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PERTURBATIONS OF GROUP INVERSES OF QUATERNION TENSORS UNDER THE QT-PRODUCT

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Abstract: In this paper, we study the properties and perturbations of group inverses of third-order quaternion tensors under the QT-product (called QT-group inverse). First, we give the definition of the index of quaternion tensors under the QT-product. And we define drazin inverses and group inverses of quaternion tensors and prove their existence and uniqueness. Secondly, the core nilpotent decomposition and Jordan decomposition of quaternion tensors under the QT-product are given. By virtue of the QT-Jordan decomposition, the expression of the group inverses of quaternion tensors is given. And we give the limit expression for QT-Drazin inverses of quaternion tensors. Finally, the perturbation analysis of the one-sided and two-sided conditions of the quaternion tensor is carried out. Therefore, the expression and the perturbation bounds of the QT-group inverse of the perturbed quaternion tensor are obtained.

Key words: *quaternion tensor, QT-product, QT-group inverse, QT-Jordan decomposition, perturbation*

Mathematics Subject Classification: *15A09, 15A69*

1 Introduction

In the actual measurement, due to the possible error of the measured value, the matrix elements will change, resulting in the change of the solution of the equations. It is necessary to use Drazin inverse to study linear differential equations and finite Markov chain processes, so that the solutions of the equations can be represented by Drazin inverse. In 1967, Erdelyi defined the concept of group inverses of matrices [17], which is a special case of the Drazin inverses of matrices. Wei discussed perturbation of group inverse and oblique projection of complex matrices in [49]. Then, Li and Wei derived new general upper bounds which are sharper than the results of Wei such that the continuity of the group inverse directly follows [28]. In 2017, Wei studied the acute perturbations of the group inverses of complex matrices [50]. The group inverses of matrices are applied in many fields, such as least squares problems, Markov chain, etc [1, 2, 6, 33].

Tensors are high-dimensional generalization of vectors and matrices. In recent years, generalized inverses of tensors have been playing an increasingly vital role in computational mathematics and numerical analysis. More and more scholars have begun to study the tensors and their generalized inverses [5, 6, 9, 13, 37, 35, 47, 51, 52]. Some scholars have

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studied the generalized inverses of complex tensors based on the Einstein product proposed by Einstein in [15, 30, 41, 45, 46, 48]. In 2011, Kilmer and Martin defined another product which called the T-product of complex tensors [26]. Since then, scholars have begun to study tensors and their generalized inverses under the T-product [3, 7, 8, 11, 25, 29, 32].

Quaternion, first proposed by Hamilton in 1843 [19], is associative and non-commutative division algebras over real number field. At present, quaternion and quaternion matrix play an important role in computer science, signal and color image processing, quantum mechanics and so on [4, 10, 23, 24, 27, 42, 55]. The concept of dual number was first proposed by William Clifford in 1873 [12], and its earliest application was used to represent geometric angles in spatial geometry [43]. In 2018, Falco discussed the mathematical condition for the existence of generalized inverses of dual matrices [14]. In 2021, Qi defined the right eigenvalues and subeigenvalues of the dual complex matrices, and proposed the concept of dual quaternion matrices, hoping that the existing concept can also be true for dual quaternion matrices [38]. In 2022, Zhong defined the group inverses of dual matrices, and studied the existence of dual group inverses, computational methods and applications in solving linear dual equations [56]. In 2023, Qi extended the Perron-Frobenius theory to dual number matrices with primitive and irreducible nonnegative standard parts. And he obtained the perturbation relation in the Markov chain process by the group inverses of the dual number matrices [36]. In 2023, Qi gave the definition of the dual quaternion Laplacian matrices, and proved a Gershgorin-type theorem for square dual quaternion Hermitian matrices, for studying properties of dual quaternion Laplacian matrices. The role of the dual quaternion Laplacian matrices in formation control is discussed [39]. It is possible to extend the theory of dual complex numbers to dual quaternions, due to the relationship that the dual complex number is a special case of the dual quaternion.

In present, quaternions have proven to be a very suitable framework for encoding color pixels. Quaternion technique has been widely used in color image processing and has obtained outstanding performance in various color image processing tasks, but there is no much work focusing on the color video inpainting problem [44, 34, 31, 18]. In [16], Ell used quaternions to define a Fourier transform for color images. Jia studies robust quaternion matrix completion for image restoration in [22]. He studied the eigenvalues of quaternion tensors, presented the k-means method for quaternion tensor and used it to color video clustering [20]. Jia can inpaint color videos under the condition of all of frontal slices should miss pixels at the same positions [21]. However, in 2021, Zhang proposed the QT-product of quaternion tensors based on the T-product of complex tensors, and gave the singular value decomposition of quaternion tensors [40]. Hopefully, the accompanied technique can be applied to color video inpainting problems with no extra restriction.

In this paper, we first introduce some symbols that need to be used, and briefly review some relevant knowledge. Further, we define drazin inverses and group inverses of quaternion tensors and prove their existence and uniqueness. Next, some properties about the group inverses of quaternion tensors are proved. On this basis, the QT-Jordan decomposition of quaternion tensors under the QT-product are given. Then, we give the expression of group inverses by the QT-Jordan decomposition. Further, we define the QT-core nilpotent decomposition of quaternion tensors under the QT-product. And we give the limit expression for drazin inverses of quaternion tensors. Then, some related properties about the core part of quaternion tensors are proved. Finally, under the one-sided and the two-sided perturbations, the expressions and perturbation bounds of the group inverses of quaternion tensors are given, and some numerical examples are given to verify the results.

2 Preliminaries

In this paper, $\mathbb{R}$ represents the field of real numbers, $\mathbb{C}$ represents the field of complex numbers, $\mathbb{H}$ represents the set of quaternions, $\mathbb{H}^{m\times n}$ represents the set of $m \times n$ quaternion matrices, and $\mathbb{H}^{m\times n\times p}$ represents the set of $m \times n \times p$ quaternion tensors. A quaternion x has the following form

$$x = x_0 + x_1 i + x_2 j + x_3 k,$$

where x_0, x_1, x_2, x_3 are real numbers and i, j, k satisfy

$$\begin{cases} i^2 = j^2 = k^2 = -1, \\ ij = -ji = k, jk = -kj = i, ki = -ik = j, \end{cases}$$

the conjugate of x is $x^* = x_0 - x_1 i - x_2 j - x_3 k$, and its norm $|x| = \sqrt{x^* x} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. For any quaternion x, we can write it as $x = a + jb$, where $a = x_0 + x_1 i, b = x_2 - x_3 i$.

A quaternion matrix $A \in \mathbb{H}^{m\times n}$ has the following form [53]

$$A = A_0 + A_1 i + A_2 j + A_3 k,$$

where $A_0, A_1, A_2, A_3 \in \mathbb{R}^{m\times n}$. Its conjugate transposition $A^* = A_0^* - A_1^* i - A_2^* j - A_3^* k$. Any quaternion matrix A can be written as $A = A' + jA''$, where $A' = A_0 + A_1 i, A'' = A_2 - A_3 i$.

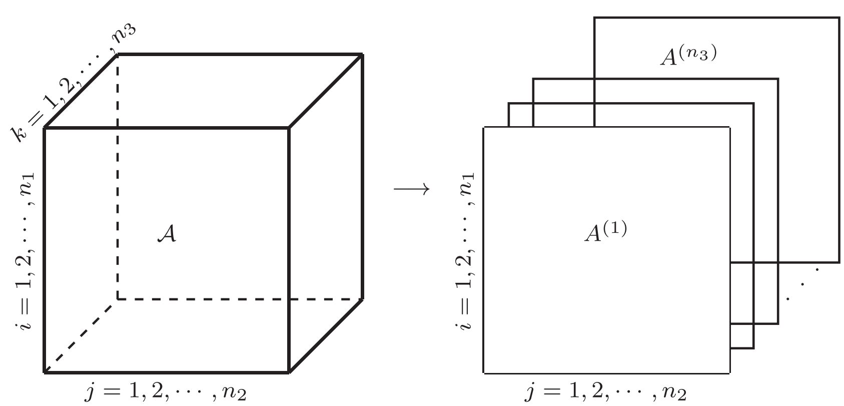
Definition 2.1 ([53, 54])**.** Given a quaternion matrix $A \in \mathbb{H}^{n\times n}$, if the non-zero quaternion vector $x \in \mathbb{H}^n$ and the constant $\lambda \in \mathbb{H}$ satisfy

$$Ax = x\lambda \ (Ax = \lambda x),$$

we define that λ is the right (left) eigenvalue of A and x is the corresponding right (left) eigenvector.

Since the quaternion does not satisfy the commutative law of multiplication, the eigenvalues of the quaternion matrix are divided into left and right eigenvalues. In general, there is no direct relationship between the left and right eigenvalues of quaternion matrices. For this paper, we use the right eigenvalues.

For any quaternion tensor $\mathcal{A} \in \mathbb{H}^{n_1\times n_2\times n_3}$ whose i-th frontal slice is denoted by $A^{(i)} \in \mathbb{H}^{n_1\times n_2}, i = 1, 2, \ldots, n_3$, and the frontal slices are shown as follows:



The following slices are all the frontal slices of the tensor, and the three operations "$circ$", "$unfold$", and "$fold$" are defined below [40]:

$$circ(\mathcal{A}) = \begin{pmatrix} A^{(1)} & A^{(n_3)} & A^{(n_3-1)} & \cdots & A^{(2)} \\ A^{(2)} & A^{(1)} & A^{(n_3)} & \cdots & A^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^{(n_3)} & A^{(n_3-1)} & A^{(n_3-2)} & \cdots & A^{(1)} \end{pmatrix} \in \mathbb{H}^{n_1 n_3 \times n_2 n_3}$$

is a block circulant matrix, "$fold$" is the inverse operation of "$unfold$",

$$unfold(\mathcal{A}) = \begin{pmatrix} A^{(1)} \\ A^{(2)} \\ \vdots \\ A^{(p)} \end{pmatrix} \in \mathbb{H}^{n_1 n_3 \times n_2}, \qquad fold(unfold(\mathcal{A})) = \mathcal{A}.$$

For any third-order complex tensor $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$, its block circulant complex matrix can be diagonalized by the Discrete Fourier Transformation (DFT) matrix , then we get

$$(F_{n_3} \otimes I_{n_1}) circ(\mathcal{A}) (F_{n_3}^* \otimes I_{n_2}) = \begin{pmatrix} \widehat{A}^{(1)} & & & \\ & \widehat{A}^{(2)} & & \\ & & \ddots & \\ & & & \widehat{A}^{(n_3)} \end{pmatrix} := diag(\widehat{\mathcal{A}}),$$

where $F_{n_3} \in \mathbb{C}^{n_3 \times n_3}$ is the DFT matrix,

$$F_{n_3} = \frac{1}{\sqrt{n_3}} \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & w^3 & \cdots & w^{n_3-1} \\ 1 & w^2 & w^4 & w^6 & \cdots & w^{2(n_3-1)} \\ 1 & w^3 & w^6 & w^9 & \cdots & w^{3(n_3-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n_3-1} & w^{2(n_3-1)} & w^{3(n_3-1)} & \cdots & w^{(n_3-1)(n_3-1)} \end{pmatrix},$$

where $\omega = e^{-2\pi i/n_3}$ is the primitive n_3-th root of unity in which $i = \sqrt{-1}$.

There are some relevant concepts for the QT-product of quaternion tensors as follows.

Definition 2.2 ([40])**.** (*QT-product*) For $\mathcal{A} = \mathcal{A}_1 + j\mathcal{A}_2 \in \mathbb{H}^{n_1 \times r \times n_3}$ and $\mathcal{B} \in \mathbb{H}^{r \times n_2 \times n_3}$, the QT-product $\mathcal{A} *_Q \mathcal{B}$ is an $n_1 \times n_2 \times n_3$ quaternion tensor defined by

$$\mathcal{A} *_Q \mathcal{B} = fold\big((circ(\mathcal{A}_1) + j circ(\mathcal{A}_2)(P_{n_3} \otimes I_r))\, unfold(\mathcal{B})\big) \in \mathbb{H}^{n_1 \times n_2 \times n_3},$$

where the third-order tensors $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{C}^{n_1 \times r \times n_3}$, the symbol "$\otimes$" represents the kronecker product, and the matrix $P_{n_3} = (P_{ij}) \in \mathbb{R}^{n_3 \times n_3}$ is a permutation matrix, where if $i + j = n_3 + 2, 2 \leq i, j \leq n_3$, then $P_{11} = P_{ij} = 1$, otherwise $P_{ij} = 0$.

The definitions of the unit quaternion tensor, invertible quaternion tensor, unitary quaternion tensor and conjugate transpose of quaternion tensor under the QT-product are reviewed below.

Definition 2.3 ([32])**.** (*Unit quaternion tensor*) The $n \times n \times n_3$ identity quaternion tensor $\mathcal{I}_{nnn_3}$ is the tensor whose first frontal slice is the identity matrix and other slices are zero matrices.

Definition 2.4 ([57]). (*Invertible quaternion tensor*) For $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$, if there exists $\mathcal{B} \in \mathbb{H}^{n\times n\times n_3}$ such that

$$\mathcal{A} *_Q \mathcal{B} = \mathcal{B} *_Q \mathcal{A} = \mathcal{I},$$

where $\mathcal{I} \in \mathbb{H}^{n\times n\times n_3}$, then $\mathcal{A}$ is said to be a invertible quaternion tensor and denoted by $\mathcal{A}^{-1} = \mathcal{B}$.

Definition 2.5 ([40]). (*Conjugate transpose*) The conjugate transpose of a quaternion tensor $\mathcal{A} = \mathcal{A}_1 + j\mathcal{A}_2 \in \mathbb{H}^{n\times n\times n_3}$ is denoted as $\mathcal{A}^* \in \mathbb{H}^{n\times n\times n_3}$, satisfying $unfold(\mathcal{A}^*) = unfold(\mathcal{A}_1^*) - (P_{n_3} \otimes I_n)unfold(\mathcal{A}_2^*)j$.

Definition 2.6 ([40]). (*Unitary quaternion tensor*) The $n \times n \times n_3$ quaternion tensor $\mathcal{U}$ is a unitary quaternion tensor if

$$\mathcal{U}^* *_Q \mathcal{U} = \mathcal{U} *_Q \mathcal{U}^* = \mathcal{I}_{nnn_3}.$$

Definition 2.7 ([40]). (*F-diagonal quaternion tensor*) The $n \times n \times n_3$ F- diagonal quaternion tensor $\mathcal{P}_{nnn_3}$ is the tensor whose every frontal slice is the diagonal quaternion matrix.

Definition 2.8. (*F-upper triangular-bidiagonal quaternion tensor*) The $n \times n \times n_3$ F-upper triangular-bidiagonal quaternion tensor $\mathcal{P}_{nnn_3}$ is the tensor whose every frontal slice is the upper triangular-bidiagonal quaternion matrix.

