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UNITARILY INVARIANT NORMS ON DUAL QUATERNION MATRICES

Sheng Chen and Haofei Hu*

Abstract: Dual quaternion matrices have recently received significant attention in research. In this paper, we primarily investigate unitarily invariant norms of dual quaternion matrices. We first introduce symmetric gauge function on dual numbers and establish a one-to-one correspondence between unitarily invariant norms of dual quaternion matrices and the symmetric gauge function of their singular values. Next, we introduce Schatten p-norm and Fan k-norm on dual quaternion matrices. These are two important types of unitarily invariant norms. We also present several inequalities related to the Schatten p-norm and Fan k-norm.

Key words: *dual quaternion matrix, unitarily invariant norm, Schatten p-norm, Fan k-norm, singular value*

Mathematics Subject Classification: *15B33, 15A60, 15A18*

1 Introduction

Dual quaternions, originally introduced by Clifford in 1873 [4], constitute a fundamental element within Clifford algebra or geometric algebra. They represent an 8-dimensional real algebra that is isomorphic to the tensor product of quaternions and dual numbers. In the realm of mechanics, dual quaternions offer an elegant and concise means to unify the representation of translations and rotations for rigid transformations in three dimensions. Just as rotations in 3D space can be represented by unit quaternions, rigid motions in 3D space can be represented by unit dual quaternions.

With the rapid advancements in artificial intelligence, dual quaternions have found successful applications in various fields, including automatic control, computer vision, and bioengineering. For instance, they have been utilized in robotics control [6, 13, 18, 22, 23] and 3D computer graphics [2, 3, 19], as well as in the domains of neuroscience [9] and biomechanics [14]. Notably, Qi, Wang, and Luo demonstrated the significant role of dual quaternion matrices in multi-agent formation control [17]. In 2011, Wang raised this topic [21]. Additionally, an unpublished manuscript by Wang, Yu, and Zheng [24] introduced three classes of dual quaternion matrices designed for the study of multi-agent formation control, specifically relative configuration adjacency matrices, logarithm adjacency matrices, and relative twist adjacency matrices. Subsequently, Qi, Wang, and Luo [17] established that the relative configuration adjacency matrix and the logarithm adjacency matrix are both dual quaternion Hermitian matrices, and they introduced dual quaternion Laplacian

*Corresponding author.

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matrices. These dual quaternion matrices assume crucial roles in the context of multi-agent formation control.

In a previous study by Qi and Luo [16], the authors investigated the right and left eigenvalues of square dual quaternion matrices. They demonstrated that if a right eigenvalue is a dual number, it is also a left eigenvalue, and such a dual number is termed an eigenvalue of the dual quaternion matrix. Moreover, it was also shown that an n-by-n dual quaternion Hermitian matrix has exactly n eigenvalues. The singular value decomposition of dual quaternion matrices was also explored. Additionally, they proposed a unitary decomposition for dual quaternion Hermitian matrices. Further exploration of the eigenvalue theory of dual quaternion Hermitian matrices was carried out in the context of multi-agent formation control [17].

Furthermore,the minimax principle and generalized inverses of dual quaternion matrices were studied [11], along with the singular values and low-rank approximations of dual quaternion matrices [10]. Qi, Wang, and Luo [17] also demonstrated that the relative configuration adjacency matrix and the logarithm adjacency matrix are dual quaternion Hermitian matrices. These discoveries hold significant implications for multi-agent formation control and provide a fundamental theoretical basis for studying stability concerns in this area.

In another work, Cui and Qi [5] proposed a power method for computing the eigenvalues of a dual quaternion Hermitian matrix, which was applied to the simultaneous localization and mapping problem. Additionally, a dual quaternion version of Cauchy-Schwarz inequality was presented [11], and the dual quaternion version of the von Neumann trace inequality and Hoffman-Wielandt inequality were considered [12].

Unitarily invariant norm is pivotal in dual quaternion matrix research as it adeptly preserves rotational characteristics and ensures geometric consistency, particularly in scenarios involving rigid body motion, relative positions, and orientations. This unitarily invariant property of norms holds significant importance in diverse mathematical contexts as well. A well-known theorem concerning unitarily invariant norms over $\mathbb{C}$ states that they can be characterized by the symmetric gauge function of the singular values of their matrix argument, as seen in [20]. In this paper, our primary focus lies on unitarily invariant norms for dual quaternion matrices. Building upon the total order of dual numbers introduced in a previous work [15], we first extend the concept of the symmetric gauge function to dual quaternions and demonstrate its equivalence with the case of dual numbers. Utilizing the von Neumann trace inequality introduced in [12], we establish that the symmetric gauge function of the singular values of dual quaternion matrices satisfies the triangle inequality. As a result, we are able to provide a characterization of the unitarily invariant property of norms on dual quaternion matrices.

Moreover, we define the exponentiation of a positive appreciable dual number to a real number power, which serves as a fundamental component for considering the Schatten p-norm and the dual number version of the Hölder inequality. Consequently, we introduce the definitions of the Schatten p-norm and the Fan k-norm, both of which are vital unitarily invariant norms. Additionally, we present several inequalities related to these norms, including Fan dominance principle for dual quaternion matrices, the dual number version of the Hölder inequality and the Hölder inequality for dual quaternion matrices.

The rest of this paper is organized as follows: In Section 2, we provide a comprehensive review of quaternions, dual numbers, dual quaternions, and dual quaternion matrices. In Section 3, we introduce the symmetric gauge function on dual quaternions and discuss inequalities related to the singular values of dual quaternion matrices. In Section 4, we utilize the symmetric gauge function to characterize the unitarily invariant property of norms of dual quaternion matrices. Additionally, we introduce the definition of exponentiation of a positive appreciable dual number to a real number power, which is crucial for establishing two important contexts - Schatten p-norm and Hölder inequality. Finally, in Section 5, we present norm inequalities that are relevant to the Schatten p-norm and Fan k-norm.

2 Preliminaries

In this section, we will provide a review of notations related to quaternions, dual numbers, dual quaternions, and dual quaternion matrices. For more comprehensive details, please refer to the references [10, 15].

2.1 Quaternions

Let $\mathbb{R}$ be the real field, and denote $\mathbb{Q}$ as the set of quaternions. A quaternion $q \in \mathbb{Q}$ has the following form:

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k},$$

where $q_0, q_1, q_2, q_3 \in \mathbb{R}$, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are three imaginary units of quaternions. These units satisfy $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ijk} = -\mathbf{ikj} = \mathbf{jik} = -\mathbf{jki} = \mathbf{kij} = -\mathbf{kji} = \mathbf{ijk} = -1$. In fact, $\mathbb{Q}$ is a 4-dimensional associative but non-commutative algebra over $\mathbb{R}$.

