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ON THE D-GAP FUNCTIONS FOR VARIATIONAL-HEMIVARIATIONAL INEQUALITIES WITH AN APPLICATION TO CONTACT MECHANICS*

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Abstract: The aim of this paper is to investigate the difference gap (for brevity, D-gap) functions and global error bounds for an abstract class of elliptic variational-hemivariational inequalities (for brevity, EVHIs). Based on the optimality condition for the concerning minimization problem, the regularized gap function for EVHIs is proposed under some suitable conditions. Accordingly, we establish the D-gap functions for EVHIs in terms of these regularized gap functions. Furthermore, we provide global error bounds for EVHIs by virtue of the regularized gap functions and the D-gap functions. An application to contact mechanic problem is given to illustrate our main results.

Key words: *elliptic variational-hemivariational inequality, regularized gap function, D-gap function, error bound, contact mechanic problem*

Mathematics Subject Classification: *47J20, 49J40, 49K40*

1 Introduction

The target problem in this article is a class of generalized variational inequalities and the main tool is the D-gap function. We start with briefly recall and review these two notions. In 1976, Auslender [1] introduced the gap function as a valuable tool for solving variational inequalities via associated optimization problems. A gap function defined by

$$\mathbf{p}(z) = \sup_{v \in D} \langle \rho(z), z - v \rangle,$$

where $z \in D \subset \mathbb{R}^n$, $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$, and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$. The function $\mathbf{p}$ satisfies the following conditions:

(i) $\mathbf{p}(z) \geq 0$, for all $z \in D$,

(ii) z^* is a solution to the variational inequality (VI) of finding $z^* \in D$ such that

$$\langle \rho(z^*), v - z^* \rangle \geq 0, \text{ for all } v \in D$$

if and only if $z^* \in D$ and $\mathbf{p}(z^*) = 0$.

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A disadvantage is that the gap function $\mathbf{p}$ is non-differentiable in general. To conquer this drawback, in 1992, Fukushima [7] originally proposed a new gap function for VI in the following form:

$$\mathbf{p}_\alpha(z) = \sup_{v\in D}\{\langle \rho(z), z - v\rangle - \alpha\|z - v\|^2\},$$

where $\alpha > 0$. The function $\mathbf{p}_\alpha$ is finite valued and differentiable as long as the mapping ρ is differentiable, and it is called the regularized gap function. Then, Peng [35] provided the notion of the D-gap (where D stands for "difference") function which leads to an unconstrained optimization reformulation of the VI. Another D-gap function derived from the difference of two regularized gap functions, given by Yamashita and Fukushima [43], is as follows:

$$\mathbf{d}_{\alpha\beta}(z) = \mathbf{p}_\alpha(z) - \mathbf{p}_\beta(z) \quad (0 < \alpha < \beta).$$

Note that $\mathbf{d}_{\alpha\beta}$ is also a gap function for VI. Peng-Fukushima [36] developed a global error bound result for variational inequalities in terms of D-gap functions using the strong monotonicity assumption. Error bound explores the upper estimation of the distance between an arbitrary feasible point and the solution set of a certain problem. So, it has been critical in analyzing the convergence of iterative methods for solving variational inequalities. Therefore, the D-gap function and error bounds have been investigated for various kinds of equilibrium problems and variational inequalities, see e.g., [2, 3, 16, 17, 21, 22, 25, 26, 38].

On the other hand, it is well known that the theory of hemivariational inequalities is an extension of variational inequalities. This theory was introduced by Panagiotopoulos for dealing with various problems of mechanical problems with nonconvex and nonsmooth energy potentials, and based on the concept of the Clarke generalized gradient for locally Lipschitz functions, see e.g., [33, 34]. Variational–hemivariational inequality is a generalization of hemivariational inequality which includes both convex and nonconvex potentials. This theory has been extensively investigated by many authors in various directions, and it has found different applications in engineering, mechanics, especially in nonsmooth analysis and optimization. Recent existence results for some types of variational–hemivariational inequalities can be found, in e.g., [20, 27, 29, 31, 32, 37], the stability in the sense of convergence and the well-posedness, in e.g., [13, 24, 28, 41, 44, 45], the gap functions and error bounds, in e.g., [8, 15, 19, 40] and the computational issues have been addressed in, e.g., [9, 12].