Lemma 2.9 ([57]). *Let $\mathcal{A} = \mathcal{A}_1 + j\mathcal{A}_2 \in \mathbb{H}^{n\times n\times n_3}, \mathcal{B} \in \mathbb{H}^{n\times n\times n_3}$, where $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{C}^{n\times n\times n_3}$, the quaternion tensor $\widehat{\mathcal{A}} \in \mathbb{H}^{n\times n\times n_3}$ is the result of DFT of $\mathcal{A}$, and $F_{n_3} \in \mathbb{C}^{n_3\times n_3}$ is a DFT matrix. Then*

(i) $\widehat{\mathcal{A}}(i,j,:) = \sqrt{n_3}F_{n_3}\mathcal{A}(i,j,:), i \in [n], j \in [n]$,

(ii) $unfold(\widehat{\mathcal{A}}) = \sqrt{n_3}(F_{n_3} \otimes I_n)unfold(\mathcal{A})$,

(iii) $unfold(\widehat{\mathcal{A}}) = unfold(\widehat{\mathcal{A}}_1) + j(P_{n_3} \otimes I_n)unfold(\widehat{\mathcal{A}}_2)$,

(iv) $unfold(\widehat{\mathcal{A} *_Q \mathcal{B}}) = diag(\widehat{\mathcal{A}})unfold(\widehat{\mathcal{B}})$,

(v) $diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{A}}_1) + j(P_{n_3} \otimes I_n)diag(\widehat{\mathcal{A}}_2)(P_{n_3} \otimes I_n)$,

(vi) $diag(\widehat{\mathcal{A}}^*) = diag(\widehat{\mathcal{A}}_1)^* - \big((P_{n_3} \otimes I_n)diag(\widehat{\mathcal{A}}_2)(P_{n_3} \otimes I_n)\big)^* j$,

(vii) $\mathcal{A} = fold\big((F_{n_3}^* \otimes I_n)\frac{1}{\sqrt{n_3}}diag(\widehat{\mathcal{A}})(e \otimes I_n)\big)$, *where e is an n_3 dimensional column vector with all elements being 1.*

Lemma 2.10 ([40]). *Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{H}^{n\times n\times n_3}$, and $\widehat{\mathcal{A}}, \widehat{\mathcal{B}}, \widehat{\mathcal{C}}$ are the DFT of $\mathcal{A}, \mathcal{B}, \mathcal{C}$, respectively. Then*

$$\mathcal{A} *_Q \mathcal{B} = \mathcal{C} \Leftrightarrow diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{B}}) = diag(\widehat{\mathcal{C}}).$$

Lemma 2.11 ([40]). *(QT-SVD) Let $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$. Then $\mathcal{A}$ can be factored as*

$$\mathcal{A} = \mathcal{U} *_Q \mathcal{S} *_Q \mathcal{V}^*,$$

where $\mathcal{U}, \mathcal{V} \in \mathbb{H}^{n\times n\times n_3}$ are unitary quaternion tensors and $\mathcal{S} \in \mathbb{H}^{n\times n\times n_3}$ is an F-diagonal quaternion tensor.

Definition 2.12 ([40]). (QT-rank) For $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$ and its QT-SVD is $\mathcal{A} = \mathcal{U} *_Q \mathcal{S} *_Q \mathcal{V}^*$, the QT-rank of $\mathcal{A}$, denoted as $\text{rank}_{QT}(\mathcal{A})$, is the number of non-zero elements of $\{\mathcal{S}(i,i,:)\}_{i=1}^n$.

Definition 2.13 ([57]). Let $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$. Then

(i) the QT-range space of $\mathcal{A}$ is defined as $\mathcal{R}(\mathcal{A}) = \text{Ran}(diag(\widehat{\mathcal{A}}))$,

(ii) the QT-null space of $\mathcal{A}$ is defined as $\mathcal{N}(\mathcal{A}) = \text{Null}(diag(\widehat{\mathcal{A}}))$,

(iii) the QT-norm of $\mathcal{A}$ is defined as $||\mathcal{A}|| = ||circ(\mathcal{A})||$, where $||circ(\mathcal{A})||$ represents the 2-norm of the quaternion matrix $circ(\mathcal{A})$.

If $\mathcal{I}$ is an arbitrary dimensional unit quaternion tensor, then $||\mathcal{I}|| = 1$. Here is a lemma.

Lemma 2.14 ([57]). *Let $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}, ||\mathcal{A}|| < 1$. Then $\mathcal{I} + \mathcal{A}$ is an invertible tensor, and*

$$||(\mathcal{I} + \mathcal{A})^{-1}|| \leq \frac{1}{1 - ||\mathcal{A}||}.$$

Proposition 2.15 ([57]). *Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{H}^{n\times n\times n_3}$. Then*

(i) $diag(\widehat{\mathcal{A}^*}) = diag(\widehat{\mathcal{A}})^*$,

(ii) $(\mathcal{A} *_Q \mathcal{B})^* = \mathcal{B}^* *_Q \mathcal{A}^*$,

(iii) $(\mathcal{A} *_Q \mathcal{B}) *_Q \mathcal{C} = \mathcal{A} *_Q (\mathcal{B} *_Q \mathcal{C})$,

(iv) $\mathcal{A} *_Q (\mathcal{B} + \mathcal{C}) = \mathcal{A} *_Q \mathcal{B} + \mathcal{A} *_Q \mathcal{C}$,

(v) $(\mathcal{B} + \mathcal{C}) *_Q \mathcal{A} = \mathcal{B} *_Q \mathcal{A} + \mathcal{C} *_Q \mathcal{A}$,

(vi) $diag(\widehat{\mathcal{A} + \mathcal{B}}) = diag(\widehat{\mathcal{A}}) + diag(\widehat{\mathcal{B}})$,

(vii) $\|\mathcal{A} *_Q \mathcal{B}\| \leq \|\mathcal{A}\| \, \|\mathcal{B}\|$,

(viii) $\|circ(\mathcal{A})\| = \left\|diag(\widehat{\mathcal{A}})\right\|$.

Lemma 2.16 ([57]). *Let $\mathcal{U} \in \mathbb{H}^{n\times n\times n_3}, \mathcal{V} \in \mathbb{H}^{n\times n\times n_3}$, and $\mathcal{I}$ be the unit quaternion tensor of the corresponding dimension, if $\mathcal{I} + \mathcal{U} *_Q \mathcal{V}$ and $\mathcal{I} + \mathcal{V} *_Q \mathcal{U}$ both are invertible quaternion tensors. Then*

(i) $\mathcal{U} *_Q (\mathcal{I} + \mathcal{V} *_Q \mathcal{U})^{-1} = (\mathcal{I} + \mathcal{U} *_Q \mathcal{V})^{-1} *_Q \mathcal{U}$,

(ii) $(\mathcal{I} + \mathcal{U} *_Q \mathcal{V})^{-1} = \mathcal{I} - \mathcal{U} *_Q (\mathcal{I} + \mathcal{V} *_Q \mathcal{U})^{-1} *_Q \mathcal{V}$.

3 Main Results

Before giving the definition of the group inverses of the quaternion tensors under the QT-product, the definition of the index of the quaternion tensor is given.

Definition 3.1. (QT-Index) For $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$, the smallest non-negative integer k, such that $\text{rank}_{QT}(\mathcal{A}^{k+1})$ =$\text{rank}_{QT}(\mathcal{A}^k)$, denoted by $\text{Ind}_{QT}(\mathcal{A}) = k$, is called the index of $\mathcal{A}$.

Definition 3.2. (QT-Drazin inverse) Let $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$ with $\mathrm{Ind}_{QT}(\mathcal{A}) = k$. The quaternion tensor $\mathcal{X} \in \mathbb{H}^{n\times n\times n_3}$ which satisfies:

$$\mathcal{A}^k *_Q \mathcal{X} *_Q \mathcal{A} = \mathcal{A}^k, \qquad \mathcal{X} *_Q \mathcal{A} *_Q \mathcal{X} = \mathcal{X}, \qquad \mathcal{A} *_Q \mathcal{X} = \mathcal{X} *_Q \mathcal{A} \tag{3.1}$$

is called the QT-Drazin inverse of $\mathcal{A}$, denoted by $\mathcal{A}^D$.

In particular, the definition of the QT-group inverse of the quaternion tensor when $k = 1$ is given below.

Definition 3.3. (QT-group inverse) Let $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$ with $\mathrm{Ind}_{QT}(\mathcal{A}) = 1$. The quaternion tensor $\mathcal{X} \in \mathbb{H}^{n\times n\times n_3}$ which satisfies:

$$\mathcal{A} *_Q \mathcal{X} *_Q \mathcal{A} = \mathcal{A}, \qquad \mathcal{X} *_Q \mathcal{A} *_Q \mathcal{X} = \mathcal{X}, \qquad \mathcal{A} *_Q \mathcal{X} = \mathcal{X} *_Q \mathcal{A} \tag{3.2}$$

is called the QT-group inverse of $\mathcal{A}$, denoted by $\mathcal{A}^{\#}$.

Theorem 3.4. *Let $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$ with $\mathrm{Ind}_{QT}(\mathcal{A}) = k$. Then the QT-Drazin inverse of $\mathcal{A}$ exists and is unique.*

Proof. For any quaternion tensor $\mathcal{A}$, by Lemma 2.9, there is

$$\mathcal{A} = fold\big((F_{n_3}^* \otimes I_n)\frac{1}{\sqrt{n_3}} diag(\widehat{\mathcal{A}})(e \otimes I_n)\big),$$

where

$$diag(\widehat{\mathcal{A}}) = \begin{pmatrix} \widehat{A}^{(1)} & & & \\ & \widehat{A}^{(2)} & & \\ & & \ddots & \\ & & & \widehat{A}^{(n_3)} \end{pmatrix}.$$

For any $\widehat{A}^{(i)} \in \mathbb{H}^{n\times n}, i = 1, 2, \cdots, n_3$, there always exists $\widehat{X}^{(i)} = (\widehat{A}^{(i)})^D \in \mathbb{H}^{n\times n}$ that satisfies

$$\begin{pmatrix} (\widehat{A}^{(1)})^k \widehat{X}^{(1)} \widehat{A}^{(1)} & & & \\ & (\widehat{A}^{(2)})^k \widehat{X}^{(2)} \widehat{A}^{(2)} & & \\ & & \ddots & \\ & & & (\widehat{A}^{(n_3)})^k \widehat{X}^{(n_3)} \widehat{A}^{(n_3)} \end{pmatrix}$$
$$= \begin{pmatrix} (\widehat{A}^{(1)})^k & & & \\ & (\widehat{A}^{(2)})^k & & \\ & & \ddots & \\ & & & (\widehat{A}^{(n_3)})^k \end{pmatrix},$$

$$\begin{pmatrix} \widehat{X}^{(1)} \widehat{A}^{(1)} \widehat{X}^{(1)} & & & \\ & \widehat{X}^{(2)} \widehat{A}^{(2)} \widehat{X}^{(2)} & & \\ & & \ddots & \\ & & & \widehat{X}^{(n_3)} \widehat{A}^{(n_3)} \widehat{X}^{(n_3)} \end{pmatrix} = \begin{pmatrix} \widehat{X}^{(1)} & & & \\ & \widehat{X}^{(2)} & & \\ & & \ddots & \\ & & & \widehat{X}^{(n_3)} \end{pmatrix},$$

$$\begin{pmatrix} \widehat{A}^{(1)}\widehat{X}^{(1)} & & & \\ & \widehat{A}^{(2)}\widehat{X}^{(2)} & & \\ & & \ddots & \\ & & & \widehat{A}^{(n_3)}\widehat{X}^{(n_3)} \end{pmatrix} = \begin{pmatrix} \widehat{X}^{(1)}\widehat{A}^{(1)} & & & \\ & \widehat{X}^{(2)}\widehat{A}^{(2)} & & \\ & & \ddots & \\ & & & \widehat{X}^{(n_3)}\widehat{A}^{(n_3)} \end{pmatrix},$$

so

$$diag(\widehat{\mathcal{A}})^D = \begin{pmatrix} \widehat{X}^{(1)} & & & \\ & \widehat{X}^{(2)} & & \\ & & \ddots & \\ & & & \widehat{X}^{(n_3)} \end{pmatrix} := diag(\widehat{\mathcal{X}}).$$

Then we can get $\mathcal{X} = fold((F_{n_3}^* \otimes I_n)\frac{1}{\sqrt{n_3}}diag(\widehat{\mathcal{X}})(e \otimes I_n))$ satisfies all three equations in (3.1), so $\mathcal{X} = \mathcal{A}^D$. Next we prove uniqueness of the QT-Drazin inverse of $\mathcal{A}$.

If $\mathcal{X}_1$ and $\mathcal{X}_2$ are both QT-Drazin inverses of $\mathcal{A}$, then $diag(\widehat{\mathcal{X}_1})$ and $diag(\widehat{\mathcal{X}_2})$ are both Drazin inverses of the quaternion matrix $diag(\widehat{\mathcal{A}})$. Now let

$$\begin{aligned} diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{X}_1}) &= diag(\widehat{\mathcal{X}_1})diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{E}}), \\ diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{X}_2}) &= diag(\widehat{\mathcal{X}_2})diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{F}}), \end{aligned}$$

obviously, $diag(\widehat{\mathcal{E}})^2 = diag(\widehat{\mathcal{E}}), diag(\widehat{\mathcal{F}})^2 = diag(\widehat{\mathcal{F}})$. So

$$\begin{aligned} diag(\widehat{\mathcal{E}}) &= diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{X}_1}) \\ &= diag(\widehat{\mathcal{A}})^k diag(\widehat{\mathcal{X}_1})^k \\ &= diag(\widehat{\mathcal{A}})^k diag(\widehat{\mathcal{X}_2})diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{X}_1})^k \\ &= diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{X}_2})diag(\widehat{\mathcal{A}})^k diag(\widehat{\mathcal{X}_1})^k \\ &= diag(\widehat{\mathcal{F}})diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{X}_1}) \\ &= diag(\widehat{\mathcal{F}})diag(\widehat{\mathcal{E}}), \end{aligned}$$

and

$$\begin{aligned} diag(\widehat{\mathcal{F}}) &= diag(\widehat{\mathcal{X}_2})diag(\widehat{\mathcal{A}}) \\ &= diag(\widehat{\mathcal{X}_2})^k diag(\widehat{\mathcal{A}})^k \\ &= diag(\widehat{\mathcal{X}_2})^k diag(\widehat{\mathcal{A}})^k diag(\widehat{\mathcal{X}_1})diag(\widehat{\mathcal{A}}) \\ &= diag(\widehat{\mathcal{X}_2})^k diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{E}}) \\ &= diag(\widehat{\mathcal{F}})diag(\widehat{\mathcal{E}}), \end{aligned}$$

it means that $diag(\widehat{\mathcal{E}}) = diag(\widehat{\mathcal{F}})$. Further, we have

$$\begin{aligned} diag(\widehat{\mathcal{X}_1}) &= diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{X}_1})^2 = diag(\widehat{\mathcal{E}})diag(\widehat{\mathcal{X}_1}) \\ &= diag(\widehat{\mathcal{F}})diag(\widehat{\mathcal{X}_1}) = diag(\widehat{\mathcal{X}_2})diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{X}_1}) \\ &= diag(\widehat{\mathcal{X}_2})diag(\widehat{\mathcal{E}}) = diag(\widehat{\mathcal{X}_2})diag(\widehat{\mathcal{F}}) \\ &= diag(\widehat{\mathcal{X}_2})^2 diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{X}_2})^2 \\ &= diag(\widehat{\mathcal{X}_2}). \end{aligned}$$

So we obtain $\widehat{\mathcal{X}_1} = \widehat{\mathcal{X}_2}$. Due to condition (vii) of Lemma 2.9, we get the desired result $\mathcal{X}_1 = \mathcal{X}_2$. □

Corollary 3.5. *Let $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$ with $\mathrm{Ind}_{QT}(\mathcal{A}) = 1$. Then the QT-group inverse of $\mathcal{A}$ exists and is unique.*

Lemma 3.6. *Let $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$. Then a quaternion tensor $\mathcal{A}$ is invertible if and only if $diag(\widehat{\mathcal{A}})$ is invertible.*

Proof. Since $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$ is invertible, there exists a quaternion tensor $\mathcal{B} \in \mathbb{H}^{n\times n\times n_3}$ such that

$$\mathcal{A} *_Q \mathcal{B} = \mathcal{B} *_Q \mathcal{A} = \mathcal{I}.$$

By using Lemma 2.10, we can obtain

$$diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{B}}) = diag(\widehat{\mathcal{B}})diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{I}}),$$

therefore, $diag(\widehat{\mathcal{A}})$ is invertible.

