposition. Zhou et al. in [16] considered the relaxed PCP model and proved the low-rank matrix estimation for the convex programming problem. To some extent, the relaxed PCP model unifies classical PCA and PCP by considering the total sparse error and small entry noise.

All of the matrices we mentioned above are real matrices. In recent years, however, many studies have shown that quaternion and quaternion matrix are widely used in color image inpainting, color image denoising and face recognition [3, 4, 6, 9, 11, 17]. This is because color images with red, blue, and green can be encoded by the three imaginary parts of quaternions. Qi et al. in [14] provided the theoretical basis for the optimality analysis of quaternion matrix optimization (QMO) problem, and summarized QMO issues and some special cases. We noted that, since the nuclear norm is a convex surrogate of the rank

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function of the quaternion matrix and the l_1-norm is a convex relaxation of the l_0-norm, the objective function of the robust quaternion matrix completion model is convex, but is not differentiable. Jia et al. in [9] studied the subgradients of quaternion matrix norms for the first time.

Motivated by the applications and theoretical analysis of QMO, we extend the relaxed PCP model from the work of Zhou et al. in [16] to the case of quaternion matrices. From [9], we know that an exact recovery of high probability quaternion matrix can be obtained from a random subset of corrupted items by solving a convex programming problem under relatively weak assumptions. The model used in this paper is composed of a low-rank quaternion matrix, a sparse quaternion matrix and a Gaussian noise part. Due to the existence of random noise terms, we cannot accurately recover the low-rank matrix and the sparse matrix. We will show that we can obtain a stable estimate of the original low-rank and sparse matrices by solving a convex optimization problem under the assumption that the low-rank matrix satisfies the incoherence condition, the sparse elements in the sparse matrix are uniformly distributed, and the influence of Gaussian noise term is not too great. Because of the noncommutativity of quaternion multiplication, the proof of exact recovery theory is quite different from that for real matrices. To this end, we derive some special formulas to solve the noncommutativity problem.

The rest of this paper is organized as follows. Section 2 recalls some preliminaries about quaternions and quaternion matrices, and introduces some definitions of quaternion matrix and some important propositions. In section 3, we give the main result and provide the proof of our quaternion matrix recovery theorem. Conclusions are given in the last section.

2 Preliminaries

In general, we denote the real field by $\mathbb{R}$ and the quaternion algebra by $\mathbb{Q}$. The collection of real and quaternion $n_1 \times n_2$ matrices are denoted by $\mathbb{R}^{n_1\times n_2}$ and $\mathbb{Q}^{n_1\times n_2}$, respectively.

Hamilton introduced the quaternion in 1843, which consists of one real part and three imaginary parts [7, 13, 15]. Let

$$a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \mathbb{Q}, \tag{2.1}$$

where $a_0,\ a_1,\ a_2,\ a_3 \in \mathbb{R}$, a_0 is the real part of a denoted by $\mathrm{Re}\,(a)$, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are three fundamental quaternion imaginary units satisfying

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{2.2}$$

The conjugate and modulus of a are respectively defined as

$$a^* = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k},\ |a| = \sqrt{{a_0}^2 + {a_1}^2 + {a_2}^2 + {a_3}^2}. \tag{2.3}$$

Let $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $b = b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \in \mathbb{Q}$, then

$$\begin{aligned} a + b &= (a_0 + b_0) + (a_1 + b_1)\,\mathbf{i} + (a_2 + b_2)\,\mathbf{j} + (a_3 + b_3)\,\mathbf{k}, \\ ab &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)\,\mathbf{i} \\ &\quad + (a_0b_2 + a_2b_0 + a_3b_1 - a_1b_3)\,\mathbf{j} + (a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1)\,\mathbf{k}. \end{aligned}$$

$A = (a_{ij}) \in \mathbb{Q}^{n_1\times n_2}$ is a quaternion matrix if

$$A = B + C\mathbf{i} + D\mathbf{j} + E\mathbf{k}, \tag{2.4}$$

where B, C, D, $E \in \mathbb{R}^{n_1 \times n_2}$ are real matrices. And B is called the real part of the quaternion matrix A, denoted by $\mathrm{Re}\,(A)$. The identity quaternion matrix I is the same as the classical identity matrix. $A^* = (a_{ji}^*)$ is the conjugate transpose of A. The quaternion matrix A is said to be unitary if $A^*A = AA^* = I$. The rank of quaternion matrix A is the maximum number of columns of A which are right linearly independent[15], denoted by rank(A). The trace of A is denoted by Tr(A).

Specially, a color image can be denoted by quaternion matrix $A = (A_{st}) \in \mathbb{Q}^{n_1 \times n_2}$ for $1 \leq s \leq n_1, 1 \leq t \leq n_2$,

$$A_{st} = R_{st}\mathbf{i} + G_{st}\mathbf{j} + B_{st}\mathbf{k}, \tag{2.5}$$

where R_{st}, G_{st}, B_{st} are the red, green, and blue pixel values at the location (s, t) in the image, respectively.

The inner product between two quaternion matrices $A = (a_{ij}) \in \mathbb{Q}^{n_1 \times n_2}$ and $B = (b_{ij}) \in \mathbb{Q}^{n_1 \times n_2}$ is defined by

$$\langle A, B \rangle = \mathrm{Tr}(A^*B) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_{ij}^* b_{ij}. \tag{2.6}$$

The singular-value decomposition of quaternion matrix (QSVD) is as follows:

Theorem 2.1 ([15, Theorem 7.2])**.** *For any quaternion matrix $A \in \mathbb{Q}^{n_1 \times n_2}$ with* rank$(A) =$ *r $(r > 0)$, there exist unitary quaternion matrices $U \in \mathbb{Q}^{n_1 \times n_1}$, $V \in \mathbb{Q}^{n_2 \times n_2}$ such that*

$$A = U \Sigma V^*, \tag{2.7}$$

where $\Sigma = \mathrm{diag}\,\{\sigma_1, \sigma_2, \ldots, \sigma_r, 0, \ldots, 0\} \in \mathbb{R}^{n_1 \times n_2}$, and $\sigma_1, \sigma_2, \ldots, \sigma_r$ are the positive singular values of A.

Thus, the rank of A is equal to the numbers of positive singular values of A.

Let $A = (a_{ij}) \in \mathbb{Q}^{n_1 \times n_2}$. Several classes of norms of quaternion matrix are defined as follows [9]: the Frobenius norm (F-norm) $\|A\|_F := \sqrt{\mathrm{Tr}\,(A^*A)} = \sqrt{\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} |a_{ij}|^2} = \left(\sum_{i=1}^{r} \sigma_i^2\right)^{\frac{1}{2}}$, the l_1-norm $\|A\|_1 := \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} |a_{ij}|$; the ∞-norm $\|A\|_\infty := \max_{i,j} |a_{ij}|$, the spectral norm $\|A\| := \max\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$, and the nuclear norm $\|A\|_* := \sum_{i=1}^{r} \sigma_i$, where $\sigma_1, \sigma_2, \ldots, \sigma_r$ are nonzero singular values of A and r is the rank of A.

For any linear operator $\mathcal{A}$: $\mathbb{Q}^{n_1 \times n_2} \to \mathbb{Q}^{n_1 \times n_2}$, $\|\mathcal{A}\| = \sup_{\|X\|_F = 1} \|\mathcal{A}X\|_F$ denotes the operator norm of $\mathcal{A}$.

Let a real function $f : \mathbb{Q}^{n_1 \times n_2} \to \mathbb{R}$. Let a quaternion matrix variable $Y = Y_0 + Y_1\mathbf{i} + Y_2\mathbf{j} + Y_3\mathbf{k} \in \mathbb{Q}^{n_1 \times n_2}$. Chen, Qi, Zhang, and Xu in [3] defined the differentiable real-valued functions of quaternion matrix variables: f is differentiable at Y if $\frac{\partial f}{\partial Y_i}$ exists at Y_i for $i = 0, 1, 2, 3$ and denote

$$\nabla f(Y) = \frac{\partial f}{\partial Y_0} + \frac{\partial f}{\partial Y_1}\mathbf{i} + \frac{\partial f}{\partial Y_2}\mathbf{j} + \frac{\partial f}{\partial Y_3}\mathbf{k}. \tag{2.8}$$

If f have more variables, then we change $\nabla f(Y)$ in (2.8) to $\frac{\partial f}{\partial Y}$.