Although D-gap functions have turned out to be efficient mathematical tools to establish error bounds for various variational inequalities and equilibrium problems, until now, there is no contribution which deals with D-gap functions for variational-hemivariational inequalities. Based on the motivation, in this paper, we develop the D-gap function and global error bounds for an abstract class of elliptic variational-hemivariational inequalities (for brevity, EVHIs). Firstly, we provide the regularized gap function introduced by Fukushima [7] for EVHIs under some suitable conditions based on the optimality condition for the concerning minimization problem. The D-gap function for EVHIs in terms of regularized gap functions is established. Furthermore, we also give some global error bounds for EVHIs by virtue of the regularized gap function and the D-gap function. Finally, the theoretical results are applied to a contact mechanic problem. To sum up, the contribution of this work and its relation to previous literature is depicted in Figure 1.

The rest of this paper is structured as follows. The basic notations and definitions that will be used throughout the study are presented in Section 2. We also introduce an abstract class of elliptic variational-hemivariational inequalities and provide their existence under some imposed hypotheses on the data. In Section 3, we investigate the regularized gap function and the D-gap function for EVHIs. In Section 4, we establish global error bounds for

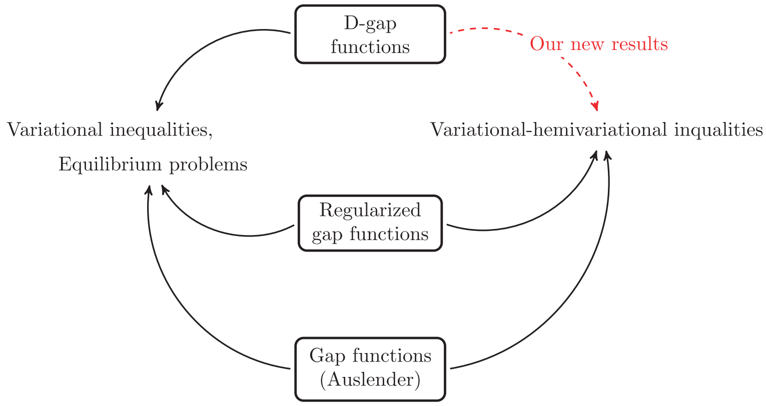

Figure 1: Illustration of the developments regarding different kinds of gap functions, regularized gap functions and D-gap functions.

EVHIs by virtue of the gap functions considered in Section 3 under some suitable conditions. Finally, an application to a contact problem is proposed in Section 5 to illustrate abstract results in the paper.

2 Preliminaries and Formulations

Let E be a normed space with its topological dual E^*. We denote by $\|\cdot\|_E$ the norm on E and $\langle \cdot, \cdot \rangle_E$ the duality pairing of E and E^*. For two normed spaces E and Z, $\mathcal{L}(E, Z)$ denotes the space of all linear continuous operators from E to Z. We recall some fundamental concepts that will be used in the sequel. For more details, please refer to [4–6, 30].

Definition 2.1. A function $\varrho\colon E \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is said to be

(a) proper, if $\varrho \not\equiv +\infty$;

(b) convex, if $\varrho(tu + (1-t)v) \leq t\varrho(u) + (1-t)\varrho(v)$ for all u, $v \in E$ and $t \in [0, 1]$;

(c) lower semicontinuous at $u_0 \in E$, if for any sequence $\{u_n\} \subset E$ such that $u_n \to u_0$, it holds $\varrho(u_0) \leq \liminf \varrho(u_n)$;

(d) upper semicontinuous at $u_0 \in E$, if for any sequence $\{u_n\} \subset E$ such that $u_n \to u_0$, it holds $\limsup \varrho(u_n) \leq \varrho(u_0)$;

(e) lower semicontinuous (resp., upper semicontinuous) on E, if ϱ is lower semicontinuous (resp., upper semicontinuous) at every $u_0 \in E$.