The real part of $q \in \mathbb{Q}$ is $\mathrm{Re}(q) = q_0$, and the imaginary part of $q \in \mathbb{Q}$ is $\mathrm{Im}(q) = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$. The conjugate of q is denoted as $q^* = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$. The magnitude of q is defined as $|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^3} = \sqrt{qq^*}$.

An important fact is that for given $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ and $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, we have

$$pq^* + qp^* = p^*q + q^*p = 2(p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3). \tag{2.1}$$

This expression is a real number.

2.2 Dual Numbers

Denote $\mathbb{D}$ as the set of dual numbers. A dual number $q \in \mathbb{D}$ has the following form:

$$q = q_{\mathrm{st}} + q_{\mathcal{I}}\epsilon,$$

where $q_{\mathrm{st}} \in \mathbb{R}$ is called the real part or standard part of q, $q_{\mathcal{I}} \in \mathbb{R}$ is called the dual part or infinitesimal part of q, ϵ is the infinitesimal unit satisfying $\epsilon \neq 0$, $\epsilon^2 = 0$, and ϵ commutes with quaternions during multiplication. If $q_{\mathrm{st}} \neq 0$, then q is appreciable; otherwise, q is infinitesimal [15].

Given $p, q \in \mathbb{D}$, the addition of p and q is defined as

$$p + q = p_{\mathrm{st}} + q_{\mathrm{st}} + (p_{\mathcal{I}} + q_{\mathcal{I}})\epsilon,$$

and the multiplication of p and q is defined as

$$pq = p_{\mathrm{st}}q_{\mathrm{st}} + (p_{\mathrm{st}}q_{\mathcal{I}} + p_{\mathcal{I}}q_{\mathrm{st}})\epsilon,$$

and the division of p and q is defined as

$$\frac{q_{\mathrm{st}} + q_{\mathcal{I}}\epsilon}{p_{\mathrm{st}} + p_{\mathcal{I}}\epsilon} = \begin{cases} \frac{q_{\mathrm{st}}}{p_{\mathrm{st}}} + \left(\frac{q_{\mathcal{I}}}{p_{\mathrm{st}}} - \frac{q_{\mathrm{st}}}{p_{\mathrm{st}}}\frac{p_{\mathcal{I}}}{p_{\mathrm{st}}}\right)\epsilon, & \text{if } p_{\mathrm{st}} \neq 0, \\ \frac{q_{\mathcal{I}}}{p_{\mathcal{I}}} + k\epsilon, & \text{if } p_{\mathrm{st}} = 0 \text{ and } q_{\mathrm{st}} = 0, \end{cases}$$

where $k \in \mathbb{R}$ is an arbitrary real number. In particular, we have

$$q^k = q_{\rm st}^k + k q_{\rm st}^{k-1} q_{\mathcal{I}} \epsilon. \tag{2.2}$$

If q is appreciable, then q is invertible and $q^{-1} = q_{\rm st}^{-1} - q_{\rm st}^{-1} q_{\mathcal{I}} q_{\rm st}^{-1} \epsilon$. In fact, $\mathbb{D}$ is a commutative algebra of dimension 2 over $\mathbb{R}$.

In [15], a total order of dual numbers was introduced. If $p_{\rm st} > q_{\rm st}$, or $p_{\rm st} = q_{\rm st}$ and $p_{\mathcal{I}} > q_{\mathcal{I}}$, then $p > q$. If $p_{\rm st} = q_{\rm st}$ and $p_{\mathcal{I}} = q_{\mathcal{I}}$, then $p = q$.

If $q \geq 0$, then we call q a nonnegative dual number and denote the set of nonnegative dual numbers by $\mathbb{D}_+$. If $q \in \mathbb{D}_+$ and is appreciable, the square root of q is defined as

$$\sqrt{q} = \sqrt{q_{\rm st}} + \frac{q_{\mathcal{I}}}{2\sqrt{q_{\rm st}}} \epsilon.$$

In particular, $\sqrt{q} = 0$ when $q = 0$.

The absolute value of $q \in \mathbb{D}$ is defined as

$$|q| = \begin{cases} |q_{\rm st}| + \operatorname{sgn}(q_{\rm st}) q_{\mathcal{I}} \epsilon, & \text{if } q_{\rm st} \neq 0, \\ |q_{\mathcal{I}}| \epsilon, & \text{otherwise}, \end{cases}$$

where sgn is the sign function.

Remark 2.1. While the symbols for the absolute value function of a dual number and the magnitude of a quaternion are the same, their distinct meanings can be easily discerned from the context. Given a nonzero quaternion q, the inverse of q can be expressed as $q^{-1} = \frac{q^*}{|q|^2}$.

There are some fundamental operational properties of the absolute value of dual numbers.

Theorem 2.2 (Theorem 2 in [15])**.** *The mapping* $|\cdot| : \mathbb{D} \to \mathbb{D}_+$. *Suppose that* $p, q \in \mathbb{D}$. *Then*

i) $|q| = 0$ *if and only if* $q = 0$*;*

ii) $|q| = q$ *if* $q \geq 0$, $|q| > q$ *otherwise;*

iii) $|q| = \sqrt{q^2}$ *if* q *is appreciable;*

iv) $|pq| = |p||q|$*;*

v) $|p + q| \leq |p| + |q|$.

2.3 Dual Quaternions

Denote $\widehat{\mathbb{Q}}$ as the set of dual quaternions. A dual quaternion $q \in \widehat{\mathbb{Q}}$ has the following form:

$$q = q_{\rm st} + q_{\mathcal{I}} \epsilon,$$

where $q_{\rm st} \in \mathbb{Q}$ is the standard part of q, and $q_{\mathcal{I}} \in \mathbb{Q}$ is the infinitesimal part of q. If $q_{\rm st} \neq 0$, then q is appreciable; otherwise, q is infinitesimal.

The conjugate of $q \in \widehat{\mathbb{Q}}$ is defined as:

$$q^* = q_{\rm st}^* + q_{\mathcal{I}}^* \epsilon.$$

Using (2.1), qq^* is a dual number, and the magnitude of q is defined as:

$$|q| = \begin{cases} |q_{\rm st}| + \frac{q_{\rm st}q_{\mathcal{I}}^* + q_{\mathcal{I}}q_{\rm st}^*}{2|q_{\rm st}|}\epsilon, & \text{if } q_{\rm st} \neq 0, \\ |q_{\mathcal{I}}|\epsilon, & \text{otherwise.} \end{cases}$$

This magnitude is also a dual number. Regarding the properties of the magnitude of dual quaternions, we have the following result

Theorem 2.3 (Theorem 4 in [15]). *The magnitude $|q|$ is a dual number for any $q \in \widehat{\mathbb{Q}}$. If q is appreciable, then*

$$|q| = \sqrt{qq^*}.$$

For any $p, q \in \widehat{\mathbb{Q}}$, we have

i) $qq^* = q^*q$;

ii) $|q| = |q^*|$;

iii) $|q| \geq 0$ *for all* q, *and* $|q| = 0$ *if and only if* $q = 0$;

iv) $|pq| = |p||q|$;

v) $|p + q| \leq |p| + |q|$.