From the definitions above, we can get some properties:

Proposition 2.2. *For $A = (a_{ij})_{n_1 \times n_2}$, $B = (b_{ij})_{n_1 \times n_2}$, $C = (c_{ij})_{n_1 \times n_2} \in \mathbb{Q}^{n_1 \times n_2}$, the following four properties hold:*

1. *Commutative law: $\langle A, B \rangle = \langle B, A \rangle^*$, and further* $\mathrm{Re}\,\langle A, B \rangle = \mathrm{Re}\,\langle B, A \rangle$.

2. *Homogeneity:* $\langle kA, B\rangle = k\langle A, B\rangle$, $k \in \mathbb{R}$.
3. *Distributive law:* $\langle A+B, C\rangle = \langle A, C\rangle + \langle B, C\rangle$, $\langle A, B+C\rangle = \langle A, B\rangle + \langle A, C\rangle$.
4. *Non-negativity:* $\langle A, A\rangle \geq 0$, $\langle A, A\rangle = 0$ *if and only if* $A = 0$.

Proof. The proofs of several results are easy, and hence, we omit them here. □

Proposition 2.3. *Let* $A, B \in \mathbb{Q}^{n_1\times n_2}$, *then* $\|A+B\|_F^2 = \|A\|_F^2 + \|B\|_F^2 + 2\mathrm{Re}\langle A, B\rangle$.

Proof. Using the F-norm of quaternion matrix and the distributive law of inner product of quaternion matrix in Proposition 2.2,

$$\begin{aligned}\|A+B\|_F^2 &= \langle A+B, A+B\rangle = \langle A+B, A\rangle + \langle A+B, B\rangle \\ &= \langle A, A\rangle + \langle B, A\rangle + \langle A, B\rangle + \langle B, B\rangle \\ &= \langle A, A\rangle + \langle B, B\rangle + \langle B, A\rangle + \langle A, B\rangle \\ &= \|A\|_F^2 + \|B\|_F^2 + \left(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} b_{ij}^* a_{ij} + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} a_{ij}^* b_{ij}\right) \\ &= \|A\|_F^2 + \|B\|_F^2 + 2\left(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \mathrm{Re}(a_{ij}^* b_{ij})\right) \\ &= \|A\|_F^2 + \|B\|_F^2 + 2\mathrm{Re}\langle A, B\rangle .\end{aligned}$$

The desired result holds. □

Proposition 2.4. *For any quaternion matrix* $A \in \mathbb{Q}^{n_1\times n_2}$ *with* $\mathrm{rank}(A) = r$ $(r > 0)$ *and* $n_1 \geq n_2$, *we have*

$$\|A\|_F \leq \|A\|_* \leq \sqrt{r}\|A\|_F, \ \|A\|_F \leq \|A\|_1 \leq n_1\|A\|_F.$$

Proof. Suppose that $\sigma_1, \sigma_2, \ldots, \sigma_r$ are positive singular values of A, it is obvious that the following inequalities are true by using the definitions of norms of quaternion matrix:

$$\left(\sum_{i=1}^{r}\sigma_i^2\right)^{\frac{1}{2}} \leq \sum_{i=1}^{r}\sigma_i \leq \sqrt{r}\left(\sum_{i=1}^{r}\sigma_i^2\right)^{\frac{1}{2}}, \tag{2.9}$$

and hence, $\|A\|_F \leq \|A\|_* \leq \sqrt{r}\|A\|_F$. Moreover, since

$$\|A\|_F = \sqrt{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}|a_{ij}|^2} \leq \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}|a_{ij}| = \|A\|_1 \tag{2.10}$$

and

$$\frac{\|A\|_1}{n_1n_2} = \frac{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}|a_{ij}|}{n_1n_2} \leq \sqrt{\frac{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}|a_{ij}|^2}{n_1n_2}} \leq \frac{\sqrt{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}|a_{ij}|^2}}{n_2} = \frac{\|A\|_F}{n_2},$$

it follows that $\|A\|_F \leq \|A\|_1 \leq n_1\|A\|_F$. □

Proposition 2.5 ([9, Proposition 4 in Appendix])**.** *For any* $A, B \in \mathbb{Q}^{n_1 \times n_2}$*,*

$$\mathrm{Re}\left(\langle A, B\rangle\right) \leq \|A\| \cdot \|B\|_*.$$

In addition, for any quaternion matrix B*, there is a matrix* A *obeying* $\|A\| = 1$*, which achieves the above equality.*

Proposition 2.6. *For any* $A = (a_{ij}) \in \mathbb{Q}^{n_1 \times n_2}$*,* $B = (b_{ij}) \in \mathbb{Q}^{n_1 \times n_2}$*,*

$$\mathrm{Re}\left(\langle A, B\rangle\right) \leq \|A\|_F \cdot \|B\|_F, \ \mathrm{Re}\left(\langle A, B\rangle\right) \leq \|A\|_\infty \cdot \|B\|_1.$$

Proof. Arrange the elements of quaternion matrix A and B into sequences of quaternions, denoted by

$$vecA = (a_{11}, a_{12}, ..., a_{1n_2}, a_{21}, ..., a_{2n_2}, ..., a_{n_1 1}, a_{n_1 2}, ..., a_{n_1 n_2}), \tag{2.11}$$

$$vecB = (b_{11}, b_{12}, ..., b_{1n_2}, b_{21}, ..., b_{2n_2}, ..., b_{n_1 1}, b_{n_1 2}, ..., b_{n_1 n_2}). \tag{2.12}$$

Using the variant of Cauchy-Schwarz's inequality for quaternions[12], we know that

$$\left|\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} a_{ij}^* b_{ij}\right|^2 \leq \left(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} |a_{ij}|^2\right)\left(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} |b_{ij}|^2\right), \tag{2.13}$$

In fact, the right-hand of (2.13) is equal to $\|A\|_F^2 \cdot \|B\|_F^2$, and it is obvious that

$$\left(\mathrm{Re}\left(\langle A, B\rangle\right)\right)^2 = \left(\mathrm{Re}\left(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} a_{ij}^* b_{ij}\right)\right)^2 \leq \left|\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} a_{ij}^* b_{ij}\right|^2.$$

Thus, $\mathrm{Re}\left(\langle A, B\rangle\right) \leq \|A\|_F \cdot \|B\|_F$.

Using the Cauchy-Schwarz's inequality for real numbers, we can get

$$\begin{aligned}
|\mathrm{Re}\left(\langle A, B\rangle\right)|^2 &= \left|\mathrm{Re}\left(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} a_{ij}^* b_{ij}\right)\right|^2 \leq \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} |a_{ij}|^2 |b_{ij}|^2 \\
&\leq \max |a_{ij}|^2 \cdot \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} |b_{ij}|^2 \leq \max |a_{ij}|^2 \cdot \left(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} |b_{ij}|\right)^2 \\
&= \|A\|_\infty^2 \cdot \|B\|_1^2.
\end{aligned}$$

Thus, $\mathrm{Re}\left(\langle A, B\rangle\right) \leq \|A\|_\infty \cdot \|B\|_1$. ☐

Proposition 2.7 ([3, Theorem 4.3])**.** *Suppose that* $f : \mathbb{Q}^{n_1 \times r} \to \mathbb{R}$ *be defined by* $f(Y) = \frac{1}{2}\|YB + C\|_F^2$*, where* $B \in \mathbb{Q}^{r \times n_2}$ *and* $C \in \mathbb{Q}^{n_1 \times n_2}$*. Then*

$$\nabla f(Y) = (YB + C)\, B^*. \tag{2.14}$$

Proposition 2.8 ([3, Theorem 4.4])**.** *Suppose that* $f : \mathbb{Q}^{r \times n_2} \to \mathbb{R}$ *be defined by* $f(Y) = \frac{1}{2}\|AY + C\|_F^2$*, where* $A \in \mathbb{Q}^{n_1 \times r}$ *and* $C \in \mathbb{Q}^{n_1 \times n_2}$*. Then*

$$\nabla f(Y) = A^*\left(AY + C\right). \tag{2.15}$$

3 Main Model and Result

3.1 Stable Quaternion PCP

The relaxed quaternion PCP problem aims to give stable estimates of the low-rank and the sparse quaternion matrix. Mathematically, suppose the measurement model we observe in the paper is

$$M{=}L_0 + S_0 + Z_0 \in \mathbb{Q}^{n_1\times n_2}, \tag{3.1}$$

where L_0 is low-rank, S_0 is sparse which acts as the some corruption data, and Z_0 is a noise term - say independent identically distributed (i.i.d) noise on each entry of the quaternion matrix M.