Definition 2.2. An operator $\mathcal{G}\colon E \to E^*$ is said to be:

(a) bounded, if $\mathcal{G}$ maps bounded sets of E into bounded sets of E^*;

(b) Lipschitz continuous, if there exists a constant $l_{\mathcal{G}} > 0$ such that

$$\|\mathcal{G}v - \mathcal{G}u\|_{E^*} \leq l_{\mathcal{G}} \|v - u\|_E \text{ for all } u, v \in E;$$

(c) pseudomonotone, if $\mathcal{G}$ is a bounded operator and for every sequence $\{u_n\} \subset E$ converging weakly to $u \in E$ such that $\limsup \langle \mathcal{G}u_n, u_n - u\rangle_E \le 0$, we have

$$\langle \mathcal{G}u, u - v\rangle_E \le \liminf \langle \mathcal{G}u_n, u_n - v\rangle_E, \quad \text{for all} \quad v \in E.$$

Definition 2.3. Let $\theta\colon E \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function. The convex subdifferential $\partial_c\theta\colon E \rightrightarrows E^*$ of θ is defined by

$$\partial_c\theta(u) = \big\{w^* \in E^* \mid \langle w^*, v - u\rangle_E \le \theta(v) - \theta(u) \;\; \text{for all} \;\; v \in E\big\} \;\; \text{for all} \;\; u \in E.$$

An element $w^* \in \partial_c\theta(u)$ is called a subgradient of θ at $u \in E$. Given a bifunction $h\colon E \times E \to \mathbb{R}$, we will denote by $\partial_2 h$ the convex subdifferential of h with respect to the second component.

Definition 2.4. A function $\varrho\colon E \to \mathbb{R}$ is said to be locally Lipschitz, if for every $u \in E$, there exist a neighbourhood $\mathcal{N}$ of u and a constant $l_u > 0$ such that

$$|\varrho(v_1) - \varrho(v_2)| \le l_u \|v_1 - v_2\|_E \;\; \text{for all} \;\; v_1, v_2 \in \mathcal{N}.$$

Given a locally Lipschitz function $\varrho\colon E \to \mathbb{R}$, we denote by $\varrho^0(u; v)$ the Clarke generalized directional derivative of ϱ at the point $u \in E$ in the direction $v \in E$ defined by

$$\varrho^0(u; v) = \limsup_{y \to u,\ t \to 0^+} \frac{\varrho(y + tv) - \varrho(y)}{t}.$$

The generalized gradient of ϱ at $u \in E$, denoted by $\partial\varrho(u)$, is a subset of E^* given by

$$\partial\varrho(u) = \big\{\zeta^* \in E^* \mid \varrho^0(u; v) \ge \langle \zeta^*, v\rangle_E \text{ for all } v \in E\big\}.$$

For convenience, some basic and useful results of the generalized gradient and directional derivative of a locally Lipschitz function are collected in the following lemma, see, e.g., [4, Proposition 2.1.1].

Lemma 2.5. *Let E be a real Banach space and $\varrho\colon E \to \mathbb{R}$ be a locally Lipschitz function. Then, the following assertions hold.*

(i) *For each $u \in E$, the function $E \ni v \mapsto \varrho^0(u; v) \in \mathbb{R}$ is finite, positively homogeneous, subadditive and Lipschitz continuous.*

(ii) *The function $E \times E \ni (u, v) \mapsto \varrho^0(u; v) \in \mathbb{R}$ is upper semicontinuous.*

(iii) *For every u, $v \in E$, it holds*

$$\varrho^0(u; v) = \max\{\langle \zeta, v\rangle_E \mid \zeta \in \partial\varrho(u)\}.$$

Next, we recall the existence and uniqueness result of solutions for uniformly convex optimization problems.

Definition 2.6 (see [23])**.** A function $\varrho\colon E \to \mathbb{R}$ is said to be uniformly convex if there exists a continuously increasing function $\pi\colon \mathbb{R} \to \mathbb{R}$ such that $\pi(0) = 0$ and that for all $u, v \in E$ and for each $t \in [0, 1]$, we have

$$\varrho(tu + (1 - t)v) \le t\varrho(u) + (1 - t)\varrho(v) - t(1 - t)\pi(\|u - v\|)\|u - v\|.$$

If $\pi(\tau) = k\tau$ for $k > 0$, then ϱ is called a strongly convex function.

Lemma 2.7 (see [42], Chapter 1, Section 3, Theorem 9)**.** *Suppose that $\mathcal{W}$ is a nonempty, convex and closed subset of a reflexive Banach space E, $\varrho\colon E \to \mathbb{R}$ is a uniformly convex and lower semicontinuous function. Then the optimization problem*

$$\min_{u\in\mathcal{W}} \varrho(u)$$

has the unique solution $u^* \in \mathcal{W}$.