2.4 Dual Quaternion Matrices

Denote the set of $m \times n$ dual quaternion matrices as $\widehat{\mathbb{Q}}^{m\times n}$. A dual quaternion matrix A can be written as $A = A_{\rm st} + A_{\mathcal{I}}\epsilon$, where $A_{\rm st}, A_{\mathcal{I}} \in \mathbb{Q}^{m\times n}$ are the standard part and the infinitesimal part of A, respectively. If $A_{\rm st} \neq O$, we call A appreciable; otherwise, A is infinitesimal. The conjugate transpose of A is defined as $A^* = A_{\rm st}^* + A_{\mathcal{I}}^*\epsilon$. If $A^* = A$, then we call A a dual quaternion Hermitian matrix, with both its standard part and infinitesimal part being quaternion Hermitian matrices. A square matrix $A \in \widehat{\mathbb{Q}}^{m\times m}$ is called unitary, if A satisfies $A^*A = I_m$. Similarly, we say $A \in \widehat{\mathbb{Q}}^{m\times n}$ with $m \geq n$ is partially unitary, if A satisfies $A^*A = I_n$. The Frobenius norm of A is defined as

$$\|A\|_F = \begin{cases} \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |A(i,j)|^2}, & \text{if } A_{\rm st} \neq O, \\ \|A_{\mathcal{I}}\|_F\epsilon, & \text{otherwise.} \end{cases}$$

Given $A \in \widehat{\mathbb{Q}}^{m\times n}$ and $B \in \widehat{\mathbb{Q}}^{n\times m}$, we have $(AB)^* = B^*A^*$. However, in general, $\mathrm{tr}(AB) \neq \mathrm{tr}(BA)$, where $\mathrm{tr}(\cdot)$ is the trace function on dual quaternion square matrices, defined as the sum of diagonal entries.

For a dual quaternion square matrix $A \in \widehat{\mathbb{Q}}^{m\times m}$, if there exists a dual quaternion $\lambda \in \widehat{\mathbb{Q}}$ and an appreciable $\mathbf{x} \in \widehat{\mathbb{Q}}^m$ such that $A\mathbf{x} = \mathbf{x}\lambda$, then we call λ a right eigenvalue of A, with $\mathbf{x}$ as the associated right eigenvector. If λ is also a left eigenvalue, that is, $A\mathbf{x} = \lambda\mathbf{x}$, then we say that λ is an eigenvalue of A. It can be observed that if the right eigenvalue λ is a dual number, then it is an eigenvalue. The following results show that an $m \times m$ dual quaternion Hermitian matrix has exactly m eigenvalues, all of which are dual numbers:

Theorem 2.4 (Theorem 4.1 in [16]). *Let $A \in \widehat{\mathbb{Q}}^{m\times m}$ be an Hermitian matrix. Then there exists a unitary matrix $U \in \widehat{\mathbb{Q}}^{m\times m}$ and a diagonal matrix $\Sigma \in \mathbb{D}^{m\times m}$ such that $A = U\Sigma U^*$, where*

$$\Sigma = \mathrm{diag}(\lambda_1 + \lambda_{1,1}\epsilon, \ldots, \lambda_1 + \lambda_{1,k_1}\epsilon, \lambda_2 + \lambda_{2,1}\epsilon, \ldots, \lambda_r + \lambda_{r,k_r}\epsilon),$$

and $\lambda_1 > \lambda_2 > \ldots > \lambda_r$ *are real numbers,* λ_i *is a* k_i*-multiple right eigenvalue of* $A_{\rm st}$*,* $\lambda_{i,1} \geq \lambda_{i,2} \geq \ldots \geq \lambda_{i,k_i}$ *are also real numbers. Counting possible multiplicities* $\lambda_{i,j}$*, the form* Σ *is unique.*

By Theorem 2.4, we have $\mathrm{tr}(A) = \mathrm{tr}(UAU^*) = \sum_{i=1}^{m} \lambda_i(A)$ for a Hermitian matrix $A \in \widehat{\mathbb{Q}}^{m\times m}$ and any unitary matrix $U \in \widehat{\mathbb{Q}}^{m\times m}$.

For a general dual quaternion matrix, there is a result regarding singular value decomposition:

Theorem 2.5 (Theorem 6.1 in [16])**.** *For given* $A \in \widehat{\mathbb{Q}}^{m\times n}$*, there exists a dual quaternion unitary matrix* $U \in \widehat{\mathbb{Q}}^{m\times m}$ *and a dual quaternion unitary matrix* $V \in \widehat{\mathbb{Q}}^{n\times n}$*, such that*

$$A = U \begin{pmatrix} \Sigma_t & O \\ O & O \end{pmatrix} V^*,$$

where $\Sigma_t \in \widehat{\mathbb{Q}}^{t\times t}$ *is a diagonal matrix, taking the form* $\Sigma_t = \mathrm{diag}(\mu_1, \ldots, \mu_r, \ldots, \mu_t)$*,* $r \leq t \leq \min\{m, n\}$*,* $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r$ *are positive appreciable dual numbers, and* $\mu_{r+1} \geq \mu_{r+2} \geq \ldots \geq \mu_t$ *are positive infinitesimal dual numbers. Counting possible multiplicities of the diagonal entries, the form* Σ_t *is unique.*

In this paper, we denote a sequence $s(A)$ as the singular values of the dual quaternion matrix A:

$$s(A) = (s_1(A), s_2(A), \ldots, s_{\min\{m,n\}}(A)),$$

where $s_1(A) \geq s_2(A) \geq \cdots \geq s_{\min\{m,n\}}(A) \geq 0$. It is clear that $s(A) \in \mathbb{D}_+^{\min\{m,n\}}$.

The spectral norm [12] of a dual quaternion matrix A is defined as the largest singular value of A, namely $s_1(A)$. Notably, both the Frobenius norm and the spectral norm of dual quaternion matrices are unitarily invariant norms.

3 Symmetric Gauge Function

Given $x \in \widehat{\mathbb{Q}}^n$, we define $|x| \in \mathbb{D}^n$ to be the coordinate-wise magnitude of x, that is,

$$|x| = (|x_1|, |x_2|, \cdots, |x_n|).$$

We write $|x| \leq |y|$ for $x, y \in \widehat{\mathbb{Q}}^n$ if $|x_j| \leq |y_j|$ for each j.

Definition 3.1. A norm on $\widehat{\mathbb{Q}}^n$ is a dual number-valued function $\|\cdot\| : \widehat{\mathbb{Q}}^n \to \mathbb{D}$ with the following properties:

(1) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \widehat{\mathbb{Q}}^n$.

