Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 19, No. 4, pp. 715–733, 2023

A SELF-ADAPTIVE INERTIAL VISCOSITY PROJECTION ALGORITHM FOR SOLVING SPLIT FEASIBILITY PROBLEM WITH MULTIPLE OUTPUT SETS*

Yazheng Dang†, Caihua Hou and Yang Liu

Abstract: We propose an inertial viscosity projection algorithm for solving the split feasibility problem with multiple output sets in the Hilbert spaces. The stepsize of the algorithm is selected via a self-adaptive technique which does not require prior information about operator norm. In addition, the inertial technique and viscosity method are combined to improve the convergence. Under suitable conditions, we show the strong convergence of the algorithm. Furthermore, we present new results on the algorithm for solving the split feasibility problem and split feasibility problem with multiple output sets. Finally, two numerical experiments are presented to illustrate the convergence behavior and the effectiveness of the algorithm.

Key words: *inertial viscosity algorithm, strong convergence, self-adaptive method, split feasibility problem with multiple output sets*

Mathematics Subject Classification: *47H09, 65K10, 47J25*

1 Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a nonzero bounded linear operator. The *split feasibility problem* (SFP) is formulated to find a point $x^* \in H_1$ satisfying

$$x^* \in C \text{ such that } Ax^* \in Q. \tag{1.1}$$

The SFP was first introduced in [5], which has broad applications in many fields such as phase retrievals and in medical image reconstruction [5, 28, 29], intensity-modulated radiation therapy (IMRT) [6], gene regulatory network inference [30], and so on.

In recent years, focusing on real world applications, many iterative methods for solving the SFP (1.1) have been introduced and analyzed. Among them, Byrne [3] introduced the first applicable and most celebrated method called the well-known CQ-algorithm as follows:

$$x^0 \in H_1; \quad x^{k+1} := P_C\,(x^k - \lambda_k\, A^T\,(I - P_Q\,)Ax^k\,), \tag{1.2}$$

where P_C and P_Q are the metric projections on to C and Q, respectively, and the stepsize $\lambda_k \in (0, \frac{2}{\|A\|^2})$, where $\|A\|^2$ is the spectral radius of the matrix A^TA (A^T stands for the

*This work was supported by National Natural Science Foundation of China (72071130).
†Corresponding author

adjoint operator of A). It is observed that, in order to achieve the convergence, one has to estimate the norm of the bounded linear operator A (or the spectral radius of the matrix A^TA in the finite-dimensional framework) for selecting the step size λ_k, which is not always possible in practice. To avoid this obstacle, there have been a number of work to guarantee convergence without any prior information of the matrix norm (see, for examples [31]). For instance, López et al. [18] introduced a new way to select the step size by replacing the parameter λ_k appeared in (1.2) with the following parameter:

$$\tau_k = \frac{\rho_k g(x^k)}{\|\nabla g(x^k)\|^2}, \quad k \geq 1, \tag{1.3}$$

where $\rho_k \in (0,\ 4)$, $g\left(x^k\right) = \frac{1}{2}\left\|(I-P_Q)Ax^k\right\|^2$ and $\nabla g\left(x^k\right) = A^T\left(I-P_Q\right)Ax^k$ for all $k \geq 1$. This method is a modification of the CQ method which is often called the self-adaptive method. Some modifications of the CQ algorithm and the self-adaptive method now have been invented for solving the SFP (see, for example [1, 12, 26, 28]).

Some generalizations of the SFP have also been studied by many authors. For example, the split common fix point problem (SFPP) [4, 23], the multiple-sets SFP (MSSFP) [10, 17, 32], the split variational inequality problem (SVIP) [16, 27] and the split common null point problem (SCNPP) [11, 21]. Recently, Reich et al. [22] considered and studied another generalized split feasibility problem with multiple output sets (SFPwMOS) as follows: Let $H,\ H_i,\ i = 1,\dots,N$ be real Hilbert spaces and let $A_i : H \to H_i,\ i = 1,\dots,N$ be bounded linear operators and let $A_i^T : H_{i+1} \to H_i,\ i = 1,\dots,N-1$ be its adjoint. Let C and $Q_i,\ i = 1,\dots,N$ be nonempty, closed and convex subsets of H and $H_i,\ i = 1,\dots,N$, respectively. Given H, H_i and A_i as above, *the split feasibility problem with multiple output sets* (SFPwMOS) is to find an element x^* such that

$$x^* \in \Gamma\ := C \cap (\cap_{i=1}^N\ A_i^{-1}(Q_i)) \neq \emptyset. \tag{1.4}$$

Reich et al. [22] defined the function $g : H \to R$ as

$$g\left(x\right) := \frac{1}{2}\sum_{i=1}^N \left\|(I-P_{Q_i})A_ix\right\|^2,\ for\ all\ x \in H. \tag{1.5}$$

It is not difficult to see that an element x^* is a solution of the SFPwMOS (1.4) if and only if it is the solution of the problem

$$\min_{x\in C} g(x), \tag{1.6}$$

this is equivalent to

$$0 \in \nabla g\left(x^*\right) + N_C\left(x^*\right), \tag{1.7}$$

where $N_C\left(x\right)$ is the normal cone of C at the point x. It implies that

$$x^* = P_C\left(x^* - \alpha\sum_{i=1}^N A_i^T\left(I-P_{Q_i}\right)A_ix^*\right),$$

where α is an arbitrary positive real number. Motivated by these characterizations, Reich et al. [24] introduced the following iterative method for solving the SFPwMOS (1.4). For any given point $x^0 \in H$, $\left\{x^k\right\}$ is a sequence generated by the iterative method

$$x^{k+1}\ := t_kf(x^k) + (1-t_k)P_C(x^k - \alpha_k\sum_{i=1}^N A_i^T(I-P_{Q_i})A_ix^k), \tag{1.8}$$

where $f: C \to C$ is a strict contraction mapping of H into itself with the contraction constant $\theta \in [0, 1)$, $\alpha_k \subset (0,\infty)$ and $t_k \subset (0,1)$. It was proved that if the sequences $\{\alpha_k\}$ and $\{t_k\}$ satisfy the conditions:

$$0 < a \le \alpha_k \le b < \frac{2}{N \max\limits_{i=1,\ldots,N}\{\|A_i\|^2\}} \text{ for all } k > 1 \text{ and } \lim_{k\to\infty} t_k = 0, \sum_{k=1}^{\infty} t_k = \infty,$$

then the sequence $\{x^k\}$ generated by (1.8) converges strongly to a solution point $x^* \in \Gamma$ of the SFPwMOS (1.4), which is a unique solution of the variational inequality

$$\langle (I-f)x^*,\ x-x^*\rangle \ge 0 \ \ \forall x \in \Gamma.$$

In optimization field, to speed up the convergence, Polyak [20] firstly proposed the inertial extrapolation for solving smooth convex minimization problem which makes use of two previous iterates to update the next iterate. Inertial type algorithms can speed up the convergence rate, due to the fact the presence of inertial term. Hence, they have been widely studied by many authors [7, 8, 9, 13, 25].

Inspired by the above works, in this paper, we propose a self-adaptive inertial viscosity algorithm for solving the SFPwMOS (1.4) in general Hilbert spaces. The main contributions of this paper are as follows:

(i) We adopt self-adaptive strategy to update the step-size based on the information of the objective function and its gradient, to improve the flexibility of the algorithm.

(ii) We combine inertial technique with the nearly contractive viscosity-type iteration, to speed up the convergence.

The rest of the paper is organized as follows. Some fundamental tools and results are presented in Section 2. In Section 3, we construct an algorithm for solving the SFPwMOS and analyze its strong convergence. Several derived results are presented in Section 4. In Section 5, we illustrate the performance of the algorithm by testing a numerical example.

2 Preliminaries

Let I be the identity operator on H. Given a sequence $\{x^k\}$ in H and $x \in H$. We use $x^k \to x$ to denote that the sequence $\{x^k\}$ converges strongly to a point x as $k \to \infty$.

Definition 2.1. Given a not necessarily linear operator $T: H \to H$, denote by $Fix(T) := \{x \in H | x = Tx\}$ the set of all fixed points of T.

(i) quasi-nonexpansive if $Fix(T) \ne \emptyset$ and $\|Tx - z\| \le \|x - z\|, \forall x \in H,\ z \in Fix(T)$.

