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PERFECT HERMITIAN MATRIX AND DUAL QUATERNION LÖWNER PARTIAL ORDERS*

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Abstract: In this article, we give characterizations of dual quaternion Moore-Penrose generalized inverse(DQMPGI for short), and get necessary and sufficient conditions for the existence of DQMPGI. By applying DQMPGI and positive semi-definite dual quaternion matrix, we consider properties and characterizations of perfect Hermitian matrix. At last, we introduce two partial orders: Lönwer-S and Lönwer-P partial orders, and discuss their properties, characterizations and relationships.

Key words: *dual quaternion Moore-Penrose generalized inverse, positive semi-definite dual quaternion Hermitian matrix, perfect Hermitian matrix, Löwner-P partial order, Löwner-S partial order*

Mathematics Subject Classification: *15A10, 15B33*

1 Introduction

In 1873, Clifford introduced dual numbers and dual quaternions. Many researches have been done on dual numbers, dual quaternions and their applications [2, 3, 4, 5, 6, 8, 18, 19, 20, 21].

In this paper, we use the following notation. The symbols $\mathbb{R}$, $\mathbb{D}$, $\mathbb{Q}$, $\mathbb{DQ}$ respectively denote the set of real numbers, dual numbers, quaternions, dual quaternions. $\mathbb{R}^{m\times n}$, $\mathbb{D}^{m\times n}$, $\mathbb{Q}^{m\times n}$, $\mathbb{DQ}^{m\times n}$ respectively denote the set of $m\times n$ real matrices, dual matrices, quaternions matrices, dual quaternions matrices. For any $q = q_{st} + \epsilon q_I \in \mathbb{D}$(or $\mathbb{DQ}$), where $q_{st}, q_I \in \mathbb{R}$(or $\mathbb{Q}$) are standard part, infinitesimal part of q, respectively. ϵ is the infinitesimal unit satisfying $\epsilon \neq 0, \epsilon^2 = 0$, and ϵ is commutative in multiplication with real numbers, dual numbers and quaternion numbers.

If $q_{st} \neq 0$, we say that q is appreciable. Otherwise, we say that q is infinitesimal. In 2021, Qi, Ling and Yan[12] introduced total order over $\mathbb{D}$. Let $\widehat{p} = p + \epsilon p_1$, $\widehat{q} = q + \epsilon q_1 \in \mathbb{D}$, then $\widehat{p} < \widehat{q}$ if $p < q$, or $p = q$ and $p_1 < q_1$; $\widehat{p} = \widehat{q}$ if $p = q$ and $p_1 = q_1$. Based on this result, Wang, Cui and Wei[16]introduce QLY total order over $\mathbb{D}^m$.

Let $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, in which $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are three imaginary units of quaternions, and q_0, q_1, q_2 and q_3 are real numbers. The conjugate of q is $\overline{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$. Denote $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{m\times n}$, where A_{st}, $A_I \in \mathbb{Q}^{m\times n}$ are the standard part, infinitesimal part of A, respectively. Denote the transpose of A as $A^T = A_{st}^T + \epsilon A_I^T$, the conjugate of A as

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$\overline{A} = \overline{A}_{st} + \epsilon\overline{A}_I$, the conjugate transpose of A as $A^* = \overline{A}^T$, and $A^* = A_{st}^* + \epsilon A_I^*$. A square dual quaternion matrices $A \in \mathbb{DQ}^{n\times n}$ is nonsingular(invertible) if there exists $B \in \mathbb{DQ}^{n\times n}$ such that $AB = BA = I_n$, we denote $A^{-1} = B$.

The dual quaternion Moore-Penrose generalized inverse(DQMPGI for short)[11] of $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{m\times n}$ is the unique matrix which satisfies

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA, \tag{1.1}$$

and is denoted by $A^\dagger$. It should be noted that, unlike the general quaternion matrix, DQMPGI does not necessarily exist for any given dual quaternion matrix. If the DQMPGI of dual quaternion matrix exists, it is unique . Furthermore, Ling, Qi and Yan[11] give necessary and sufficient conditions for $X = A_{st}^\dagger - \epsilon A_{st}^\dagger A_I A_{st}^\dagger \in \mathbb{DQ}^{n\times m}$ to be a $\{1\}$-, $\{3\}$- and $\{4\}$-dual quaternion generalized inverse of A, respectively.

An $n \times n$ dual quaternion matrix A is said to be Hermitian, if $A^* = A$. We denote the set of all dual quaternion Hermitian matrices over $\mathbb{DQ}^{n\times n}$ as $\mathbb{DQ}_{\mathrm{H}}^{n\times n}$. Let $A = A_{st} + \epsilon A_I \in \mathbb{DQ}_{\mathrm{H}}^{n\times n}$. If for any $x = x_{st} + \epsilon x_I \in \mathbb{DQ}^n$ with x being appreciable, we have $x^*Ax > 0$ and it is appreciable, then A is called positive definite. A is called positive semi-definite if for any $x = x_{st} + \epsilon x_I \in \mathbb{DQ}^n$, we have $x^*Ax \geq 0$. Furthermore, let $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{n\times n}$ be a positive semi-definite dual quaternion Hermitian matrix, then A is called perfect Hermitian if there is a positive semi-definite dual quaternion Hermitian matrix $L \in \mathbb{DQ}^{n\times n}$ such that $A = L^2$, [13]. The symbols $\mathbb{DQ}_{>}^{n\times n}$, $\mathbb{DQ}_{\geq}^{n\times n}$ and $\mathbb{DQ}_{\mathrm{PH}}^{n\times n}$ stand for the set of all dual Hermitian positive definite, dual Hermitian positive semi-definite and perfect Hermitian matrices over $\mathbb{DQ}_{\mathrm{H}}^{n\times n}$, respectively.

For more details about the theory of dual quaternion matrix, such as the properties and applications of the left eigenvalue, right eigenvalue, singular value decomposition of dual quaternion matrix, orthogonal dual quaternion matrix, unitary dual quaternion matrix, you can refer to [10, 11, 12, 13]. These results provide a solid foundation for the following research on dual quaternion theory and its applications.

In this paper, we consider the conditions for the existence of DQMPGI. By applying DQMPGI, we get characterizations of perfect Hermitian matrix. At last, we introduce two partial orders over $\mathbb{DQ}^{m\times n}$.