(2) $\|x \cdot q\| = |q| \cdot \|x\|$ for all $x \in \widehat{\mathbb{Q}}^n$ and $q \in \widehat{\mathbb{Q}}$.

(3) $\|x\| \geq 0$ for all $x \in \widehat{\mathbb{Q}}^n$ and $\|x\| = 0$ if and only if $x = 0$.

A norm $\|\cdot\|$ on $\widehat{\mathbb{Q}}^n$ is called monotone if, whenever $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. A norm $\|\cdot\|$ on $\widehat{\mathbb{Q}}^n$ is called absolute if $\|x\| = \||x|\|$ for each $x \in \widehat{\mathbb{Q}}^n$.

Next, we will demonstrate an important property of the symmetric gauge function, which will be utilized to establish the positiveness of the symmetric gauge function of the singular values of dual quaternion matrices in Section 4.

Proposition 3.2. *A norm on $\widehat{\mathbb{Q}}^n$ is monotone if and only if it is absolute.*

Proof. Suppose that $\|\cdot\|$ is a monotone norm. Given $x \in \widehat{\mathbb{Q}}^n$, let $y = |x|$, then we have $|y| = |x|$. By monotonicity, we have $\|x\| \le \|y\|$ and $\|y\| \le \|x\|$. Therefore, $\|\cdot\|$ is absolute.

Now suppose that $\|\cdot\|$ is an absolute norm. Let $x \in \mathbb{D}^n$. Taking $q \in \mathbb{D}$ with $|q| \le 1$, we have $|q_{\mathrm{st}}| \le 1$. By the absolute norm property, we have

$$\begin{aligned}\|(x_1, \dots, x_j q, \dots, x_n)\| &= \left\| \frac{1+q}{2} x + \frac{1-q}{2}(x_1, \dots, -x_j, \dots, x_n) \right\| \\ &\le \left| \frac{1+q}{2} \right| \cdot \|x\| + \left| \frac{1-q}{2} \right| \cdot \|(x_1, \dots, -x_j, \dots, x_n)\| \\ &= \left| \frac{1+q}{2} \right| \cdot \|x\| + \left| \frac{1-q}{2} \right| \cdot \||(x_1, \dots, -x_j, \dots, x_n)|\| \\ &= \left(\left| \frac{1+q}{2} \right| + \left| \frac{1-q}{2} \right| \right) \|x\|.\end{aligned}$$

When $|q_{\mathrm{st}}| = 1$, we have $q = \pm 1 \cdot (1 - k\epsilon)$, where k is a nonnegative real number. If $q = 1 - k\epsilon$, we have

$$\begin{aligned}\left(\left| \frac{1+q}{2} \right| + \left| \frac{1-q}{2} \right| \right) \|x\| &= \frac{1}{2}(|2 - k\epsilon| + |k\epsilon|)\|x\| \\ &= \frac{1}{2}(2 - k\epsilon + k\epsilon)\|x\| \\ &= \|x\|.\end{aligned}$$

If $q = -(1 - k\epsilon)$, we also have

$$\left(\left| \frac{1+q}{2} \right| + \left| \frac{1-q}{2} \right| \right) \|x\| = \frac{1}{2}(|k\epsilon| + |2 - k\epsilon|)\|x\| = \|x\|.$$

When $|q_{\mathrm{st}}| < 1$, we have

$$\begin{aligned}&\left(\left| \frac{1+q}{2} \right| + \left| \frac{1-q}{2} \right| \right) \|x\| \\ &= \frac{1}{2}[(|1 + q_{\mathrm{st}}| + \mathrm{sgn}(1 + q_{\mathrm{st}}) q_{\mathcal{I}}\epsilon) + (|1 - q_{\mathrm{st}}| + \mathrm{sgn}(1 - q_{\mathrm{st}})(-q_{\mathcal{I}}\epsilon))]\|x\| \\ &= \frac{1}{2}(1 + q_{\mathrm{st}} + 1 - q_{\mathrm{st}})\|x\| = \|x\|.\end{aligned}$$

Therefore, we have

$$\|(x_1, \dots, x_j q, \dots, x_n)\| \le \|x\|$$

for each j. Iterating this implies that $\|\cdot\|$ is a monotone norm on $\mathbb{D}^n$.

If $x \in \widehat{\mathbb{Q}}^n$ and $q \in \widehat{\mathbb{Q}}$ with $|q| \le 1$, by the absolute norm property, we have

$$\|(x_1, \dots, x_j q, \dots, x_n)\| = \||(x_1, \dots, x_j q, \dots, x_n)|\| = \|(|x_1|, \dots, |x_j q|, \dots, |x_n|)\|.$$

Therefore, by Theorem 2.3, we have $|x_j q| = |x_j||q|$, and by the result that $\|\cdot\|$ is a monotone norm on $\mathbb{D}^n$, we have $\|\cdot\|$ is a monotone norm on $\widehat{\mathbb{Q}}^n$. That is,

$$\begin{aligned}\|(x_1, \dots, x_j q, \dots, x_n)\| &= \||(x_1, \dots, x_j q, \dots, x_n)|\| \\ &= \|(|x_1|, \dots, |x_j||q|, \dots, |x_n|)\| \\ &\le \|(|x_1|, \dots, |x_j|, \dots, |x_n|)\| \\ &= \|x\|.\end{aligned}$$

□

Definition 3.3. A norm $\|\cdot\|$ on $\widehat{\mathbb{Q}}^n$ is called a symmetric gauge function if it is an absolute norm and satisfies $\|x\| = \|Px\|$ for every $x \in \widehat{\mathbb{Q}}^n$ and permutation matrix P, where a permutation matrix P is defined as an $n \times n$ square binary matrix with exactly one entry of 1 in each row and each column and 0s elsewhere.

Based on the proof of Proposition 3.2, we observe that symmetric gauge functions on $\mathbb{D}^n$ and $\widehat{\mathbb{Q}}^n$ are essentially identical. Therefore, we will focus on symmetric gauge functions on $\mathbb{D}^n$ for the subsequent analysis. Additionally, we present a proposition on the symmetric gauge function on $\mathbb{D}^n$ without providing a proof, which is similar to the case of real numbers as shown on Page 45 in [1].

Proposition 3.4. *Suppose that $\|\cdot\|$ is a symmetric gauge function on $\mathbb{D}^n$. If $x, y \in \mathbb{D}^n_+$ and $x \prec_w y$, then $\|x\| \leq \|y\|$.*

Theorem 3.5 (Theorem 4.4 in [12])**.** *For any $A, B \in \widehat{\mathbb{Q}}^{m\times n}$, it holds that*

$$\operatorname{tr}(A^*B + B^*A) \leq 2 \sum_{i=1}^{\min\{m,n\}} s_i(A)s_i(B),$$

where $s_1(A) \geq s_2(A) \geq \ldots \geq s_{\min\{m,n\}}(A) \geq 0$ and $s_1(B) \geq s_2(B) \geq \ldots \geq s_{\min\{m,n\}}(B) \geq 0$ are singular values of A and B, respectively.