Throughout the paper, unless otherwise specified, for each $i \in \{1,\dots,k\}$, let E be a Hilbert space and $E_{\mathcal{P}}$, E_{Υ_i} be Banach spaces, $\mathcal{W} \subset E$ and $\mathcal{K}_{\mathcal{P}} \subset E_{\mathcal{P}}$. In addition, let $\mathcal{G}\colon E \to E^*$, $\delta\colon E \to E_{\mathcal{P}}$, $\gamma_i\colon E \to E_{\Upsilon_i}$ be operators, $\mathcal{P}\colon \mathcal{K}_{\mathcal{P}} \times \mathcal{K}_{\mathcal{P}} \to \mathbb{R}$, $\Upsilon_i\colon E_{\Upsilon_i} \to \mathbb{R}$ be functions and $f \in E^*$. We now consider the abstract elliptic variational-hemivariational inequality:

Problem 2.1. Find $u^* \in \mathcal{W}$ such that

$$\langle \mathcal{G}u^*, v - u^*\rangle_E + \mathcal{P}(\delta u^*, \delta v) - \mathcal{P}(\delta u^*, \delta u^*) + \sum_{i=1}^{k} \Upsilon_i^0\left(\gamma_i u^*; \gamma_i v - \gamma_i u^*\right) \geq \langle f, v - u^*\rangle_E$$

for all $v \in \mathcal{W}$.

To proceed, the following hypotheses are imposed on the data of Problem 2.1.
$\underline{\mathfrak{H}(\mathcal{G})}$: For the operator $\mathcal{G}\colon E \to E^*$,

(a) $\mathcal{G}$ is Lipschitz continuous, i.e., there exists $l_{\mathcal{G}} > 0$ such that

$$\|\mathcal{G}v_1 - \mathcal{G}v_2\|_{E^*} \leq l_{\mathcal{G}}\|v_1 - v_2\|_E, \quad \forall v_1, v_2 \in E;$$

(b) $\mathcal{G}$ is strongly monotone, i.e., there exists $m_{\mathcal{G}} > 0$ such that

$$\langle \mathcal{G}v_1 - \mathcal{G}v_2, v_1 - v_2\rangle_E \geq m_{\mathcal{G}}\|v_1 - v_2\|_E^2, \quad \forall v_1, v_2 \in E.$$

$\underline{\mathfrak{H}(\mathcal{P})}$: For the function $\mathcal{P}\colon \mathcal{K}_{\mathcal{P}} \times \mathcal{K}_{\mathcal{P}} \to \mathbb{R}$,

(a) for each $u \in \mathcal{K}_{\mathcal{P}}$, $\mathcal{P}(u,\cdot)\colon \mathcal{K}_{\mathcal{P}} \to \mathbb{R}$ is convex and lower semicontinuous;

(b) there exists $\alpha_{\mathcal{P}} > 0$ such that

$$\begin{aligned} &\mathcal{P}(u_1, v_2) - \mathcal{P}(u_1, v_1) + \mathcal{P}(u_2, v_1) - \mathcal{P}(u_2, v_2) \\ &\qquad \leq \alpha_{\mathcal{P}}\|u_1 - u_2\|_{E_{\mathcal{P}}}\|v_1 - v_2\|_{E_{\mathcal{P}}}, \quad \forall u_1, u_2, v_1, v_2 \in \mathcal{K}_{\mathcal{P}}. \end{aligned}$$

$\underline{\mathfrak{H}(\Upsilon)}$: For each $i \in \{1,\dots,k\}$, for the locally Lipschitz function $\Upsilon_i\colon E_{\Upsilon_i} \to \mathbb{R}$,

(a) $\|\xi\|_{E^*_{\Upsilon_i}} \leq c_0 + c_1\|v\|_{E_{\Upsilon_i}}$, $\forall v \in E_{\Upsilon_i}$, $\xi \in \partial\Upsilon_i(v)$ with some $c_0, c_1 \geq 0$;

(b) there exists $L_{\Upsilon_i} \geq 0$ such that

$$\Upsilon_i^0(w_1; v_2 - v_1) + \Upsilon_i^0(w_2; v_1 - v_2) \leq L_{\Upsilon_i}\|w_1 - w_2\|_{E_{\Upsilon_i}}\|v_1 - v_2\|_{E_{\Upsilon_i}}, \tag{2.1}$$

for all $w_1, w_2, v_1, v_2 \in E_{\Upsilon_i}$.