(ii) firmly nonexpansive if $\|Tx - Ty\|^2 \le \langle Tx - Ty,\ x - y\rangle,\ \forall x,\ y \in H,$ or equivalently,

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I-T)x - (I-T)y\|^2,\ \forall x,\ y \in H.$$

Lemma 2.2. *Let Q be a nonempty closed convex subset of H. P_Q denotes the projection onto the convex set Q, that is,*

$$P_Q(x) = arg\ \min_{y\in Q} \|x - y\|.$$

It has the following well-know properties:

(i) $x \in Q \Leftrightarrow P_Q(x) = x$;

(ii) $\langle x - P_Q(x), z - P_Q(x)\rangle \le 0, \forall x \in H$ *and* $\forall z \in Q$;

(iii) $\langle P_Q(y) - P_Q(x), y - x\rangle \ge \|P_Q(y) - P_Q(x)\|^2, \ \forall x,\ y \in H$;

(iv) $\|P_Q(x) - z\|^2 \le \|x - z\|^2 - \|P_Q(x) - x\|^2, \ \forall x \in H$ *and* $\forall z \in Q$;

(v) $\|P_Q(x) - P_Q(y)\|^2 \le \|x - y\|^2 \ \forall x,\ y \in H$.

From Definition 2.1, we know that P_Q *is firmly nonexpansive.*

Lemma 2.3 ([2])**.** *Let* $x, y \in H$ *and* $t, s \in R$. R *is the set of real numbers. Then*

(i) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$;

(ii) $\|tx + sy\|^2 = t(t+s)\|x\|^2 + s(t+s)\|y\|^2 - st\|x - y\|^2$;

(iii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$.

Lemma 2.4 ([14])**.** *Assume* $\{\omega_k\}$ *is a sequence of non-negative real numbers such that*

$$\begin{cases} \omega_{k+1} \le (1 - v_k)\omega_k + v_k u_k, & k \ge 1, \\ \omega_{k+1} \le \omega_k - \tau_k + \sigma_k, & k \ge 1, \end{cases}$$

where $\{v_k\}, \{u_k\}$ *and* $\{\sigma_k\}$ *are sequences of real numbers such that*

(i) $\{v_k\} \subset (0, 1)$ *and* $\sum_{k=1}^{\infty} v_k = \infty$;

(ii) $\lim_{k\to\infty} \sigma_k = 0$;

(iii) $\limsup_{j\to\infty} u_{k_j} \le 0$ *whenever* $\limsup_{j\to\infty} l_{k_j} = 0$ *for any subsequence* $\{k_j\}$ *of* $\{k\}$.

Then $\lim_{k\to\infty} \omega_k = 0$.

Lemma 2.5 ([19])**.** *Let* h *be a contraction on* H*. The viscosity approximation method proposed by Moudafi generates a strongly convergent sequence:*

$$\begin{cases} x^0 \in H, \\ x^{k+1} = t_k h\left(x^k\right) + (1 - t_k) T x^k \ for \ k \in \ N, \end{cases}$$

which converges strongly to a fixed point x^* *of* T*. In [31], Xu further proved that the above* x^* *also satisfies the following variational inequality:*

$$\langle h(x^*) - x^*,\ x - x^*\rangle \le 0,\ \forall x \in Fix(T),$$

provided that $\{t_k\} \in (0, 1)$.

Recall that the sequence of mappings h_k *from* H *into* H *is called a nearly contractive mappings with sequence* $\{(\zeta_k, a_k)\}$ *in* $[0, 1) \times [0, \infty)$*. A useful and simple norm inequality is the following*

$$\|h_k(x) - h_k(y)\| \le \zeta_k \|x - y\| + a_k,$$

for $a_k \to 0$*, all* $x,\ y \in H$ *and* $k \in N$.

3 Inertial Viscosity Projection Algorithm and Its Convergence

In this section, we propose a self-adaptive inertial viscosity projection algorithm for split feasibility problem with multiple output sets, and prove its strong convergence.

Algorithm 3.1 (Self-adaptive inertial viscosity projection algorithm)**.** Let $\{t_k\}$ $\{\lambda_k\}$ be two sequences in $(0,1)$, $\{\rho_k\} \subset (0,4)$, $\theta_k \in [0,\theta]$ with $\theta \in [0,1)$, and $\{h_k\}$ be a nearly contractive mapping with $\{(\zeta_k, a_k)\}$. Set

$$\begin{cases} x^0,\ x^1 \in H, \\ z^k = x^k + \theta_k \left(x^k - x^{k-1}\right), \\ \tau_k := \frac{\rho_k g_k(z^k)}{\|\nabla g_k(z^k)\|^2}, \\ x^{k+1} = (1-\lambda_k)\, z^k + \lambda_k \left(t_k h_k\left(z^k\right) + (1-t_k)\, P_C\left(z^k - \tau_k \nabla g_k\left(z^k\right)\right)\right), for\ k \in N. \end{cases} \tag{3.1}$$

where $g_k\left(z^k\right) := \frac{1}{2}\sum_{i=1}^N \left\|(I-P_{Q_i})A_i z^k\right\|^2$, $\nabla g_k(z^k) := \sum_{i=1}^N A_i^*(I-P_{Q_i})A_i z^k$.

The following lemmas play an important role in the convergence proof of Algorithm 3.1.

Lemma 3.2. *Let Γ be the solution set of the problem (1.4). Define an operator $S: H_1 \to H_2$ as follows:*

$$S(z^k) = P_C\left(z^k - \tau_k \nabla g_k\left(z^k\right)\right). \tag{3.2}$$

For $z^k \in H$ and $z \in \Gamma$, the following inequality holds:

$$\left\|S(z^k) - z\right\|^2 \le \left\|z^k - z\right\|^2 - r(z^k), \tag{3.3}$$

where $r(z^k) = \rho_k(4-\rho_k)\frac{g_k^2(z^k)}{\|\nabla g_k(z^k)\|^2} + \left\|w^k - P_C(w^k)\right\|^2$, $w^k = z^k - \tau_k \nabla g_k\left(z^k\right)$.

Proof. Assume that the sequence z^k is infinite, that is, Algorithm 3.1 does not terminate in a finite number of iterations. Thus $\nabla g_k\left(z^k\right) \neq \emptyset$ for all $k \ge 0$. P_Γ denotes the metric projection, set $z \in \Gamma$. Note that $I - P_{Q_i}$ for each $i = 1, \ldots, N$ is firmly nonexpansive and $\nabla g_k\left(z\right) = 0$. Hence, we have from Lemma 2.2 that

$$\begin{aligned} \left\langle \nabla g_k(z^k),\ z^k - z\right\rangle &= \left\langle \sum_{i=1}^N A_i^T(I-P_{Q_i})A_i z^k,\ z^k - z\right\rangle \\ &= \sum_{i=1}^N \left\langle A_i^T(I-P_{Q_i})A_i z^k,\ z^k - z\right\rangle \\ &= \sum_{i=1}^N \left\langle (I-P_{Q_i})A_i z^k,\ A_i z^k - A_i z\right\rangle \\ &\ge \sum_{i=1}^N \left\|(I-P_{Q_i})A_i z^k\right\|^2 \\ &= 2g_k(z^k), \end{aligned}$$

which implies that

$$\begin{aligned}
\left\|w^k - z\right\|^2 &= \left\|(z^k - z) - \tau_k \nabla g_k(z^k)\right\|^2 \\
&= \left\|z^k - z\right\|^2 + \tau_k^2 \left\|\nabla g_k(z^k)\right\|^2 - 2\tau_k \left\langle \nabla g_k(z^k),\ z^k - z\right\rangle \\
&\le \left\|z^k - z\right\|^2 + \frac{\rho_k^2 g_k^2\left(z^k\right)}{\|\nabla g_k(z^k)\|^2} - \frac{2\rho_k g_k\left(z^k\right)}{\|\nabla g_k(z^k)\|^2}(2g_k(z^k)) \\
&= \left\|z^k - z\right\|^2 + \frac{\rho_k^2 g_k^2\left(z^k\right)}{\|\nabla g_k(z^k)\|^2} - \frac{4\rho_k g_k^2\left(z^k\right)}{\|\nabla g_k(z^k)\|^2} \\
&= \left\|z^k - z\right\|^2 - \rho_k\left(4 - \rho_k\right)\frac{g_k^2\left(z^k\right)}{\|\nabla g_k(z^k)\|^2}. \qquad (3.4)
\end{aligned}$$

By the definition of ρ_k, we obtain

$$\left\|w^k - z\right\|^2 \le \left\|z^k - z\right\|^2,\ \forall k \ge 0. \qquad (3.5)$$

From Lemma 2.2 (iv) and (2.3), it is easy to get that

$$\begin{aligned}
\left\|S(z^k) - z\right\|^2 &= \left\|P_C(w^k) - z\right\|^2 \\
&\le \left\|w^k - z\right\|^2 - \left\|w^k - P_C(w^k)\right\|^2 \\
&\le \left\|z^k - z\right\|^2 - \rho_k\left(4 - \rho_k\right)\frac{g_k^2\left(z^k\right)}{\|\nabla g_k(z^k)\|^2} - \left\|w^k - P_C(w^k)\right\|^2 \\
&= \left\|z^k - z\right\|^2 - r(z^k). \qquad (3.6)
\end{aligned}$$

Thus, the proof is completed. □

Remark 3.3. Since $\{\rho_k\} \subset (0, 4)$, we observe that $r\left(x^k\right) \ge 0$ for all $x \in H$. Therefore, the operator S is quasi-nonexpansive.