Proposition 3.6 (Proposition 4.5 in [12])**.** *Let $A \in \widehat{\mathbb{Q}}^{m\times n}$. We have*

$$\max_{X\in\mathbb{P}\widehat{\mathbb{Q}}^{m\times n}} \operatorname{tr}(A^*X + X^*A) = 2 \sum_{i=1}^{\min\{m,n\}} s_i(A),$$

where $\mathbb{P}\widehat{\mathbb{Q}}^{m\times n} = \{X \in \widehat{\mathbb{Q}}^{m\times n} | XX^* = I_m\}$ and $s_i(A)$ is the i-th singular value of A for $1 \leq i \leq \min\{m, n\}$.

Given $x, y \in \mathbb{D}^n$, if

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad 1 \leq k \leq n,$$

then we say that x is weakly majorized by y, denoted as $x \prec_w y$. Furthermore, if

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,$$

we say that x is majorized by y, denoted as $x \prec y$. With respect to the sum of singular values of dual quaternion matrices, we have the following result:

Lemma 3.7. *For any $A, B \in \widehat{\mathbb{Q}}^{m\times n}$, we have*

$$s(A + B) \prec_w s(A) + s(B).$$

Proof. Denote $\mathbb{P}\widehat{\mathbb{Q}}^{m\times n}_k$ as the set of $X \in \widehat{\mathbb{Q}}^{m\times n}$, where X satisfies that the singular values of X are $s_1(X) = \cdots = s_k(X) = 1$ and $s_{k+1}(X) = \cdots = s_{\min\{m,n\}}(X) = 0$.

Given $A \in \widehat{\mathbb{Q}}^{m\times n}$, by Theorem 3.5, we have $\operatorname{tr}(A^*B + B^*A) \le 2\sum_{i=1}^{\min\{m,n\}} s_i(A)s_i(B) = 2\sum_{i=1}^{k} s_i(A)$ for $B \in \mathbb{P}\widehat{\mathbb{Q}}_k^{m\times n}$. Let $A = U\Sigma V^*$ be the SVD of A. Similar to the proof of Proposition 3.6, taking $\widetilde{X} = USV^*$, where $S = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$, we have $\widetilde{X} \in \mathbb{P}\widehat{\mathbb{Q}}_k^{m\times n}$ and $\operatorname{tr}(A^*\widetilde{X} + \widetilde{X}^*A) = 2\sum_{i=1}^{k} s_i(A)$. Thus, we have

$$\max_{X\in\mathbb{P}\widehat{\mathbb{Q}}_k^{m\times n}} \operatorname{tr}(A^*X + X^*A) = 2\sum_{i=1}^{k} s_i(A).$$

Therefore, we have

$$\begin{aligned}\sum_{i=1}^{k} s_i(A+B) &= \frac{1}{2}\max_{X\in\mathbb{P}\widehat{\mathbb{Q}}_k^{m\times n}} \operatorname{tr}((A+B)^*X + X^*(A+B)) \\ &= \frac{1}{2}\max_{X\in\mathbb{P}\widehat{\mathbb{Q}}_k^{m\times n}} (\operatorname{tr}(A^*X + X^*A) + \operatorname{tr}(B^*X + X^*B)) \\ &\le \frac{1}{2}\max_{X\in\mathbb{P}\widehat{\mathbb{Q}}_k^{m\times n}} \operatorname{tr}(A^*X + X^*A) + \frac{1}{2}\max_{X\in\mathbb{P}\widehat{\mathbb{Q}}_k^{m\times n}} \operatorname{tr}(B^*X + X^*B) \\ &= \sum_{i=1}^{k} s_i(A) + \sum_{i=1}^{k} s_i(B)\end{aligned}$$

for $k = 1, 2, \ldots, \min\{m, n\}$. So we have

$$s(A+B) \prec_w s(A) + s(B).$$

□

Theorem 3.8. *Suppose that* $\|\cdot\|$ *is a symmetric gauge function on* $\mathbb{D}^{\min\{m,n\}}$. *If* $A, B \in \widehat{\mathbb{Q}}^{m\times n}$, *then*

$$\|s(A+B)\| \le \|s(A) + s(B)\| \le \|s(A)\| + \|s(B)\|.$$

Proof. This follows immediately from Proposition 3.4 and Lemma 3.7. □

4 Unitarily Invariant Norms

A norm $\|\cdot\|$ on $\widehat{\mathbb{Q}}^{m\times n}$ is called unitarily invariant if $\|UAV^*\| = \|A\|$ for every $A \in \widehat{\mathbb{Q}}^{m\times n}$, and unitary matrices $U \in \widehat{\mathbb{Q}}^{m\times m}$, $V \in \widehat{\mathbb{Q}}^{n\times n}$. Unitarily invariant norms are particularly valuable as they satisfy submultiplicativity and other desirable properties.

The following theorem states that any unitarily invariant norm on $\widehat{\mathbb{Q}}^{m\times n}$ can be represented in terms of a norm of the sequence of singular values $s(A) \in \mathbb{D}_+^{\min\{m,n\}}$.

Theorem 4.1. *Suppose that* $\|\cdot\|$ *is a unitarily invariant norm on* $\widehat{\mathbb{Q}}^{m\times n}$. *There exists a symmetric gauge function on* $\mathbb{D}^{\min\{m,n\}}$, *also denoted by* $\|\cdot\|$, *such that*

$$\|A\| = \begin{cases} \|s(A)\|, & \text{if } A_{\rm st} \neq 0, \\ \|s(A_{\mathcal{I}})\|\epsilon, & \text{otherwise}. \end{cases}$$

Conversely, given a symmetric gauge function on $\mathbb{D}^{\min\{m,n\}}$*, the formula*

$$\|A\| = \begin{cases} \|s(A)\|, & \textit{if } A_{\rm st} \neq 0, \\ \|s(A_{\mathcal{I}})\|\epsilon, & \text{otherwise}, \end{cases}$$

defines a unitarily invariant norm on $\widehat{\mathbb{Q}}^{m\times n}$*.*

Proof. "$\Longrightarrow$" Suppose $\|\cdot\|$ is a unitarily invariant norm on $\widehat{\mathbb{Q}}^{m\times n}$. Without loss of generality, we assume $m \le n$.