2 Some Properties and Characterizations of Dual Euaternion Moore-Penrose Generalized Inverse

It is well known that generalized inverse is one of the powerful tools to study the least-squares problem, and it is widely used in numerical computation, control theory, optimization, etc. Dual generalized inverses (DGIs) are also applied to the study of dual analog of least-squares problem, and to kinematic analysis and synthesis of spatial mechanisms. In [11], Ling, Qi and Yan introduce the DQMPGI $A^\dagger$ of dual quaternion matrix $A \in \mathbb{DQ}^{m\times n}$. When the DQMPGI of A exists, Ling, Qi and Yan[11] get a characterization of $A^\dagger$ by applying the SVD of dual quaternion matrix A. In this section, we consider properties and characterizations of DQMPGI.

Lemma 2.1 ([11, 13])**.** *Suppose that $A \in \mathbb{DQ}^{m\times n}$, there exist dual quaternion unitary matrices $U \in \mathbb{DQ}^{m\times m}$ and $V \in \mathbb{DQ}^{n\times n}$, such that*

$$U^*AV = \begin{bmatrix} \Sigma_t & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.1}$$

where $\Sigma_t \in \mathbb{D}^{t\times t}$ *has the form*

$$\Sigma_t = \operatorname{diag}\left(\lambda_1, \dots, \lambda_r, \dots, \lambda_t\right), \tag{2.2}$$

$r \le t \le \min\{m, n\}$, $\lambda_1 \ge \cdots \ge \lambda_r$ *are positive appreciable dual numbers, and* $\lambda_{r+1} \ge \cdots \ge \lambda_t$ *are positive infinitesimal dual numbers.*

When the DQMPGI of A *exists, the diagonal elements of* Σ_t *are all appreciable, the rank of* A *is equal to* t*. Furthermore,*

$$A^{\dagger} = V \begin{bmatrix} \Sigma_t^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^{*}. \tag{2.3}$$

Theorem 2.2. *Let* $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{m\times n}$*, then a necessary and sufficient conditioning for* A *has the DQMPGI* $A^{\dagger} = Y - \epsilon Z \in \mathbb{DQ}^{n\times m}$ *is:*

$$\begin{cases} Y = A_{st}^{\dagger}, \\ A_I = A_{st}A_{st}^{\dagger}A_I - A_{st}ZA_{st} + A_IA_{st}^{\dagger}A_{st}, \\ Z = A_{st}^{\dagger}A_{st}Z - A_{st}^{\dagger}A_IA_{st}^{\dagger} + ZA_{st}A_{st}^{\dagger}, \\ (A_IA_{st}^{\dagger} - A_{st}Z) \text{ and } (A_{st}^{\dagger}A_I - ZA_{st}) \text{ are Hermitian.} \end{cases} \tag{2.4}$$

Proof. Let $X = Y - \epsilon Z \in \mathbb{DQ}^{n\times m}$, where Y and Z satisfy (2.4). Then $AXA = A$ is equivalent to $(A_{st} + \epsilon A_I)(Y - \epsilon Z)(A_{st} + \epsilon A_I) = A_{st} + \epsilon A_I$, that is,

$$\begin{cases} A_{st}YA_{st} = A_{st}, & (2.5) \\ A_{st}YA_I - A_{st}ZA_{st} + A_IYA_{st} = A_I; & (2.6) \end{cases}$$

$XAX = X$ is equivalent to $(Y - \epsilon Z)(A_{st} + \epsilon A_I)(Y - \epsilon Z) = Y - \epsilon Z$, that is,

$$\begin{cases} YA_{st}Y = Y, & (2.7) \\ YA_{st}Z - YA_IY + ZA_{st}Y = Z. & (2.8) \end{cases}$$

$AX = (AX)^*$ is equivalent to $(A_{st} + \epsilon A_I)(Y - \epsilon Z) = \left((A_{st} + \epsilon A_I)(Y - \epsilon Z)\right)^*$, that is,

$$\begin{cases} A_{st}Y = (A_{st}Y)^*, & (2.9) \\ (A_IY - A_{st}Z) = (A_IY - A_{st}Z)^*; & (2.10) \end{cases}$$

$XA = (XA)^*$ is equivalent to $(Y - \epsilon Z)(A_{st} + \epsilon A_I) = \left((Y - \epsilon Z)(A_{st} + \epsilon A_I)\right)^*$, that is,

$$\begin{cases} YA_{st} = (YA_{st})^*, & (2.11) \\ (YA_I - ZA_{st}) = (YA_I - ZA_{st})^*. & (2.12) \end{cases}$$

Since Y is the Moore-Penrose inverse of A_{st} is equivalent to that it satisfies (2.5), (2.7), (2.9), (2.11). According to (2.6), (2.8), (2.10) and (2.12) we get that a necessary and sufficient condition for A has the DQMPGI $A^{\dagger} = Y - \epsilon Z \in \mathbb{DQ}^{n\times m}$ is (2.4). □

Theorem 2.3. *Let* $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{m\times n}$*. Then the following conditions are equivalent:*

(1) *the DQMPGI* $A^{\dagger}$ *of* A *exists;*

(2) $\left(I_m - A_{st}A_{st}^{\dagger}\right) A_I \left(I_n - A_{st}^{\dagger}A_{st}\right) = 0$*;*

(3) $\operatorname{rank}\begin{bmatrix} A_I & A_{st} \\ A_{st} & 0 \end{bmatrix} = 2\operatorname{rank}(A_{st})$.

Furthermore, when the DQMPGI $A^\dagger$ of A exists,

$$A^\dagger = A_{st}^\dagger - \epsilon R, \tag{2.13}$$

where

$$R = A_{st}^\dagger A_I A_{st}^\dagger - (A_{st}^* A_{st})^\dagger A_I^* (I_m - A_{st} A_{st}^\dagger) - (I_n - A_{st}^\dagger A_{st}) A_I^* (A_{st} A_{st}^*)^\dagger. \tag{2.14}$$

Proof. Let the rank of A_{st} be r and the singular value decomposition of A_{st} be

$$A_{st} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^*, \tag{2.15}$$

where $U \in \mathbb{Q}^{m\times m}$ and $V \in \mathbb{Q}^{n\times n}$ are unitary quaternion matrices, $\Sigma \in \mathbb{Q}^{r\times r}$ is a diagonal positive quaternion matrix [22]. Then

$$A_{st}^\dagger = V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* \tag{2.16}$$

and

$$A_{st}^\dagger A_{st} = V \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} V^*, \; A_{st} A_{st}^\dagger = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{2.17}$$

Furthermore, write

$$U^* A_I V = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \; V^* Z U = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}, \tag{2.18}$$

where $A_1 \in \mathbb{Q}^{r\times r}$ and $Z_1 \in \mathbb{Q}^{r\times r}$.