Lemma 3.4. *Set $z^k := x^k + \theta_k(x^k - x^{k-1})$, where $0 \le \theta_k < 1$ for all $k \in N$. Then for all $z \in H$,*

$$\left\|z^k - z\right\|^2 \le \left\|x^k - z\right\|^2 + \theta_k(\left\|x^k - z\right\|^2 - \left\|x^{k-1} - z\right\|) + 2\theta_k \left\|x^k - x^{k-1}\right\|^2.$$

Proof. Using the identity $2\langle a, b\rangle = \|a\|^2 + \|b\|^2 - \|a - b\|^2$, we have

$$\begin{aligned}
\left\|z^k - z\right\|^2 &= \left\|x^k - z + \theta_k(x^k - x^{k-1})\right\|^2 \\
&= \left\|x^k - z\right\|^2 + 2\theta_k\left\langle x^k - z,\ x^k - x^{k-1}\right\rangle + \theta_k^2\left\|x^k - x^{k-1}\right\|^2 \\
&= \left\|x^k - z\right\|^2 + \theta_k(\left\|x^k - z\right\|^2 + \left\|x^k - x^{k-1}\right\|^2 - \left\|x^{k-1} - z\right\|^2) \\
&\quad + \theta_k^2\left\|x^k - x^{k-1}\right\|^2 \\
&= \left\|x^k - z\right\|^2 + \theta_k(\left\|x^k - z\right\|^2 - \left\|x^{k-1} - z\right\|^2) + \theta_k(1 + \theta_k)\left\|x^k - x^{k-1}\right\|^2 \\
&\le \left\|x^k - z\right\|^2 + \theta_k(\left\|x^k - z\right\|^2 - \left\|x^{k-1} - z\right\|^2) + 2\theta_k\left\|x^k - x^{k-1}\right\|^2. \qquad (3.7)
\end{aligned}$$

□

Lemma 3.5. *Given a contraction h with $\zeta \in (0,1)$, $x^* = P_\Gamma h(x^*)$ and $x^* \in \Gamma$. $\{h_k\}$ is a nearly contractive mapping with $\{(\zeta_k, a_k)\}$ such that $\zeta_k \to \zeta$. Assume that $\{\lambda_k\}$, $\{t_k\}$, and $\{\theta_k\}$ satisfy the following conditions:*

(i) $\lim_{k\to\infty} h_k(x^*) = h(x^*)$;

(ii) $0 < \lambda_k \leq 1$ *and* $k \in N$;

(iii) $t_k \in (0,1)$ *such that* $\lim_{k\to\infty} t_k = 0$ *and* $\sum_{k=1}^{\infty} t_k = \infty$;

(iv) $\lim_{k\to\infty} \frac{\theta_k}{t_k} \left\|x^k - x^{k+1}\right\| = 0$.

Then the sequence $\{x^k\}$ generated by Algorithm 3.1 is bounded.

Proof. Let $y^k = t_k h_k(z^k) + (1-t_k) S(z^k)$ and from Lemma 2.5, we have

$$\begin{aligned}
\left\|y^k - x^*\right\| &= \left\|t_k h_k(z^k) + (1-t_k) S(z^k) - x^*\right\| \\
&\leq t_k \left\|h_k(z^k) - h(x^*)\right\| + (1-t_k)\left\|S(z^k) - x^*\right\| \\
&\leq t_k(\left\|h_k(z^k) - h(x^*)\right\| + \|h_k(x^*) - x^*\|) + (1-t_k)\left\|S(z^k) - x^*\right\| \\
&\leq t_k(\zeta_k \left\|z^k - x^*\right\| + a_k) + t_k\|h_k(x^*) - x^*\| + (1-t_k)\left\|S(z^k) - x^*\right\| \\
&\leq (1-(1-\zeta_k)t_k)\left\|z^k - x^*\right\| + t_k(\|h_k(x^*) - x^*\| + a_k). \qquad (3.8)
\end{aligned}$$

From (3.8), $\zeta_k \in (0,1)$, $t_k \in (0,1)$ and (ii), we have

$$\begin{aligned}
&\left\|x^{k+1} - x^*\right\| \\
&= \left\|(1-\lambda_k)(z^k - x^*) + \lambda_k(y^k - x^*)\right\| \\
&\leq (1-\lambda_k)\left\|z^k - x^*\right\| + \lambda_k\left\|y^k - x^*\right\| \\
&\leq (1-\lambda_k)\left\|z^k - x^*\right\| + \lambda_k(1-(1-\zeta_k)t_k)\left\|z^k - x^*\right\| + \lambda_k t_k(\|h_k(x^*) - x^*\| + a_k) \\
&= (1-(1-\zeta_k)\lambda_k t_k)\left\|z^k - x^*\right\| + \lambda_k t_k(\|h_k(x^*) - x^*\| + a_k) \\
&= (1-(1-\zeta_k)\lambda_k t_k)\left\|z^k - x^*\right\| + (1-\zeta_k)\lambda_k t_k\Big(\frac{\|h_k(x^*) - x^*\| + a_k}{1-\zeta_k}\Big) \\
&= (1-(1-\zeta_k)\lambda_k t_k)\left\|x^k - x^* + \theta_k(x^k - x^{k-1})\right\| \\
&\quad + (1-\zeta_k)\lambda_k t_k\Big(\frac{\|h_k(x^*) - x^*\| + a_k}{1-\zeta_k}\Big) \\
&\leq (1-(1-\zeta_k)\lambda_k t_k)\left\|x^k - x^*\right\| + \theta_k\left\|x^k - x^{k-1}\right\| \\
&\quad + (1-\zeta_k)\lambda_k t_k\Big(\frac{\|h_k(x^*) - x^*\| + a_k}{1-\zeta_k}\Big) \\
&= (1-(1-\zeta_k)\lambda_k t_k)\left\|x^k - x^*\right\| \\
&\quad + (1-\zeta_k)\lambda_k t_k\Big(\frac{\|h_k(x^*) - x^*\| + a_k}{1-\zeta_k} + \frac{\theta_k}{(1-\zeta_k)\lambda_k t_k}\left\|x^k - x^{k-1}\right\|\Big). \qquad (3.9)
\end{aligned}$$

Since $\lim_{k\to\infty} h_k(x^*) = h(x^*)$, $\lim_{k\to\infty} \zeta_k = \zeta$ and $\lim_{k\to\infty} a_k = 0$, we conclude that the sequence $\{\frac{\left\|h_k(x^*) - x^*\right\| + a_k}{1-\zeta_k}\}$ is bounded. On the other hand, the conditions (ii) and (iv) imply the sequence $\{\frac{\theta_k}{(1-\zeta_k)\lambda_k t_k}\left\|x^k - x^{k-1}\right\|\}$ is also bounded. Hence, we obtain an upper bound and let

$$M = \limsup_{k\to\infty} \left\{ \frac{\|h_k(x^*) - x^*\| + a_k}{1-\zeta_k} + \frac{\theta_k}{(1-\zeta_k)\lambda_k t_k} \left\|x^k - x^{k-1}\right\| \right\}.$$

Then we rewrite (3.9) as

$$\left\|x^{k+1} - x^*\right\| \le max\left\{\left\|x^k - x^*\right\|,\ M\right\},$$

by induction, we have

$$\left\|x^{k+1} - x^*\right\| \le max\left\{\left\|x^1 - x^*\right\|,\ M\right\}.$$

Thus, the sequence $\left\{\left\|x^k - x^*\right\|\right\}$ is bounded. The proof is completed. □

Remark 3.6. Since $\left\{\left\|x^k - x^*\right\|\right\}$ is bounded, so we can obtain that $\left\{\left\|z^k - x^*\right\|\right\}$ and $\left\{\left\|y^k - x^*\right\|\right\}$ are bounded. By Lemma 2.5, we know

$$\left\|h_k(z^k)\right\| \le \left\|h_k(z^k) - h_k(x^*)\right\| + \|h_k(x^*)\| \le \zeta_k \left\|z^k - x^*\right\| + a_k + \|h_k(x^*)\|.$$

This shows that $\left\{h_k\left(z^k\right)\right\}$ is also bounded.
Next, we will give the convergence analysis of Algorithm 3.1.