For any $x \in \mathbb{D}^m$, there exists a matrix $A \in \widehat{\mathbb{Q}}^{m\times n}$, and unitary matrices $U \in \widehat{\mathbb{Q}}^{m\times m}$ and $V \in \widehat{\mathbb{Q}}^{n\times n}$ such that

$$M(|x|) = \begin{pmatrix} |x_1| & \cdots & 0 & O \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & |x_m| & O \end{pmatrix} = UAV^* \in \mathbb{D}^{m\times n}.$$

Assuming x is appreciable, we define $\|x\| = \|M(x)\|$. All the properties of a norm follow easily. This essentially amounts to observing that the restriction of a norm to a subspace is still a norm.

By Theorem 2.2, we have $|x_i| = x_i$ if $x_i \ge 0$ and $|x_i| = -x_i$ if $x_i < 0$ for $x_i \in \mathbb{D}$. Let $D = \operatorname{diag}\{d_1, d_2, \dots, d_m\}$, where

$$d_j = \begin{cases} 1, & \text{if } x_i \ge 0, \\ -1, & \text{if } x_i < 0. \end{cases}$$

D is a unitary matrix, and we have

$$\||x|\| = \|M(|x|)\| = \|DM(x)\| = \|M(x)\| = \|x\|.$$

Thus, $\|\cdot\|$ is absolute on $\mathbb{D}^m$.

For any permutation matrix P, taking $Q = \begin{pmatrix} P & 0 \\ 0 & I_{n-m} \end{pmatrix}$, we have Q is a unitary matrix, and

$$\|Px\| = \|M(Px)\| = \|PM(x)Q\| = \|M(x)\| = \|x\|.$$

Therefore, when x is appreciable, $\|\cdot\|$ is a symmetric gauge function on $\mathbb{D}^m$.

Assuming x is infinitesimal, we define $\|x_{\mathcal{I}}\|\epsilon = \|M(x_{\mathcal{I}})\|\epsilon = \|M(x)\|$. The argument is similar to the case when x is appreciable. Thus, $\|\cdot\|$ is a symmetric gauge function on $\mathbb{D}^m$, such that $\|A\| = \|s(A)\|$.

"$\Longleftarrow$" Now given a symmetric gauge function on $\mathbb{D}^m$, we define the norm $\|A\|$ as follows:

$$\|A\| = \begin{cases} \|s(A)\|, & \text{if } A_{\rm st} \neq 0, \\ \|s(A_{\mathcal{I}})\|\epsilon, & \text{otherwise}. \end{cases}$$

By Proposition 3.2, which shows the symmetric gauge function is monotone, we have $\|A\| \ge 0$, and $\|A\| = 0$ if and only if $A = 0$.

Now we will prove the property of absolute homogeneity for this norm:

Case 1. A is appreciable.

For any $q \in \widehat{\mathbb{Q}}$, if $q_{\rm st} \neq 0$, then by $(Aq)\cdot(Aq)^* = |q|^2 \cdot AA^*$, we have $s(Aq) = |q|\cdot s(A)$, thus

$$\|Aq\| = \|s(Aq)\| = \||q|\cdot s(A)\| = |q|\cdot\|s(A)\| = |q|\cdot\|A\|.$$

If $q_{\mathrm{st}} = 0$, let $A = U\Sigma V^*$ be the singular value decomposition of A, by computation, we have $A_{\mathrm{st}} = U_{\mathrm{st}}\Sigma_{\mathrm{st}}V_{\mathrm{st}}^*$, which means that the appreciable part of $\|s(A)\|$ equals $\|s(A_{\mathrm{st}})\|$, then we have $\|s(A_{st})\| = \|s(A)\| + k\epsilon$ for some $k \in \mathbb{R}$. Thus we have

$$\begin{aligned}\|Aq\| &= \|s(A_{\mathrm{st}}q_{\mathcal{I}})\|\epsilon = \||q_{\mathcal{I}}| \cdot s(A_{\mathrm{st}})\|\epsilon = |q_{\mathcal{I}}| \cdot \|s(A_{\mathrm{st}})\|\epsilon = |q_{\mathcal{I}}| \cdot (\|s(A)\| + k\epsilon)\epsilon \\ &= |q_I| \cdot \|s(A)\|\epsilon = |q| \cdot \|A\|.\end{aligned}$$

Case 2. A is infinitesimal.

For any $q \in \widehat{\mathbb{Q}}$, if $q_{\mathrm{st}} \neq 0$, then we have

$$\begin{aligned}\|Aq\| &= \|s(A_{\mathcal{I}}q_{\mathrm{st}})\|\epsilon = \||q_{\mathrm{st}}| \cdot s(A_{\mathcal{I}})\|\epsilon = |q_{\mathrm{st}}| \cdot \|s(A_{\mathcal{I}})\|\epsilon \\ &= \left(|q| - \frac{q_{\mathrm{st}}q_{\mathcal{I}}^* + q_{\mathcal{I}}q_{\mathrm{st}}^*}{2|q_{\mathrm{st}}|}\epsilon\right) \cdot \|s(A_{\mathcal{I}})\|\epsilon = |q| \cdot \|s(A_{\mathcal{I}})\|\epsilon = |q| \cdot \|A\|.\end{aligned}$$

If $q_{\mathrm{st}} = 0$, $Aq = 0$,

$$\|Aq\| = 0 = |q_{\mathcal{I}}|\epsilon \cdot \|s(A_{\mathcal{I}})\|\epsilon = |q| \cdot \|A\|.$$

Hence, the property of absolute homogeneity holds.

By Theorem 3.8, which guarantees the triangle inequality for $\|\cdot\|$ on $\widehat{\mathbb{Q}}^{m\times n}$, we conclude that $\|\cdot\|$ is a norm on $\widehat{\mathbb{Q}}^{m\times n}$.

Taking unitary matrices $U \in \widehat{\mathbb{Q}}^{m\times m}$, $V \in \widehat{\mathbb{Q}}^{n\times n}$, we have $s(A) = s(UAV^*)$. Therefore

$$\|A\| = \|UAV^*\|.$$

Thus, $\|\cdot\|$ on $\widehat{\mathbb{Q}}^{m\times n}$ is unitarily invariant. □

Next, we will define the exponentiation of a nonnegative appreciable dual number to a real number power, which plays a crucial role in establishing Schatten p-norm and Hölder inequality. Following that, we will introduce two important unitarily invariant norms: the Schatten p-norm and the Fan k-norm.