(1) $\Rightarrow$ (2) Suppose that the DQMPGI $A^\dagger$ of A exists and $A^\dagger = Y - \epsilon Z$. Then by applying (2.4), we have $Y = A_{st}^\dagger$, and $A_I = A_I A_{st}^\dagger A_{st} - A_{st} Z A_{st} + A_{st} A_{st}^\dagger A_I$. It follows that

$$\begin{aligned} \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ A_3 & 0 \end{bmatrix} - \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2A_1 - \Sigma Z_1 \Sigma & A_2 \\ A_3 & 0 \end{bmatrix}. \end{aligned}$$

Therefore, $A_4 = 0$, that is, $\left(I_m - A_{st} A_{st}^\dagger\right) A_I \left(I_n - A_{st}^\dagger A_{st}\right) = 0$.

(2) $\Rightarrow$ (1) If $\left(I_m - A_{st} A_{st}^\dagger\right) A_I \left(I_n - A_{st}^\dagger A_{st}\right) = 0$, it is easy to check that $A_4 = 0$. Then from (2.18) we get

$$A_I = U \begin{bmatrix} A_1 & A_2 \\ A_3 & 0 \end{bmatrix} V^*. \tag{2.19}$$

It follows from (2.15) that

$$A = A_{st} + \epsilon A_I = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^* + \epsilon U \begin{bmatrix} A_1 & A_2 \\ A_3 & 0 \end{bmatrix} V^*.$$

Write

$$X = V\begin{bmatrix}\Sigma^{-1} & 0\\ 0 & 0\end{bmatrix}U^* - \epsilon V\begin{bmatrix}\Sigma^{-1}A_1\Sigma^{-1} & -\Sigma^{-2}A_3^*\\ -A_2^*\Sigma^{-2} & 0\end{bmatrix}U^*. \tag{2.20}$$

Then

$$\begin{aligned} AX &= U\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix}U^* + \epsilon U\begin{bmatrix}0 & \Sigma^{-1}A_3^*\\ A_3\Sigma^{-1} & 0\end{bmatrix}U^*,\\ XA &= V\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix}V^* + \epsilon V\begin{bmatrix}0 & \Sigma^{-1}A_2\\ A_2^*\Sigma^{-1} & 0\end{bmatrix}V^*. \end{aligned}$$

It shows that AX and XA are Hermitian. And it is easy to check that $AXA = A$ and $XAX = X$. Therefore, by applying (1.1) we conclude that the DQMPGI of A exists and $X = A^\dagger$.

Furthermore, applying (2.15), (2.16) and (2.19), we get

$$\begin{aligned} V\begin{bmatrix}\Sigma^{-1}A_1\Sigma^{-1} & 0\\ 0 & 0\end{bmatrix}U^* &= A_{st}^\dagger A_I A_{st}^\dagger,\\ V\begin{bmatrix}0 & \Sigma^{-2}A_3^*\\ 0 & 0\end{bmatrix}U^* &= (A_{st}^*A_{st})^\dagger A_I^*(I_m - A_{st}A_{st}^\dagger),\\ V\begin{bmatrix}0 & 0\\ A_2^*\Sigma^{-2} & 0\end{bmatrix}U^* &= (I_n - A_{st}^\dagger A_{st})A_I^*(A_{st}A_{st}^*)^\dagger. \end{aligned}$$

Therefore, from (2.20) and $X = A^\dagger$ we get (2.13).

(2) $\Leftrightarrow$ (3) By applying (2.15), (2.16) and (2.17), we have

$$\operatorname{rank}\begin{bmatrix}A_I & A_{st}\\ A_{st} & 0\end{bmatrix} = \operatorname{rank}(A_{st}) + \operatorname{rank}(A_{st}) + \operatorname{rank}\left(\left(I_m - A_{st}A_{st}^\dagger\right)A_I\left(I_n - A_{st}^\dagger A_{st}\right)\right),$$

Therefore, we can get that $\operatorname{rank}\left(\left(I_m - A_{st}A_{st}^\dagger\right)A_I\left(I_n - A_{st}^\dagger A_{st}\right)\right) = 0$ if and only if condition (3) holds. □

Remark 2.4. A matrix X is called a $\{1\}$-dual quaternion generalized inverse of $A \in \mathbb{DQ}^{m\times n}$, in which X satisfies equation $AXA = A$. In [11], Ling, Qi and Yan get that a necessary and sufficient conditions for the matrix $X = A_{st}^\dagger - \epsilon A_{st}^\dagger A_I A_{st}^\dagger$ to be a $\{1\}$-dual quaternion generalized inverse of $A \in \mathbb{DQ}_{m\times n}$ is

$$\left(I_m - A_{st}A_{st}^\dagger\right)A_I\left(I_n - A_{st}^\dagger A_{st}\right) = 0.$$

Therefore, from Theorem 2.3 we get that the DQMPGI $A^\dagger$ of A exists if and only if $\{1\}$-dual quaternion generalized inverse of A exists.

The solvability of quaternion matrix equations is one of the important topics in quaternion matrix analysis and applications.

Lemma 2.5 ([17])**.** *Let $A_{st} \in \mathbb{Q}^{m\times n}$, $X_{st} \in \mathbb{Q}^{n\times l}$, $C_{st} \in \mathbb{Q}^{m\times l}$. Then the equation $A_{st}X_{st} = C_{st}$ is consistent if and only if*

$$\left(I_m - A_{st}A_{st}^\dagger\right)C_{st} = 0 \tag{2.21}$$

if and only if

$$\operatorname{rank}\begin{bmatrix}A_{st} & C_{st}\end{bmatrix} = \operatorname{rank}(A_{st}). \tag{2.22}$$

Furthermore, the general solution of $A_{st}X_{st} = C_{st}$ *is* $X_{st} = A_{st}^{\dagger}C_{st} + (I - A_{st}^{\dagger}A_{st})Y_{st}$*, where* $Y_{st} \in \mathbb{Q}^{n\times l}$.

Next we consider the dual quaternion matrix equation

$$AX = C, \tag{2.23}$$

where $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{m\times n}$, $C = C_{st} + \epsilon C_I \in \mathbb{DQ}^{m\times l}$, $X = X_{st} + \epsilon X_I \in \mathbb{DQ}^{n\times l}$.