Since $diag(\widehat{\mathcal{A}})$ is invertible, there exists a quaternion matrix $diag(\widehat{\mathcal{B}})$ such that

$$diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{B}}) = diag(\widehat{\mathcal{B}})diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{I}}).$$

According to Lemma 2.10, we can get

$$\mathcal{A} *_Q \mathcal{B} = \mathcal{B} *_Q \mathcal{A} = \mathcal{I},$$

so $\mathcal{A}$ is invertible. □

Theorem 3.7. (*QT-Jordan decomposition*) *Let $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$. Then there exists an invertible quaternion tensor $\mathcal{P} \in \mathbb{H}^{n\times n\times n_3}$ such that*

$$\mathcal{A} = \mathcal{P} *_Q \mathcal{J} *_Q \mathcal{P}^{-1}, \tag{3.3}$$

where $\mathcal{A}$ can be transformed into $diag(\widehat{\mathcal{A}})$, the form is as follows,

$$diag(\widehat{\mathcal{A}}) = \begin{pmatrix} \widehat{A}^{(1)} & & & \\ & \widehat{A}^{(2)} & & \\ & & \ddots & \\ & & & \widehat{A}^{(n_3)} \end{pmatrix}, \widehat{A}^{(i)} \in \mathbb{H}^{n\times n}, i = 1, \ldots, n_3,$$

the distinct right eigenvalues of $\widehat{A}^{(i)}$ are $\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_s^{(i)}, s \le n$, whose imaginary parts are nonnegative, and $\mathcal{J} \in \mathbb{H}^{n\times n\times n_3}$ is the F-upper triangular-bidiagonal quaternion tensor, and the main diagonal elements of $\widehat{J}^{(i)}$ are $\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_s^{(i)}$.

Proof. Since quaternion tensor $\mathcal{A}$ can be converted to $diag(\widehat{\mathcal{A}})$, there exist invertible quaternion matrices $\widehat{P}^{(i)} \in \mathbb{H}^{n\times n}$ such that

$$\widehat{A}^{(i)} = \widehat{P}^{(i)}\widehat{J}^{(i)}(\widehat{P}^{(i)})^{-1}, \quad i = 1, 2, \ldots, n,$$

where $\widehat{J}^{(i)}$ are upper triangular-bidiagonal quaternion matrices, and the main diagonal elements of its are $\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_s^{(i)}$, respectively. That is

$$\begin{aligned} diag(\widehat{\mathcal{A}}) &= \begin{pmatrix} \widehat{P}^{(1)}\widehat{J}^{(1)}(\widehat{P}^{(1)})^{-1} & & & \\ & \widehat{P}^{(2)}\widehat{J}^{(2)}(\widehat{P}^{(2)})^{-1} & & \\ & & \ddots & \\ & & & \widehat{P}^{(n_3)}\widehat{J}^{(n_3)}(\widehat{P}^{(n_3)})^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \widehat{P}^{(1)} & & & \\ & \widehat{P}^{(2)} & & \\ & & \ddots & \\ & & & \widehat{P}^{(n_3)} \end{pmatrix} \begin{pmatrix} \widehat{J}^{(1)} & & & \\ & \widehat{J}^{(2)} & & \\ & & \ddots & \\ & & & \widehat{J}^{(n_3)} \end{pmatrix} \\ &\qquad \begin{pmatrix} (\widehat{P}^{(1)})^{-1} & & & \\ & (\widehat{P}^{(2)})^{-1} & & \\ & & \ddots & \\ & & & (\widehat{P}^{(n_3)})^{-1} \end{pmatrix} \\ &= diag(\widehat{\mathcal{P}})diag(\widehat{\mathcal{J}})diag(\widehat{\mathcal{P}^{-1}}). \end{aligned} \tag{3.4}$$

So we get $diag(\widehat{\mathcal{J}})$ is an upper triangular-bidiagonal quaternion matrices, and the main diagonal elements are $\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_s^{(i)}$. By condition (vii) of Lemma 2.9, we get $\mathcal{J}$ is an F-upper triangular-bidiagonal quaternion tensor. Because $\widehat{P}^{(i)}$ is invertible, we have $diag(\widehat{\mathcal{P}})$ is invertible. Using Lemma 3.6, we get $\mathcal{P}$ is invertible. By Lemma 2.10 for (3.4), we obtain

$$\mathcal{A} = \mathcal{P} *_Q \mathcal{J} *_Q \mathcal{P}^{-1}.$$

The conclusion is valid. □

Corollary 3.8. *Let $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$ with $\mathrm{Ind}_{QT}(\mathcal{A}) = 1$. Then the form of $\mathcal{J}$ in the QT-Jordan decomposition of $\mathcal{A}$ can be written as*

$$diag(\widehat{\mathcal{J}}) = \begin{pmatrix} \widehat{J}^{(1)} & & & \\ & \widehat{J}^{(2)} & & \\ & & \ddots & \\ & & & \widehat{J}^{(n_3)} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \widehat{C}^{(1)} & O \\ O & O \end{pmatrix} & & & \\ & \begin{pmatrix} \widehat{C}^{(2)} & O \\ O & O \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} \widehat{C}^{(n_3)} & O \\ O & O \end{pmatrix} \end{pmatrix},$$

where $\widehat{C}^{(i)}$ corresponds to the non-zero eigenvalue of $\widehat{A}^{(i)}$, $i = 1, 2, \ldots, n_3$.

Proof. According to Theorem 3.7, we can know that there exists a invertible quaternion tensor $\mathcal{P} \in \mathbb{H}^{n\times n\times n_3}$ such that $\mathcal{A} = \mathcal{P} *_Q \mathcal{J} *_Q \mathcal{P}^{-1}$. Then by Lemma 2.10, we get $diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{P}})diag(\widehat{\mathcal{J}})diag(\widehat{\mathcal{P}^{-1}})$, so

$$\widehat{A}^{(i)} = \widehat{P}^{(i)}\widehat{J}^{(i)}(\widehat{P}^{(i)})^{-1} = \widehat{P}^{(i)} \begin{pmatrix} \widehat{C}^{(i)} & O \\ O & \widehat{N}^{(i)} \end{pmatrix} (\widehat{P}^{(i)})^{-1},$$

where $\widehat{C}^{(i)}$ corresponds to the non-zero eigenvalue of $\widehat{A}^{(i)}$, $\widehat{N}^{(i)}$ corresponds to the zero eigenvalue of $\widehat{A}^{(i)}$. Because $\mathrm{Ind}_{\mathrm{QT}}(\mathcal{A}) = 1$, we have $\mathrm{rank}_{\mathrm{QT}}(\mathcal{A}^2) = \mathrm{rank}_{\mathrm{QT}}(\mathcal{A})$. Further, we obtain

$$\mathrm{rank}_{\mathrm{QT}}(\mathcal{J}^2) = \mathrm{rank}_{\mathrm{QT}}(\mathcal{J}).$$

So $\mathrm{rank}_{\mathrm{QT}}(\widehat{\mathcal{J}}^2) = \mathrm{rank}_{\mathrm{QT}}(\widehat{\mathcal{J}})$. If $\mathrm{rank}(\widehat{J}^{(i)}) \neq \mathrm{rank}\Big((\widehat{J}^{(i)})^2\Big)$, and $\mathrm{rank}\Big((\widehat{J}^{(i)})^2\Big) \leq \mathrm{rank}(\widehat{J}^{(i)})$, that is $\mathrm{rank}\Big((\widehat{J}^{(i)})^2\Big) < \mathrm{rank}(\widehat{J}^{(i)})$. So we have $\mathrm{rank}_{\mathrm{QT}}(\widehat{\mathcal{J}}^2) < \mathrm{rank}_{\mathrm{QT}}(\widehat{\mathcal{J}})$, this is contradictory to $\mathrm{rank}_{\mathrm{QT}}(\widehat{\mathcal{J}}^2) = \mathrm{rank}_{\mathrm{QT}}(\widehat{\mathcal{J}})$, so

$$\mathrm{rank}\Big((\widehat{J}^{(i)})^2\Big) = \mathrm{rank}\Big(\widehat{J}^{(i)}\Big).$$

This means that in the Jordan decomposition of $\widehat{A}^{(i)}$, $\widehat{N}^{(i)} = O$, so

$$\widehat{A}^{(i)} = \widehat{P}^{(i)} \begin{pmatrix} \widehat{C}^{(i)} & O \\ O & O \end{pmatrix} (\widehat{P}^{(i)})^{-1},$$

that is

$$\begin{pmatrix} \widehat{A}^{(1)} & & \\ & \ddots & \\ & & \widehat{A}^{(n_3)} \end{pmatrix} = \begin{pmatrix} \widehat{P}^{(1)} & & \\ & \ddots & \\ & & \widehat{P}^{(n_3)} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \widehat{C}^{(1)} & O \\ O & O \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \widehat{C}^{(n_3)} & O \\ O & O \end{pmatrix} \end{pmatrix} \begin{pmatrix} \widehat{P}^{(1)} & & \\ & \ddots & \\ & & \widehat{P}^{(n_3)} \end{pmatrix}^{-1}.$$

According to Theorem 3.7 and Lemma 2.10, we get

$$\begin{pmatrix} \widehat{J}^{(1)} & & \\ & \ddots & \\ & & \widehat{J}^{(n_3)} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \widehat{C}^{(1)} & O \\ O & O \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \widehat{C}^{(n_3)} & O \\ O & O \end{pmatrix} \end{pmatrix}.$$

Hence the proof is complete. □

Proposition 3.9. *Let* $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$ *with* $\mathrm{Ind}_{QT}(\mathcal{A}) = 1$. *Then*

(i) *if A is invertible* $diag(\widehat{\mathcal{A}})^{-1} = diag(\widehat{\mathcal{A}^{-1}})$,

(ii) $diag(\widehat{\mathcal{A}})^{\#} = diag(\widehat{\mathcal{A}^{\#}})$,

(iii) $(\mathcal{A}^{\#})^{\#} = \mathcal{A}$,

(iv) $\mathcal{A} + \mathcal{B} = \mathcal{C} \Leftrightarrow diag(\widehat{\mathcal{A}}) + diag(\widehat{\mathcal{B}}) = diag(\widehat{\mathcal{C}})$, *for* $\mathcal{B}, \mathcal{C} \in \mathbb{H}^{n\times n\times n_3}$.

Proof. For any quaternion tensor $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$, the quaternion tensor $\widehat{\mathcal{A}} \in \mathbb{H}^{n\times n\times n_3}$ is the result of DFT of $\mathcal{A}$, we can get

$$diag(\widehat{\mathcal{A}}) = \begin{pmatrix} \widehat{A}^{(1)} & & \\ & \ddots & \\ & & \widehat{A}^{(n_3)} \end{pmatrix}.$$

(i) For a quaternion matrix $A^{(i)} \in H^{n\times n}$, there exists an invertible quaternion matrix $P^{(i)} \in H^{n\times n}, i = 1, 2, \ldots, n_3$, such that

$$\begin{aligned} diag(\widehat{\mathcal{A}}) &= \begin{pmatrix} \widehat{P}^{(1)}\widehat{J}^{(1)}(\widehat{P}^{(1)})^{-1} & & \\ & \ddots & \\ & & \widehat{P}^{(n_3)}\widehat{J}^{(n_3)}(\widehat{P}^{(n_3)})^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \widehat{P}^{(1)} & & \\ & \ddots & \\ & & \widehat{P}^{(n_3)} \end{pmatrix}\begin{pmatrix} \widehat{J}^{(1)} & & \\ & \ddots & \\ & & \widehat{J}^{(n_3)} \end{pmatrix}\begin{pmatrix} (\widehat{P}^{(1)})^{-1} & & \\ & \ddots & \\ & & (\widehat{P}^{(n_3)})^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \widehat{P}^{(1)} & & \\ & \ddots & \\ & & \widehat{P}^{(n_3)} \end{pmatrix}\begin{pmatrix} \widehat{J}^{(1)} & & \\ & \ddots & \\ & & \widehat{J}^{(n_3)} \end{pmatrix}\begin{pmatrix} \widehat{P}^{(1)} & & \\ & \ddots & \\ & & \widehat{P}^{(n_3)} \end{pmatrix}^{-1}. \end{aligned}$$

Let $diag(\widehat{\mathcal{P}}) = \begin{pmatrix} \widehat{P}^{(1)} & & \\ & \ddots & \\ & & \widehat{P}^{(n_3)} \end{pmatrix}, diag(\widehat{\mathcal{J}}) = \begin{pmatrix} \widehat{J}^{(1)} & & \\ & \ddots & \\ & & \widehat{J}^{(n_3)} \end{pmatrix}$, so we have

$$diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{P}})diag(\widehat{\mathcal{J}})diag(\widehat{\mathcal{P}})^{-1}.$$

Because $\mathcal{A} = \mathcal{P} *_Q \mathcal{J} *_Q \mathcal{P}^{-1}$, and by Lemma 2.10, we can get

$$diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{P}})diag(\widehat{\mathcal{J}})diag(\widehat{\mathcal{P}^{-1}}).$$

Therefore, $diag(\widehat{\mathcal{P}})^{-1} = diag(\widehat{\mathcal{P}^{-1}})$. Further, we obtain $diag(\widehat{\mathcal{A}})^{-1} = diag(\widehat{\mathcal{A}^{-1}})$.

(ii) Because $\mathrm{Ind}_{QT}(\mathcal{A}) = 1$, $\mathrm{Ind}(diag(\widehat{\mathcal{A}})) = 1$. For the quaternion matrix $diag(\widehat{\mathcal{A}})$, we have

$$diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{A}})^{\#}diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{A}}),\ diag(\widehat{\mathcal{A}})^{\#}diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{A}})^{\#} = diag(\widehat{\mathcal{A}})^{\#},$$
$$diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{A}})^{\#} = diag(\widehat{\mathcal{A}})^{\#}diag(\widehat{\mathcal{A}}).$$

For the Drazin inverse $\mathcal{A}^{\#}$ of the quaternion tensor $\mathcal{A}$ satisfies the equation (3.2), and using Lemma 2.10 for the equation (3.2), we obtain

$$diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{A}^{\#}})diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{A}}),\ diag(\widehat{\mathcal{A}^{\#}})diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{A}^{\#}}) = diag(\widehat{\mathcal{A}^{\#}}),$$
$$diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{A}^{\#}}) = diag(\widehat{\mathcal{A}^{\#}})diag(\widehat{\mathcal{A}}).$$

Because the Drazin inverse of the quaternion matrix $diag(\widehat{\mathcal{A}})$ is unique, we can get $diag(\widehat{\mathcal{A}})^{\#} = diag(\widehat{\mathcal{A}^{\#}})$.