Theorem 3.7. *Let H, $H_i, i = 1, \ldots, N$ be real Hilbert spaces and $A_i : H \to H_i$, $i = 1, \ldots, N$ be bounded linear operators. Given a contraction h with $\zeta \in (0,1)$, $x^* = P_\Gamma h(x^*)$ and $x^* \in \Gamma$. $\{h_k\}$ is a nearly contractive mapping with $\{(\zeta_k, a_k)\}$ such that $\zeta_k \to \zeta$. Let C and $Q_i, i = 1, \ldots, N$ be nonempty, closed convex subsets of H and $H_i, i = 1, \ldots, N$ respectively. Suppose the sequences $\{\lambda_k\}$, $\{t_k\}$ and $\{\theta_k\}$ in Algorithm 3.1 satisfy the following conditions:*

(i) $t_k \in (0,1)$ *such that* $\lim_{k\to\infty} t_k = 0$ *and* $\sum_{k=1}^{\infty} t_k = \infty$*;*

(ii) $0 < \lambda_k \le 1$ *and* $k \in N$*;*

(iii) $\lim_{k\to\infty} \frac{\theta_k}{t_k}\left\|x^k - x^{k+1}\right\| = 0$*;*

(iv) $\lim_{k\to\infty} h_k(x^*) = h(x^*)$.

Then the sequence $\left\{x^k\right\}$ generated by Algorithm 3.1 converges strongly to $x^* = P_\Gamma h(x^*)$.

Proof. From the inequality

$$\langle u,\ v\rangle \le \|u\|\,\|v\| \le \frac{1}{2}(\|u\|^2 + \|v\|^2),$$

together with Lemma 2.5 and $\zeta_k \in (0,1)$, we obtain

$$\begin{aligned}
\left\|y^k - x^*\right\|^2 &= \left\|t_k(h_k(z^k) - x^*) + (1-t_k)(S(z^k) - x^*)\right\|^2 \\
&= t_k \left\langle h_k(z^k) - x^*,\ y^k - x^* \right\rangle + (1-t_k) \left\langle S(z^k) - x^*,\ y^k - x^* \right\rangle \\
&= t_k \left(\left\langle h_k(z^k) - h_k(x^*),\ y^k - x^* \right\rangle + \left\langle h_k(x^*) - h(x^*),\ y^k - x^* \right\rangle \right) \\
&+ t_k \left\langle h(x^*) - x^*,\ y^k - x^* \right\rangle + (1-t_k) \left\langle S(z^k) - x^*,\ y^k - x^* \right\rangle \\
&\le t_k \left(\left\|h_k(z^k) - h_k(x^*)\right\| + \|h_k(x^*) - h(x^*)\| \right) \left\|y^k - x^*\right\| \\
&+ t_k \left\langle h(x^*) - x^*,\ y^k - x^* \right\rangle + (1-t_k) \left\|S(z^k) - x^*\right\| \left\| y^k - x^*\right\| \\
&\le t_k \left[\left(\zeta_k \left\|z^k - x^*\right\| + a_k\right) + \|h_k(x^*) - h(x^*)\| \right] \left\|y^k - x^*\right\| \\
&+ t_k \left\langle h(x^*) - x^*,\ y^k - x^* \right\rangle + (1-t_k) \left\|S(z^k) - x^*\right\| \left\| y^k - x^*\right\| \\
&\le \frac{t_k\zeta_k}{2} \left(\left\|z^k - x^*\right\|^2 + \left\|y^k - x^*\right\|^2 \right) + \frac{1-t_k}{2} \left(\left\|S(z^k) - x^*\right\|^2 + \left\|y^k - x^*\right\|^2 \right) \\
&+ t_k \left\langle h(x^*) - x^*,\ y^k - x^* \right\rangle + t_k(a_k + \|h_k(x^*) - h(x^*)\|) \left\| y^k - x^*\right\| \\
&\le \frac{t_k\zeta_k}{2} \left\|z^k - x^*\right\|^2 + \frac{1}{2} \left\|y^k - x^*\right\|^2 + \frac{1-t_k}{2} \left\|S(z^k) - x^*\right\|^2 \\
&+ t_k \left\langle h(x^*) - x^*,\ y^k - x^* \right\rangle + t_k(a_k + \|h_k(x^*) - h(x^*)\|) \left\| y^k - x^*\right\|.
\end{aligned}$$

This means that

$$\begin{aligned}
\left\|y^k - x^*\right\|^2 &\leqslant t_k\zeta_k \left\|z^k - x^*\right\|^2 + (1-t_k) \left\|S(z^k) - x^*\right\|^2 + 2t_k \left\langle h(x^*) - x^*,\ y^k - x^* \right\rangle \\
&+ 2t_k(a_k + \|h_k(x^*) - h(x^*)\|) \left\| y^k - x^*\right\|.
\end{aligned}$$

Then, together with the inequality

$$\left\|S(z^k) - x^*\right\|^2 \le \left\|z^k - x^*\right\|^2 - r(z^k),$$

we have

$$\begin{aligned}
\left\|y^k - x^*\right\|^2 &\le t_k\zeta_k \left\|z^k - x^*\right\|^2 + 2t_k \left\langle h(x^*) - x^*,\ y^k - x^* \right\rangle \\
&+ 2t_k(a_k + \|h_k(x^*) - h(x^*)\|) \left\|y^k - x^*\right\| + (1-t_k) \left\|z^k - x^*\right\|^2 - (1-t_k)r(z^k) \\
&= [1 - (1-\zeta_k)t_k] \left\|z^k - x^*\right\|^2 + 2t_k \left\langle h(x^*) - x^*,\ y^k - x^* \right\rangle \\
&+ 2t_k(a_k + \|h_k(x^*) - h(x^*)\|) \left\|y^k - x^*\right\| - (1-t_k)r(z^k). \qquad (3.10)
\end{aligned}$$

By Lemma 2.3 (iii) and (3.10), we obtain

$$\begin{aligned}
\left\|x^{k+1} - x^*\right\|^2 &= \left\|(1-\lambda_k)\left(z^k - x^*\right) + \lambda_k \left(y^k - x^*\right)\right\|^2 \\
&= (1-\lambda_k) \left\|z^k - x^*\right\|^2 + \lambda_k \left\|y^k - x^*\right\|^2 - \lambda_k(1-\lambda_k) \left\|z^k - y^k\right\|^2 \\
&\leqslant (1-\lambda_k) \left\|z^k - x^*\right\|^2 + \lambda_k [1 - (1-\zeta_k)t_k] \left\|z^k - x^*\right\|^2 \\
&+ 2\lambda_k t_k \left\langle h\left(x^*\right) - x^*,\ y^k - x^* \right\rangle + 2\lambda_k t_k(a_k + \|h_k(x^*) - h(x^*)\|) \left\|y^k - x^*\right\| \\
&- \lambda_k(1-\lambda_k)r(z^k) - \lambda_k(1-\lambda_k) \left\|z^k - y^k\right\|^2 \\
&= [1 - (1-\zeta_k)\lambda_k t_k] \left\|z^k - x^*\right\|^2 \\
&+ 2\lambda_k t_k \left\langle h\left(x^*\right) - x^*,\ y^k - x^* \right\rangle + 2\lambda_k t_k(a_k + \|h_k(x^*) - h(x^*)\|) \left\|y^k - x^*\right\| \\
&- \lambda_k(1-\lambda_k)r(z^k) - \lambda_k(1-\lambda_k) \left\|z^k - y^k\right\|^2. \qquad (3.11)
\end{aligned}$$

From Lemma 2.3 (i), we get

$$\begin{aligned}\left\|z^k-x^*\right\|^2&=\left\|x^k-x^*+\theta_k\left(x^k-x^{k-1}\right)\right\|^2\\&\le\left\|x^k-x^*\right\|^2+2\theta_k\left\langle x^k-x^{k-1},\ z^k-x^*\right\rangle\\&\leqslant\left\|x^k-x^*\right\|^2+2\theta_k\left\|x^k-x^{k-1}\right\|\left\|\ z^k-x^*\right\|.\end{aligned}\tag{3.12}$$

Putting (3.12) into (3.11) and $k\to\infty$, we have

$$\begin{aligned}\left\|x^{k+1}-x^*\right\|^2\leqslant{}&[1-(1-\zeta_k)\lambda_kt_k]\left(\left\|x^k-x^*\right\|^2+2\theta_k\left\|x^k-x^{k-1}\right\|\left\|\ z^k-x^*\right\|\right)\\&+2\lambda_kt_k\left\langle h\left(x^*\right)-x^*,y^k-x^*\right\rangle+2\lambda_kt_k(a_k+\|h_k(x^*)-h(x^*)\|)\left\|y^k-x^*\right\|\\&-\lambda_k(1-t_k)r(z^k)-\lambda_k(1-\lambda_k)\left\|z^k-y^k\right\|^2\\&\leqslant[1-(1-\zeta_k)\lambda_kt_k]\left\|x^k-x^*\right\|^2+2\theta_k\left\|x^k-x^{k-1}\right\|\left\|\ z^k-x^*\right\|\\&+2\lambda_kt_k\left\langle h\left(x^*\right)-x^*,y^k-x^*\right\rangle+2\lambda_kt_k(a_k+\|h_k(x^*)-h(x^*)\|)\left\|y^k-x^*\right\|\\&-\lambda_k(1-t_k)r(z^k)-\lambda_k(1-\lambda_k)\left\|z^k-y^k\right\|^2.\end{aligned}\tag{3.13}$$