Theorem 2.6. *The matrix equation (2.23) is consistent if and only if*

$$\operatorname{rank}\begin{bmatrix}A_{st} & 0 & C_{st}\\ A_I & A_{st} & C_I\end{bmatrix} = \operatorname{rank}\begin{bmatrix}A_{st} & 0\\ A_I & A_{st}\end{bmatrix}. \tag{2.24}$$

Furthermore, if the DQMPGI of A *exists, then the equation* $AX = C$ *is consistent if and only if*

$$\operatorname{rank}\begin{bmatrix}A_{st} & 0 & C_{st}\\ A_I & A_{st} & C_I\end{bmatrix} = 2\operatorname{rank}(A_{st}). \tag{2.25}$$

In this case, the general solution to this equation is

$$X = A^{\dagger}C + (I - A^{\dagger}A)Y, \tag{2.26}$$

where $Y \in \mathbb{DQ}^{n\times l}$.

Proof. Let $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{m\times n}$, $C = C_{st} + \epsilon C_I \in \mathbb{DQ}^{m\times l}$ and $X = X_{st} + \epsilon X_I \in \mathbb{DQ}^{n\times l}$. Since

$$AX = C \Leftrightarrow \begin{bmatrix}A_{st}\\ A_I\end{bmatrix} X_{st} + \begin{bmatrix}0\\ A_{st}\end{bmatrix} X_I = \begin{bmatrix}C_{st}\\ C_I\end{bmatrix}, \tag{2.27}$$

then $AX = C$ is consistent if and only if $\begin{bmatrix}A_{st}\\ A_I\end{bmatrix} X_{st} + \begin{bmatrix}0\\ A_{st}\end{bmatrix} X_I = \begin{bmatrix}C_{st}\\ C_I\end{bmatrix}$ is consistent. By applying Lemma 2.5 we get that (2.24) is true.

Furthermore, if the DQMPGI of A exists, then by applying Theorem 2.3 we get

$$\operatorname{rank}\begin{bmatrix}A_{st} & 0\\ A_I & A_{st}\end{bmatrix} = 2\operatorname{rank}(A_{st}).$$

It follows from (2.24) that (2.25) holds.

Let the DQMPGI of A exists, and $AX = C$ is consistent. Then $AX = C$ is consistent if and only if there exist X_{st} and X_I such that

$$\begin{cases} A_{st}X_{st} = C_{st}\\ A_I X_{st} + A_{st}X_I = C_I. \end{cases} \tag{2.28}$$

It is obvious that $X_{st} = A_{st}^{\dagger}C_{st}$ is a special solution of $A_{st}X_{st} = C_{st}$. Substituting $X_{st} = A_{st}^{\dagger}C_{st}$ into $A_I X_{st} + A_{st}X_I = C_I$ we get $A_{st}X_I = C_I - A_I A_{st}^{\dagger}C_{st}$. It is obvious that

$X_I = A_{st}^{\dagger}C_I - A_{st}^{\dagger}A_I A_{st}^{\dagger}C_{st}$ is a special solution of $A_{st}X_I = C_I - A_I A_{st}^{\dagger}C_{st}$. Therefore, we get a special solution of (2.28) as follows,

$$\begin{cases} X_{st} = A_{st}^{\dagger}C_{st} \\ X_I = A_{st}^{\dagger}C_I - A_{st}^{\dagger}A_I A_{st}^{\dagger}C_{st}, \end{cases} \tag{2.29}$$

and

$$\begin{cases} A_{st}A_{st}^{\dagger}C_{st} = C_{st} \\ A_{st}A_{st}^{\dagger}\left(C_I - A_I A_{st}^{\dagger}C_{st}\right) = C_I - A_I A_{st}^{\dagger}C_{st}. \end{cases} \tag{2.30}$$

Next, when the DQMPGI of A exists, applying Theorem 2.3 and (2.30) we get

$$\begin{aligned} AA^{\dagger}C &= (A_{st} + \epsilon A_I)\left(A_{st}^{\dagger} - \epsilon R\right)(C_{st} + \epsilon C_I) \\ &= A_{st}A_{st}^{\dagger}C_{st} + \epsilon\left(-A_{st}RC_{st} + A_I A_{st}^{\dagger}C_{st} + A_{st}A_{st}^{\dagger}C_I\right) \\ &= C_{st} + \epsilon\Big(-A_{st}A_{st}^{\dagger}A_I A_{st}^{\dagger}C_{st} + A_{st}\left(A_{st}^{*}A_{st}\right)^{\dagger}A_I^{*}\left(I_m - A_{st}A_{st}^{\dagger}\right)C_{st} \\ &\qquad + A_I A_{st}^{\dagger}C_{st} + A_{st}A_{st}^{\dagger}C_I\Big) \\ &= C_{st} + \epsilon\left(-A_{st}A_{st}^{\dagger}A_I A_{st}^{\dagger}C_{st} + A_I A_{st}^{\dagger}C_{st} + A_{st}A_{st}^{\dagger}C_I\right) \\ &= C_{st} + \epsilon C_I = C. \end{aligned} \tag{2.31}$$

It follows from $A(I - A^{\dagger}A) = 0$ that $X = A^{\dagger}C + (I - A^{\dagger}A)Y$ is a general solution of $AX = C$, in which $Y \in \mathbb{DQ}^{n\times l}$.

Suppose that X is a solution of $AX = C$, then $X = A^{\dagger}C + X - A^{\dagger}C = A^{\dagger}C + X - A^{\dagger}AX = A^{\dagger}C + (I - A^{\dagger}A)X$. Therefore, the general solution of $AX = C$ is (2.26). □

In [7], Cui and Qi introduce the 2-norm of a dual quaternion vector.

Lemma 2.7 ([7]). *Let $x \in \mathbb{DQ}^{n\times 1}$ is a dual quaternion vector, then the 2-norm of x is defined by*

$$\|x\|_2 = \begin{cases} \sqrt{\sum_{i=1}^{n} |x_i|^2}, & if\ x\ is\ appreciable \\ \sqrt{\sum_{i=1}^{n} |x_{i,I}|^2}\epsilon, & otherwise. \end{cases} \tag{2.32}$$

It is obvious that $\|Ux\|_2 = \|x\|_2$, where U is an arbitrary $n \times n$ dual quaternion unitary matrix.