$\underline{\mathfrak{H}(\mathcal{W})}$: $\mathcal{W}$ is a nonempty, closed and convex subset of E with $\mathbf{0}_E \in \mathcal{W}$.

$\underline{\mathfrak{H}(\mathcal{K})}$: $\mathcal{K}_\mathcal{P}$ is a nonempty, closed and convex subset of $E_\mathcal{P}$ with $\delta(\mathcal{W}) \subset \mathcal{K}_\mathcal{P}$.

$\underline{\mathfrak{H}(\delta)}$: For the operator $\delta \in \mathcal{L}(E, E_\mathcal{P})$, there exists $c_\mathcal{P} > 0$,

$$\|\delta v\|_{E_\mathcal{P}} \le c_\mathcal{P} \|v\|_E.$$

$\underline{\mathfrak{H}(\gamma)}$: For each $i \in \{1, \ldots, k\}$, for the operator $\gamma_i \in \mathcal{L}(E, E_{\Upsilon_i})$, there exists $c_{\Upsilon_i} > 0$,

$$\|\gamma_i v\|_{E_{\Upsilon_i}} \le c_{\Upsilon_i} \|v\|_E.$$

$\underline{\mathfrak{H}(f)}$: $f \in E^*$.
$\underline{\mathfrak{H}(const.)}$:

$$m_\mathcal{G} - \alpha_\mathcal{P} c_\mathcal{P}^2 - \sum_{i=1}^{k} L_{\Upsilon_i} c_{\Upsilon_i}^2 > 0.$$

Remark 2.8. (i) It is easily seen that $\mathfrak{H}(\mathcal{G})$(b) implies that $\mathcal{G}$ is pseudomonotone.

(ii) If $w_1 = v_1, w_2 = v_2$, then the condition (2.1) reduces to

$$\Upsilon_i^0(v_1; v_2 - v_1) + \Upsilon_i^0(v_2; v_1 - v_2) \le L_{\Upsilon_i} \|v_1 - v_2\|_{E_{\Upsilon_i}}^2, \quad \forall v_1, v_2 \in E_{\Upsilon_i}.$$

The following example illustrates that the case where the hypotheses $\mathfrak{H}(\mathcal{P})$ and $\mathfrak{H}(\Upsilon)$ are satisfied.

Example 2.9. For each $i \in \{1, 2\}$, let $E_{\Upsilon_i} = E_\mathcal{P} = E = \mathbb{R}$, $\mathcal{K}_\mathcal{P} = [0, \frac{7}{3}]$, $\mathcal{P} \colon \mathcal{K}_\mathcal{P} \times \mathcal{K}_\mathcal{P} \to \mathbb{R}$ and $\Upsilon_i \colon E_{\Upsilon_i} \to \mathbb{R}$ be the functions defined by

$$\mathcal{P}(u, v) = \frac{5 + uv^2}{3} \quad \text{and} \quad \Upsilon_i(u) = \begin{cases} \left(\frac{1}{2} - i\right) u^2 + iu & \text{if } u \ge 0 \\ 0 & \text{if } u < 0. \end{cases}$$

It is not difficult to show that the condition $\mathfrak{H}(\mathcal{P})(a)$ holds. For any $u_1, u_2, v_1, v_2 \in \mathcal{K}_\mathcal{P}$, we have

$$\begin{aligned} &\mathcal{P}(u_1, v_2) - \mathcal{P}(u_1, v_1) + \mathcal{P}(u_2, v_1) - \mathcal{P}(u_2, v_2) \\ &= \frac{1}{3}\left(u_1 v_2^2 - u_1 v_1^2 + u_2 v_1^2 - u_2 v_2^2\right) \\ &= \frac{1}{3}(v_1 + v_2)(u_2 - u_1)(v_1 - v_2) \\ &\le \frac{14}{9}|u_1 - u_2||v_1 - v_2|, \end{aligned}$$

which implies that the condition $\mathfrak{H}(\mathcal{P})(b)$ is satisfied with $\alpha_\mathcal{P} = \frac{14}{9}$. Thus, $\mathfrak{H}(\mathcal{P})$ is valid.