Set

$$\begin{aligned}&\phi_k=\left\|x^k-x^*\right\|^2,\\&l_k=\lambda_k\left(1-t_k\right)r\left(z^k\right)+\lambda_k\left(1-\lambda_k\right)\left\|z^k-y^k\right\|^2,\\&v_k=\left(1-\zeta_k\right)\lambda_kt_k,\\&u_k=\frac{2}{(1-\zeta_k)}(\left\langle h\left(x^*\right)-x^*,\ y^k-x^*\right\rangle+(a_k+\left\|h_k\left(x^*\right)-h\left(x^*\right)\right\|)\left\|y^k-x^*\right\|\\&\quad+\frac{\theta_k}{\lambda_kt_k}\left\|x^k-x^{k-1}\right\|\left\|z^k-x^*\right\|).\end{aligned}$$

Then, we rewrite (3.13) as

$$\phi_{k+1}\leqslant\left[1-v_k\right]\phi_k+v_ku_k,\tag{3.14}$$
$$\phi_{k+1}\leqslant\phi_k-l_k+\sigma_k,\tag{3.15}$$

where

$$\begin{aligned}\sigma_k={}&2\theta_k\left\|x^k-x^{k-1}\right\|\left\|z^k-x^*\right\|+2\lambda_kt_k\left\langle h(x^*)-x^*,\ y^k-x^*\right\rangle\\&+2\lambda_kt_k(a_k+\|h_k(x^*)-h(x^*)\|)\left\|y^k-x^*\right\|.\end{aligned}$$

Since $\sum_{k=1}^{\infty}t_k=\infty$ and Theorem 3.7 (ii) hold, $\lim_{k\to\infty}\zeta_k=\zeta$. It follows that

$$\sum_{k=1}^{\infty}\left(1-\zeta_k\right)\lambda_kt_k=\infty.$$

Because $t_k\in(0,1)$ and Theorem 3.7 (iii), we see that $\lim_{k\to\infty}\theta_k\left\|x^k-x^{k-1}\right\|=0$. Together with the boundedness of $\left\{\left\|z^k-x^*\right\|\right\}$, $\left\{\left\|y^k-x^*\right\|\right\}$ and $\lim_{k\to\infty}t_k=0$, we have $\lim_{k\to\infty}\sigma_k=0$.

We now show that $\phi_k\to 0$ as $k\to\infty$ by considering two possible cases.

Case 1 $\{\phi_k\}$ is eventually decreasing (i.e., there exists $j \geq 0$ such that $\phi_k > \phi_{k+1}$ holds for all $k \geq j$). In this case, ϕ_k must be convergent, and from (3.15) it follows that

$$l_k \leq (\phi_k - \phi_{k+1}) + \sigma_k. \tag{3.16}$$

Noting $\lim_{k\to\infty} \sigma_k = 0$, letting $k \to \infty$ in (3.16) yields $l_k \to 0$ as $k \to \infty$. Since $\lim_{k\to\infty} l_k = 0$, $\lim_{k\to\infty}\ t_k = 0$ and $0 < \lambda_k \leq 1$, we have

$$\lim_{k\to\infty} \left\|z^k - y^k\right\|^2 = 0,$$

and

$$\lim_{k\to\infty} r(z^k) = \lim_{k\to\infty} \left[\rho_k(4-\rho_k)\frac{g_k^2(z^k)}{\|\nabla g_k(x^k)\|^2} + \left\|w^k - P_C(w^k)\right\|\right] = 0,$$

which implies

$$\lim_{k\to\infty}\ \left\|w^k - P_C(w^k)\right\| = 0, \tag{3.17}$$

$$\frac{g_k^2(z^k)}{\|\nabla g_k(x^k)\|^2} \to 0 \ as\ k \to \infty. \tag{3.18}$$

We note that for each $i = 1, 2, \ldots, N$, $A_i^T(I - P_{Q_i})A_i(\cdot)$ is Lipschitz continuous. Since the sequence $\left\{z^k\right\}$ is bounded and

$$\left\|A_i^T(I-P_{Q_i})A_iz^k\right\| = \left\|A_i^T(I-P_{Q_i})A_iz^k - A_i^T(I-P_{Q_i})A_ix^*\right\| \leq \left(\max_{1\leq i\leq N} \|A_i\|\right)\left\|z^k - x^*\right\|,$$

for all $i = 1, 2, \ldots, N$ we have the sequence $\left\{\left\|A_i^T(I-P_{Q_i})A_iz^k\right\|\right\}_{k=1}^{\infty}$ is bounded. Hence, $\left\{\left\|\nabla g_k(x^k)\right\|\right\}_{k=1}^{\infty}$ is bounded. Consequently, we have from (3.18) that

$$\lim_{k\to\infty}\ \left\|(I-P_{Q_i})A_iz^k\right\|^2 = 0, \tag{3.19}$$

for each $i = 1, 2, \ldots, N$. Since $w^k = z^k - \tau_k\nabla g_k(z^k)$, then we have from (3.19) that

$$\left\|w^k - z^k\right\| = \tau_k\left\|\nabla g_k(z^k)\right\| \to 0, \ \ as\ k \to \infty. \tag{3.20}$$

On the other hand, using (i) and (iii), since $z^k = x^k + \theta_k(x^k - x^{k-1})$, we have

$$\left\|z^k - x^k\right\| = \theta_k\left\|x^k - x^{k-1}\right\| \to 0, as\ k \to \infty. \tag{3.21}$$

From (3.20) and (3.21), we obtain

$$\left\|w^k - x^k\right\| = \left\|w^k - z^k + z^k - x^k\right\| \leq \left\|w^k - z^k\right\| + \left\|z^k - x^k\right\| \to 0. \tag{3.22}$$

Since $\left\{x^k\right\}$ is bounded, there exists a subsequence $\left\{x^{k_j}\right\}$ of $\left\{x^k\right\}$ such that $\left\{x^{k_j}\right\} \to \widehat{x}^*$. Now we show that $\widehat{x}^* \in \Gamma$. That is, we need to show $\widehat{x}^* \in C$ and $A_i\widehat{x}^* \in Q_i, i = 1, 2, \ldots, N$. From (3.22) and (3.17) we can conclude that

$$\lim_{j\to\infty}\left\|x^{k_j} - P_C(x^{k_j})\right\| = \lim_{j\to\infty}\left\|w^{k_j} - P_C(w^{k_j})\right\| = \|\widehat{x}^* - P_C(\widehat{x}^*)\| \to 0.$$

Thus, $\widehat{x}^* \in C$. From (3.19) and (3.21) we can obtain that

$$\lim_{j\to\infty}\left\|(I-P_{Q_i})A_iz^{k_j}\right\|^2 = \lim_{k\to\infty}\left\|(I-P_{Q_i})A_ix^{k_j}\right\|^2 = \|(I-P_{Q_i})A_i\widehat{x}^*\|^2 = 0.$$

That is, $A_i\widehat{x}^* \in Q_i, i = 1, 2, \ldots, N$ for all $i = 1, 2, \ldots, N$. Hence $\widehat{x}^* \in \Gamma$.

Moreover, for $x^* = P_\Gamma h(x^*)$, we can see that

$$\mathop{lim\ sup}_{k\to\infty} \left\langle x^k - x^*,\ h(x^*) - x^* \right\rangle = \lim_{k\to\infty} \left\langle x^k - P_\Gamma h(x^*),\ h(x^*) - P_\Gamma h(x^*) \right\rangle \le 0.$$

Together with $\lim_{k\to\infty}\ a_k = 0$, $\lim_{k\to\infty}\ h_k(x^*) = h(x^*)$ and $\lim_{k\to\infty}\ \frac{\theta_k}{t_k}\left\|x^k - x^{k-1}\right\| = 0$, we conclude that

$$\mathop{lim\ sup}_{k\to\infty} u_k \ = \mathop{lim\ sup}_{k\to\infty} \frac{2}{(1-\zeta_k)} \left\langle h(x^*) - x^*,\ y^k - x^* \right\rangle \le 0.$$

By Lemma 2.5, we get as $k \to \infty$, $\phi_k \to 0$.