Suppose that quaternion matrix $L_0 \in \mathbb{Q}^{n_1\times n_2}$ with $\operatorname{rank}(L_0) = r$ has the following QSVD form:

$$L_0{=}U\Sigma_r V^* = \sum_{i=1}^{r} \sigma_i u_i v_i^*, \tag{3.2}$$

where $U{=}(u_1,\ldots,u_r) \in \mathbb{Q}^{n_1\times r}$, $V{=}(v_1,\ldots,v_r) \in \mathbb{Q}^{n_2\times r}$, Σ_r is a real positive $r\times r$ diagonal matrix, and $\sigma_1,\sigma_2,\ldots,\sigma_r$ are singular values of L_0. The matrix L_0 is said to obey *incoherence condition* means [9]:

$$\max_i \|U^* e_i\|^2 \le \frac{\mu r}{n_1}, \quad \max_i \|V^* e_i\|^2 \le \frac{\mu r}{n_2}, \quad \text{and} \quad \|UV^*\|_\infty \le \sqrt{\frac{\mu r}{n_1 n_2}}, \tag{3.3}$$

where $e_i \in \mathbb{R}^n$ $(i = 1,2,\ldots,n)$ are the canonical basis vectors and μ is a positive constant.

Suppose that L_0 and S_0 satisfy the following two basic assumptions.

Assumption 3.1. The low-rank matrix L_0 obeys incoherence condition.

Assumption 3.2. The support set of sparse matrix S_0 (all the nonzero entries of S_0) is uniformly distributed among all subsets of size m.

Based on Assumptions 3.1 and 3.2, we hope to obtain stable estimates of L_0 and S_0 by solving the following convex minimization problem:

$$\begin{aligned} \min \quad & \|L\|_* + \lambda\|S\|_1 \\ s.t. \quad & \|M - L - S\|_F \le \delta. \end{aligned} \tag{3.4}$$

The problem (3.4) is a generalization of relaxed PCP model in the case of real matrix[16] to the quaternion matrix. Denote $n_{(1)} = \max\{n_1,n_2\}$ and $n_{(2)} = \min\{n_1,n_2\}$, respectively. We will show that the following result holds.

Theorem 3.1. *Suppose that $L_0 \in \mathbb{Q}^{n_1\times n_2}$ obeys incoherence condition (3.3), the support set of $S_0 \in \mathbb{Q}^{n_1\times n_2}$ is uniformly distributed, and $Z_0 \in \mathbb{Q}^{n_1\times n_2}$ satisfies $\|Z_0\|_F \le \delta$. If L_0 and S_0 satisfy*

$$\operatorname{rank}(L_0) \le \rho_r n_{(2)} \mu^{-1} (\log n_{(1)})^{-2} \quad \text{and} \quad m \le \rho_s n_1 n_2,$$

where $\rho_r,\rho_s > 0$ are sufficiently small numerical constants, then there exists a positive constant C such that the solution $\left(\hat{L},\hat{S}\right)$ to the convex program (3.4) satisfies

$$\left\|\hat{L} - L_0\right\|_F^2 + \left\|\hat{S} - S_0\right\|_F^2 \le C n_{(1)}^2 \delta^2.$$

This is the main result in this paper. It shows that the stable estimates of L_0 and S_0 can be given by the convex optimization problem (3.4) based on Assumptions 3.1 and 3.2. We will prove it in detail later.

In order to simplify the analysis and proof of Theorem 3.1, similar to the real matrix case in [1, 16, 2] and the quaternion matrix case in [9], *Bernoulli model* is considered to replace uniform sampling for the support set Ω of sparse matrix S_0. We suppose $\Omega = \{ (i,j) |\, \delta_{ij} = 1\}$ sampled according to the Bernoulli model, where $\{\delta_{ij}\}_{1\leq i\leq n_1, 1\leq j\leq n_2}$ is the sequence of independent identically distributed (0 or 1) Bernoulli random variables with parameter

$$\rho = P\left(\delta_{ij} = 1\right) = \frac{m}{n_1 n_2}. \tag{3.5}$$

Let T denote the subspace generated by quaternion matrices with the same row space or column space as L_0, which is given by

$$T = \{UY^* + ZV^* |\, Y \in \mathbb{Q}^{n_2\times r}, Z \in \mathbb{Q}^{n_1\times r}\} \subset \mathbb{Q}^{n_1\times n_2},$$

where U, V are defined in (3.2). Let P_T be the unitary projection operator onto the quaternion matrix space T, then

$$P_T(A) = \min_{E\in T} \|E - A\|_F^2\,, \tag{3.6}$$

where A is an arbitrary quaternion matrix.

Let P_Ω be the unitary projection onto the quaternion matrix space supported on $\Omega \subseteq [n_1] \times [n_2]$, defined as

$$(P_\Omega(S))_{ij} = \begin{cases} S_{ij}, & (i,j) \in \Omega, \\ 0, & (i,j) \notin \Omega. \end{cases} \tag{3.7}$$

For any quaternion matrix pair $X = (L, S) \in \mathbb{Q}^{n_1\times n_2}$, we denote

$$\|X\|_F \doteq \left(\|L\|_F^2 + \|S\|_F^2\right)^{1/2} \quad \text{and} \quad \|X\|_\dagger = \|L\|_* + \lambda\|S\|_1. \tag{3.8}$$

Define the projection operator $P_T \times P_\Omega : (L, S) \mapsto (P_T(L), P_\Omega(S))$, and $(P_{T^\perp} \times P_{\Omega^\perp})$ is the complement of $P_T \times P_\Omega$. Define the subspace $\Gamma \doteq \{(Q,Q)\,|Q \in \mathbb{Q}^{n_1\times n_2}\}$, $\Gamma^\perp$ is the orthogonal complement to Γ, and P_Γ and $P_{\Gamma^\perp}$ are their respective projection operators.

3.2 Subgradients of quaternion matrix norm

Let $A = (a_{ij}) \in \mathbb{Q}^{n_1\times n_2}$, the subgradient of any quaternion matrix norm of A in [9] is defined by:

$$\partial|||A||| = \left\{D \in \mathbb{Q}^{n_1\times n_2} : |||B||| \geq |||A||| + \mathrm{Re}\left(\langle D, B - A\rangle\right), \forall\, B \in \mathbb{Q}^{n_1\times n_2}\right\}, \tag{3.9}$$

where $|||A|||$ denote any quaternion matrix norm of A.

For the definition of subgradient of quaternion matrix norm, Jia, Ng, and Song defined it in [9] by using the real part of the inner product of D and $B - A$. While Qi, Luo, Wang and Zhang defined it by R-product in [14]. Moreover, Qi, Luo, Wang and Zhang proved that these two definitions are consistent, and also revealed that the subgradient of the quaternion matrix norm could be considered as the subgradient of the norm of the real matrix variables. We use the definition in [9] here because Jia, Ng, and Song further gave the concrete form of the quaternion matrix l_1-norm and the nuclear norm subgradients, which is necessary for us to prove the main result.

Proposition 3.2 ([9, Lemma 2]). *Let $A \in \mathbb{Q}^{n_1 \times n_2}$, then the subgradient of the l_1-norm $\|\cdot\|_1$ at A supported on Ω is given by*

$$\partial\|A\|_1 = \left\{D \in \mathbb{Q}^{n_1 \times n_2} : D = \operatorname{direct}(A) + F, P_\Omega(F) = 0, \|F\|_\infty \leq 1\right\}, \tag{3.10}$$

where $\operatorname{direct}(A)$ *is an $n_1 \times n_2$ matrix with its entries being given by* $\left[\frac{a_{ij}}{|a_{ij}|}\right]_{n_1 \times n_2}$.

Proposition 3.3 ([9, Lemma 3]). *Let $A \in \mathbb{Q}^{n_1 \times n_2}$ have QSVD as in (2.7), then D is a subgradient of the nuclear norm $\|\cdot\|_*$ at A if*

$$D = \sum_{1 \leq k \leq r} u_k v_k^* + W, \tag{3.11}$$

where u_k and v_k $(1 \leq k \leq r)$ are the column vectors of U and V, respectively, and W obeys: (1) the right column space of W is unitary to U, and the left row space of W is unitary to V; (2) $\|W\| \leq 1$.