On the other hand, it is obvious that for each $i \in \{1, 2\}$, Υ_i is a locally Lipschitz nonconvex function. Moreover, its generalized gradient and Clarke generalized directional derivative are given by

$$\partial \Upsilon_i(u) = \begin{cases} (1 - 2i)u + i & \text{if } u > 0 \\ [0, i] & \text{if } u = 0 \\ 0 & \text{if } u < 0, \end{cases}$$

and

$$\Upsilon_i^0(u;d) = \begin{cases} (1-2i)ud + id & \text{if } u > 0 \\ \max\{0, id\} & \text{if } u = 0 \\ 0 & \text{if } u < 0 \end{cases}$$

for all $d \in \mathbb{R}$ and $i \in \{1,2\}$.

Hence, $|w| \leq i + (2i-1)|u|$ for all $w \in \partial\Upsilon_i(u)$ and $u \in \mathbb{R}$ and $i \in \{1,2\}$ and so the condition $\mathfrak{H}(\Upsilon)(a)$ holds with $c_0 = i, c_1 = 2i-1$ for $i \in \{1,2\}$. Furthermore, we also obtain

$$\Upsilon_i^0(w_1; v_2 - v_1) + \Upsilon_i^0(w_2; v_1 - v_2) \leq (2i-1)|w_1 - w_2||v_1 - v_2|$$

for all $w_1, w_2, v_1, v_2 \in \mathbb{R}$ and so the assumption $\mathfrak{H}(\Upsilon)(b)$ holds with $L_{\Upsilon_i} = 2i-1$ for $i \in \{1,2\}$.

By slightly modifying the arguments in [10, 31], we obtain the existence and uniqueness result for Problem 2.1.

Theorem 2.10. *Assume that the assumptions $\mathfrak{H}(\mathcal{G})$, $\mathfrak{H}(\mathcal{P})$, $\mathfrak{H}(\Upsilon)$, $\mathfrak{H}(\mathcal{W})$, $\mathfrak{H}(\mathcal{K})$, $\mathfrak{H}(\delta)$, $\mathfrak{H}(\gamma)$, $\mathfrak{H}(f)$ and $\mathfrak{H}(const.)$ hold. Then Problem 2.1 has a unique solution.*

We point out that there are various problems investigated in the literature which are included as special cases in Problem 2.1.

Special case (a): When $k = 1$, $\Upsilon_1 = \Upsilon$ and $\gamma_1 = \gamma$, Problem 2.1 is equivalent to the following class of variational-hemivariational inequalities studied by Han et al. [12]:

Problem 2.2. Find $u \in \mathcal{W}$ such that

$$\langle \mathcal{G}u, v-u\rangle_E + \mathcal{P}(\delta u, \delta v) - \mathcal{P}(\delta u, \delta u) + \Upsilon^0(\gamma u; \gamma v - \gamma u) \geq \langle f, v-u\rangle_E, \ \forall v \in \mathcal{W}.$$

Special case (b): When $\Upsilon_i \equiv 0$ for all $i \in \{1, \ldots, k\}$, Problem 2.1 reduces to the following variational inequality considered in Hung and Tam [18]:

Problem 2.3. Find $u \in \mathcal{W}$ such that

$$\langle \mathcal{G}u, v-u\rangle_E + \mathcal{P}(\delta u, \delta v) - \mathcal{P}(\delta u, \delta u) \geq \langle f, v-u\rangle_E, \quad \forall v \in \mathcal{W}.$$

Special case (c): When $k = 2$, $\mathcal{P} \equiv 0$, Problem 2.1 has the below form, which was introduced by Han et al. [10].

Problem 2.4. Find $u \in \mathcal{W}$ such that

$$\langle \mathcal{G}u, v-u\rangle_E + \Upsilon_1^0(\gamma_1 u; \gamma_1 v - \gamma_1 u) + \Upsilon_2^0(\gamma_2 u; \gamma_2 v - \gamma_2 u) \geq \langle f, v-u\rangle_E, \quad \forall v \in \mathcal{W}.$$

3 Different Gap Functions

In this section, we construct the gap functions in the regularized form of the Fukushima type for Problem 2.1 using some suitable conditions. Furthermore, based on these regularized gap functions, the D-gap function for Problem 2.1 is established. Since the existence of solutions have been considered in Theorem 2.10, we always assume that the solution set of Problem 2.1 is nonempty.

First, we propose the exact definition of gap functions of Problem 2.1 as below.