Next, we consider one application of DQMPGI in least-squares problems.

Theorem 2.8. *Let $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{m\times n}$, $x = x_{st} + \epsilon x_I \in \mathbb{DQ}^{n\times 1}$, $b = b_{st} + \epsilon b_I \in \mathbb{DQ}^{m\times 1}$, the DQMPGI $A^{\dagger}$ of A exists. Then the least-squares solution of the inconsistent dual quaternion matrix equation*

$$Ax \approx b \tag{2.33}$$

is

$$x = A^{\dagger}b + \left(I_n - A^{\dagger}A\right)w, \tag{2.34}$$

where $w \in \mathbb{DQ}^{n\times 1}$.

Proof. Let the decomposition of A be as in (2.1) and $\text{rank}(A) = t$. Denote

$$V^*x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; U^*b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{2.35}$$

where $x_1 \in \mathbb{DQ}^{t\times 1}$ $x_2 \in \mathbb{DQ}^{(n-t)\times 1}$, $b_1 \in \mathbb{DQ}^{t\times 1}$ and $b_2 \in \mathbb{DQ}^{(m-t)\times 1}$. Therefore, we have

$$\|Ax-b\|^2 = \left\| U \begin{bmatrix} \Sigma_t & 0 \\ 0 & 0 \end{bmatrix} V^*x - b \right\|^2 = \left\| \begin{bmatrix} \Sigma_t x_1 - b_1 \\ -b_2 \end{bmatrix} \right\|^2 = \|\Sigma_t x_1 - b_1\|^2 + \|b_2\|^2.$$

So, $\|Ax-b\|^2 = \min$ if and only if $\|\Sigma_t x_1 - b_1\|^2 = 0$, that is $x_1 = \Sigma_t^{-1} b_1$. Furthermore, applying (2.3) and (2.35) we get

$$x = V \begin{bmatrix} \Sigma_t^{-1} b_1 \\ x_2 \end{bmatrix} = A^\dagger b + \left(I_n - A^\dagger A\right) \begin{bmatrix} w_1 \\ x_2 \end{bmatrix},$$

where $w_1 \in \mathbb{DQ}^{t\times 1}$ is arbitrary. Therefore, we get (2.34). □

3 Perfect Hermitian Matrix

It is well known that positive definite (positive semi-definite) matrix is one of the most important special matrices in the study of matrix theory. In [13], on the basis of positive semi-definite dual quaternion Hermitian matrices, perfect Hermitian matrix is introduced and its properties are studied over $\mathbb{DQ}^{n\times n}$.

Lemma 3.1 ([11, 13]). *Let $A \in \mathbb{DQ}^{n\times n}$ is a positive semi-definite dual quaternion Hermitian matrix. Then there are unitary matrix $U \in \mathbb{DQ}^{n\times n}$ and a diagonal matrix $\Sigma \in \mathbb{D}^{n\times n}$ such that $U^*AU = \Sigma$, where*

$$\Sigma = \begin{bmatrix} M & 0 \\ 0 & \epsilon N \end{bmatrix}, \tag{3.1}$$

M and N are diagonal, the diagonal entries of M are positive and appreciable, and the diagonal entries of N are nonnegative real numbers.

Furthermore, when $N = 0$, A is a perfect Hermitian matrix.

Lemma 3.2 ([13]). *For any $B \in \mathbb{DQ}^{m\times n}$, $M = B^*B$ is a perfect Hermitian matrix.*

Theorem 3.3. *Let $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{n\times n}$ is a positive semi-definite dual quaternion Hermitian matrix. Then A is a perfect Hermitian matrix if and only if the DQMPGI of A exists.*

Proof. " $\Rightarrow$ ": Let A be a perfect Hermitian matrix. By applying Lemma 3.2 we see that there exists $B = B_{st} + \epsilon B_I \in \mathbb{DQ}^{n\times n}$ such that $A = B^*B$. Then $A_{st} + \epsilon A_I = (B_{st}^* + \epsilon B_I^*)(B_{st} + \epsilon B_I) = B_{st}^* B_{st} + \epsilon (B_{st}^* B_I + B_I^* B_{st})$, that is,

$$\begin{cases} A_{st} = B_{st}^* B_{st}, \\ A_I = B_{st}^* B_I + B_I^* B_{st}. \end{cases} \tag{3.2}$$

Since

$$\begin{aligned}
\text{rank} \begin{bmatrix} A_I & A_{st} \\ A_{st} & 0 \end{bmatrix} &= \text{rank} \begin{bmatrix} B_{st}^* B_I + B_I^* B_{st} & B_{st}^* B_{st} \\ B_{st}^* B_{st} & 0 \end{bmatrix} \\
&= \text{rank} \begin{bmatrix} 0 & B_{st}^* B_{st} \\ B_{st}^* B_{st} & 0 \end{bmatrix} = 2\text{rank}\left(B_{st}^* B_{st}\right) \\
&= 2\text{rank}\left(A_{st}\right),
\end{aligned}$$

applying Theorem 2.3, we get that the DQMPGI of A exists.

" $\Leftarrow$ ": Let $A = A_{st} + \epsilon A_I$ be a positive semi-definite dual quaternion Hermitian matrix, then it is as the form in (3.1). Write $U = U_{st} + \epsilon U_I$ and $M = M_{st} + \epsilon M_I$. Then

$$\begin{aligned} A &= (U_{st} + \epsilon U_I)\left(\begin{bmatrix} M_{st} & 0 \\ 0 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} M_I & 0 \\ 0 & N \end{bmatrix}\right)(U_{st}^* + \epsilon U_I^*) \\ &= U_{st}\begin{bmatrix} M_{st} & 0 \\ 0 & 0 \end{bmatrix}U_{st}^* + \epsilon\left(U_{st}\begin{bmatrix} M_{st} & 0 \\ 0 & 0 \end{bmatrix}U_I^* + U_{st}\begin{bmatrix} M_I & 0 \\ 0 & N \end{bmatrix}U_{st}^* + U_I\begin{bmatrix} M_{st} & 0 \\ 0 & 0 \end{bmatrix}U_{st}^*\right). \end{aligned}$$