(iii) For the group inverse $\mathcal{A}^{\#}$ of $\mathcal{A}$, there is

$$\mathcal{A}^{\#} *_Q \mathcal{A} *_Q \mathcal{A}^{\#} = \mathcal{A}^{\#}\ ,\ \mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{A} = \mathcal{A}\ ,\ \mathcal{A}^{\#} *_Q \mathcal{A} = \mathcal{A} *_Q \mathcal{A}^{\#}.$$

Therefore, we have $(\mathcal{A}^{\#})^{\#} = \mathcal{A}$.
(iv) Using Lemma 2.9, we can know

$$\mathcal{A} = fold\big((F_{n_3}^* \otimes I_n)\frac{1}{\sqrt{n_3}}diag(\widehat{\mathcal{A}})(e \otimes I_n)\big).$$

Now, we impose "*unfold*" on both sides of the equation $\mathcal{A}+\mathcal{B}=\mathcal{C}$, then

$$\begin{aligned}\left(F_{n_3}^* \otimes I_n\right)\frac{1}{\sqrt{n_3}}diag(\widehat{\mathcal{A}})(e \otimes I_n) + \left(F_{n_3}^* \otimes I_n\right)\frac{1}{\sqrt{n_3}}diag(\widehat{\mathcal{B}})(e \otimes I_n)\\ = \left(F_{n_3}^* \otimes I_n\right)\frac{1}{\sqrt{n_3}}diag(\widehat{\mathcal{C}})(e \otimes I_n).\end{aligned} \tag{3.5}$$

Multiply both sides by $\sqrt{n_3}\left(F_{n_3}^* \otimes I_n\right)^{-1}$ on the left hand side and $(e \otimes I_n)^*$ on the right hand side of (3.5), we get

$$diag(\widehat{\mathcal{A}}) + diag(\widehat{\mathcal{B}}) = diag(\widehat{\mathcal{C}}),$$

where e is an n_3 dimensional column vector with all elements being 1.

For $diag(\widehat{\mathcal{A}})+diag(\widehat{\mathcal{B}}) = diag(\widehat{\mathcal{C}})$, multiply both sides by $\frac{1}{\sqrt{n_3}}\left(F_{n_3}^* \otimes I_n\right)$ on the left hand side and $e \otimes I_n$ on the right hand side of it, then we get (3.5). According to Lemma 2.9, we impose "*fold*" on both sides of (3.5), so we get

$$\mathcal{A}+\mathcal{B}=\mathcal{C}.$$

The proof is complete. □

Some properties that will be used in subsequent proofs are given below.

Theorem 3.10. *Let $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$ with $\mathrm{Ind}_{QT}(\mathcal{A}) = 1$. Then the QT-group inverse of $\mathcal{A}$ has the following form*

$$\mathcal{A}^{\#} = \mathcal{P} *_Q \mathcal{J}^{\dagger} *_Q \mathcal{P}^{-1}.$$

Proof. By Lemma 2.10 for (3.3), we can get $diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{P}})diag(\widehat{\mathcal{J}})diag(\widehat{\mathcal{P}})^{-1}$, where

$$diag(\widehat{\mathcal{P}}) = \begin{pmatrix}\widehat{P}^{(1)} & & \\ & \ddots & \\ & & \widehat{P}^{(n_3)}\end{pmatrix},$$

and

$$diag(\widehat{\mathcal{J}}) = \begin{pmatrix}\widehat{J}^{(1)} & & \\ & \ddots & \\ & & \widehat{J}^{(n_3)}\end{pmatrix}, \widehat{J}^{(i)} = \begin{pmatrix}\widehat{C}^{(i)} & O\\ O & O\end{pmatrix},$$

here, $\widehat{C}^{(i)}$ corresponds to the non-zero eigenvalue, $i = 1, 2, \ldots, n_3$. Further, we get

$$diag(\widehat{\mathcal{A}})^{\#} = diag(\widehat{\mathcal{P}})diag(\widehat{\mathcal{J}})^{\dagger}diag(\widehat{\mathcal{P}})^{-1},$$

where

$$diag(\widehat{\mathcal{A}})^{\#} = diag(\widehat{\mathcal{A}^{\#}}) = \begin{pmatrix} \widehat{A^{\#}}^{(1)} & & \\ & \ddots & \\ & & \widehat{A^{\#}}^{(n_3)} \end{pmatrix},$$

and

$$diag(\widehat{\mathcal{J}})^{\dagger} = \begin{pmatrix} \widehat{J^{\dagger}}^{(1)} & & \\ & \ddots & \\ & & \widehat{J^{\dagger}}^{(n_3)} \end{pmatrix}, \widehat{J^{\dagger}}^{(i)} = \begin{pmatrix} (\widehat{C}^{(i)})^{-1} & O \\ O & O \end{pmatrix}, i = 1, \ldots, n_3,$$

that is

$$\begin{pmatrix} \widehat{A^{\#}}^{(1)} & & \\ & \ddots & \\ & & \widehat{A^{\#}}^{(n_3)} \end{pmatrix} = \begin{pmatrix} \widehat{P}^{(1)}\widehat{J^{\dagger}}^{(1)}(\widehat{P}^{(1)})^{-1} & & \\ & \ddots & \\ & & \widehat{P}^{(n_3)}\widehat{J^{\dagger}}^{(n_3)}(\widehat{P}^{(1)})^{-1} \end{pmatrix}. \quad (3.6)$$

Multiplying both sides of (3.5) by $e \otimes I_n$ to the right hand side, where e is an n_3 dimensional column vector with all elements being 1, then we get the following equation

$$\begin{pmatrix} \widehat{A^{\#}}^{(1)} \\ \vdots \\ \widehat{A^{\#}}^{(n_3)} \end{pmatrix} = \begin{pmatrix} \widehat{P}^{(1)}\widehat{J^{\dagger}}^{(1)} & & \\ & \ddots & \\ & & \widehat{P}^{(n_3)}\widehat{J^{\dagger}}^{(n_3)} \end{pmatrix} \begin{pmatrix} (\widehat{P}^{(1)})^{-1} \\ \vdots \\ (\widehat{P}^{(n_3)})^{-1} \end{pmatrix}.$$

So we have

$$unfold(\widehat{\mathcal{A}}^{\#}) = diag(\widehat{\mathcal{P}})diag(\widehat{\mathcal{J}^{\dagger}})unfold(\widehat{\mathcal{P}^{-1}}). \quad (3.7)$$

By Lemma 2.10, we get

$$diag(\widehat{\mathcal{P}})diag(\widehat{\mathcal{J}^{\dagger}}) = diag(\widehat{\mathcal{P} *_Q \mathcal{J}^{\dagger}}). \quad (3.8)$$

According to condition (iv) of Lemma 2.9, we have

$$diag(\widehat{\mathcal{P} *_Q \mathcal{J}^{\dagger}})unfold(\widehat{\mathcal{P}^{-1}}) = unfold(\mathcal{P} *_Q \widehat{\mathcal{J}^{\dagger} *_Q} \mathcal{P}^{-1}). \quad (3.9)$$

By equation (3.6), (3.7) and (3.8), we get

$$unfold(\widehat{\mathcal{A}}^{\#}) = unfold(\mathcal{P} *_Q \widehat{\mathcal{J}^{\dagger} *_Q} \mathcal{P}^{-1}).$$

Using condition (ii) of Lemma 2.9, we have

$$unfold(\widehat{\mathcal{A}}^{\#}) = \sqrt{n_3}(F_{n_3} \otimes I_n)unfold(\mathcal{A}^{\#}),$$

and

$$unfold(\mathcal{P} *_Q \widehat{\mathcal{J}^{\dagger} *_Q} \mathcal{P}^{-1}) = \sqrt{n_3}(F_{n_3} \otimes I_n)unfold(\mathcal{P} *_Q \mathcal{J}^{\dagger} *_Q \mathcal{P}^{-1}).$$

Therefore,

$$unfold(\mathcal{A}^{\#}) = unfold(\mathcal{P} *_Q \mathcal{J}^{\dagger} *_Q \mathcal{P}^{-1}).$$

In conclusion, $\mathcal{A}^{\#} = \mathcal{P} *_Q \mathcal{J}^{\dagger} *_Q \mathcal{P}^{-1}$. □

Remark 3.11. Let $A \in \mathbb{H}^{n\times n}$, $\mathrm{Ind}(A) = k$ if and only if $C^n = R(A^k) \oplus N(A^k)$.

For $A \in \mathbb{H}^{n\times n}$, we can have

$$N(A^k) \subset N(A^{k+1}) \Leftrightarrow R(A^{k+1}) \subset R(A^k) \Leftrightarrow \mathrm{rank}(A^{k+1}) \le \mathrm{rank}(A^k).$$

Then we can use proof by contradiction to reach the conclusion. Assume $\mathrm{Ind}(A) \neq k$, we get $\mathrm{rank}(A^{k+1}) < \mathrm{rank}(A^k) \Leftrightarrow$ There exists x such that $A^{k+1}x = O, A^k x \neq O$. Let $y = A^k x$, then $A^k y = A^{2k} x = O$. That is $y \in N(A^k)$. It follows that $O \neq y = A^k x \in R(A^k) \cap N(A^k)$ contradicts $C^n = R(A^k) \oplus N(A^k)$. Therefore, $\mathrm{Ind}(A) = k$.

Remark 3.12. Let $A \in \mathbb{H}^{n\times n}$, $\mathrm{Ind}(A) = k$. Then there exists $P \in \mathbb{H}^{n\times n}, C \in \mathbb{H}^{r\times r}$ such that

$$A^D = P \begin{pmatrix} C^{-1} & O \\ O & O \end{pmatrix} P^{-1}.$$

Since $\mathrm{Ind}(A) = k$, we know from Remark 3.11 that $C^n = R(A^k) \oplus N(A^k)$. Let $P = (v_1, \ldots, v_r, v_{r+1}, \ldots, v_n)$, where $\{v_1, \ldots, v_r\}$ and $\{v_{r+1}, \ldots, v_n\}$ are basis of $R(A^k)$ and $N(A^k)$, respectively. Let

$$\begin{aligned} P &= (P_1, P_2), \\ P_1 &= (v_1, \ldots, v_r), \\ P_2 &= v_{r+1}, \ldots, v_n. \end{aligned}$$

Since $R(A^k)$ and $N(A^k)$ both are invariant subspaces of A, there are $C \in \mathbb{H}^{r\times r}$ and $N \in \mathbb{H}^{(n-r)\times(n-r)}$ such that

$$AP_1 = P_1 C, \quad AP_2 = P_2 N$$

So A has the decomposition

$$A = P \begin{pmatrix} C & O \\ O & N \end{pmatrix} P^{-1}.$$

Because $A^k N(A^k) = O$, we get $O = A^k P_2 = P_2 N^k$. Therefore $N^k = O$. Further, we can get

$$A^k = P \begin{pmatrix} C^k & O \\ O & O \end{pmatrix} P^{-1}.$$

And because $r = \mathrm{rank}(A^k) = \mathrm{rank}(C^k) \le \mathrm{rank}(C) \le r$, we can know $\mathrm{rank}(C) = r$. So C is a non-singular quaternion matrix of order r. Let

$$X = P \begin{pmatrix} C^{-1} & O \\ O & O \end{pmatrix} P^{-1},$$

it is easy to get that x satisfies the definition of drazin inverse of the quaternion matrix A. So $X = A^D$.

Definition 3.13. Let $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$ with $\mathrm{Ind}_{QT}(\mathcal{A}) = k$. The quaternion tensor

$$\mathcal{C}_{\mathcal{A}} = \mathcal{A} *_Q \mathcal{A}^D *_Q \mathcal{A} = \mathcal{A}^2 *_Q \mathcal{A}^D = \mathcal{A}^D *_Q \mathcal{A}^2 \in \mathbb{H}^{n\times n\times n_3}$$

is called the QT-core part of $\mathcal{A}$.

Definition 3.14. (QT-core nilpotent decomposition) Let $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$ with $\mathrm{Ind}_{QT}(\mathcal{A}) = k$. The quaternion tensor

$$\mathcal{N}_{\mathcal{A}} = \mathcal{A} - \mathcal{C}_{\mathcal{A}} = (\mathcal{I} - \mathcal{A} *_Q \mathcal{A}^D) *_Q \mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$$

is called the QT-nilpotent part of $\mathcal{A}$, and $\mathcal{A} = \mathcal{C}_{\mathcal{A}} + \mathcal{N}_{\mathcal{A}}$ is the QT-core nilpotent decomposition of $\mathcal{A}$.