3.3 Main result

First of all, we can obtain the specific expression of $P_T(A)$ in the following Lemma by solving optimization problem (3.6):

Lemma 3.4. *For an arbitrary quaternion matrix $A \in \mathbb{Q}^{n_1 \times n_2}$,*

$$P_T(A) = P_U A + A P_V - P_U A P_V,$$

where $P_U = UU^*$ and $P_V = VV^*$ are unitary projections onto U and V, respectively.

Proof. From (3.6), we know that

$$P_T(A) = \min_{E \in T} \|E - A\|_F^2 = \min_{Y,Z} \|UY^* + ZV^* - A\|_F^2 = \min_{Y,Z} \|YU^* + VZ^* - A^*\|_F^2.$$

Let

$$f(Y, Z) = \|UY^* + ZV^* - A\|_F^2 = \|YU^* + VZ^* - A^*\|_F^2,$$

Based on the first-order optimality condition of QMO problem in [14, Theorem 4.3], suppose that $E_\diamond = UY^* + ZV^* \in \mathbb{Q}^{n_1 \times n_2}$ is an optimal solution of (3.6), then we can obtain

$$\frac{\partial f}{\partial Y} = 0 \quad \textit{and} \quad \frac{\partial f}{\partial Z} = 0.$$

By Proposition 2.7 and 2.8, we have

$$\frac{\partial f}{\partial Y} = 2\left(YU^* + VZ^* - A^*\right)U = 0$$

and

$$\frac{\partial f}{\partial Z} = 2\left(UY^* + ZV^* - A\right)V = 0,$$

Then we can derive

$$Y = A^*U - VZ^*U \quad \textit{and} \quad Z = AV - UY^*V.$$

Therefore,

$$
\begin{aligned}
E_{\diamondsuit} &= UY^* + ZV^* \\
&= U(A^*U - VZ^*U)^* + (AV - UY^*V)\,V^* \\
&= UU^*A - UU^*ZV^* + AVV^* - UY^*VV^* \\
&= UU^*A + AVV^* - UU^*ZV^* - U(A^*U - VZ^*U)^*VV^* \\
&= UU^*A + AVV^* - UU^*ZV^* - (UU^*AVV^* - UU^*ZV^*) \\
&= UU^*A + AVV^* - UU^*AVV^* \\
&= P_UA + AP_V - P_UAP_V,
\end{aligned}
$$

where $P_U = UU^*$ and $P_V = VV^*$ are unitary projections onto U and V.

In other words, $P_T(A) = P_UA + AP_V - P_UAP_V$. □

The following lemma comes from [9, Lemma 4], and we give its proof for completeness.

Lemma 3.5. *Let $A, B \in \mathbb{Q}^{n_1 \times n_2}$. Then,* $\mathrm{Re}\,(\langle A, P_T(B)\rangle) = \mathrm{Re}\,(\langle P_T(A), B\rangle)$.

Proof. By using the definition of the inner product between two quaternion matrices and Lemma 3.4, we have

$$
\begin{aligned}
\mathrm{Re}\,(\langle A, P_T(B)\rangle) &= \mathrm{Re}\,[\mathrm{Tr}\,(A^*\,(P_UB + BP_V - P_UBP_V))] \\
&= \mathrm{Re}\,[\mathrm{Tr}\,(A^*P_UB + A^*BP_V - A^*P_UBP_V)] \\
&= \mathrm{Re}\,[\mathrm{Tr}\,(A^*P_UB)] + \mathrm{Re}\,[\mathrm{Tr}\,(A^*BP_V)] - \mathrm{Re}\,[\mathrm{Tr}\,(A^*P_UBP_V)]\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Re}\,(\langle P_T(A), B\rangle) &= \mathrm{Re}\,\left[\mathrm{Tr}\,\left((P_UA + AP_V - P_UAP_V)^*B\right)\right] \\
&= \mathrm{Re}\,[\mathrm{Tr}\,(A^*P_UB + P_VA^*B - P_VA^*P_UB)] \\
&= \mathrm{Re}\,[\mathrm{Tr}\,(A^*P_UB)] + \mathrm{Re}\,[\mathrm{Tr}\,(P_VA^*B)] - \mathrm{Re}\,[\mathrm{Tr}\,(P_VA^*P_UB)]\,,
\end{aligned}
$$

and

$$
\mathrm{Re}\,[\mathrm{Tr}\,(A^*BP_V)] = \mathrm{Re}\,[\mathrm{Tr}\,(P_VA^*B)]\,, \quad \mathrm{Re}\,[\mathrm{Tr}\,(A^*P_UBP_V)] = \mathrm{Re}\,[\mathrm{Tr}\,(P_VA^*P_UB)]\,.
$$

Thus, we can get $\mathrm{Re}\,(\langle A, P_T(B)\rangle) = \mathrm{Re}\,(\langle P_T(A), B\rangle)$. □

Generalizing [1, Lemma 2.5] to the quaternion matrix, we can get the following result.

Lemma 3.6. *Suppose that $\|P_\Omega P_T\| \leq 1/2$ and $\lambda \leq 1$. If there is a matrix triple (W, F, D) satisfying*

$$
UV^* + W = \lambda\,(\mathrm{direct}(S_0) + F + P_\Omega(D))\,, \tag{3.12}
$$

where U, V as in (3.2), $P_T(W) = 0$, $\|W\| \leq 1/2$, $P_\Omega F = 0$, $\|F\|_\infty \leq 1/2$, $\|P_\Omega(D)\|_F \leq 1/4$. Then (L_0, S_0) is the unique solution to the minimization problem

$$
\begin{aligned}
\min \quad & \|L\|_* + \lambda\|S\|_1 \\
s.t. \quad & M' = L + S.
\end{aligned} \tag{3.13}
$$

Proof. We consider a feasible perturbation $(L_0 + H, S_0 - H)$ with $H \neq 0$, and prove (L_0, S_0) is the optimal solution of problem (3.13). Let $UV^* + W_0$ be an arbitrary subgradient of L_0, and $\mathrm{direct}(S_0) + F_0$ be an arbitary subgradient of l_1-norm of S_0. From Proposition 3.3, we know that W_0 obeys $\|W_0\| \leq 1$, $U^*W_0 = 0$, and $W_0V = 0$ which equal to $P_T(W_0) = 0$. Similarly, from Proposition 3.2, it follows that F_0 obeys $P_\Omega(F_0) = 0$ and $\|F_0\|_\infty \leq 1$. Using the definition of subgradients of quarternion matrix in (3.9) and concrete expressions of the nuclear norm and l_1-norm of quaternion matrix, we have

$$\begin{aligned}&\|L_0+H\|_*+\lambda\|S_0-H\|_1\\ \geq&\|L_0\|_*+\|S_0\|_1+\mathrm{Re}\,\langle UV^*+W_0,H\rangle-\lambda\mathrm{Re}\,\langle \mathrm{direct}(S_0)+F_0,H\rangle\\ =&\|L_0\|_*+\|S_0\|_1+\mathrm{Re}\,\langle UV^*-\lambda\mathrm{direct}(S_0),H\rangle+\mathrm{Re}\,\langle W_0,H\rangle-\lambda\mathrm{Re}\,\langle F_0,H\rangle\,.\end{aligned} \tag{3.14}$$

Now let's narrow down the last three terms in (3.14). Because of the properties of W_0 and F_0, we can choose $W_0=P_{T^\perp}(W_0)$, where $\|W_0\|=1$ and $F_0=-\mathrm{direct}(P_{\Omega^\perp}(H))$. Then, using Lemma 3.5, Proposition 2.5 and Proposition 2.6, we have

$$\mathrm{Re}\,\langle W_0,H\rangle=\mathrm{Re}\,\langle P_{T^\perp}(W_0),H\rangle=\mathrm{Re}\,\langle W_0,P_{T^\perp}(H)\rangle=\|P_{T^\perp}(H)\|_* \tag{3.15}$$

and

$$\mathrm{Re}\,\langle F_0,H\rangle=\mathrm{Re}\,\langle -\mathrm{direct}(P_{\Omega^\perp}(H)),H\rangle=-\left\|P_{\Omega^\perp}(H)\right\|_1. \tag{3.16}$$