It follows from $A = A_{st} + \epsilon A_I$ that

$$\begin{cases} A_{st} = U_{st}\begin{bmatrix} M_{st} & 0 \\ 0 & 0 \end{bmatrix}U_{st}^*, \\ A_I = U_{st}\begin{bmatrix} M_{st} & 0 \\ 0 & 0 \end{bmatrix}U_I^* + U_{st}\begin{bmatrix} M_I & 0 \\ 0 & N \end{bmatrix}U_{st}^* + U_I\begin{bmatrix} M_{st} & 0 \\ 0 & 0 \end{bmatrix}U_{st}^* \end{cases}$$

and

$$\begin{aligned} &\operatorname{rank}\begin{bmatrix} A_I & A_{st} \\ A_{st} & 0 \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} U_{st}\begin{bmatrix} M_{st} & 0 \\ 0 & 0 \end{bmatrix}U_I^* + U_{st}\begin{bmatrix} M_I & 0 \\ 0 & N \end{bmatrix}U_{st}^* + U_I\begin{bmatrix} M_{st} & 0 \\ 0 & 0 \end{bmatrix}U_{st}^* & U_{st}\begin{bmatrix} M_{st} & 0 \\ 0 & 0 \end{bmatrix}U_{st}^* \\ U_{st}\begin{bmatrix} M_{st} & 0 \\ 0 & 0 \end{bmatrix}U_{st}^* & 0 \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} \begin{bmatrix} M_{st} & 0 \\ 0 & 0 \end{bmatrix}U_I^*U_{st} + \begin{bmatrix} M_I & 0 \\ 0 & N \end{bmatrix} + U_{st}^*U_I\begin{bmatrix} M_{st} & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} M_{st} & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} M_{st} & 0 \\ 0 & 0 \end{bmatrix} & 0 \end{bmatrix} \\ &= 2\operatorname{rank}(M_{st}) + \operatorname{rank}(N). \end{aligned}$$

When the DQMPGI of A exists, by applying Theorem 2.3 and $\operatorname{rank}(A_{st}) = \operatorname{rank}(M_{st})$, we get $N = 0$. Furthermore, by applying Lemma 3.1, we get that A is a perfect Hermitian matrix. □

Lemma 3.4. *Let $A = A_{st} + \epsilon\,(A_I A_{st} + A_{st} A_I^*) \in \mathbb{DQ}^{n\times n}$. If A_{st} is a positive semi-definite quaternion Hermitian matrix, then A is perfect Hermitian matrix.*

Proof. Since A_{st} is a positive semi-definite quaternion Hermitian matrix, then $A = A_{st} + \epsilon\,(A_I A_{st} + A_{st} A_I^*)$ is Hermitian. Write $A_{st} = \left(A_{st}^{\frac{1}{2}}\right)^2$, where $A_{st}^{\frac{1}{2}}$ is positive semi-definite.

Denote $B^* = A_{st}^{\frac{1}{2}} + \epsilon\left(A_I A_{st}^{\frac{1}{2}}\right)$. Then $B = A_{st}^{\frac{1}{2}} + \epsilon\left(A_{st}^{\frac{1}{2}} A_I^*\right)$ and

$$B^*B = \left(A_{st}^{\frac{1}{2}} + \epsilon\left(A_I A_{st}^{\frac{1}{2}}\right)\right)\left(A_{st}^{\frac{1}{2}} + \epsilon\left(A_{st}^{\frac{1}{2}} A_I^*\right)\right) = A_{st} + \epsilon\,(A_{st}A_I^* + A_I A_{st}) = A.$$

Furthermore, applying Lemma 3.2 gives that A is a perfect Hermitian matrix. □

Theorem 3.5. *Let $A = A_{st} + \epsilon A_I \in \mathbb{DQ}_{\mathrm{H}}^{n\times n}$. If*

$$\begin{cases} A_{st} \text{ is positive semi-definite}, \\ \left(I_n - A_{st}A_{st}^\dagger\right) A_I \left(I_n - A_{st}^\dagger A_{st}\right) = 0. \end{cases} \tag{3.3}$$

then A is a perfect Hermitian matrix.

Proof. Let $A = A_{st} + \epsilon A_I \in \mathbb{DQ}_{\mathrm{H}}^{n\times n}$. If A_{st} and A_I satisfy (3.3) then

$$\begin{aligned} A &= A_{st} + \epsilon A_I \\ &= A_{st} + \epsilon \left(A_{st} A_{st}^{\dagger} A_I + A_I A_{st}^{\dagger} A_{st} - A_{st} A_{st}^{\dagger} A_I A_{st}^{\dagger} A_{st} \right) \\ &\qquad + \epsilon \left(I_n - A_{st} A_{st}^{\dagger} \right) A_I \left(I_n - A_{st}^{\dagger} A_{st} \right) \\ &= A_{st} + \epsilon \left(A_{st} A_{st}^{\dagger} A_I + A_I A_{st}^{\dagger} A_{st} - A_{st} A_{st}^{\dagger} A_I A_{st}^{\dagger} A_{st} \right) \\ &= A_{st} + \epsilon \left(\left(\left(A_I - \frac{1}{2} A_{st} A_{st}^{\dagger} A_I \right) A_{st}^{\dagger} \right) A_{st} + A_{st} \left(A_{st}^{\dagger} \left(A_I - \frac{1}{2} A_I A_{st}^{\dagger} A_{st} \right) \right) \right). \end{aligned}$$

Since A_{st} is positive semi-definite, applying $A_I = A_I^*$ we get

$$\left(\left(A_I - \frac{1}{2} A_{st} A_{st}^{\dagger} A_I \right) A_{st}^{\dagger} \right)^* = A_{st}^{\dagger} \left(A_I - \frac{1}{2} A_{st} A_{st}^{\dagger} A_I \right)^* = A_{st}^{\dagger} \left(A_I - \frac{1}{2} A_I A_{st}^{\dagger} A_{st} \right).$$

Applying Lemma 3.4, it follows that A is a perfect Hermitian matrix. □

Theorem 3.6. *Let $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{n\times n}$. Then A is perfect Hermitian matrix if and only if D^*AD is perfect Hermitian matrix for any $D = D_{st} + \epsilon D_I \in \mathbb{DQ}^{n\times m}$.*

Proof. "⇒" Since A is perfect Hermitian matrix, we see that there exists a positive semi-definite dual quaternion Hermitian matrix $L \in \mathbb{DQ}^{n\times n}$ such that $A = L^*L$.

For any $D \in \mathbb{DQ}^{n\times m}$, denote $P = LD$. Then $D^*AD = D^*L^*LD = P^*P$. Therefore, applying Lemma 3.2 gives that D^*AD is a perfect Hermitian matrix.