Definition 3.15. If $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$, $\mathcal{C}_{\mathcal{A}}$ and $\mathcal{N}_{\mathcal{A}}$ are the QT-core and QT-nilpotent part of $\mathcal{A}$, respectively, then for integers $m \geq -1$, we define

$$\mathcal{C}_{\mathcal{A}}^{(m)} = \mathcal{A}^{m+1} *_Q \mathcal{A}^D = \begin{cases} \mathcal{A}^D, & if\ m = -1 \\ \mathcal{A} *_Q \mathcal{A}^D, & if\ m = 0 \\ \mathcal{C}_{\mathcal{A}}^m, & if\ m \geq 1 \end{cases},$$

and

$$\mathcal{N}_{\mathcal{A}}^{(m)} = \begin{cases} \mathcal{O}, & if\ m = -1 \\ \mathcal{A}^m - \mathcal{C}_{\mathcal{A}}^{(m)}, & if\ m \geq 0 \end{cases} = \begin{cases} \mathcal{O}, & if\ m = -1 \\ \mathcal{I} - \mathcal{A} *_Q \mathcal{A}^D, & if\ m = 0 \\ \mathcal{N}_{\mathcal{A}}^m, & if\ m \geq 1 \end{cases}.$$

Theorem 3.16. *Let $\mathcal{A}, \mathcal{I} \in \mathbb{H}^{n\times n\times n_3}$ with $\mathrm{Ind}_{QT}(\mathcal{A}) = k$. For every integer $l \geq k$,*

$$\mathcal{A}^D = \lim_{\varepsilon\to 0} (\mathcal{A}^{l+1} + \varepsilon\mathcal{I})^{-1} *_Q \mathcal{A}^l. \tag{3.10}$$

For every non-negative integer l,

$$\mathcal{A}^D = \lim_{\varepsilon\to 0} (\mathcal{A}^{l+1} + \varepsilon\mathcal{I})^{-1} *_Q \mathcal{C}_{\mathcal{A}}^{(l)}. \tag{3.11}$$

Proof. If $k = 0$, then $\mathcal{A}$ is non-singular, and the result is evident. For $k > 0$, use Lemma 2.9, Lemma 2.10 and Remark 3.12, we can get

$$diag\big(\widehat{(\mathcal{A}^{l+1} + \varepsilon\mathcal{I})^{-1} *_Q \mathcal{C}_{\mathcal{A}}^{(l)}}\big) = P \begin{pmatrix} (C^{l+1} + \varepsilon I)^{-1} C^l & O \\ O & O \end{pmatrix} P^{-1}.$$

And we can obtain

$$\lim_{\varepsilon\to 0} (C^{l+1} + \varepsilon I)^{-1} C^l = C^{-1}.$$

Using (vi) of Proposition 2.15, (i) of Proposition 3.1 and Lemma 2.10 for $diag\big(\widehat{(\mathcal{A}^{l+1} + \varepsilon\mathcal{I})^{-1} *_Q \mathcal{C}_{\mathcal{A}}^{(l)}}\big)$, we get

$$\begin{aligned} diag\big(\widehat{(\mathcal{A}^{l+1} + \varepsilon\mathcal{I})^{-1} *_Q \mathcal{C}_{\mathcal{A}}^{(l)}}\big) &= diag\big(\widehat{(\mathcal{A}^{l+1} + \varepsilon\mathcal{I})^{-1}}\big) diag\big(\widehat{\mathcal{C}_{\mathcal{A}}^{(l)}}\big) \\ &= diag\big(\widehat{\mathcal{A}^{l+1} + \varepsilon\mathcal{I}}\big)^{-1} diag\big(\widehat{\mathcal{C}_{\mathcal{A}}^{(l)}}\big) \\ &= [diag(\widehat{\mathcal{A}^{l+1}}) + diag(\widehat{\varepsilon\mathcal{I}})]^{-1} diag\big(\widehat{\mathcal{C}_{\mathcal{A}}^{(l)}}\big) \\ &= [diag(\widehat{\mathcal{A}^{l+1}}) + \varepsilon diag(\widehat{\mathcal{I}})]^{-1} diag\big(\widehat{\mathcal{C}_{\mathcal{A}}^{(l)}}\big). \end{aligned}$$

Then we get

$$\lim_{\varepsilon\to 0} diag\big(\widehat{(\mathcal{A}^{l+1}+\varepsilon\mathcal{I})^{-1} *_Q \mathcal{C}_{\mathcal{A}}^{(l)}}\big) = \lim_{\varepsilon\to 0}[diag(\widehat{\mathcal{A}^{l+1}}) + \varepsilon diag(\widehat{\mathcal{I}})]^{-1} diag\big(\widehat{\mathcal{C}_{\mathcal{A}}^{(l)}}\big)$$
$$= P\begin{pmatrix} C^{-1} & O \\ O & O \end{pmatrix} P^{-1}$$
$$= diag(\widehat{\mathcal{A}})^D.$$

Further, by virtue of Lemma 2.10, we have

$$\mathcal{A}^D = \lim_{\varepsilon\to 0}(\mathcal{A}^{l+1}+\varepsilon\mathcal{I})^{-1} *_Q \mathcal{C}_{\mathcal{A}}^{(l)}.$$

That is, equation (3.11) is true. When $l \geq k$, we obtain

$$\mathcal{C}_{\mathcal{A}}^{(l)} = \mathcal{A}^{l+1} *_Q \mathcal{A}^D = \mathcal{A}^l.$$

So equation (3.10) is also true. This proof is complete. □

Since it is obvious that $n \geq k$, we immediately get the following inference.

Corollary 3.17. *Let* $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$*. Then*

$$\mathcal{A}^D = \lim_{\varepsilon\to 0}(\mathcal{A}^{n+1}+\varepsilon\mathcal{I})^{-1} *_Q \mathcal{A}^n.$$

Lemma 3.18. *Let* $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$ *with* $\mathrm{Ind}_{QT}(\mathcal{A}) = 1$*. Then*

$$||\mathcal{I} - \mathcal{A} *_Q \mathcal{A}^{\#}|| = ||\mathcal{A} *_Q \mathcal{A}^{\#}||.$$

Proof. By Lemma 2.10, condition (iii) of Definition 2.13, and conditions (vi), (viii) of Proposition 2.15, we can get

$$\begin{aligned}
||\mathcal{I} - \mathcal{A} *_Q \mathcal{A}^{\#}|| &= ||diag(\widehat{\mathcal{I} - \mathcal{A} *_Q \mathcal{A}^{\#}})|| \\
&= ||diag(\widehat{\mathcal{I}}) - diag(\widehat{\mathcal{A} *_Q \mathcal{A}^{\#}})|| \\
&= ||diag(\widehat{\mathcal{I}}) - diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{A}^{\#}})|| \\
&= ||diag(\widehat{\mathcal{I}}) - diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{A}})^{\#}|| \\
&= ||diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{A}})^{\#}|| \\
&= ||diag(\widehat{\mathcal{A}})diag(\widehat{\mathcal{A}^{\#}})|| \\
&= ||diag(\widehat{\mathcal{A} *_Q \mathcal{A}^{\#}})|| \\
&= ||\mathcal{A} *_Q \mathcal{A}^{\#}||.
\end{aligned}$$

Hence the proof is complete. □

Theorem 3.19. *Let* $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$ *with* $\mathrm{Ind}_{QT}(\mathcal{A}) = 1$*. Then*

(i) $\mathcal{C}_{\mathcal{A}} *_Q \mathcal{A} *_Q \mathcal{A}^{\#} = \mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{C}_{\mathcal{A}} = \mathcal{C}_{\mathcal{A}}$,

(ii) $(\mathcal{A}^{\#})^{\#} = \mathcal{C}_{\mathcal{A}}$,

(iii) $(\mathcal{A}^{\#})^{*} = (\mathcal{A}^{*})^{\#}$.

Proof. (i) By the Definition 3.13, we get $\mathcal{C}_{\mathcal{A}} = \mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{A}$, then

$$\begin{aligned}\mathcal{C}_{\mathcal{A}} *_Q \mathcal{A} *_Q \mathcal{A}^{\#} &= \mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{A} *_Q \mathcal{A} *_Q \mathcal{A}^{\#} \\ &= \mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{A} \\ &= \mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{A} \\ &= \mathcal{C}_{\mathcal{A}}.\end{aligned}$$

Similarly, we can prove that

$$\mathcal{C}_{\mathcal{A}} = \mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{C}_{\mathcal{A}}.$$

(ii) By virtue of Definition 3.13 and condition (iii) of Proposition 3.6, we get

$$\mathcal{C}_{\mathcal{A}} = \mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{A} = \mathcal{A} = (\mathcal{A}^{\#})^{\#}.$$

Therefore, $(\mathcal{A}^{\#})^{\#} = \mathcal{C}_{\mathcal{A}}$.
(iii) For the group inverse $\mathcal{A}^{\#}$ of $\mathcal{A}$, since

$$\mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{A} = \mathcal{A}\,,\ \mathcal{A}^{\#} *_Q \mathcal{A} *_Q \mathcal{A}^{\#} = \mathcal{A}^{\#}\,,\ \mathcal{A} *_Q \mathcal{A}^{\#} = \mathcal{A}^{\#} *_Q \mathcal{A},$$

we have

$$\begin{aligned}&\mathcal{A}^{*} *_Q (\mathcal{A}^{\#})^{*} *_Q \mathcal{A}^{*} = (\mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{A})^{*} = \mathcal{A}^{*},\\ &(\mathcal{A}^{\#})^{*} *_Q \mathcal{A}^{*} *_Q (\mathcal{A}^{\#})^{*} = (\mathcal{A}^{\#} *_Q \mathcal{A} *_Q \mathcal{A}^{\#})^{*} = (\mathcal{A}^{\#})^{*},\\ &\mathcal{A}^{*} *_Q (\mathcal{A}^{\#})^{*} = (\mathcal{A}^{\#} *_Q \mathcal{A})^{*} = (\mathcal{A} *_Q \mathcal{A}^{\#})^{*} = (\mathcal{A}^{\#})^{*} *_Q \mathcal{A}^{*}.\end{aligned}$$

Therefore, $(\mathcal{A}^{\#})^{*} = (\mathcal{A}^{*})^{\#}$. □

Next we will discuss perturbations of group inverses of quaternion tensors under the one-sided and the two-sided conditions based on the QT-product.

Theorem 3.20. *Suppose that* $\mathcal{A} \in \mathbb{H}^{n\times n\times n_3}$ *with* $\mathrm{Ind}_{QT}(\mathcal{A}) = 1$*,* $\mathcal{E} \in \mathbb{H}^{n\times n\times n_3}$*,* $\mathcal{B} = \mathcal{A} + \mathcal{E} \in \mathbb{H}^{n\times n\times n_3}$*. Let* $\mathcal{E} = \mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{E} = \mathcal{E} *_Q \mathcal{A}^{\#} *_Q \mathcal{A}$ *and* $\| \mathcal{E} \| \| \mathcal{A}^{\#} \| < 1$*. Then*

$$\begin{aligned}&\mathcal{B}^{\#} = \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} = \mathcal{A}^{\#} *_Q \left(\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#}\right)^{-1},\\ &\mathcal{B} *_Q \mathcal{B}^{\#} = \mathcal{A} *_Q \mathcal{A}^{\#},\\ &\mathcal{B}^{\#} - \mathcal{A}^{\#} = -\mathcal{B}^{\#} *_Q \mathcal{E} *_Q \mathcal{A}^{\#} = -\mathcal{A}^{\#} *_Q \mathcal{E} *_Q \mathcal{B}^{\#},\end{aligned}$$

and

$$\begin{aligned}&\frac{\| \mathcal{A}^{\#} \|}{1+ \| \mathcal{A}^{\#} *_Q \mathcal{E} \|} \le \| \mathcal{B}^{\#} \| \le \frac{\| \mathcal{A}^{\#} \|}{1- \| \mathcal{A}^{\#} *_Q \mathcal{E} \|},\\ &\frac{\| \mathcal{B}^{\#} - \mathcal{A}^{\#} \|}{\| \mathcal{A}^{\#} \|} \le \frac{\| \mathcal{A}^{\#} *_Q \mathcal{E} \|}{1- \| \mathcal{A}^{\#} *_Q \mathcal{E} \|}.\end{aligned}$$

Proof. For $\| \mathcal{A}^{\#} *_Q \mathcal{E} \| < 1$, by virtue of Lemma 2.14, we have $\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}$ is invertible. So we obtain

$$\begin{aligned}
&\mathcal{B} *_Q \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} *_Q \mathcal{B} \\
&= (\mathcal{A} + \mathcal{E}) *_Q \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} *_Q (\mathcal{A} + \mathcal{E}) \\
&= \left(\mathcal{A} + \mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{E}\right) *_Q \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} *_Q (\mathcal{A} + \mathcal{E}) \\
&= \mathcal{A} *_Q \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right) *_Q \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} *_Q (\mathcal{A} + \mathcal{E}) \\
&= \mathcal{A} *_Q \mathcal{A}^{\#} *_Q (\mathcal{A} + \mathcal{E}) \\
&= \mathcal{A} + \mathcal{E} \\
&= \mathcal{B},
\end{aligned}$$

$$\begin{aligned}
&\left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} *_Q \mathcal{B} *_Q \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} \\
&\quad = \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} *_Q (\mathcal{A} + \mathcal{E}) *_Q \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} \\
&\quad = \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \left(\mathcal{A}^{\#} *_Q \mathcal{A} + \mathcal{A}^{\#} *_Q \mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{E}\right) *_Q \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} \\
&\quad = \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} *_Q \mathcal{A} *_Q \mathcal{A}^{\#} \\
&\quad = \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B} *_Q \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} &= \left(\mathcal{A} + \mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{E}\right) *_Q \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} \\
&= \mathcal{A} *_Q \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right) *_Q \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} \\
&= \mathcal{A} *_Q \mathcal{A}^{\#} \\
&= \mathcal{A}^{\#} *_Q \mathcal{A}.
\end{aligned}$$

Using conditions (vii) of Proposition 2.15, we get

$$||\mathcal{E} *_Q \mathcal{A}^{\#}|| \leq ||\mathcal{E}||||\mathcal{A}^{\#}|| < 1.$$

By Lemma 2.14, we obtain $\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#}$ is an invertible quaternion tensor. From Lemma 2.16, we have

$$\begin{aligned}
\left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} *_Q \mathcal{B} &= \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} *_Q (\mathcal{A} + \mathcal{E}) \\
&= \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} *_Q \left(\mathcal{A} + \mathcal{E} *_Q \mathcal{A}^{\#} *_Q \mathcal{A}\right) \\
&= \mathcal{A}^{\#} *_Q \left(\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#}\right)^{-1} *_Q \mathcal{A}^{\#} *_Q \left(\mathcal{A} + \mathcal{E} *_Q \mathcal{A}^{\#} *_Q \mathcal{A}\right) \\
&= \mathcal{A}^{\#} *_Q \mathcal{A}.
\end{aligned}$$

Therefore,

$$\mathcal{B} *_Q \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} = \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} *_Q \mathcal{B}.$$

By Definition 3.3, we have $\left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#}$ is the group inverse of $\mathcal{B}$. Similarly, we can prove that $\mathcal{A}^{\#} *_Q \left(\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#}\right)^{-1}$ is also the group inverse of $\mathcal{B}$. According to Lemma 2.16, we obtain

$$\mathcal{B}^{\#} = \left(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E}\right)^{-1} *_Q \mathcal{A}^{\#} = \mathcal{A}^{\#} *_Q \left(\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#}\right)^{-1}.$$

Let $\mathcal{E} = \mathcal{P} *_Q \widetilde{\mathcal{E}} *_Q \mathcal{P}^{-1}$ is the QT-Jordan decomposition of $\mathcal{E}$, by Lemma 2.10, we have

$$diag(\widehat{\mathcal{E}}) = diag(\widehat{\mathcal{P}})diag(\widehat{\widetilde{\mathcal{E}}})diag(\widehat{\mathcal{P}^{-1}}),$$

where

$$diag(\widehat{\widetilde{\mathcal{E}}}) = \begin{pmatrix} \widehat{\widetilde{E}}^{(1)} & & \\ & \ddots & \\ & & \widehat{\widetilde{E}}^{(n_3)} \end{pmatrix}, \widehat{\widetilde{E}}^{(i)} = \begin{pmatrix} E_{11}^{(i)} & E_{12}^{(i)} \\ E_{21}^{(i)} & E_{22}^{(i)} \end{pmatrix}, i = 1, 2, \ldots, n_3.$$

By virtue of Theorem 3.10, there is

$$\mathcal{A} *_Q \mathcal{A}^{\#} = \mathcal{P} *_Q \mathcal{J} *_Q \mathcal{J}^{\dagger} *_Q \mathcal{P}^{-1}.$$

Since $\mathcal{E} = \mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{E}$, we have $\mathcal{E} = \mathcal{P} *_Q \mathcal{J} *_Q \mathcal{J}^{\dagger} *_Q \widetilde{\mathcal{E}} *_Q \mathcal{P}^{-1}$, that is,

$$diag(\widehat{\mathcal{E}}) = diag(\widehat{\mathcal{P}})diag(\widehat{\mathcal{J}})diag(\widehat{\mathcal{J}^{\dagger}})diag(\widehat{\widetilde{\mathcal{E}}})diag(\widehat{\mathcal{P}})^{-1}.$$

Hence,

$$diag(\widehat{\widetilde{\mathcal{E}}}) = diag(\widehat{\mathcal{J}})diag(\widehat{\mathcal{J}^{\dagger}})diag(\widehat{\widetilde{\mathcal{E}}}).$$

It means that

$$\begin{pmatrix} \widehat{\widetilde{E}}^{(1)} & & \\ & \ddots & \\ & & \widehat{\widetilde{E}}^{(n_3)} \end{pmatrix} = \begin{pmatrix} \widehat{J}^{(1)} & & \\ & \ddots & \\ & & \widehat{J}^{(n_3)} \end{pmatrix} \begin{pmatrix} \widehat{J^{\dagger}}^{(1)} & & \\ & \ddots & \\ & & \widehat{J^{\dagger}}^{(n_3)} \end{pmatrix} \begin{pmatrix} \widehat{\widetilde{E}}^{(1)} & & \\ & \ddots & \\ & & \widehat{\widetilde{E}}^{(n_3)} \end{pmatrix}.$$

By $\widehat{\widetilde{E}}^{(i)} = \widehat{J}^{(i)}\widehat{J^{\dagger}}^{(i)}\widehat{\widetilde{E}}^{(i)}$, $i = 1, 2, \ldots, n_3$, we get

$$\begin{aligned} \begin{pmatrix} E_{11}^{(i)} & E_{12}^{(i)} \\ E_{21}^{(i)} & E_{22}^{(i)} \end{pmatrix} &= \begin{pmatrix} \widehat{C}^{(i)} & O \\ O & O \end{pmatrix} \begin{pmatrix} (\widehat{C}^{(i)})^{-1} & O \\ O & O \end{pmatrix} \begin{pmatrix} E_{11}^{(i)} & E_{12}^{(i)} \\ E_{21}^{(i)} & E_{22}^{(i)} \end{pmatrix} \\ &= \begin{pmatrix} E_{11}^{(i)} & \widehat{E}_{12}^{(i)} \\ O & O \end{pmatrix}, \end{aligned}$$

Therefore, $E_{21}^{(i)} = 0, E_{22}^{(i)} = 0$.