Also, we have

$$\begin{aligned}&|\mathrm{Re}\,\langle UV^*-\lambda\mathrm{direct}(S_0),H\rangle|\\ =&\,|\mathrm{Re}\,\langle \lambda F-W+\lambda P_\Omega(D),H\rangle|\\ \leq&\lambda\,|\mathrm{Re}\,\langle F,H\rangle|+|\mathrm{Re}\,\langle W,H\rangle|+\lambda\,|\mathrm{Re}\,\langle P_\Omega(D),H\rangle|\\ =&\lambda\,|\mathrm{Re}\,\langle P_{\Omega^\perp}(F),H\rangle|+|\mathrm{Re}\,\langle P_{T^\perp}(W),H\rangle|+\lambda\,|\mathrm{Re}\,\langle P_\Omega P_\Omega(D),H\rangle|\\ =&\lambda\,|\mathrm{Re}\,\langle F,P_{\Omega^\perp}(H)\rangle|+|\mathrm{Re}\,\langle W,P_{T^\perp}(H)\rangle|+\lambda\,|\mathrm{Re}\,\langle P_\Omega(D),P_\Omega(H)\rangle|\\ \leq&\lambda\|F\|_\infty\cdot\|P_{\Omega^\perp}(H)\|_1+\|W\|\cdot\|P_{T^\perp}(H)\|_*+\lambda\|P_\Omega(D)\|_F\cdot\|P_\Omega(H)\|_F\\ \leq&\frac{\lambda}{2}\|P_{\Omega^\perp}(H)\|_1+\frac{1}{2}\|P_{T^\perp}(H)\|_*+\frac{\lambda}{4}\|P_\Omega(H)\|_F,\end{aligned}$$

where the second inequality follows from distributive law of inner product of quaternion matrix, the fourth equality follows from Lemma 3.5, and the fifth inequality follows from Proposition 2.5 and Proposition 2.6. From the above inequality, it is obvious that

$$\mathrm{Re}\,\langle UV^*-\lambda\mathrm{direct}(S_0),H\rangle\geq-\frac{\lambda}{2}\|P_{\Omega^\perp}(H)\|_1-\frac{1}{2}\|P_{T^\perp}(H)\|_*-\frac{\lambda}{4}\|P_\Omega(H)\|_F. \tag{3.17}$$

Now, substituting (3.15), (3.16) and (3.17) into (3.14), we obtain that

$$\begin{aligned}&\|L_0+H\|_*+\lambda\|S_0-H\|_1\\ \geq&\|L_0\|_*+\|S_0\|_1+\frac{1}{2}\left(\|P_{T^\perp}(H)\|_*+\lambda\|P_{\Omega^\perp}(H)\|_1\right)-\frac{\lambda}{4}\|P_\Omega(H)\|_F.\end{aligned} \tag{3.18}$$

Furthermore, since

$$\begin{aligned}\|P_\Omega(H)\|_F=&\left\|P_\Omega\left(P_T+P_{T^\perp}\right)(H)\right\|_F\leq\|P_\Omega P_T(H)\|_F+\|P_\Omega P_{T^\perp}(H)\|_F\\ \leq&\frac{1}{2}\|H\|_F+\|P_{T^\perp}(H)\|_F\leq\frac{1}{2}\left(\|P_\Omega(H)\|_F+\|P_{\Omega^\perp}(H)\|_F\right)+\|P_{T^\perp}(H)\|_F,\end{aligned}$$

we have

$$\|P_\Omega(H)\|_F\leq\|P_{\Omega^\perp}(H)\|_F+2\|P_{T^\perp}(H)\|_F\leq\|P_{\Omega^\perp}(H)\|_1+2\|P_{T^\perp}(H)\|_*, \tag{3.19}$$

where the second inequality follows from Proposition 2.4.

Finally, substituting (3.19) into (3.18), we have

$$\|L_0 + H\|_* + \lambda\|S_0 - H\|_1 \geq \|L_0\|_* + \|S_0\|_1 + \frac{1}{2}\left((1-\lambda)\|P_{T^\perp}(H)\|_* + \frac{\lambda}{2}\|P_{\Omega^\perp}(H)\|_1\right).$$

Since $(1-\lambda)\|P_{T^\perp}(H)\|_* + \frac{\lambda}{2}\|P_{\Omega^\perp}(H)\|_1 > 0$ if $H \neq 0$, the above inequality implies the desired result. □

Next, we discuss the conditions used in Lemma 3.6.

If there is a matrix triple (W, F, D) obeying

$$UV^* + W = \lambda\left(\text{direct}(S_0) + F + P_\Omega(D)\right), \tag{3.20}$$

where $P_T(W) = 0$, $\|W\| \leq 1/2$, $P_\Omega(F) = 0$, $\|F\|_\infty \leq 1/2$, and $\|P_\Omega(D)\|_F \leq 1/4$, then, by noting that $P_{\Omega^\perp}(\text{direct}(S_0)) = 0$ (since Ω is the support set of S_0), we have

$$\|P_\Omega(UV^* - \lambda\text{direct}(S_0) + W)\|_F = \lambda\|P_\Omega(F + P_\Omega(D))\|_F \leq \lambda\|P_\Omega(D)\|_F \leq \lambda/4,$$

$$\|P_{\Omega^\perp}(UV^* + W)\|_\infty = \lambda\|P_{\Omega^\perp}(\text{direct}(S_0) + F + P_\Omega(D))\|_\infty = \lambda\|F\|_\infty \leq \lambda/2,$$

and hence, W satisfies

$$\begin{cases} W \in T^\perp, \|W\| \leq 1/2, \\ \|P_\Omega(UV^* - \lambda\text{direct}(S_0) + W)\|_F \leq \lambda/4, \\ \|P_{\Omega^\perp}(UV^* + W)\|_\infty \leq \lambda/2. \end{cases} \tag{3.21}$$

Conversely, suppose that W satisfies (3.21), we show that the conditions used in Lemma 3.6 hold. For this purpose, we need to show that there is a matrix pair (F, D) which satisfies (3.12), i.e.,

$$UV^* + W = \lambda\left(\text{direct}(S_0) + F + P_\Omega(D)\right),$$

and $P_\Omega(F) = 0$, $\|F\|_\infty \leq 1/2$, and $\|P_\Omega(D)\|_F \leq 1/4$. Firstly, we choose a matrix F supported on $\Omega^\perp$, then $P_\Omega(F) = 0$. Secondly, from the second inequality of (3.21), we have

$$\lambda/4 \geq \|P_\Omega(UV^* - \lambda\text{direct}(S_0) + W)\|_F = \lambda\|P_\Omega(F + P_\Omega(D))\|_F = \lambda\,\|P_\Omega(D)\|_F\,, \tag{3.22}$$

which yields $\|P_\Omega(D)\|_F \leq 1/4$. Thirdly, from the third inequality of (3.21), we have

$$\begin{aligned} \lambda/2 &\geq \|P_{\Omega^\perp}(UV^* + W)\|_\infty \\ &= \lambda\|P_{\Omega^\perp}(\text{direct}(S_0) + F + P_\Omega(D))\|_\infty \\ &= \lambda\|P_{\Omega^\perp}(F)\|_\infty = \lambda\,\|F\|_\infty\,, \end{aligned}$$

and hence, $\|F\|_\infty \leq 1/2$. Therefore, the desired result holds.

Therefore, as a corollary of Lemma 3.6, if W satisfies (3.21), (L_0, S_0) is the unique solution to the minimization problem (3.13).

Suppose the observed quaternion matrix M is decomposed without the i.i.d noise term Z_0 in (3.1). If there exists matrix triple (W, F, D) such that (3.12) holds (or there exists a matrix W such that (3.21) holds), then (L_0, S_0) is the unique solution to the problem (3.13). In this paper, however, we discuss that M can be decomposed into a form like (3.1), which contains the i.i.d. noise term Z_0 with $\|Z_0\|_F \leq \delta$ for some $\delta > 0$. In this case, it is difficult to directly recover (L_0, S_0) exactly from problem (3.4). Suppose that $\left(\hat{L}, \hat{S}\right)$ is the optimal solution to (3.4), however, it is expected that the difference between $(\hat{L}, \hat{S})$ and (L_0, S_0) is very small when δ is very small. This is exactly what Theorem 3.1 proves.

Next, we use dual certificate in (3.21) to give the following theorem.