"⇐" Let $m = n$ and $D = I$. It is obvious that A is a perfect Hermitian matrix. □

Corollary 3.7. *Let $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{n\times n}$, then A is perfect Hermitian matrix if and only if kA is a perfect Hermitian matrix for any nonnegative and appreciable dual number k.*

Theorem 3.8. *Let $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{n\times n}$ be a perfect Hermitian matrix, then $A^{\dagger}$ is a perfect Hermitian matrix.*

Proof. Let $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{n\times n}$ be a perfect Hermitian matrix. Then A_{st} and A_I are positive semi-definite dual quaternion Hermitian matrices. From Theorem 3.3 we get that the DQMPGI of A exists.

We denote $P = I_n - A_{st}A_{st}^{\dagger}$ and $Q = A_{st}^*A_{st}$. Applying Theorem 2.3 gives

$$A^{\dagger} = A_{st}^{\dagger} + \epsilon \left(-A_{st}^{\dagger} A_I A_{st}^{\dagger} + Q^{\dagger} A_I^* P + P A_I^* Q^{\dagger} \right). \tag{3.4}$$

Applying Theorem 2.3, it is easy to check that DQMPGI of $A^{\dagger}$ exists. And since $-A_{st}^{\dagger}A_IA_{st}^{\dagger} + Q^{\dagger}A_I^*P + PA_I^*Q^{\dagger}$ is Hermitian, applying Theorem 3.5 gives that $A^{\dagger}$ is a perfect Hermitian matrix.

If the DQMPGI of A exists, then the DQMPGI of $A^{\dagger}$ exists and $\left(A^{\dagger}\right)^{\dagger} = A$. Therefore, we get that if A is a perfect Hermitian matrix, then $A^{\dagger}$ is a perfect Hermitian matrix. □

Write

$$X = A_{st}^{\dagger} - \epsilon A_{st}^{\dagger} A_I A_{st}^{\dagger}. \tag{3.5}$$

we call it the Moore-Penrose dual quaternion generalized inverse(MPDQGI for short) of A, and denote it as A^{P}. Since every quaternion matrix has a unique Moore-Penrose inverse, similarly, dual quaternion matrix has a unique MPDQGI. In [11], Ling, Qi and Yan check that

$$A^{\mathrm{P}}AA^{\mathrm{P}} = \left(A_{st}^{\dagger} - \epsilon A_{st}^{\dagger}A_I A_{st}^{\dagger}\right) A \left(A_{st}^{\dagger} - \epsilon A_{st}^{\dagger}A_I A_{st}^{\dagger}\right) = A_{st}^{\dagger} - \epsilon A_{st}^{\dagger}A_I A_{st}^{\dagger} = A^{\mathrm{P}},$$

that is, A^{P} is a $\{2\}$-inverse of A.

Theorem 3.9. *Let $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{n\times n}$ be a positive semi-definite dual quaternion Hermitian matrix, then A^{P} is a perfect Hermitian matrix.*

Proof. Let $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{n\times n}$ be a positive semi-definite dual quaternion Hermitian matrix. Then A_{st} and $A_{st}^{\dagger}$ are positive semi-definite quaternion Hermitian matrices.

Applying equation (3.5) gives $A^{\mathrm{P}} = A_{st}^{\dagger} - \epsilon A_{st}^{\dagger}A_I A_{st}^{\dagger}$. Since

$$\operatorname{rank}\begin{bmatrix} -A_{st}^{\dagger}A_I A_{st}^{\dagger} & A_{st}^{\dagger} \\ A_{st}^{\dagger} & 0 \end{bmatrix} = 2\operatorname{rank}\left(A_{st}^{\dagger}\right),$$

applying Theorem 2.3 gives that the DQMPGI of A^{P} exists. And since $A_{st}^{\dagger}A_I A_{st}^{\dagger}$ is Hermitian, it follows from applying Theorem 2.3 and Theorem 3.5 that A^{P} is a perfect Hermitian matrix. □

Corollary 3.10. *Let $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{n\times n}$, A be a perfect Hermitian matrix. Then A^{P} is a perfect Hermitian matrix.*

Let A^{P} is a perfect Hermitian matrix, From Theorem 2.3, we conclude that the DQMPGI of A^{P} exists, even if DQMPGI of A may not exist. Therefore, the converse of Corollary 3.10 does not hold.

Example 3.11. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $A^{\mathrm{P}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$. It is easy to check that A^{P} is a perfect Hermitian matrix and A is not a perfect Hermitian matrix.

Corollary 3.12. *Let $A = A_{st} + \epsilon A_I \in \mathbb{DQ}^{n\times n}$, A is a positive definite dual quaternion matrix. Then A^{-1} is a positive definite dual quaternion matrix.*

Proof. Suppose A is non-singular, then we can get $A^{-1} = A_{st}^{-1} - \epsilon A_{st}^{-1}A_I A_{st}^{-1}$. Because A is a positive definite dual quaternion matrix, A_{st}^{-1} also is a positive definite quaternion matrix. Furthermore,

$$\left(A_{st}^{-1}A_I A_{st}^{-1}\right)^* = \left(A_{st}^{-1}\right)^* A_I^* \left(A_{st}^{-1}\right)^* = \left(A_{st}^*\right)^{-1} A_I^* \left(A_{st}^*\right)^{-1} = A_{st}^{-1}A_I A_{st}^{-1},$$

then $A_{st}^{-1}A_I A_{st}^{-1}$ is Hermitian. Since there is $x^*A^{-1}x > 0$ for any x being appreciale, A^{-1} is a positive definite dual quaternion matrix. □

4 Dual Quaternion Partial Order

In this section, we introduce two Lönwer-type partial orders on $\mathbb{DQ}^{n\times n}$.

4.1 Löwner-P partial order

Consider the binary relation:

$$M \overset{\text{L-P}}{\leq} N : N - M \text{ is a perfect Hermitian matrix}, \tag{4.1}$$

in which $M, N \in \mathbb{DQ}^{n\times n}$. We call the binary operation (4.1) the Lönwer-P order.

Theorem 4.1. *The Lönwer-P order "* $\overset{\text{L-P}}{\leq}$ *" is a partial order.*

Proof. (i) Reflexivity is self-evident.