Proposition 4.2. *Suppose that u, v are positive integers, if $q \in \mathbb{D}_+$ is appreciable. Then we have*

$$q^{\frac{v}{u}} = q_{\mathrm{st}}^{\frac{v}{u}} + \frac{v}{u}q_{\mathrm{st}}^{\frac{v-u}{u}}q_{\mathcal{I}}\epsilon.$$

Proof. Let $z = x + y\epsilon \in \mathbb{D}$. Then by equation (2.2), we have

$$z^u = x^u + ux^{u-1}y\epsilon.$$

If z is a u-th root of appreciable $q \in \mathbb{D}_+$, then we have

$$\begin{cases} x^u = q_{\mathrm{st}}, \\ ux^{u-1}y = q_{\mathcal{I}}. \end{cases}$$

Thus we have

$$q^{\frac{1}{u}} = \sqrt[u]{q_{\mathrm{st}}} + \frac{q_{\mathcal{I}}}{u\sqrt[u]{q_{\mathrm{st}}^{u-1}}}\epsilon.$$

By equation (2.2) again, we have

$$q^{\frac{v}{u}} = q_{\mathrm{st}}^{\frac{v}{u}} + \frac{v}{u}q_{\mathrm{st}}^{\frac{v-u}{u}}q_{\mathcal{I}}\epsilon.$$

□

In particular, if $q = 0$, we have $q^{\frac{v}{u}} = 0$ for any positive integers u and v. If q is a positive appreciable dual number, then $q^{\frac{v}{u}}$ is appreciable for any positive integers u and v. Thus, we have $q^{-\frac{v}{u}} = (q^{\frac{v}{u}})^{-1} = q_{\rm st}^{-\frac{v}{u}} - \frac{v}{u} q_{\rm st}^{\frac{-v-u}{u}} q_{\mathcal{I}}\epsilon$.

We can naturally extend the definition of exponentiation of a nonnegative appreciable dual number from rational number powers to real number powers using the framework established for rational exponents.

Definition 4.3. Let q be a positive appreciable dual number. If $\alpha = \frac{v}{u}$, where u, v are integers, we have

$$q^{\alpha} = q_{\rm st}^{\frac{v}{u}} + \frac{v}{u} q_{\rm st}^{\frac{v-u}{u}} q_{\mathcal{I}}\epsilon.$$

If α is an irrational number, there exists a sequence of rational numbers $\{r_n\}$ such that $\lim\limits_{n\to\infty} r_n = \alpha$, define

$$q^{\alpha} = \lim_{n\to\infty} q^{r_n} = q_{\rm st}^{\alpha} + \alpha q_{\rm st}^{\alpha-1} q_{\mathcal{I}}\epsilon.$$

Now, we can give the definition of Schatten p-norm.

Definition 4.4. For $1 \le p < \infty$, the Schatten p-norm of $A \in \widehat{\mathbb{Q}}^{m\times n}$ is given by

$$\|A\|_p = \begin{cases} \left(\sum\limits_{j=1}^{\min\{m,n\}} s_j(A)^p \right)^{1/p}, & \text{if } A_{\rm st} \neq 0, \\ \left(\sum\limits_{j=1}^{\min\{m,n\}} s_j(A_I)^p \right)^{1/p} \epsilon, & \text{otherwise.} \end{cases}$$

Remark 4.5. Given $A \in \widehat{\mathbb{Q}}^{m\times n}$, there exist unitary matrices U and V such that $A = U\Sigma V^*$, where Σ is a diagonal matrix containing the singular values of A. For a dual quaternion Hermitian matrix $B \in \widehat{\mathbb{Q}}^{m\times n}$, we have $\mathrm{tr}(B) = \mathrm{tr}(KBK^*)$ for any unitary matrix $K \in \widehat{\mathbb{Q}}^{m\times m}$. Thus, we have

$$\mathrm{tr}(AA^*) = \mathrm{tr}(U\Sigma V^* \cdot V\Sigma U^*) = \mathrm{tr}(U\Sigma \cdot \Sigma U^*) = \mathrm{tr}(\Sigma^2) = \sum_{i=1}^{\min\{m,n\}} s_i(A)^2.$$

When A is appreciable, we have

$$\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\mathrm{tr}(AA^*)} = \sqrt{\sum_{i=1}^{\min\{m,n\}} s_i(A)^2}.$$

When A is infinitesimal, we have

$$\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |(a_{ij})_{\mathcal{I}}|^2\epsilon} = \sqrt{\mathrm{tr}(AA^*)} = \sqrt{\sum_{i=1}^{\min\{m,n\}} s_i(A_{\mathcal{I}})^2\epsilon}.$$

That is, the Schatten 2-norm of A equals the Frobenius norm of A.

Remark 4.6. For $p = \infty$, if A is appreciable, the Schatten ∞-norm of A is $\|A\|_\infty = s_1(A)$. If A is infinitesimal, $\|A\|_\infty = s_1(A_{\mathcal{I}})\epsilon = s_1(A_{\mathcal{I}}\epsilon) = s_1(A)$.

That is, the Schatten ∞-norm of A equals the spectral norm of A.

Definition 4.7. For $1 \leq k \leq \min\{m, n\}$, the Fan k-norm of $A \in \widehat{\mathbb{Q}}^{m\times n}$ is given by

$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A).$$

In particular, $\|A\|_{(1)} = \|A\|_\infty = s_1(A)$.

5 Inequalities for Unitarily Invariant Norms

The Fan k-norm and Schatten p-norms find applications in matrix theory as well as quantum information theory [7, 8]. Both of these norms are important unitarily invariant norms.

One property of the Fan k-norm that may not be immediately obvious, however, is the following result, which demonstrates its significance in the general theory of unitarily invariant norms:

Theorem 5.1. *Let $A, B \in \widehat{\mathbb{Q}}^{m\times n}$. If $\|A\|_{(k)} \leq \|B\|_{(k)}$ for each $1 \leq k \leq \min\{m, n\}$, then $\|A\| \leq \|B\|$ for every unitarily invariant norm $\|\cdot\|$ on $\widehat{\mathbb{Q}}^{m\times n}$.*

Proof. This follows from Proposition 3.4 and Theorem 4.1. □

The Theorem 5.1 can be equivalently stated as follows: Let $A, B \in \widehat{\mathbb{Q}}^{m\times n}$. $\|A\| \leq \|B\|$ for every unitarily invariant norm holds if and only if $s(A) \prec_w s(B)$.

Next, we will present a Hölder inequality for dual quaternion matrices, which is relevant to the Schatten p-norms. Before that, we introduce a lemma that will be used to prove the dual number version of the Hölder inequality.

Lemma 5.2. *If the real numbers u and v are both greater than 1 and satisfy $\frac{1}{u} + \frac{1}{v} = 1$, then for any nonnegative dual numbers p and q, we have*

$$pq \leq \frac{p^u}{u} + \frac{q^v}{v}.$$

Proof. If p or q equals 0, the inequality obviously holds. Now, assume that $p > 0$ and $q > 0$.