Similarly, since $\mathcal{E} = \mathcal{E} *_Q \mathcal{A}^{\#} *_Q \mathcal{A}$, we also get $E_{12}^{(i)} = 0, E_{22}^{(i)} = 0$. Thus, we obtain

$$\widehat{\widetilde{E}}^{(i)} = \begin{pmatrix} E_{11}^{(i)} & O \\ O & O \end{pmatrix}, i = 1, 2, \ldots, n_3.$$

Since $\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#}$ is invertible, we know $diag(\mathcal{I} + \widehat{\mathcal{E} *_Q} \mathcal{A}^{\#})$ is also invertible by condition (iii) of Lemma 3.6. According to

$$\begin{aligned} & diag(\mathcal{I} + \widehat{\mathcal{E} *_Q} \mathcal{A}^{\#}) \\ &= diag(\widehat{\mathcal{I}}) + diag(\widehat{\mathcal{E}})diag(\widehat{\mathcal{A}^{\#}}) \\ &= diag(\widehat{\mathcal{P}})diag(\widehat{\mathcal{I}})diag(\widehat{\mathcal{P}})^{-1} + diag(\widehat{\mathcal{P}})diag(\widehat{\widetilde{\mathcal{E}}})diag(\widehat{\mathcal{P}})^{-1}diag(\widehat{\mathcal{P}})diag(\widehat{\mathcal{J}})^{\dagger}diag(\widehat{\mathcal{P}})^{-1} \\ &= diag(\widehat{\mathcal{P}}) \left(diag(\widehat{\mathcal{I}}) + diag(\widehat{\widetilde{\mathcal{E}}})diag(\widehat{\mathcal{J}})^{\dagger} \right) diag(\widehat{\mathcal{P}})^{-1}, \end{aligned}$$

we get $diag(\widehat{\mathcal{I}}) + diag(\widehat{\widehat{\mathcal{E}}})diag(\widehat{\mathcal{J}})^{\dagger}$ is invertible. Due to

$$
\begin{aligned}
&diag(\widehat{\mathcal{I}}) + diag(\widehat{\widehat{\mathcal{E}}})diag(\widehat{\mathcal{J}})^{\dagger} \\
&= \begin{pmatrix} \begin{pmatrix} \widehat{I}_{r_1} & O \\ O & \widehat{I}_{n-r_1} \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \widehat{I}_{r_{n_3}} & O \\ O & I_{n-r_{n_3}} \end{pmatrix} \end{pmatrix} \\
&\quad + \begin{pmatrix} \begin{pmatrix} E_{11}^{(1)} & O \\ O & O \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} E_{11}^{(n_3)} & O \\ O & O \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \left(\widehat{C}^{(1)}\right)^{-1} & O \\ O & O \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \left(\widehat{C}^{(n_3)}\right)^{-1} & O \\ O & O \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} \widehat{I}_{r_1} + E_{11}^{(1)} \left(\widehat{C}^{(1)}\right)^{-1} & O \\ O & \widehat{I}_{n-r_1} \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \widehat{I}_{r_{n_3}} + E_{11}^{(n_3)} \left(\widehat{C}^{(n_3)}\right)^{-1} & O \\ O & \widehat{I}_{n-r_{n_3}} \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} \left(\widehat{C}^{(1)} + E_{11}^{(1)}\right) \left(\widehat{C}^{(1)}\right)^{-1} & O \\ O & \widehat{I}_{n-r_1} \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \left(\widehat{C}^{(n_3)} + E_{11}^{(n_3)}\right) \left(\widehat{C}^{(n_3)}\right)^{-1} & O \\ O & \widehat{I}_{n-r_{n_3}} \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} \widehat{C}^{(1)} + E_{11}^{(1)} & O \\ O & \widehat{I}_{n-r_1} \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \widehat{C}^{(n_3)} + E_{11}^{(n_3)} & O \\ O & \widehat{I}_{n-r_{n_3}} \end{pmatrix} \end{pmatrix} \\
&\qquad \begin{pmatrix} \begin{pmatrix} \left(\widehat{C}^{(1)}\right)^{-1} & O \\ O & \widehat{I}_{n-r_1} \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \left(\widehat{C}^{(n_3)}\right)^{-1} & O \\ O & \widehat{I}_{n-r_{n_3}} \end{pmatrix} \end{pmatrix},
\end{aligned}
$$

where r_i is the number of non-zero eigenvalues of $\widehat{A^{(i)}}, i = 1, 2, \ldots, n_3$. So we know $\widehat{C}^{(i)} + E_{11}^{(i)}$ is invertible. Because

$$\mathcal{B} = \mathcal{A} + \mathcal{E} = \mathcal{P} *_Q \left(\mathcal{J} + \widetilde{\mathcal{E}}\right) *_Q \mathcal{P}^{-1},$$

according to Lemma 2.10, we get

$$diag(\widehat{\mathcal{B}}) = diag(\widehat{\mathcal{P}}) diag(\widehat{\mathcal{J} + \widetilde{\mathcal{E}}}) diag(\widehat{\mathcal{P}})^{-1},$$

where

$$diag(\widehat{\mathcal{J} + \widetilde{\mathcal{E}}}) = \begin{pmatrix} \widehat{J}^{(1)} + \widehat{\widetilde{E}}^{(1)} & & \\ & \ddots & \\ & & \widehat{J}^{(n_3)} + \widehat{\widetilde{E}}^{(n_3)} \end{pmatrix}, \widehat{J}^{(i)} + \widehat{\widetilde{E}}^{(i)} = \begin{pmatrix} \widehat{C}^{(i)} + E_{11}^{(i)} & O \\ O & O \end{pmatrix},$$

$$i = 1, 2, \ldots, n_3.$$

By Theorem 3.10, we can get $\mathcal{B}^{\#} = \mathcal{A} + \mathcal{E} = \mathcal{P} *_Q \left(\mathcal{J} + \widetilde{\mathcal{E}}\right) *_Q \mathcal{P}^{-1}$. Since

$$\begin{aligned} diag(\widehat{\mathcal{B}}) diag(\widehat{\mathcal{B}^{\#}}) &= diag(\widehat{\mathcal{P}}) diag(\widehat{\mathcal{J} + \widetilde{\mathcal{E}}}) diag(\widehat{\mathcal{P}})^{-1} diag(\widehat{\mathcal{P}}) diag(\widehat{\mathcal{J} + \widetilde{\mathcal{E}}})^{\dagger} diag(\widehat{\mathcal{P}})^{-1} \\ &= diag(\widehat{\mathcal{P}}) \begin{pmatrix} \begin{pmatrix} \widehat{C}^{(1)} + E_{11}^{(1)} & O \\ O & O \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \widehat{C}^{(n_3)} + E_{11}^{(n_3)} & O \\ O & O \end{pmatrix} \end{pmatrix} \\ &\quad \begin{pmatrix} \begin{pmatrix} (\widehat{C}^{(1)} + E_{11}^{(1)})^{-1} & O \\ O & O \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} (\widehat{C}^{(n_3)} + E_{11}^{(n_3)})^{-1} & O \\ O & O \end{pmatrix} \end{pmatrix} diag(\widehat{\mathcal{P}})^{-1} \\ &= diag(\widehat{\mathcal{P}}) \begin{pmatrix} \begin{pmatrix} \widehat{I}_{r_1} & O \\ O & O \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \widehat{I}_{r_{n_3}} & O \\ O & O \end{pmatrix} \end{pmatrix} diag(\widehat{\mathcal{P}})^{-1} \\ &= diag(\widehat{\mathcal{P}}) diag(\widehat{\mathcal{J}}) diag(\widehat{\mathcal{P}})^{-1} diag(\widehat{\mathcal{P}}) diag(\widehat{\mathcal{J}})^{\dagger} diag(\widehat{\mathcal{P}})^{-1} \\ &= diag(\widehat{\mathcal{A}}) diag(\widehat{\mathcal{A}^{\#}}), \end{aligned}$$

we have $\mathcal{B} *_Q \mathcal{B}^{\#} = \mathcal{A} *_Q \mathcal{A}^{\#}$. According to

$$\begin{aligned} \mathcal{B}^{\#} - \mathcal{A}^{\#} &= \mathcal{B}^{\#} - \mathcal{A}^{\#} + \mathcal{B}^{\#} *_Q (\mathcal{B} - \mathcal{A}) *_Q \mathcal{A}^{\#} - \mathcal{B}^{\#} *_Q \mathcal{E} *_Q \mathcal{A}^{\#} \\ &= \mathcal{B}^{\#} - \mathcal{A}^{\#} + \mathcal{B}^{\#} *_Q \mathcal{B} *_Q \mathcal{A}^{\#} - \mathcal{B}^{\#} *_Q \mathcal{A} *_Q \mathcal{A}^{\#} - \mathcal{B}^{\#} *_Q \mathcal{E} *_Q \mathcal{A}^{\#} \\ &= \mathcal{B}^{\#} - \mathcal{A}^{\#} + \mathcal{A}^{\#} *_Q \mathcal{A} *_Q \mathcal{A}^{\#} - \mathcal{B}^{\#} *_Q \mathcal{B} *_Q \mathcal{B}^{\#} - \mathcal{B}^{\#} *_Q \mathcal{E} *_Q \mathcal{A}^{\#} \\ &= -\mathcal{B}^{\#} *_Q \mathcal{E} *_Q \mathcal{A}^{\#}, \end{aligned}$$

and

$$\begin{aligned}\mathcal{B}^{\#} - \mathcal{A}^{\#} &= \mathcal{B}^{\#} - \mathcal{A}^{\#} + \mathcal{A}^{\#} *_Q (\mathcal{B} - \mathcal{A}) *_Q \mathcal{B}^{\#} - \mathcal{A}^{\#} *_Q \mathcal{E} *_Q \mathcal{B}^{\#} \\ &= \mathcal{B}^{\#} - \mathcal{A}^{\#} + \mathcal{A}^{\#} *_Q \mathcal{B} *_Q \mathcal{B}^{\#} - \mathcal{A}^{\#} *_Q \mathcal{A} *_Q \mathcal{B}^{\#} - \mathcal{A}^{\#} *_Q \mathcal{E} *_Q \mathcal{B}^{\#} \\ &= \mathcal{B}^{\#} - \mathcal{A}^{\#} + \mathcal{A}^{\#} *_Q \mathcal{A} *_Q \mathcal{A}^{\#} - \mathcal{B}^{\#} *_Q \mathcal{B} *_Q \mathcal{B}^{\#} - \mathcal{A}^{\#} *_Q \mathcal{E} *_Q \mathcal{B}^{\#} \\ &= -\mathcal{A}^{\#} *_Q \mathcal{E} *_Q \mathcal{B}^{\#},\end{aligned}$$

we can obtain $\mathcal{B}^{\#} - \mathcal{A}^{\#} = -\mathcal{B}^{\#} *_Q \mathcal{E} *_Q \mathcal{A}^{\#} = -\mathcal{A}^{\#} *_Q \mathcal{E} *_Q \mathcal{B}^{\#}$. By condition (vii) of Proposition 2.15, we have

$$\left\|\mathcal{B}^{\#} - \mathcal{A}^{\#}\right\| = \left\|-\mathcal{A}^{\#} *_Q \mathcal{E} *_Q \mathcal{B}^{\#}\right\| \le \left\|\mathcal{A}^{\#} *_Q \mathcal{E}\right\| \left\|\mathcal{B}^{\#}\right\|,$$

so

$$\frac{\left\|\mathcal{B}^{\#} - \mathcal{A}^{\#}\right\|}{\left\|\mathcal{A}^{\#}\right\|} \le \frac{\left\|\mathcal{A}^{\#} *_Q \mathcal{E}\right\| \left\|\mathcal{B}^{\#}\right\|}{\left\|\mathcal{A}^{\#}\right\|}.$$

By the equation $\mathcal{B}^{\#} = (\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E})^{-1} *_Q \mathcal{A}^{\#}$, Lemma 2.14 and condition (vii) of Proposition 2.15, we have

$$\begin{aligned}\left\|\mathcal{B}^{\#}\right\| &= \left\|(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E})^{-1} *_Q \mathcal{A}^{\#}\right\| \\ &\le \left\|(\mathcal{I} + \mathcal{A}^{\#} *_Q \mathcal{E})^{-1}\right\| \left\|\mathcal{A}^{\#}\right\| \\ &\le \frac{\left\|\mathcal{A}^{\#}\right\|}{1 - \left\|\mathcal{A}^{\#} *_Q \mathcal{E}\right\|}.\end{aligned}$$

Therefore,

$$\frac{\left\|\mathcal{B}^{\#} - \mathcal{A}^{\#}\right\|}{\left\|\mathcal{A}^{\#}\right\|} \le \frac{\left\|\mathcal{A}^{\#} *_Q \mathcal{E}\right\|}{1 - \left\|\mathcal{A}^{\#} *_Q \mathcal{E}\right\|}.$$

Since $\mathcal{B}^{\#} - \mathcal{A}^{\#} = -\mathcal{B}^{\#} *_Q \mathcal{E} *_Q \mathcal{A}^{\#}$, $\mathcal{A}^{\#} = \mathcal{B}^{\#} *_Q (\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#})$ and condition (vii) of Proposition 2.15, we get

$$\left\|\mathcal{A}^{\#}\right\| = \left\|\mathcal{B}^{\#} *_Q (\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#})\right\| \le \left\|\mathcal{B}^{\#}\right\| \left\|\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#}\right\|.$$