Theorem 3.7. *Suppose that* $\|P_\Omega P_T\| \le 1/2$, $\lambda \le 1/2$, *and there is a matrix* W *satisfying (3.21). Let* $\hat{X} = (\hat{L}, \hat{S})$ *be the solution to minimization problem (3.4) with* $\lambda = \frac{1}{\sqrt{n_{(1)}}}$. *Then,* $\hat{X}$ *satisfies*

$$\left\|X_0 - \hat{X}\right\|_F \le \left(8\sqrt{5}n_{(1)} + 1\right)\delta, \tag{3.23}$$

where $X_0 = (L_0, S_0)$.

Proof. Since $X_0 = (L_0, S_0)$ is a feasible solution of (3.4), we have that $\|\hat{X}\|_\dagger \le \|X_0\|_\dagger$ and

$$\begin{aligned}\left\|\hat{L} + \hat{S} - L_0 - S_0\right\|_F &= \left\|(\hat{L} + \hat{S} - M) - (L_0 + S_0 - M)\right\|_F \\ &\le \left\|\hat{L} + \hat{S} - M\right\|_F + \|L_0 + S_0 - M\|_F \le 2\delta.\end{aligned} \tag{3.24}$$

Let $\hat{X} = X_0 + H$, where $H := (H_L, H_S)$. Denote $H^\Gamma := P_\Gamma(H)$ and $H^{\Gamma^\perp} := P_{\Gamma^\perp}(H)$. Then, we need to estimate the upper bound of $\|H\|_F$. It is easy to see that

$$\begin{aligned}\|H\|_F^2 &= \|(P_\Gamma + P_{\Gamma^\perp})(H)\|_F^2 \\ &= \|P_\Gamma(H)\|_F^2 + \|P_{\Gamma^\perp}(H)\|_F^2 + 2\mathrm{Re}\left(\langle P_\Gamma(H), P_{\Gamma^\perp}(H)\rangle\right) \\ &= \left\|H^\Gamma\right\|_F^2 + \left\|H^{\Gamma^\perp}\right\|_F^2 \\ &= \left\|H^\Gamma\right\|_F^2 + \left\|(P_{T^\perp} \times P_{\Omega^\perp})\left(H^{\Gamma^\perp}\right)\right\|_F^2 + \left\|(P_T \times P_\Omega)\left(H^{\Gamma^\perp}\right)\right\|_F^2,\end{aligned} \tag{3.25}$$

where the second equality follows from Proposition 2.3. Next, we estimate the upper bounds of three terms lies in the right-hand of (3.25), which are discussed respectively in 1), 2) and 3) below.

1) The upper bound of $\|H^\Gamma\|_F^2$ (the first term in the right-hand of (3.25)).

For the matrix pair $H = (H_L, H_S)$, we have

$$H^\Gamma = P_\Gamma(H) = \left(\frac{H_L + H_S}{2}, \frac{H_L + H_S}{2}\right),$$

from the definition of Γ. Then, by using (3.8) and (3.24), we can get

$$\left\|H^\Gamma\right\|_F^2 = \left\|\frac{H_L + H_S}{2}\right\|_F^2 + \left\|\frac{H_L + H_S}{2}\right\|_F^2 = \frac{1}{2}\|H_L + H_S\|_F^2 \le \frac{1}{2}\cdot(2\delta)^2 = 2\delta^2. \tag{3.26}$$

2) The upper bound of $\|(P_{T^\perp} \times P_{\Omega^\perp})(H^{\Gamma^\perp})\|_F^2$ (the second term in the right-hand of (3.25)).

Write $\Lambda = UV^* + W$, where W satisfies (3.21). For any perturbation $(H_L^{\Gamma^\perp}, H_S^{\Gamma^\perp})$ satisfying $H_L^{\Gamma^\perp} + H_S^{\Gamma^\perp} = 0$, and any $Z = (Z_L, Z_S) \in \partial\|X_0\|_\dagger$ (that is, $Z_L \in \partial\|L_0\|_*$ and $Z_S \in \partial(\lambda\|S_0\|_1)$), it follows from the definition of the subgradient of any matrix norm of quaternion matrix that

$$\begin{aligned}\left\|L_0 + H_L^{\Gamma^\perp}\right\|_* &\ge \|L_0\|_* + \mathrm{Re}\left(\left\langle Z_L, H_L^{\Gamma^\perp}\right\rangle\right), \\ \lambda\left\|S_0 + H_S^{\Gamma^\perp}\right\|_1 &\ge \lambda\|S_0\|_1 + \mathrm{Re}\left(\left\langle Z_S, H_S^{\Gamma^\perp}\right\rangle\right).\end{aligned} \tag{3.27}$$

From Proposition 3.3, we have $Z_L = UV^* + W_0$ where $W_0 \in T^\perp$, and hence,

$$Z_L = UV^* + W_0 = UV^* + W + W_0 - W = \Lambda + P_{T^\perp}(Z_L - \Lambda).$$

From Lemma 3.6 and Proposition 3.2, we have

$$\begin{aligned}Z_S &= \lambda\left(direct(S_0)+F\right) = UV^* + W - \lambda P_\Omega(D) = \Lambda - \lambda P_\Omega(D)\\ &= \Lambda - \lambda P_\Omega(D) + P_{\Omega^\perp}P_\Omega(D) = \Lambda - \lambda P_\Omega(D) + P_{\Omega^\perp}\left(Z_S - \Lambda\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\mathrm{Re}\left(\left\langle Z_L, H_L^{\Gamma^\perp}\right\rangle\right) + \mathrm{Re}\left(\left\langle Z_S, H_S^{\Gamma^\perp}\right\rangle\right)\\ =&\mathrm{Re}\left(\left\langle \Lambda, H_L^{\Gamma^\perp} + H_S^{\Gamma^\perp}\right\rangle\right) - \lambda\mathrm{Re}\left(\left\langle P_\Omega(D), P_\Omega(H_S^{\Gamma^\perp})\right\rangle\right)\\ &+ \mathrm{Re}\left(\left\langle Z_L - \Lambda, P_{T^\perp}(H_L^{\Gamma^\perp})\right\rangle\right) + \mathrm{Re}\left(\left\langle Z_S - \Lambda, P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\rangle\right)\\ \geq&0 - \lambda\|P_\Omega(D)\|_F \cdot \left\|P_\Omega(H_S^{\Gamma^\perp})\right\|_F + \mathrm{Re}\left(\left\langle Z_L - \Lambda, P_{T^\perp}(H_L^{\Gamma^\perp})\right\rangle\right)\\ &+ \mathrm{Re}\left(\left\langle Z_S - \Lambda, P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\rangle\right)\\ \geq&-\lambda/4\left\|P_\Omega(H_S^{\Gamma^\perp})\right\|_F + \mathrm{Re}\left(\left\langle Z_L - \Lambda, P_{T^\perp}(H_L^{\Gamma^\perp})\right\rangle\right)\\ &+ \mathrm{Re}\left(\left\langle Z_S - \Lambda, P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\rangle\right),\end{aligned} \tag{3.28}$$

where the second inequality holds by Proposition 2.5.

Next, we give the estimate of three terms in the right-hand of (3.28).

Firstly, by using the triangle inequality of quaternion matrix norm, we have

$$\begin{aligned}\left\|P_\Omega(H_S^{\Gamma^\perp})\right\|_F \leq&\left\|P_\Omega P_T(H_S^{\Gamma^\perp})\right\|_F + \left\|P_\Omega P_{T^\perp}(H_S^{\Gamma^\perp})\right\|_F\\ \leq&\|P_\Omega P_T\|\left\|H_S^{\Gamma^\perp}\right\|_F + \left\|P_{T^\perp}(H_S^{\Gamma^\perp})\right\|_F \leq \frac{1}{2}\left\|H_S^{\Gamma^\perp}\right\|_F + \left\|P_{T^\perp}(H_S^{\Gamma^\perp})\right\|_F\\ \leq&\frac{1}{2}\left\|P_\Omega(H_S^{\Gamma^\perp})\right\|_F + \frac{1}{2}\left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_F + \left\|P_{T^\perp}(H_S^{\Gamma^\perp})\right\|_F,\end{aligned}$$

which yields

$$\begin{aligned}\left\|P_\Omega(H_S^{\Gamma^\perp})\right\|_F &\leq \left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_F + 2\left\|P_{T^\perp}(H_S^{\Gamma^\perp})\right\|_F\\ &\leq \left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_1 + 2\left\|P_{T^\perp}(H_S^{\Gamma^\perp})\right\|_*,\end{aligned} \tag{3.29}$$

where the last inequality holds by Proposition 2.4.