If both p and q are appreciable, consider a real number α with $0 < \alpha < 1$, and let $f(x) = x^\alpha - \alpha x$ for positive dual numbers x. We have

$$\begin{aligned} f(x) = x^\alpha - \alpha x &= x_{\rm st}^\alpha + \alpha x_{\rm st}^{\alpha-1} x_{\mathcal{I}}\epsilon - \alpha x_{\rm st} - \alpha x_{\mathcal{I}}\epsilon \\ &= (x_{\rm st}^\alpha - \alpha x_{\rm st}) + \alpha x_{\mathcal{I}}(x_{\rm st}^{\alpha-1} - 1)\epsilon. \end{aligned}$$

To maximize $f(x)$, we need $x_{\rm st}^\alpha - \alpha x_{\rm st}$ to reach its maximum. This occurs when $x_{\rm st} = 1$, resulting in $x_{\rm st}^\alpha - \alpha x_{\rm st}$ achieving its maximum value of $1 - \alpha$. Simultaneously, we have $\alpha x_{\mathcal{I}}(x_{\rm st}^{\alpha-1} - 1)\epsilon = 0$. Therefore, we have

$$f(x) \leq 1 - \alpha$$

for any positive dual numbers x, i.e.,

$$x^\alpha \leq 1 - \alpha + \alpha x.$$

For any positive dual numbers M and N, let $x = \frac{M}{N}$, $\alpha = \frac{1}{u}$, and $1 - \alpha = \frac{1}{v}$. Then we have

$$M^{\frac{1}{u}} N^{\frac{1}{v}} \leq \frac{M}{u} + \frac{N}{v}.$$

Now, set $p = M^{\frac{1}{u}}$ and $q = N^{\frac{1}{v}}$, yielding

$$pq \le \frac{p^u}{u} + \frac{q^v}{v}.$$

If one of p or q is appreciable and the other is infinitesimal, without loss of generality, assume that p is appreciable and q is infinitesimal. In this case, pq is infinitesimal, but $\frac{p^u}{u}$ is appreciable. Therefore, $pq \le \frac{p^u}{u} + \frac{q^v}{v}$.

If both p and q are infinitesimal, we have $pq = 0$, and the inequality holds as well. □

Now, we can present the dual number version of the Hölder inequality.

Lemma 5.3. *Let $x = (x_1, \ldots, x_n)^{\mathrm{T}}, y = (y_1, \ldots, y_n)^{\mathrm{T}} \in \mathbb{D}^n$. Then, for real numbers u and v satisfying $u > 1$, $v > 1$, and $\frac{1}{u} + \frac{1}{v} = 1$, we have*

$$\sum_{i=1}^{n} |x_i y_i| \le \|x\|_u \|y\|_v,$$

where

$$\|x\|_u = \begin{cases} (\sum_{i=1}^{n} |x_i|^u)^{1/u}, & \text{if } x_{\mathrm{st}} \ne 0, \\ (\sum_{i=1}^{n} |(x_i)_{\mathcal{I}}|^u)^{1/u}\epsilon, & \text{otherwise.} \end{cases}$$

Proof. If $x = 0$ or $y = 0$, the inequality obviously holds. Assume that $x \ne 0$ and $y \ne 0$. If both x and y are appreciable, then $(\sum_{i=1}^{n} |x_i|^u)^{1/u}$ and $(\sum_{i=1}^{n} |y_i|^v)^{1/v}$ are appreciable. Let $p = \frac{|x_i|}{(\sum_{i=1}^{n} |x_i|^u)^{1/u}}$ and $q = \frac{|y_i|}{(\sum_{i=1}^{n} |y_i|^v)^{1/v}}$. By Lemma 5.2 and Theorem 2.2, we have

$$\frac{|x_i| \cdot |y_i|}{(\sum_{i=1}^{n} |x_i|^u)^{1/u}(\sum_{i=1}^{n} |y_i|^v)^{1/v}} = \frac{|x_i y_i|}{(\sum_{i=1}^{n} |x_i|^u)^{1/u}(\sum_{i=1}^{n} |y_i|^v)^{1/v}} \le \frac{|x_i|^u}{u(\sum_{i=1}^{n} |x_i|^u)} + \frac{|y_i|^v}{v(\sum_{i=1}^{n} |y_i|^v)}.$$

Thus,

$$\frac{\sum_{i=1}^{n} |x_i y_i|}{(\sum_{i=1}^{n} |x_i|^u)^{1/u}(\sum_{i=1}^{n} |y_i|^v)^{1/v}} \le \frac{\sum_{i=1}^{n} |x_i|^u}{u(\sum_{i=1}^{n} |x_i|^u)} + \frac{\sum_{i=1}^{n} |y_i|^v}{v(\sum_{i=1}^{n} |y_i|^v)} = \frac{1}{u} + \frac{1}{v} = 1.$$

Therefore, we have $\sum_{i=1}^{n} |x_i y_i| \le \|x\|_u \|y\|_v$.

If x is appreciable and y is infinitesimal, then $x_i y_i$ is infinitesimal, and $|x_i y_i| = |(x_i y_i)_{\mathcal{I}}|\epsilon = |(x_i)_{\mathrm{st}}(y_i)_{\mathcal{I}}|\epsilon$. By the Hölder inequality for real numbers, we have

$$\sum_{i=1}^{n} |(x_i)_{\mathrm{st}}(y_i)_{\mathcal{I}}| \le (\sum_{i=1}^{n} |(x_i)_{\mathrm{st}}|^u)^{1/u}(\sum_{i=1}^{n} |(y_i)_{\mathcal{I}}|^v)^{1/v}.$$

Thus, we have

$$\sum_{i=1}^{n} |(x_i)_{\mathrm{st}}(y_i)_{\mathcal{I}}|\epsilon \le (\sum_{i=1}^{n} |(x_i)_{\mathrm{st}}|^u)^{1/u}(\sum_{i=1}^{n} |(y_i)_{\mathcal{I}}|^v)^{1/v}\epsilon = (\|x\|_u + k\epsilon) \cdot \|y_{\mathcal{I}}\|_v \epsilon = \|x\|_u \|y\|_v$$

for some real number k. The case of x being infinitesimal and y being appreciable is similar.

If both x and y are infinitesimal, we have $|x_i y_i| = 0$, and the inequality holds as well. □

Now, we present the following theorem, which establishes the Hölder inequality for dual quaternion matrices.

Theorem 5.4. *If $A, B \in \widehat{\mathbb{Q}}^{m\times n}$, then*

$$\mathrm{tr}(A^*B + B^*A) \leq 2\|A\|_u\|B\|_v,$$

where $\frac{1}{u} + \frac{1}{v} = 1$.

Proof. The inequality follows from the combination of Theorem 3.5 and Lemma 5.3. □

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Sheng Chen
Department of Mathematics, Harbin Institute of Technology
Harbin 150001, China
E-mail address: schen@hit.edu.cn

Haofei Hu
Department of Mathematics, Harbin Institute of Technology
Harbin 150001, China
E-mail address: hfhumath@gmail.com