Hence,

$$\frac{\left\|\mathcal{A}^{\#}\right\|}{\left\|\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#}\right\|} \le \left\|\mathcal{B}^{\#}\right\|.$$

Because $\left\|\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#}\right\| \le 1 + \left\|\mathcal{E} *_Q \mathcal{A}^{\#}\right\| \le 1 + \left\|\mathcal{E}\right\| \left\|\mathcal{A}^{\#}\right\|$, then we obtain

$$\frac{\left\|\mathcal{A}^{\#}\right\|}{1 + \left\|\mathcal{E}\right\| \left\|\mathcal{A}^{\#}\right\|} \le \frac{\left\|\mathcal{A}^{\#}\right\|}{\left\|\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#}\right\|}.$$

So

$$\left\|\mathcal{B}^{\#}\right\| \ge \frac{\left\|\mathcal{A}^{\#}\right\|}{\left\|\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#}\right\|} \ge \frac{\left\|\mathcal{A}^{\#}\right\|}{1 + \left\|\mathcal{E}\right\| \left\|\mathcal{A}^{\#}\right\|} = \frac{\left\|\mathcal{A}^{\#}\right\|}{1 + \left\|\mathcal{A}^{\#}\right\| \left\|\mathcal{E}\right\|} \ge \frac{\left\|\mathcal{A}^{\#}\right\|}{1 + \left\|\mathcal{A}^{\#} *_Q \mathcal{E}\right\|}.$$

Therefore,

$$\frac{\left\|\mathcal{A}^{\#}\right\|}{1 + \left\|\mathcal{A}^{\#} *_Q \mathcal{E}\right\|} \le \left\|\mathcal{B}^{\#}\right\| \le \frac{\left\|\mathcal{A}^{\#}\right\|}{1 - \left\|\mathcal{A}^{\#} *_Q \mathcal{E}\right\|}.$$

□

Example 3.21. Suppose $\mathcal{A} \in \mathbb{H}^{2\times 2\times 3}$ with elements of the frontal slices as follows,

$$\mathcal{A}^{(1)} = \begin{pmatrix} 3+4i+j-k & 2+7i+3j+4k \\ 1-2i-2j+8k & 4+i-j-3k \end{pmatrix},$$

$$\mathcal{A}^{(2)} = \begin{pmatrix} 2-i+5j+3k & 11+7i-3j+9k \\ 4+5i+2j-4k & 8+i+4j-3k \end{pmatrix},$$

$$\mathcal{A}^{(3)} = \begin{pmatrix} 5+7i+4j-3k & 4+4i-j+6k \\ 7+3i+4j-2k & 2+5i-8j \end{pmatrix}.$$

By calculating, it is obtained that $\text{rank}_{QT}(\mathcal{A}) = 2$ and $\text{rank}_{QT}(\mathcal{A}^2) = 2$, then $\text{Ind}_{QT}(\mathcal{A}) = 1$. Next, by the QT-Jordan decomposition, we obtain

$$\mathcal{A} = \mathcal{P} *_Q \mathcal{J} *_Q \mathcal{P}^{-1}.$$

where the elements of frontal slices of $\mathcal{P}$, $\mathcal{J}$ and $\mathcal{P}^{-1}$ are

$$\mathcal{P}^{(1)} = \begin{pmatrix} -0.2079+0.0763i+0.0181j-0.2243k & -3.9419-2.4556i-0.3133j-1.1588k \\ 0.1550+0.3316i+1.0000j & -2.8008-0.9598i+1.0000j \end{pmatrix},$$

$$\mathcal{P}^{(2)} = \begin{pmatrix} -0.6419+0.5482i-0.8847j-0.2128k & -3.4450-2.5968i+0.1659j-1.1271k \\ 0.4675+0.1400i-0.0000j+0.0000k & -2.6713-0.8138i-0.0000j+0.0000k \end{pmatrix},$$

$$\mathcal{P}^{(3)} = \begin{pmatrix} -0.6949-0.0141i-0.0830j-0.0382k & -3.2246-2.8653i-0.1954j-1.6562k \\ 0.0675-0.3561i+0.0000j+0.0000k & -3.0447-1.3790i+0.0000j+0.0000k \end{pmatrix},$$

$$\mathcal{J}^{(1)} = \begin{pmatrix} -8.6366+7.3073i & 0 \\ 0 & 15.6366+15.5425i \end{pmatrix},$$

$$\mathcal{J}^{(2)} = \begin{pmatrix} 3.7230-0.2942i & 0 \\ 0 & 2.9438-0.3547i \end{pmatrix},$$

$$\mathcal{J}^{(3)} = \begin{pmatrix} 1.6142+4.4115i & 0 \\ 0 & 8.7189+6.9395i \end{pmatrix},$$

and

$$(\mathcal{P}^{-1})^{(1)} = \begin{pmatrix} 0.1640+0.0663i-0.0949j+0.0255k & 0.0432-0.0757i-0.3048j+0.0166k \\ -0.2304-0.1045i+0.0990j-0.0507k & 0.0582-0.0201i-0.2534j-0.0022k \end{pmatrix},$$

$$(\mathcal{P}^{-1})^{(2)} = \begin{pmatrix} -0.2356+0.0211i+0.2728j+0.0846k & 0.1236+0.1721i-0.0378j+0.0434k \\ 0.1395-0.0180i-0.1948j-0.0729k & -0.1926+0.1148i+0.1833j-0.0826k \end{pmatrix},$$

$$(\mathcal{P}^{-1})^{(3)} = \begin{pmatrix} -0.1078-0.1399i-0.0019j-0.0230k & 0.1081-0.0483i+0.0444j-0.0244k \\ 0.0629+0.1383i+0.0904j+0.1394k & 0.0846-0.0656i+0.0683j+0.0752k \end{pmatrix}.$$

According to the Theorem 3.10, it is obvious that $\mathcal{A}^{\#} = \mathcal{P} *_Q \mathcal{J}^{\dagger} *_Q \mathcal{P}^{-1}$, where

$$(\mathcal{J}^{\dagger})^{(1)} = \begin{pmatrix} -0.0520-0.0546i & 0 \\ 0 & 0.0283-0.0323i \end{pmatrix},$$

$$(\mathcal{J}^{\dagger})^{(2)} = \begin{pmatrix} -0.0017-0.0106i & 0 \\ 0 & 0.0063-0.0045i \end{pmatrix},$$

$$(\mathcal{J}^{\dagger})^{(3)} = \begin{pmatrix} 0.0303-0.0156i & 0 \\ 0 & -0.0125+0.0188i \end{pmatrix}.$$

Therefore, it is obtained that the QT-group inverse of $\mathcal{A}$ and its elements of frontal slices are

$$(\mathcal{A}^{\#})^{(1)} = \begin{pmatrix} -0.0135+0.0032i-0.0066j+0.0243k & 0.0190+0.0203i-0.0046j-0.0425k \\ -0.0165-0.0034i-0.0263j+0.0100k & -0.0102+0.0165i-0.0078j+0.0142k \end{pmatrix},$$

$$(\mathcal{A}^{\#})^{(2)} = \begin{pmatrix} 0.0278-0.0244i+0.0078j-0.0018k & 0.0267-0.0124i-0.0158j+0.0034k \\ -0.0071-0.0043i+0.0304j-0.0194k & 0.0107-0.0036i-0.0081j+0.0261k \end{pmatrix},$$

$$(\mathcal{A}^{\#})^{(3)} = \begin{pmatrix} -0.0166+0.0078i-0.0046j+0.0149k & 0.0103-0.0034i-0.0005j-0.0055k \\ 0.0242-0.0035i-0.0006j-0.0223k & 0.0006-0.0175i+0.0181j+0.0010k \end{pmatrix}.$$

Further, the frontal slices of perturbation $\mathcal{E}$ are

$$\mathcal{E}^{(1)} = \begin{pmatrix} 0.08+0.008i+0j+0k & 0+0i+0j+0k \\ 0+0.04i+0.0028j+0k & 0.0008+0.0064i+0j+0k \end{pmatrix},$$

$$\mathcal{E}^{(2)} = \begin{pmatrix} 0+0.28i+0.0068j+0.024k & -0.0084+0i+0j+0k \\ 0+0i+0j-0.01k & 0.046+0i+0.004j+0k \end{pmatrix},$$

$$\mathcal{E}^{(3)} = \begin{pmatrix} 0.003+0.0001i+0.036j+0k & 0.005+0i+0.007j+0.026k \\ 0+0.012i+0.03j-0.004k & 0.0001+0i+0.075j-0.05k \end{pmatrix}.$$

Then by computing, it is obtained that the perturbation $\mathcal{E}$ satisfies the condition $\mathcal{E} = \mathcal{A} *_Q \mathcal{A}^{\#} *_Q \mathcal{E} = \mathcal{E} *_Q \mathcal{A}^{\#} *_Q \mathcal{A}$ and $\| \mathcal{E} \| \| \mathcal{A}^{\#} \| = 0.0385 < 1$.

Let $\mathcal{B} = \mathcal{A} + \mathcal{E}$, by Theorem 3.20, it can be obtained that $\mathcal{B}^{\#}$ and its frontal slices are

$$(\mathcal{B}^{\#})^{(1)} = \begin{pmatrix} -0.0138+0.0029i-0.0071j+0.0242k & 0.0187+0.0202i-0.0044j-0.0428k \\ -0.0166-0.0031i-0.0264j+0.0101k & -0.0098+0.0164i-0.0078j+0.0141k \end{pmatrix},$$

$$(\mathcal{B}^{\#})^{(2)} = \begin{pmatrix} 0.0282-0.0243i+0.0078j-0.0020k & 0.0265-0.0122i-0.0160j+0.0034k \\ -0.0074-0.0043i+0.0304j-0.0200k & 0.0107-0.0039i-0.0078j+0.0262k \end{pmatrix},$$

$$(\mathcal{B}^{\#})^{(3)} = \begin{pmatrix} -0.0165+0.0078i-0.0042j+0.0150k & 0.0102-0.0033i-0.0009j-0.0052k \\ 0.0245-0.0034i-0.0004j-0.0219k & 0.0003-0.0173i+0.0182j+0.0012k \end{pmatrix}.$$

By calculation, we can have

$$\frac{||\mathcal{A}^{\#}||}{1+||\mathcal{A}^{\#} *_Q \mathcal{E}||} = 0.1067, \quad ||\mathcal{B}^{\#}|| = 0.1090, \quad \frac{||\mathcal{A}^{\#}||}{1-||\mathcal{A}^{\#} *_Q \mathcal{E}||} = 0.1121.$$

So

$$\frac{\| \mathcal{A}^{\#} \|}{1+ \| \mathcal{A}^{\#} *_Q \mathcal{E} \|} \leq \| \mathcal{B}^{\#} \| \leq \frac{\| \mathcal{A}^{\#} \|}{1- \| \mathcal{A}^{\#} *_Q \mathcal{E} \|}.$$

And we can calculate that

$$\frac{||\mathcal{B}^{\#} - \mathcal{A}^{\#}||}{||\mathcal{A}^{\#}||} = 0.0159, \quad \frac{||\mathcal{A}^{\#} *_Q \mathcal{E}||}{1-||\mathcal{A}^{\#} *_Q \mathcal{E}||} = 0.0254.$$

That is

$$\frac{\| \mathcal{B}^{\#} - \mathcal{A}^{\#} \|}{\| \mathcal{A}^{\#} \|} \leq \frac{\| \mathcal{A}^{\#} *_Q \mathcal{E} \|}{1- \| \mathcal{A}^{\#} *_Q \mathcal{E} \|}.$$

Theorem 3.22. *Suppose that* $\mathcal{A}, \mathcal{E} \in \mathbb{H}^{n\times n\times n_3}, \mathrm{Ind}_{QT}(\mathcal{A}) = 1, \mathcal{B} = \mathcal{A} + \mathcal{E} \in \mathbb{H}^{n\times n\times n_3}$, $\mathcal{E} = \mathcal{E} *_Q \mathcal{A} *_Q \mathcal{A}^{\#}$ *and* $\| \mathcal{E} *_Q \mathcal{A}^{\#} \| < 1$. *Then*

$$\begin{aligned} \mathcal{B}^{\#} =& \mathcal{A}^{\#} - \mathcal{A}^{\#} *_Q \mathcal{E} *_Q \mathcal{A}^{\#} *_Q (\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#})^{-1} \\ &- (\mathcal{I} - \mathcal{A} *_Q \mathcal{A}^{\#}) *_Q (\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#})^{-1} *_Q \mathcal{A}^{\#} *_Q (\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#})^{-1}, \end{aligned} \tag{3.12}$$

and

$$\frac{\| \mathcal{B}^{\#} - \mathcal{A}^{\#} \|}{\| \mathcal{A}^{\#} \|} \leq \frac{\| \mathcal{E} *_Q \mathcal{A}^{\#} \|}{1- \| \mathcal{E} *_Q \mathcal{A}^{\#} \|} + \frac{\| \mathcal{A} *_Q \mathcal{A}^{\#} \|}{(1- \| \mathcal{E} *_Q \mathcal{A}^{\#} \|)^2}. \tag{3.13}$$

Proof. For any quaternion tensor $\mathcal{B}$, on the one hand, we know

$$diag(\widehat{\mathcal{B}})^{\#} = \begin{pmatrix} \widehat{B}_1^{\#} & & \\ & \ddots & \\ & & \widehat{B}_{n_3}^{\#} \end{pmatrix}.$$

On the other hand, we can obtain

$$diag(\widehat{\mathcal{B}})^{\#} = diag(\widehat{\mathcal{A}+\mathcal{E}})^{\#} = \Big(diag(\widehat{\mathcal{A}}) + diag(\widehat{\mathcal{E}})\Big)^{\#} = \begin{pmatrix} (\widehat{A}_1 + \widehat{E}_1)^{\#} & & \\ & \ddots & \\ & & (\widehat{A}_{n_3} + \widehat{E}_{n_3})^{\#} \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} \widehat{B}_1^{\#} & & \\ & \ddots & \\ & & \widehat{B}_{n_3}^{\#} \end{pmatrix} = \begin{pmatrix} (\widehat{A}_1 + \widehat{E}_1)^{\#} & & \\ & \ddots & \\ & & (\widehat{A}_{n_3} + \widehat{E}_{n_3})^{\#} \end{pmatrix},$$

that is $\widehat{B}_i = \widehat{A}_i + \widehat{E}_i,\ i = 1, 2, \dots, n_3$. By $\mathrm{Ind}_{QT}(\mathcal{A}) = 1$, we have $\mathrm{rank}_{QT}(\mathcal{A})^2 = \mathrm{rank}_{QT}(\mathcal{A})$. Further more, $\mathrm{rank}_{QT}(\widehat{\mathcal{A}})^2 = \mathrm{rank}_{QT}(\widehat{\mathcal{A}})$, we have $\mathrm{rank}\big((\widehat{A}^{(i)})^2\big) = \mathrm{rank}(\widehat{A}^{(i)})$, it means that $\mathrm{Ind}(\widehat{A}^{(i)}) = 1$. Since $\mathcal{E} = \mathcal{E} *_Q \mathcal{A} *_Q \mathcal{A}^{\#}$, we get $diag(\widehat{\mathcal{E}}) = diag(\widehat{\mathcal{E}}) diag(\widehat{\mathcal{A}}) diag(\widehat{\mathcal{A}})^{\#}$. Therefore, $\widehat{E}^{(i)} = \widehat{E}^{(i)} \widehat{A}^{(i)} (\widehat{A}^{(i)})^{\#}$.