Secondly, there exists $Z_L \in \partial\|L_0\|_*$ with $\|Z_L\| = 1$ such that

$$\mathrm{Re}\left(\left\langle Z_L, P_{T^\perp}(H_L^{\Gamma^\perp})\right\rangle\right) = \left\|P_{T^\perp}(H_L^{\Gamma^\perp})\right\|_*$$

and

$$\left|\mathrm{Re}\left(\left\langle \Lambda, P_{T^\perp}(H_L^{\Gamma^\perp})\right\rangle\right)\right| = \left|\mathrm{Re}\left(\left\langle P_{T^\perp}(\Lambda), P_{T^\perp}(H_L^{\Gamma^\perp})\right\rangle\right)\right| \leq \|P_{T^\perp}(\Lambda)\|\left\|P_{T^\perp}(H_L^{\Gamma^\perp})\right\|_*.$$

So,

$$\mathrm{Re}\left(\left\langle Z_L - \Lambda, P_{T^\perp}(H_L^{\Gamma^\perp})\right\rangle\right) \geq \left(1 - \|P_{T^\perp}(\Lambda)\|\right) \cdot \left\|P_{T^\perp}(H_L^{\Gamma^\perp})\right\|_*. \tag{3.30}$$

Similarly, the third term in the right-hand of (3.28) satisfies

$$\operatorname{Re}\left(\left\langle Z_S-\Lambda, P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\rangle\right) \geq (\lambda - \|P_{\Omega^\perp}(\Lambda)\|_\infty)\cdot \left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_1. \tag{3.31}$$

Now, by using (3.29), (3.30) and (3.31), it follows from (3.28) that

$$\begin{aligned}
&\operatorname{Re}\left(\left\langle Z_L, H_L^{\Gamma^\perp}\right\rangle\right) + \operatorname{Re}\left(\left\langle Z_S, H_S^{\Gamma^\perp}\right\rangle\right)\\
\geq& -\lambda/4\left\|P_\Omega(H_S^{\Gamma^\perp})\right\|_F + (1-\|P_{T^\perp}(\Lambda)\|)\cdot\left\|P_{T^\perp}(H_L^{\Gamma^\perp})\right\|_*\\
&+(\lambda-\|P_{\Omega^\perp}(\Lambda)\|_\infty)\cdot\left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_1\\
\geq& -\lambda/4\left(\left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_1 + 2\left\|P_{T^\perp}(H_S^{\Gamma^\perp})\right\|_*\right) + (1-\|P_{T^\perp}(\Lambda)\|)\cdot\left\|P_{T^\perp}(H_L^{\Gamma^\perp})\right\|_*\\
&+(\lambda-\|P_{\Omega^\perp}(\Lambda)\|_\infty)\cdot\left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_1\\
=&(1-\lambda/2-\|P_{T^\perp}(\Lambda)\|)\left\|P_{T^\perp}(H_L^{\Gamma^\perp})\right\|_* + (3\lambda/4-\|P_{\Omega^\perp}(\Lambda)\|_\infty)\left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_1\\
\geq&(3/4-\|P_{T^\perp}(\Lambda)\|)\left\|P_{T^\perp}(H_L^{\Gamma^\perp})\right\|_* + (3\lambda/4-\|P_{\Omega^\perp}(\Lambda)\|_\infty)\left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_1.
\end{aligned} \tag{3.32}$$

Furthermore, by using $\|P_{T^\perp}(\Lambda)\| \leq \frac{1}{2}$, $\|P_{\Omega^\perp}(\Lambda)\|_\infty \leq \frac{\lambda}{2}$, (3.27) and (3.32), we can get

$$\begin{aligned}
\left\|X_0+H^{\Gamma^\perp}\right\|_\dagger \geq&\|X_0\|_\dagger + \operatorname{Re}\left(\left\langle Z_L, H_L^{\Gamma^\perp}\right\rangle\right) + \operatorname{Re}\left(\left\langle Z_S, H_S^{\Gamma^\perp}\right\rangle\right)\\
\geq&\|X_0\|_\dagger + (3/4-\|P_{T^\perp}(\Lambda)\|)\left\|P_{T^\perp}(H_L^{\Gamma^\perp})\right\|_*\\
&+(3\lambda/4-\|P_{\Omega^\perp}(\Lambda)\|_\infty)\left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_1\\
\geq&\|X_0\|_\dagger + 1/4\left(\left\|P_{T^\perp}(H_L^{\Gamma^\perp})\right\|_* + \lambda\left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_1\right).
\end{aligned} \tag{3.33}$$

Moreover, by using the triangle inequality of quaternion matrix norm, we have

$$\left\|X_0+H^{\Gamma^\perp}\right\|_\dagger = \left\|X_0+H-H^\Gamma\right\|_\dagger \leq \|X_0+H\|_\dagger + \left\|H^\Gamma\right\|_\dagger, \tag{3.34}$$

Thus, combining (3.33) with (3.34), we have

$$\begin{aligned}
&\left\|P_{T^\perp}(H_L^{\Gamma^\perp})\right\|_* + \lambda\left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_1 \leq 4\left(\left\|X_0+H^{\Gamma^\perp}\right\|_\dagger - \|X_0\|_\dagger\right)\\
\leq&4\left(\left\|X_0+H^{\Gamma^\perp}\right\|_\dagger - \|X_0+H\|_\dagger\right) \leq 4\left\|H^\Gamma\right\|_\dagger.
\end{aligned} \tag{3.35}$$

Therefore,

$$\begin{aligned}
&\left\|(P_{T^\perp} \times P_{\Omega^\perp})\left(H^{\Gamma^\perp}\right)\right\|_F \\
\le&\left(\left\|P_{T^\perp}(H_L^{\Gamma^\perp})\right\|_F^2 + \left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_F^2\right)^{1/2} \\
\le&\left\|P_{T^\perp}(H_L^{\Gamma^\perp})\right\|_F + \left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_F \le \left\|P_{T^\perp}(H_L^{\Gamma^\perp})\right\|_* + \left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_1 \\
\le&\sqrt{n_{(1)}}\left(\left\|P_{T^\perp}(H_L^{\Gamma^\perp})\right\|_* + \lambda\left\|P_{\Omega^\perp}(H_S^{\Gamma^\perp})\right\|_1\right) \\
\le&4\sqrt{n_{(1)}}\|H^\Gamma\|_\dagger = 4\sqrt{n_{(1)}}(\|H_L^\Gamma\|_* + \lambda\|H_S^\Gamma\|_1) \\
\le&4n_{(1)}(\|H_L^\Gamma\|_F + \lambda\|H_S^\Gamma\|_F) \le 4n_{(1)}(\|H_L^\Gamma\|_F + \|H_S^\Gamma\|_F) \\
=&4n_{(1)}\sqrt{2\left(\|H_L^\Gamma\|_F^2 + \|H_S^\Gamma\|_F^2\right)} = 4\sqrt{2}n_{(1)}\|H^\Gamma\|_F \le 8n_{(1)}\delta,
\end{aligned}$$

where the third and seventh inequalities hold by Proposition 2.4, the fifth inequality holds by (3.35), the ninth equality holds from the fact that $\|H_L^\Gamma\|_F = \|H_S^\Gamma\|_F$, and the last equality holds by (3.8).

3) The upper bound of $\|(P_T \times P_\Omega)(H^{\Gamma^\perp})\|_F^2$ (the third term in the right-hand of (3.25)).