Case 2 $\{\phi_k\}$ is not eventually decreasing. Hence, we can find an integer k_0 such that $\phi_{k_0} \le \phi_{k_0+1}$. Define

$$J_k := \{k_0 \le i \le k : \phi_i \le \phi_{i+1}\}, \quad k > k_0.$$

Obviously, J_k is nonempty and satisfies $J_k \subseteq J_{k+1}$. Let

$$\tau(k) \ := \ max\ J_k, \quad k > k_0. \tag{3.23}$$

It is clear that $\tau(k) \ \to \infty$ as $k \to \infty$ (otherwise, $\{\phi_k\}$ is eventually decreasing). It is also clear that $s_{\tau(k)} \ \le \ s_{\tau(k)+1}$ for all $k > k_0$. Moreover,

$$\phi_k \ \le \phi_{\tau(k)+1}\ ,\ \forall k > k_0. \tag{3.24}$$

In fact, if $\tau_k = k$, then inequity (3.24) is trivial; if $\tau_k = k - 1$, then $\tau(k) + 1 = k$, and (3.24) is also trivial; If $\tau(k) < k - 1$, then there exists an integer $i \ge 2$ such that $\tau(k) + i = k$. Thus we deduce from the definition of $\tau(k)$ that

$$\phi_{\tau(k)+1} > \phi_{\tau(k)+2} > \cdots > \phi_{\tau(k)+i} = \phi_k, \tag{3.25}$$

and inequity (3.24) holds again. Since $\phi_{\tau(k)} < \phi_{\tau(k)+1}$ all $k > k_0$, it follows from (3.16) that

$$0 \le l_{\tau(k)} \le \delta_{\tau(k)} \ \to \ 0, \tag{3.26}$$

so that $l_{\tau(k)} \to \ 0$ as $k \to \infty$. The rest part of the proof is similar to that of case 1, hence, it is omitted.

The proof is completed. □

4 Derived Results

For the SFPwMOS (1.4), when $N = 1$, it becomes the SFP (1.1). Thus, we have the following corollary for solving the SFP (1.1), which is an immediate consequence of Theorem 3.7.

Corollary 4.1. *Let H_1 and H_2 be two real Hilbert spaces and let $A : H_1 \to H_2$ be bounded linear operator. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Given a contraction h with $\zeta \in (0,1)$ and $x^* = P_\Gamma h\left(x^*\right)$. Assume that $\Omega = C \cap A^{-1}(Q) \neq \emptyset$, let$\{h_k\}$ be a nearly contractive mapping with $\{(\zeta_k, a_k)\}$ such that $\zeta_k \to \zeta$. For any starting point x^0, $x^1 \in H_1$, let $\left\{x^k\right\}$ be the sequence generated by*

$$\begin{cases} z^k = x^k + \theta_k\left(x^k - x^{k-1}\right), \\ x^{k+1} := (1-\lambda_k)\, z^k + \lambda_k\left(t_k h_k\left(z^k\right) + (1-t_k)\, P_C\left(z^k - \tau_k \nabla f_k\left(z^k\right)\right)\right), \end{cases} \tag{4.1}$$

where $\tau_k := \frac{\rho_k f_k(z^k)}{\|\nabla f_k(z^k)\|^2}$, and $\nabla f_k\left(x^k\right) := A^T(I - P_Q)Ax^k$. Suppose the sequences $\{\lambda_k\}$, $\{t_k\}$ and $\{\theta_k\}$ satisfy the conditions in Theorem 3.7. Then, the sequence $\left\{x^k\right\}$ converges strongly to the solution $x^* \in \Omega$, where $x^* = P_\Omega h(x^*)$.

When we take $h(x) = x^0$ in Algorithm 3.1, we note also the following results regarding to the SFPwMOS (1.4).

Corollary 4.2. *Let H, H_i, $i = 1, \dots, N$ be real Hilbert spaces and let $A_i : H \to H_i$, $i = 1, \dots, N$ be bounded linear operators. Let C and Q_i, $i = 1, \dots, N$ be nonempty, closed and convex subsets of H and H_i, $i = 1, \dots, N$ respectively. Assume that the problem (1.4) is consistent. For any initial guess x^0, $x^1 \in H_1$, let $\{x^k\}$ be the sequence generated by*

$$\begin{cases} z^k = x^k + \theta_k \left(x^k - x^{k-1}\right), \\ x^{k+1} := (1-\lambda_k) z^k + \lambda_k^k \left(t_k x^0 + (1-t_k) P_C \left(z^k - \tau_k \nabla g_k \left(z^k\right)\right)\right), \end{cases} \tag{4.2}$$

where τ_k, and ∇g_k are given by (3.1). Suppose the sequences $\{\lambda_k\}$, $\{t_k\}$ and $\{\theta_k\}$ satisfy the conditions in Theorem 3.7. Then, the sequence $\{x^k\}$ generated by (4.2) strongly converges to the solution $x = P_\Omega(x^0) \in \Gamma$.

When we take $\lambda_k \equiv 1$ in Algorithm 3.1, we obtain the following result regarding the SFPwMOS (1.4).

Corollary 4.3. *Let H, H_i, $i = 1, \dots, N$ be real Hilbert spaces and let $T_i : H \to H_i$, $i = 1, \dots, N$ be bounded linear operators. Let C and Q_i, $i = 1, \dots, N$ be nonempty, closed and convex subsets of H and H_i, $i = 1, \dots, N$ respectively. Given a contraction h with $\zeta \in (0,1)$ and $x^* = P_\Gamma h(x^*)$. Assume that the problem (1.4) is consistent. Let $\{h_k\}$ be a nearly contractive mapping with $\{(\zeta_k, a_k)\}$ such that $\zeta_k \to \zeta$. and any initial guess $x^0 \in H$, let $\{x^k\}$ be the sequence generated by*

$$\begin{cases} z^k = x^k + \theta_k \left(x^k - x^{k-1}\right), \\ x^{k+1} := t_k h_k \left(z^k\right) + (1-t_k) P_C \left(z^k - \tau_k \nabla g_k \left(z^k\right)\right), \end{cases} \tag{4.3}$$

where τ_k, and ∇g_k are given by (3.1). Suppose the sequences satisfies the conditions (i) and (iv). Then, the sequence $\{x^k\}$ generated by (4.3) strong converges to the solution $x^* = P_\Omega h(x^*) \in \Gamma$.

Of course, when we take $h(x) = x^0$, we get the following result regarding the SFPwMOS (1.4).

Corollary 4.4. *Let H, H_i, $i = 1, \dots, N$ be real Hilbert spaces and let $A_i : H \to H_i$, $i = 1, \dots, N$ be bounded linear operators. Let C and Q_i, $i = 1, \dots, N$ be nonempty, closed and convex subsets of H and H_i, $i = 1, \dots, N$ respectively. Assume that the problem (1.4) is consistent. For any initial guess $x^0 \in H$, let $\{x^k\}$ be the sequence generated by*

$$\begin{cases} z^k = x^k + \theta_k \left(x^k - x^{k-1}\right), \\ x^{k+1} := t_k x^0 + (1-t_k) P_C \left(z^k - \tau_k \nabla g_k \left(z^k\right)\right), \end{cases} \tag{4.4}$$

where τ_k, and ∇g_k are given by (3.1). Suppose the sequences $\{t_k\}$ satisfies the conditions (i). Then, the sequence $\{x^k\}$ generated by (4.4) strongly converges to the solution $x^* = P_\Omega h(x^*) \in \Gamma$.

Remark 4.5. In Corollary 4.4 above, for special case, where $N = 1$, the iterative scheme (4.4) reduced exactly to iterative scheme proposed by He et al. [15, Theorem 3.2].

5 Numerical Experiment

In this section, we test two numerical experiments to demonstrate the performance and convergence of Algorithm 3.1. All the codes are written in MATLAB and are performed on a DELL laptop with RAM 8 GB and Intel(R) Core (TM) i5-8265U CPU @ 1.60GHz. Consider the following Problem:

Find a point x^* such that

$$x^* \in \Gamma := C \cap (\cap_{i=1}^{N} A_i^{-1}(Q_i)) \neq \emptyset, \tag{5.1}$$

where the sets C and $Q_i, i = 1, \ldots, N$ and the linear bounded operators $A_i, i = 1, \ldots, N$.

Example 5.1. Let

$$A_1 = \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.9 & 0.6 & 0.5 \\ 0.1 & 0.2 & 0.9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.7 & 0.5 & 0.3 \\ 0.3 & 0.5 & 0.8 \end{bmatrix},$$

$$\begin{aligned} C &= \left\{x \in R^3 | x_1 - \ x_2^2 + 2x_3 \le 0\right\}, \\ Q_1 &= \left\{x \in R^3 \middle| \ x_1^2 + x_2 - x_3 \ \le 0\right\}, \\ Q_2 &= \left\{x \in R^3 \middle| \ x_1 + x_2^2 - x_3 \ \le 0\right\}. \end{aligned}$$

In the experiments we use $E_k < \varepsilon$ as the stopping criteria, where $E_k := TOL_k$, ε is a small enough positive number, $TOL_k := \frac{1}{3}\big(\|x^k - P_C(x^k)\|^2 + \|A_1x^k - P_{Q_1}(A_1x^k)\|^2 + \|A_2x^k - P_{Q_2}(A_2x^k)\|^2\big)$. Note that if $E_k = 0$, then $x^k \in \Gamma$.