According to $\| \mathcal{E} *_Q \mathcal{A}^{\#} \| < 1$, we get

$$\| \mathcal{E} *_Q \mathcal{A}^{\#} \| = \| diag(\widehat{\mathcal{E} *_Q \mathcal{A}^{\#}}) \| = ||diag(\widehat{\mathcal{E}}) diag(\widehat{\mathcal{A}^{\#}})|| < 1,$$

then $||\widehat{E}^{(i)} \widehat{A^{\#}}^{(i)}|| < 1$. So

$$\widehat{B}_i^{\#} = \widehat{A}_i^{\#} - \widehat{A}_i^{\#} \widehat{E}_i \widehat{A}_i^{\#} (\widehat{I}_i + \widehat{E}_i \widehat{A}_i^{\#})^{-1} - (\widehat{I}_i - \widehat{A}_i \widehat{A}_i^{\#})(\widehat{I}_i + \widehat{E}_i \widehat{A}_i^{\#})^{-1} \widehat{A}_i^{\#} (\widehat{I}_i + \widehat{E}_i \widehat{A}_i^{\#})^{-1}.$$

Thus, we can get

$$\begin{aligned} diag(\widehat{\mathcal{B}})^{\#} =& diag(\widehat{\mathcal{A}})^{\#} - diag(\widehat{\mathcal{A}})^{\#} diag(\widehat{\mathcal{E}}) diag(\widehat{\mathcal{A}})^{\#} diag(\widehat{\mathcal{A}})^{\#} diag(\mathcal{I} + \widehat{\mathcal{E} *_Q} \mathcal{A}^{\#})^{-1} \\ &- diag(\mathcal{I} - \widehat{\mathcal{A} *_Q} \mathcal{A}^{\#}) diag(\mathcal{I} + \widehat{\mathcal{E} *_Q} \mathcal{A}^{\#})^{-1} diag(\widehat{\mathcal{A}})^{\#} diag(\mathcal{I} + \widehat{\mathcal{E} *_Q} \mathcal{A}^{\#})^{-1}. \end{aligned}$$

By virtue of Lemma 2.10, we have

$$\begin{aligned}\mathcal{B}^{\#} =& \mathcal{A}^{\#} - \mathcal{A}^{\#} *_Q \mathcal{E} *_Q \mathcal{A}^{\#} *_Q (\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#})^{-1} \\ &- (\mathcal{I} - \mathcal{A} *_Q \mathcal{A}^{\#}) *_Q (\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#})^{-1} *_Q \mathcal{A}^{\#} *_Q (\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#})^{-1}.\end{aligned}$$

So according to Proposition 2.15, we get

$$\begin{aligned}\| \mathcal{B}^{\#} - \mathcal{A}^{\#} \| =& \| \mathcal{A}^{\#} - \mathcal{A}^{\#} *_Q \mathcal{E} *_Q \mathcal{A}^{\#} *_Q (\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#})^{-1} \\ &- (\mathcal{I} - \mathcal{A} *_Q \mathcal{A}^{\#}) *_Q (\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#})^{-1} *_Q \mathcal{A}^{\#} *_Q (\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#})^{-1} - \mathcal{A}^{\#} \| \\ =& \| -\mathcal{A}^{\#} *_Q \mathcal{E} *_Q \mathcal{A}^{\#} *_Q (\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#})^{-1} \\ &- (\mathcal{I} - \mathcal{A} *_Q \mathcal{A}^{\#}) *_Q (\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#})^{-1} *_Q \mathcal{A}^{\#} *_Q (\mathcal{I} + \mathcal{E} *_Q \mathcal{A}^{\#})^{-1} \| \\ \leq& \frac{\| \mathcal{A}^{\#} \| \| \mathcal{E} *_Q \mathcal{A}^{\#} \|}{1 - \| \mathcal{E} *_Q \mathcal{A}^{\#} \|} + \frac{\| \mathcal{I} - \mathcal{A} *_Q \mathcal{A}^{\#} \| \| \mathcal{A}^{\#} \|}{(1 - \| \mathcal{E} *_Q \mathcal{A}^{\#} \|)^2} \\ =& \frac{\| \mathcal{A}^{\#} \| \| \mathcal{E} *_Q \mathcal{A}^{\#} \|}{1 - \| \mathcal{E} *_Q \mathcal{A}^{\#} \|} + \frac{\| \mathcal{A} *_Q \mathcal{A}^{\#} \| \| \mathcal{A}^{\#} \|}{(1 - \| \mathcal{E} *_Q \mathcal{A}^{\#} \|)^2}.\end{aligned}$$

Hence, we obtain

$$\frac{\| \mathcal{B}^{\#} - \mathcal{A}^{\#} \|}{\| \mathcal{A}^{\#} \|} \leq \frac{\| \mathcal{E} *_Q \mathcal{A}^{\#} \|}{1 - \| \mathcal{E} *_Q \mathcal{A}^{\#} \|} + \frac{\| \mathcal{A} *_Q \mathcal{A}^{\#} \|}{(1 - \| \mathcal{E} *_Q \mathcal{A}^{\#} \|)^2}.$$

Therefore, the conclusion holds. □

Example 3.23. Suppose $\mathcal{A} \in \mathbb{H}^{2\times2\times3}$ with elements of the frontal slices as follows,

$$\begin{aligned}\mathcal{A}^{(1)} &= \begin{pmatrix} 3+4i+3j-k & 2+0i+2j+0k \\ 1-2i-2j+0k & 0+i+0j+0k \end{pmatrix}, \\ \mathcal{A}^{(2)} &= \begin{pmatrix} 2-i+0j-4k & 0+0i+0j+0k \\ 0+0i+2j+0k & 3+i+4j-3k \end{pmatrix}, \\ \mathcal{A}^{(3)} &= \begin{pmatrix} -1+0i+4j+0k & 0+0i+0j+0k \\ 0+0i+4j-2k & -4+6i+3j+0k \end{pmatrix}.\end{aligned}$$

By calculating, it is obtained that $\mathrm{rank}_{QT}(\mathcal{A}) = 2$ and $\mathrm{rank}_{QT}(\mathcal{A}^2) = 2$, then $\mathrm{Ind}_{QT}(\mathcal{A}) = 1$. Next, by the QT-Jordan decomposition, we obtain

$$\mathcal{A} = \mathcal{P} *_Q \mathcal{J} *_Q \mathcal{P}^{-1}.$$

where the elements of frontal slices of $\mathcal{P}$, $\mathcal{J}$ and $\mathcal{P}^{-1}$ are

$$\begin{aligned}&\mathcal{P}^{(1)} \\ &= \begin{pmatrix} 0.8772-0.9741i-0.2899j+0.6332k & 0.4920+0.2815i+0.2608j+0.6960k \\ -0.3292+1.4692i+1.0000j & -0.6638+0.2659i+1.0000j \end{pmatrix}, \\ &\mathcal{P}^{(2)} \\ &= \begin{pmatrix} -0.4994-0.7683i+0.0524j+0.2324k & 0.4131+0.0614i+0.2736j+0.3393k \\ 0.8109+1.3616i-0.0000j+0.0000k & 0.0591+1.2040i-0.0000j+0.0000k \end{pmatrix}, \\ &\mathcal{P}^{(3)} \\ &= \begin{pmatrix} 0.4401-0.0202i-0.2668j+0.3343k & 1.4308+0.1619i+0.7411j+0.1280k \\ -0.1267+0.6227i+0.0000j+0.0000k & 0.2262+0.2311i+0.0000j+0.0000k \end{pmatrix},\end{aligned}$$

$$\mathcal{J}^{(1)} = \begin{pmatrix} -1.1975 + 8.7451i & 0 \\ 0 & 4.1975 + 7.5811i \end{pmatrix},$$
$$\mathcal{J}^{(2)} = \begin{pmatrix} 0.6384 + 1.8666i & 0 \\ 0 & 0.3283 + 3.2028i \end{pmatrix},$$
$$\mathcal{J}^{(3)} = \begin{pmatrix} 0.3754 - 0.9960i & 0 \\ 0 & -1.3421 + 0.0654i \end{pmatrix},$$

and

$$(\mathcal{P}^{-1})^{(1)} = \begin{pmatrix} 0.1819 + 0.2144i + 0.0609j + 0.1403k & -0.0143 - 0.0399i - 0.3356j + 0.0979k \\ -0.1721 - 0.1600i - 0.1013j - 0.1929k & -0.0675 + 0.0553i - 0.0562j - 0.1079k \end{pmatrix},$$
$$(\mathcal{P}^{-1})^{(2)} = \begin{pmatrix} -0.0851 - 0.0737i + 0.1711j - 0.0390k & 0.0924 - 0.1501i + 0.2346j + 0.2485k \\ 0.0321 - 0.0030i - 0.1674j + 0.0926k & 0.0890 - 0.0942i - 0.0959j - 0.0242k \end{pmatrix},$$
$$(\mathcal{P}^{-1})^{(3)} = \begin{pmatrix} -0.2377 - 0.0361i - 0.1655j + 0.0036k & -0.1596 - 0.0919i + 0.0226j - 0.2612k \\ 0.3591 - 0.0305i + 0.0370j - 0.0011k & 0.2054 + 0.0312i + 0.1024j + 0.0394k \end{pmatrix}.$$

According to the Theorem 3.10, it is obvious that $\mathcal{A}^{\#} = \mathcal{P} *_Q \mathcal{J}^{\dagger} *_Q \mathcal{P}^{-1}$, where

$$(\mathcal{J}^{\dagger})^{(1)} = \begin{pmatrix} -0.0122 - 0.1070i & 0 \\ 0 & 0.0524 - 0.0902i \end{pmatrix},$$
$$(\mathcal{J}^{\dagger})^{(2)} = \begin{pmatrix} 0.0132 + 0.0183i & 0 \\ 0 & -0.0329 + 0.0213i \end{pmatrix},$$
$$(\mathcal{J}^{\dagger})^{(3)} = \begin{pmatrix} -0.0030 - 0.0152i & 0 \\ 0 & 0.0054 - 0.0160i \end{pmatrix}.$$

Therefore, it is obtained that the QT-group inverse of $\mathcal{A}$ and its elements of frontal slices are

$$(\mathcal{A}^{\#})^{(1)} = \begin{pmatrix} 0.0245 - 0.0490i - 0.0181j - 0.0136k & 0.0139 + 0.0134i + 0.0179j + 0.0024k \\ 0.0359 + 0.0171i - 0.0132j + 0.0127k & 0.0158 - 0.0251i - 0.0058j - 0.0166k \end{pmatrix},$$
$$(\mathcal{A}^{\#})^{(2)} = \begin{pmatrix} -0.0017 + 0.0075i + 0.0200j + 0.0642k & 0.0120 + 0.0048i + 0.0036j - 0.0000k \\ -0.0078 - 0.0126i + 0.0022j - 0.0215k & -0.0471 - 0.0611i - 0.0397j + 0.0352k \end{pmatrix},$$
$$(\mathcal{A}^{\#})^{(3)} = \begin{pmatrix} 0.0104 + 0.0028i - 0.0774j - 0.0048k & -0.0080 - 0.0041i - 0.0131j - 0.0117k \\ -0.0264 + 0.0394i + 0.0159j + 0.0215k & 0.0211 + 0.0144i - 0.0028j + 0.0096k \end{pmatrix}.$$

Further, the frontal slices of perturbation $\mathcal{E}$ are

$$\mathcal{E}^{(1)} = \begin{pmatrix} 0.01+0i+0j-0.004k & 0-0.05i+0j+0k \\ 0+0i+0.01j-0.02k & 0.03+0i+0j+0k \end{pmatrix},$$
$$\mathcal{E}^{(2)} = \begin{pmatrix} 0+0i+0j+0k & 0+0i+0.096j+0k \\ 0+0.02i+0j-0.62k & 0.028+0.04i+0j+0k \end{pmatrix},$$
$$\mathcal{E}^{(3)} = \begin{pmatrix} 0.039-0.05i+0j+0.048k & 0.04+0i+0.002j+0k \\ 0+0i+0j-0.012k & 0+0.03i-0.002j-0.006k \end{pmatrix}.$$

Then by computing, it is obtained that the perturbation $\mathcal{E}$ satisfies the condition $\mathcal{E} = \mathcal{E} *_Q \mathcal{A} *_Q \mathcal{A}^{\#}$ and $\| \mathcal{E} *_Q \mathcal{A}^{\#} \| = 0.0936 < 1$.

Let $\mathcal{B} = \mathcal{A} + \mathcal{E}$, by Theorem 3.20, it can be obtained that $\mathcal{B}^{\#}$ and its frontal slices are

$$(\mathcal{B}^{\#})^{(1)} = \begin{pmatrix} 0.0247-0.0494i-0.0192j-0.0126k & 0.0138+0.0139i+0.0176j+0.0023k \\ 0.0355+0.0165i-0.0128j+0.0133k & 0.0166-0.0264i-0.0050j-0.0166k \end{pmatrix},$$
$$(\mathcal{B}^{\#})^{(2)} = \begin{pmatrix} -0.0020+0.0069i+0.0190j+0.0629k & 0.0123+0.0041i+0.0033j-0.0000k \\ -0.0064-0.0088i+0.0020j-0.0218k & -0.0477-0.0614i-0.0392j+0.0337k \end{pmatrix},$$
$$(\mathcal{B}^{\#})^{(3)} = \begin{pmatrix} 0.0098+0.0042i-0.0769j-0.0050k & -0.0076-0.0029i-0.0125j-0.0117k \\ -0.0253+0.0374i+0.0201j+0.0242k & 0.0212+0.0148i-0.0029j+0.0100k \end{pmatrix}.$$

By calculation, we can have

$$\frac{||\mathcal{B}^{\#} - \mathcal{A}^{\#}||}{||\mathcal{A}^{\#}||} = 0.0540, \quad \frac{\| \mathcal{E} *_Q \mathcal{A}^{\#} \|}{1 - \| \mathcal{E} *_Q \mathcal{A}^{\#} \|} = 0.1033, \quad \frac{\| \mathcal{A} *_Q \mathcal{A}^{\#} \|}{(1 - \| \mathcal{E} *_Q \mathcal{A}^{\#} \|)^2} = 1.2173.$$

That is

$$\frac{\| \mathcal{B}^{\#} - \mathcal{A}^{\#} \|}{\| \mathcal{A}^{\#} \|} \leq \frac{\| \mathcal{E} *_Q \mathcal{A}^{\#} \|}{1 - \| \mathcal{E} *_Q \mathcal{A}^{\#} \|} + \frac{\| \mathcal{A} *_Q \mathcal{A}^{\#} \|}{(1 - \| \mathcal{E} *_Q \mathcal{A}^{\#} \|)^2}.$$

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