Firstly, since $P_\Gamma(P_T \times P_\Omega)(H^{\Gamma^\perp}) + P_\Gamma(P_{T^\perp} \times P_{\Omega^\perp})(H^{\Gamma^\perp}) = P_\Gamma(H^{\Gamma^\perp}) = 0$, we have

$$\left\|P_\Gamma(P_T \times P_\Omega)(H^{\Gamma^\perp})\right\|_F = \left\|P_\Gamma(P_{T^\perp} \times P_{\Omega^\perp})(H^{\Gamma^\perp})\right\|_F \le \left\|(P_{T^\perp} \times P_{\Omega^\perp})(H^{\Gamma^\perp})\right\|_F. \tag{3.36}$$

Secondly, for quaternion matrix pair $H^{\Gamma^\perp} = (H_L^{\Gamma^\perp}, H_S^{\Gamma^\perp})$, we have

$$\begin{aligned}
&\left\|P_\Gamma\left(P_T \times P_\Omega\right)(H^{\Gamma^\perp})\right\|_F^2 \\
=&\left\|P_\Gamma\left(P_T(H_L^{\Gamma^\perp}), P_\Omega(H_S^{\Gamma^\perp})\right)\right\|_F^2 = \frac{1}{2}\left\|P_T(H_L^{\Gamma^\perp}) + P_\Omega(H_S^{\Gamma^\perp})\right\|_F^2 \\
=&\frac{1}{2}\left(\left\|P_T(H_L^{\Gamma^\perp})\right\|_F^2 + \left\|P_\Omega(H_S^{\Gamma^\perp})\right\|_F^2 + 2\mathrm{Re}\left(\left\langle P_T(H_L^{\Gamma^\perp}), P_\Omega(H_S^{\Gamma^\perp})\right\rangle\right)\right),
\end{aligned}\tag{3.37}$$

where the last equality holds by Proposition 2.3, while

$$\begin{aligned}
\mathrm{Re}\left(\left\langle P_T(H_L^{\Gamma^\perp}), P_\Omega(H_S^{\Gamma^\perp})\right\rangle\right) &= \mathrm{Re}\left(\left\langle P_T P_T(H_L^{\Gamma^\perp}), P_\Omega P_\Omega(H_S^{\Gamma^\perp})\right\rangle\right) \\
&= \mathrm{Re}\left(\left\langle P_T(H_L^{\Gamma^\perp}), (P_T P_\Omega)P_\Omega(H_S^{\Gamma^\perp})\right\rangle\right) \\
&\ge -\|P_T P_\Omega\| \cdot \left\|P_T(H_L^{\Gamma^\perp})\right\|_F \cdot \left\|P_\Omega(H_S^{\Gamma^\perp})\right\|_F,
\end{aligned}\tag{3.38}$$

where the inequality holds because of Proposition 2.5 and compatibility of matrix norm, hence,

$$\begin{aligned}
&\left\|P_\Gamma\left(P_T \times P_\Omega\right)\left(H^{\Gamma^\perp}\right)\right\|_F^2 \\
\ge&\frac{1}{2}\left(\left\|P_T(H_L^{\Gamma^\perp})\right\|_F^2 + \left\|P_\Omega(H_S^{\Gamma^\perp})\right\|_F^2 - \left\|P_T(H_L^{\Gamma^\perp})\right\|_F \cdot \left\|P_\Omega(H_S^{\Gamma^\perp})\right\|_F\right) \\
\ge&\frac{1}{4}\left(\left\|P_T(H_L^{\Gamma^\perp})\right\|_F^2 + \left\|P_\Omega(H_S^{\Gamma^\perp})\right\|_F^2\right) = \frac{1}{4}\left\|(P_T \times P_\Omega)\left(H^{\Gamma^\perp}\right)\right\|_F^2,
\end{aligned}\tag{3.39}$$

where the first inequality follows from (3.38), (3.37) and the condition $\|P_T P_\Omega\| \leq \frac{1}{2}$, and the second inequality follows from the fact that $a^2+b^2-ab \geq \frac{a^2+b^2}{2}$ for any a and b.

Thus, by combining (3.36) with (3.39), we obtain that

$$\left\|(P_T \times P_\Omega)(H^{\Gamma^\perp})\right\|_F^2 \leq 4\left\|(P_{T^\perp} \times P_{\Omega^\perp})(H^{\Gamma^\perp})\right\|_F. \tag{3.40}$$

Now, with the help of 1), 2) and 3), it follows from (3.25) that

$$\begin{aligned}\|H\|_F^2 =& \left\|H^\Gamma\right\|_F^2 + \left\|(P_T \times P_\Omega)(H^{\Gamma^\perp})\right\|_F^2 + \left\|(P_{T^\perp} \times P_{\Omega^\perp})(H^{\Gamma^\perp})\right\|_F^2 \\ \leq& \left\|H^\Gamma\right\|_F^2 + 5\left\|(P_{T^\perp} \times P_{\Omega^\perp})(H^{\Gamma^\perp})\right\|_F^2 \\ \leq& 2\delta^2 + 5 \times \left(8n_{(1)}\delta\right)^2,\end{aligned}$$

and hence,

$$\|H\|_F = \left\|X_0 - \hat{X}\right\|_F \leq \left(8\sqrt{5}n_{(1)} + 1\right)\delta. \tag{3.41}$$

That is, the desired result holds. □

Up to now, we can complete the proof of Theorem 3.1. On the one hand, under the conditions of Theorem 3.1 (Assumptions 3.1 and 3.2), it follows from [9, Lemma 7] that $\|P_\Omega P_T\| \leq 1/2$. On the other hand, similar to Section 2.4 in [1], by using the golfing scheme introduced in [5] and the method of least squares, it is easy to show that there exists a matrix W such that (3.21) holds. These demonstrate that we can obtain the result of Theorem 3.1 by using Theorem 3.7.

4 Conclusion

In this paper, we studied the relaxed quaternion PCP model. In the quaternion framework, we obtained stable estimates of the original low-rank and sparse terms by solving the weighted minimization problem of the nuclear norm and the l_1-norm of quaternion matrices. This provides a theoretical basis for solving the quaternion PCP problem.

A further issue is to design effective algorithms for the quaternion optimization problem to solve the quaternion PCP problem and apply the proposed algorithms to solve some related practical problems. Moreover, it is possible to apply the analysis method in this paper to extend some models of real matrix completion to the case of quaternion matrices.

References

[1] E.J. Candès, X. Li, Y. Ma and J. Wright, Robust principal component analysis?, *J. ACM* 58 (2011) 1–37.

[2] E.J. Candès and B. Recht, Exact matrix completion via convex optimization, *Found. Comput. Math.* 9 (2009) 717–772.

[3] Y. Chen, L. Qi, X. Zhang and Y. Xu, A Low Rank Quaternion Decomposition Algorithm and Its Application in Color Image Inpainting, *arXiv preprint arXiv:2009.12203*, 2020.

[4] Y. Chen, X. Xiao and Y. Zhou, Low-rank quaternion approximation for color image processing, *IEEE Trans. Image Process.* 29 (2019) 1426–1439.

[5] D. Gross, Recovering low-rank matrices from few coefficients in any basis, *IEEE Trans. Inform. Theory*, 57 (2011) 1548–1566.

[6] X. Han, J. Wu, L. Yan, L. Senhadji and H. Shu, Color image recovery via quaternion matrix completion, *2013 6th International Congress on Image and Signal Processing (CISP)*, 2013, pp. 358–362.

[7] W.R. Hamilton, Theory of Quaternions, *Math. Proc. R. Ir. Acad.* 3 (1844) 1–16.

[8] H. Hotelling, Analysis of a complex of statistical variables into principal components, *Journal of Educational Psychology* 24 (1933) 417–441.

[9] Z. Jia, M.K. Ng and G. Song, Robust quaternion matrix completion with applications to image inpainting, *Numer. Linear Algebra Appl.* 26 (2019): e2245.

[10] I.T. Jolliffe, Principal components in regression analysis, in: *Principal component analysis*, 2002, pp. 167–198.

[11] J. Miao and K.I. Kou, Quaternion-based bilinear factor matrix norm minimization for color image inpainting, *IEEE Trans. Signal Process.* 68 (2020) 5617–5631.

[12] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, Cauchy's and Related Inequalities, in: *Classical and New Inequalities in Analysis*, Springer, Dordrecht, 1993, pp. 83–98.

[13] M. Wei, Y. Li, F. Zhang and J. Zhao, *Quaternion Matrix Computations*, Nova Science Publishers, New York, 2018.

[14] L. Qi, Z. Luo, Q. Wang and X. Zhang, Quaternion matrix optimization: Motivation and analysis, *J. Optim. Theory Appl.* 193 (2022) 621–648.

[15] F. Zhang, Quaternions and matrices of quaternions, *Linear Algebra Appl.* 251 (1997) 21–57.

[16] Z. Zhou, X. Li, J. Wright, E.J. Candès and Y. Ma, Stable Principal Component Pursuit, in: *2010 IEEE International Symposium on Information Theory*, 2010, pp. 1518–1522.

[17] C. Zou, K.I. Kou and Y. Wang, Quaternion collaborative and sparse representation with application to color face recognition, *IEEE Trans. Image Process.* 25 (2016) 3287–3302.

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