Firstly, we select different ρ_k and take $\lambda_k = 0.8$, $\theta_k = 0.6$ to explore the influence of ρ_k on Algorithm 3.1. The results are listed in Table 1.

Table 1: The iterative numbers of Algorithm 3.1 under different choices of ρ_k and ε

ε	$\rho_k = 3.98$	$\rho_k = 3.00$	$\rho_k = 1.50$
10^{-3}	$Iter. = 22$	$Iter. = 26$	$Iter. = 33$
	$TOL_k = 9.06E-04$	$TOL_k = 7.96E-04$	$TOL_k = 9.17E-04$
10^{-4}	$Iter. = 24$	$Iter. = 31$	$Iter. = 51$
	$TOL_k = 6.29E-05$	$TOL_k = 7.60E-05$	$TOL_k = 9.05E-05$

The behavior of the function E_k in Table 1 is described in the following Figures 1.

It can be observed from Table 1 and Figure 1 that Algorithm 3.1 has better performance as ρ_k converges to 4, but $\rho_k \neq 4$.

Secondly, we carry out Algorithm 3.1 with different θ_k to test the effect of inertial on Algorithm 3.1. Throughout the process, we take $\lambda_k = 0.8$, $\rho_k = 3.98$. The results are reported in Table 2.

The behavior of the function E_k in Table 2 is described in Figure 2 as follows.

From Table 2 and Figure 2, it can be seen that the bigger the θ_k, the faster the convergence.

Next, we test the performance of Algorithm 3.1 in higher dimensions.

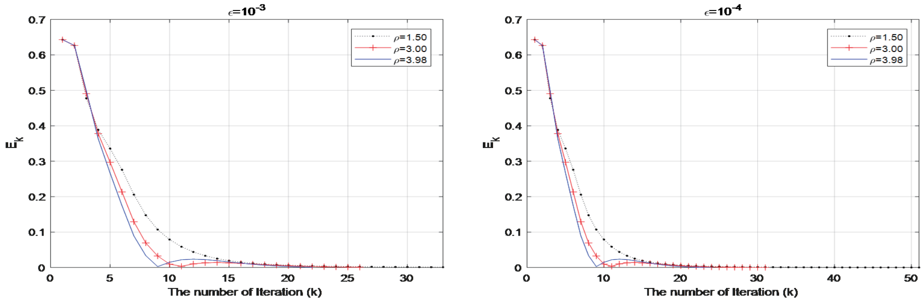

Figure 1: Numerical behavior of Algorithm 3.1 under different choices of ρ_k and ε

Table 2: The iterative numbers of Algorithm 3.1 under different choices of θ_k and ε

ε	$\theta_k = 0.6$	$\theta_k = 0.4$	$\theta_k = 0.3$	$\theta_k = 0.1$
10^{-3}	$Iter. = 22$ $TOL_k = 9.06E-04$	$Iter. = 27$ $TOL_k = 9.96E-04$	$Iter. = 34$ $TOL_k = 8.49E-04$	$Iter. = 43$ $TOL_k = 9.71E-04$
10^{-4}	$Iter. = 24$ $TOL_k = 6.29E-05$	$Iter. = 39$ $TOL_k = 8.47E-05$	$Iter. = 48$ $TOL_k = 8.51E-05$	$Iter. = 62$ $TOL_k = 9.58E-05$
10^{-5}	$Iter. = 37$ $TOL_k = 8.08E-06$	$Iter. = 50$ $TOL_k = 9.90E-06$	$Iter. = 62$ $TOL_k = 8.64E-06$	$Iter. = 81$ $TOL_k = 9.54E-06$

Example 5.2. Let $A_1 = (a_{ij})_{N\times N}$, $A_2 = (b_{ij})_{N\times N}$, $a_{ij} \in (0, 0.1)$ and $b_{ij} \in (0, 0.1)$ are a random matrix,respectively. N be a positive integer.

$$C = \left\{x \in R^N | x_1 - \ x_2^2 + 2x_3 + x_4 + \cdots + x_N \le 0\right\},$$
$$Q_1 = \left\{x \in R^N \middle| \ x_1^2 + x_2 - x_3 + x_4 + \cdots + x_N \ \le 0\right\},$$
$$Q_2 = \left\{x \in R^N \middle| \ x_1 + x_2^2 - x_3 + x_4 + \cdots + x_N \ \le 0\right\}.$$

In the experiments we use $E_k < \varepsilon$ as the stopping criteria, where $E_k := TOL_k$, ε is a small enough positive number, $TOL_k := \frac{1}{3}\left(\left\|x^k - P_C\left(x^k\right)\right\|^2 + \left\|A_1x^k - P_{Q_1}\left(A_1x^k\right)\right\|^2 + \left\|A_2x^k - P_{Q_2}\left(A_2x^k\right)\right\|^2\right)$. Note that if $E_k = 0$, then $x^k \in \Gamma$.

We take $\lambda_k = 0.7$, $\rho_k = 3.0$, $\theta_k = 0.4$ in Algorithm 3.1 to study the behavior of Algorithm 3.1 and compare it with the scheme (1.8). In the processes, we take x^1 as the initial point, where $x^1 = rand(1, N)$. Let $t_k = \frac{2}{k}$, $h\left(x^k\right) = f\left(x^k\right) = 0.1x^k$ in both Algorithm 3.1 and Scheme (1.8). Here, we select the other initial point in Algorithm 3.1 as $x^0 = rand(1, N)$. The behavior values of the function E_k with different ε and N are listed in Table 3 and Table 4, where "Iter.", "Cpu" denote the number of iterations and cpu time in seconds, respectively.

Table 3: The iterative numbers of Algorithm 3.1 and Scheme (1.8) for $N = 10$ and different choices of ε

ε	10^{-3}	10^{-4}	10^{-5}
Algorithm 3.1	$Iter. = 7$ $TOL_k = 4.62E-04$	$Iter. = 9$ $TOL_k = 9.60E-05$	$Iter. = 12$ $TOL_k = 5.23E-06$
Scheme (1.8)	$Iter. = 10$ $TOL_k = 8.0E-04$	$Iter. = 17$ $TOL_k = 9.24E-05$	$Iter. = 26$ $TOL_k = 9.98E-06$

We also plot E_k versus the number of iterations in the following Figure 3.

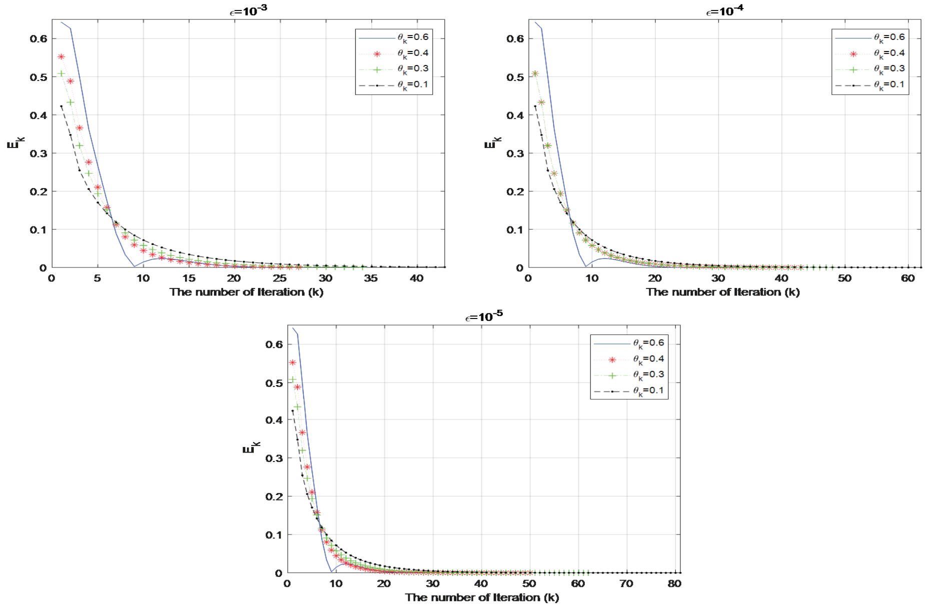

Figure 2: Numerical behavior of Algorithm 3.1 under different choices of θ_k and ε

Table 4: The iterative numbers of Algorithm 3.1 and Scheme (1.8) for $N = 30$ and different choices of ε

ε	10^{-3}	10^{-4}	10^{-5}
Algorithm 3.1	$Iter. = 8$ $TOL_k = 4.20E-04$	$Iter. = 10$ $TOL_k = 7.95E-05$	$Iter. = 13$ $TOL_k = 4.46E-06$
Scheme (1.8)	$Iter. = 19$ $TOL_k = 9.25E-04$	$Iter. = 41$ $TOL_k = 9.75E-05$	$Iter. = 84$ $TOL_k = 9.90E-06$

The results are shown in Table 3, Table 4 and Figure 3, which shows that algorithm 3.1 has better performance than scheme (1.8) no matter $N = 10$ or $N = 30$, by reason of taking much less iterations.