(ii) For anti-symmetry, let $M \overset{\text{L-P}}{\leq} N$ and $N \overset{\text{L-P}}{\leq} M$, where $M, N \in \mathbb{DQ}^{n\times n}$. Then

$$\begin{cases} M \overset{\text{L-P}}{\leq} N \\ N \overset{\text{L-P}}{\leq} M \end{cases} \Rightarrow \begin{cases} N - M \text{ is a perfect Hermitian matrix} \\ M - N \text{ is a perfect Hermitian matrix} \end{cases} \Rightarrow N = M.$$

Therefore, the anti-symmetry holds.

(iii) For transitivity, let $M \overset{\text{L-P}}{\leq} N$ and $N \overset{\text{L-P}}{\leq} K$, and DQMPGIs of $N - M$ and $K - N$ all exist. Then applying Lemma 3.2 and (4.1), we get

$$\begin{cases} M \overset{\text{L-P}}{\leq} N \\ N \overset{\text{L-P}}{\leq} K \end{cases} \Rightarrow \begin{cases} N - M = T_1^*T_1 \\ K - N = T_2^*T_2 \end{cases} \Rightarrow K - M = T_1^*T_1 + T_2^*T_2 = \begin{bmatrix} T_1^* & T_2^* \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix},$$

where $T_1, T_2 \in \mathbb{DQ}^{n\times n}$. It follows from Lemma 3.2 that $K - M$ is a perfect Hermitian matrix. Therefore, $M \overset{\text{L-P}}{\leq} K$ and the transitivity holds. □

Theorem 4.2. *Let $M, N \in \mathbb{DQ}^{n\times n}$, and $M \overset{\text{L-P}}{\leq} N$. Then the following hold:*

(1) *$M + S \overset{\text{L-P}}{\leq} N + S$ for any $S \in \mathbb{DQ}^{n\times n}$;*

(2) *$kM \overset{\text{L-P}}{\leq} kN$ for any nonnegative and appreciable dual number k;*

(3) *$T^*MT \overset{\text{L-P}}{\leq} T^*NT$ for any $T \in \mathbb{DQ}^{n\times n}$.*

Proof. Applying $M \overset{\text{L-P}}{\leq} N$ gives that $N - M$ is a perfect Hermitian matrix.

For any $S \in \mathbb{DQ}^{n\times n}$, $(N + S) - (M + S) = N - M$ is a perfect Hermitian matrix. Therefore, $M + S \overset{\text{L-P}}{\leq} N + S$.

For any nonnegative and appreciable dual number k, by applying Corollary 3.7, we get that $k(N - M)$ is a perfect Hermitian matrix, that is, $kM \overset{\text{L-P}}{\leq} kN$.

For any $T \in \mathbb{DQ}^{n\times n}$, by applying Theorem 3.6, we get that $T^*(N - M)T$ is a perfect Hermitian matrix, that is, $T^*MT \overset{\text{L-P}}{\leq} T^*NT$. □

4.2 Löwner-S partial order

Consider the binary relation: Now we define the Lönwer-S partial order.

$$M \overset{\text{L-S}}{\leq} N : \; x^* (N - M) x \geq 0, \; x \in \mathbb{DQ}^{n\times 1}, \tag{4.2}$$

in which $M, N \in \mathbb{DQ}^{n\times n}$. We call the binary operation (4.2) the Lönwer-S order.

Theorem 4.3. *The Lönwer-S order "$\overset{\text{L-S}}{\leq}$" is a partial order.*

Proof. (i) Reflexivity is obvious.

(ii) For anti-symmetry, let $M \overset{\text{L-S}}{\leq} N$ and $N \overset{\text{L-S}}{\leq} M$. Applying (4.2) gives $N - M$ is positive semi-definite. Similarly, $M - N$ is also positive semi-definite. Then $N = M$ and the anti-symmetry is holds.

(iii) For transitivity, let $M \overset{\text{L-S}}{\leq} N$ and $N \overset{\text{L-S}}{\leq} K$. By applying (4.2), we obtain

$$\begin{cases} M \overset{\text{L-S}}{\leq} N \\ N \overset{\text{L-S}}{\leq} K \end{cases} \Rightarrow \begin{cases} x^*(N-M)x \geq 0 \\ x^*(K-N)x \geq 0 \end{cases} \Rightarrow x^*(K-M)x \geq 0$$

where x is arbitrary $n \times 1$ dual quaternion matrix. Since $x^*(K-M)x \geq 0$, then $K - M$ is positive semi-definite, Applying (4.2), we get $M \overset{\text{L-P}}{\leq} K$, the transitivity holds. □

Theorem 4.4. *Let $M, N \in \mathbb{DQ}^{n\times n}$. Then $M \overset{\text{L-S}}{\leq} N$ if and only if $T^*MT \overset{\text{L-S}}{\leq} T^*NT$ for any $T \in \mathbb{DQ}^{n\times n}$.*

Proof. "⇐" It is obvious.

"⇒" Since $M \overset{\text{L-S}}{\leq} N$, that is, $x^*(N-M)x \geq 0$ for any x. Denote $y = Tx$, then $x^*T^*(N-M)Tx = (Tx)^*(N-M)(Tx) \geq 0$. Therefore, $T^*MT \overset{\text{L-S}}{\leq} T^*NT$. □

Lönwer-P partial order is different from Lönwer-S partial order. Next, we consider their relations.

Remark 4.5. Let $M, N \in \mathbb{DQ}^{n\times n}$. Then $M \overset{\text{L-P}}{\leq} N$ implies $M \overset{\text{L-S}}{\leq} N$. But the converse is not true.

Example 4.6. Let

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \; N = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}.$$

It is easy to check that $M \overset{\text{L-S}}{\leq} N$. By applying Theorem 2.3, we get M is not below N under the Lönwer-P partial order " $\overset{\text{L-P}}{\leq}$ ".

Remark 4.7. Partial order is a class of matrix inequality, which has wide range of applications in statistics, for example, semidefinite optimization problem [9], strictly convex problem[1], etc. The Lönwer-type partial order can also be seen as dual quaternion matrix inequality. This will provide some interesting dual quaternion optimization problems, for example,

$$Q \overset{\text{L-P}}{\leq} XPX^* \text{ s.t. } AX = B; \quad Q \overset{\text{L-S}}{\leq} XPX^* \text{ s.t. } AX = B;$$
$$Q \overset{\text{L-P}}{\leq} X^*PX \text{ s.t. } AX = B; \quad Q \overset{\text{L-S}}{\leq} X^*PX \text{ s.t. } AX = B.$$

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