From these numerical results, we can see that our algorithm is effective and promising for solving SFPwMOS. The results of Example 5.2 also show that Algorithm 3.1 has good performance in higher dimensions. Notice that the application of inertial technique and self-adaptive method can improve the performance of the algorithm.

6 Conclusions

In this paper, we study the self-adaptive inertial viscosity projection algorithm for solving split feasibility problem with multiple output sets. The proposed algorithm shows that the sequence converges to a solution of this problem with a simple and novel way. The algorithm uses an adaptive strategy to update the step size and combines inertia technology with approximate compression technology to improve the efficiency of the algorithm. Preliminary numerical results show that the proposed method is practical and promising for SFPwMOS. It has the potential to analyze and design other algorithms for fixed point problems, as well as a more comprehensive computational study is researched in the further.

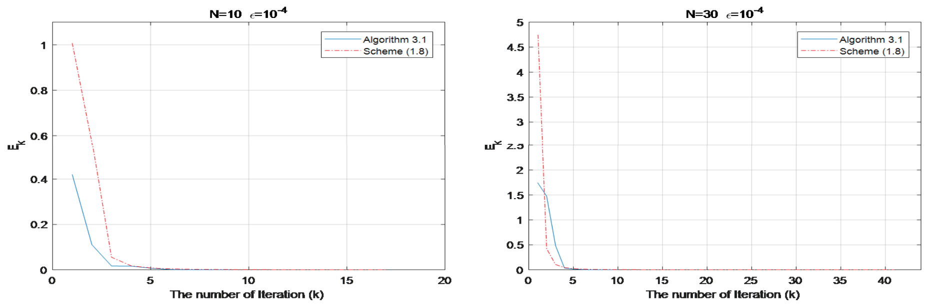

Figure 3: Numerical behavior of Algorithm 3.1 with Scheme (1.8) under different choices of N and ε

Acknowledgment

The authors would like to thank the editor and the anonymous referees for their valuable suggestions and constructive comments that greatly improve this paper.

References

[1] M. Abdellatif and B.S. Thakur, Solving proximal split feasibility problems without prior knowledge of operator norms, *Optim. Lett.* 8 (2013) 2099–2110.

[2] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Space*, Springer Ser. Comput. Math., Springer, 2011.

[3] C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problem, *Inverse Problems* 18 (2002) 441–453.

[4] Y. Censor and A. Segal, The split common fixed point problem for directed operators, *J. Convex Anal.* 16 (2009) 587–600.

[5] Y. Censor and T. Elfving, A multiprojection algorithm using Bregman projections in a product space, *Numer. Algorithms* 8 (1994) 221–239.

[6] Y. Censor, T. Elfving, N. Kopf and T. Bortfeld, The multiple-sets split feasibility problem and its applications for inverse problems, *Inverse Problems* 21 (2005) 2071–2084.

[7] Y.Z. Dang, Y. Gao and Y. Han, A perturbed projection algorithm with inertial technique for split feasibility problem, *J. Appl. Math.* 2012 (2012) Article ID 207323.

[8] Y.Z. Dang, J. Sun and H. Xu, Inertial accelerated algorithms for solving a split feasibility problem, *J. Ind. Manag. Optim.* 13 (2016) 78–78.

[9] Y.Z. Dang, J. Sun and S. Zhang, Double projection algorithms for solving the split feasibility problems, *J. Ind. Manag. Optim.* 15 (2019) 2023–2034.

[10] Y.Z. Dang, J. Yao and Y. Gao, Relaxed two points projection method for solving the multiple-sets split equality problem, *Numer. Algorithms* 78 (2017) 1–13.

[11] M. Dilshad, M. Akram and I.A. Ahmad. Algorithms for split common null point problem without pre-existing estimation of operator norm, *J. Math. Inequa.* 4 (2020) 1151–1163.

[12] A.G. Gebrie and A. Bekele, Viscosity self-adaptive method for generalized split system of variational inclusion problem, *Bull. Iranian Math. Soc.* 47 (2021) 897–917.

[13] E.C. Godwin, C. lzuchukwu and O.T. Mewomo, An inertial extrapolation method for solving generalized split feasibility problems in real Hilbert spaces, *Boll. Unione Mat. Ital.* 14 (2021) 379–401.

[14] S. He and C. Yang, Solving the variational inequality problem defined on intersection of finite level sets, *Abstr. Appl. Anal.* 2013 (2013) 94–121.

[15] S. He and Z. Zhao, Strong convergence of a relaxed CQ algorithm for the split feasibility problem, *J. Inequal. Appl.* 2013 (2013) Article number 197.

[16] P.V. Huy, H.M.V. Le, N.D. Hien and T.V. Anh, Modified Tseng's extragradient methods with self-adaptive step size for solving bilevel split variational inequality problems, *Optimization***71** (2021) 1721–1748.

[17] K. Kankam, P. Srinak, P. Cholamjiak and N. Pholasa, Solving the multiple-set split feasibility problem and the equilibrium problem by a new relaxed CQ algorithm, *Rend. Circ. Mat. Palermo (2).* 69 (2020) 1131–1148.

[18] G. López, V. Martín-Márquez, F. Wang and H.K. Xu, Solving the split feasibility problem without prior knowledge of matrix norms, *Inverse Problems* 28 (2012) 085004.

[19] A. Moudafi, Viscosity Approximation Methods for Fixed-Points Problems, *J. Math. Anal. Appl.* 241 (2000) 46–55.

[20] B. T. Polyak, *Some methods of speeding up the convergence of iteration methods*, Ussr Computational Mathematics & Mathematical Physics 4 (1964) 1–17.

[21] S. Reich and T.M. Tuyen, A new algorithm for solving the split common null point problem in Hilbert spaces, *Numer. Algorithms* 83 (2020) 789–805.

[22] S. Reich and T.M. Tuyen, Iterative methods for solving the generalized split common null point problem in Hilbert spaces, *Optimization* 69 (2020) 1013–1038.

[23] S. Reich and T.M. Tuyen, Two projection methods for solving the multiple-set split common null point problem in Hilbert spaces, *Optimization* 69 (2020) 1913–1934.

[24] S. Reich, M.T. Truong and T. Mai, The split feasibility problem with multiple output sets in Hilbert spaces, *Optim. Lett.* 14 (2020) 2335–2353.

[25] D.R. Sahu, Y.J. Cho, Q.L. Dong, M.R. Kashyap and X.H. Li, Inertial relaxed CQ algorithms for solving a split feasibility problem in Hilbert spaces, *Numer. Algorithms* 87 (2021) 1075–1095.

[26] Y. Shehu, P.T. Vuong and P. Cholamjiak, A self-adaptive projection method with an inertial technique for split feasibility problems in Banach spaces with applications to image restoration problems, *J. Fixed Point Theory Appl.* 21 (2019) Article number 50.

[27] A. Sripattanet and A. Kangtunyakarn, Convergence theorem for solving a new concept of the split variational inequality problems and application, *Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp).* 114 (2020) 1–33.

[28] S. Suantai and P. Jailoka, A self-adaptive algorithm for split null point problems and fixed point problems for demicontractive multivalued mappings, *Acta Appl. Math.* 170 (2020) 883–901.

[29] G. Wang and N. Xiu, Modified projection method for linear split feasibility problems applied to medical image reconstruction, in: 2010 International Conference of Medical Image Analysis and Clinical Application, IEEE Trans.Image Process. 2010, pp. 139–142.

[30] J. Wang, Y. Hu, C. Li and J.C. Yao, Linear convergence of CQ algorithms andapplications in gene regulatory network inference, *Inverse Problems.* 33 (2017) 055017.

[31] H. K. Xu, Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces, *Inverse Problems.* 26 (2010) 105018.

[32] Y. Yao, M. Postolache and Z. Zhu, Gradient methods with selection technique for the multiple-sets split feasibility problem, *Optimization* 69 (2020) 269–281.

Manuscript received 22 April 2021
revised 30 July 2021
accepted for publication 14 October 2021

Yazheng Dang

Department of Management
University of Shanghai for Science and Technology
Shanghai 200093, China
E-mail address: jgdyz@163.com

Caihua Hou
Department of Management
University of Shanghai for Science and Technology
Shanghai 200093, China
E-mail address: houcaihua1234@163.com

Yang Liu
Department of Information Science and Technology
East China University of Political Science and Law
Beijing 102249, China
E-mail address: lyang@ecupl.edu.cn