Definition 3.1. A real-valued function $\mathbf{m}\colon \mathcal{W} \to \mathbb{R}$ is said to be a gap function for Problem 2.1, if it satisfies the following properties:

(a) $\mathbf{m}(u) \geq 0$ for all $u \in \mathcal{W}$.

(b) $u^* \in \mathcal{W}$ is such that $\mathbf{m}(u^*) = 0$ if and only if u^* is a solution to Problem 2.1.

For each $\omega > 0$, let the function $\Xi_{\omega,f} : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ be defined by

$$\Xi_{\omega,f}(u,v) = \langle \mathcal{G}u - f, v - u\rangle_E + \mathcal{P}(\delta u, \delta v) - \mathcal{P}(\delta u, \delta u) + \sum_{i=1}^{k} \Upsilon_i^0(\gamma_i u; \gamma_i v - \gamma_i u) + \frac{\omega}{2}\|v - u\|_E^2.$$

Lemma 3.2. *For each $i \in \{1, \ldots, k\}$, suppose that $\Upsilon_i\colon E_{\Upsilon_i} \to \mathbb{R}$ is a locally Lipschitz function and $\gamma_i \in \mathcal{L}(E, E_{\Upsilon_i})$. Then, the function $\varphi_i\colon E_{\Upsilon_i} \times E_{\Upsilon_i} \to \mathbb{R}$ defined by*

$$\varphi_i(u_i, v_i) = \Upsilon_i^0(u_i; v_i - u_i) \tag{3.1}$$

satisfies the following properties:

(i) *For each $u_i \in E_{\Upsilon_i}$, the function $v \mapsto \varphi_i(u_i, \gamma_i v)$ is continuous and convex;*

(ii) *For each $u \in \mathcal{W}$, $\partial_2(\varphi_i \circ \gamma_i)(u, v) \subset \gamma_i^* \partial_2 \Upsilon_i^0(\gamma_i u; \gamma_i v - \gamma_i u)$, where $\gamma_i^*\colon E_i^* \to E^*$ is the adjoint operator to γ_i and $\varphi_i \circ \gamma_i$ denotes the composition of the function φ_i with the operator γ_i, for all $i \in \{1, \ldots, k\}$.*

Proof. (i) It follows from the property (i) of Lemma 2.5 and $\gamma_i \in \mathcal{L}(E, E_{\Upsilon_i})$ for all $i \in \{1, \ldots, k\}$.

(ii) Using the chain rule for generalized subgradient in [30, Proposition 3.37(ii)] with the condition $\gamma_i \in \mathcal{L}(E, E_{\Upsilon_i})$ for all $i \in \{1, \ldots, k\}$, we obtain that

$$\partial_2(\varphi_i \circ \gamma_i)(u, v) \subset \gamma_i^* \partial_2 \varphi_i(\gamma_i u, \gamma_i v) = \gamma_i^* \partial_2 \Upsilon_i^0(\gamma_i u; \gamma_i v - \gamma_i u)$$

for all $i \in \{1, \ldots, k\}$ and $u \in \mathcal{W}$. □

Lemma 3.3. *Suppose that all the assumptions of Lemma 3.2, $\mathfrak{H}(\mathcal{P})(a)$, $\mathfrak{H}(\mathcal{W})$ and $\mathfrak{H}(f)$ hold, and $\delta \in \mathcal{L}(E, E_{\mathcal{P}})$. Then, for each $u \in \mathcal{W}$ and $\omega > 0$ fixed, the optimization problem*

$$\min_{v \in \mathcal{W}} \Xi_{\omega,f}(u, v) \tag{3.2}$$

attains a unique solution $v_{\omega,f}(u) \in \mathcal{W}$.

Proof. For each $i \in \{1, \ldots, k\}$, by the condition $\mathfrak{H}(\mathcal{P})(a)$ and Lemma 3.2(i), we achieve that functions $v \mapsto \Upsilon_i^0(\gamma_i u; \gamma_i v - \gamma_i u)$ and $v \mapsto \mathcal{P}(\delta u, \delta v)$ are convex for all $u \in \mathcal{W}$. Then, it is easy to prove that the function $\Xi_{\omega,f}(u, \cdot)$ is a strongly convex function for all $u \in \mathcal{W}$. Furthermore, functions $v \mapsto \Upsilon_i^0(\gamma_i u; \gamma_i v - \gamma_i u)$ and $v \mapsto \mathcal{P}(\delta u, \delta v)$ are also lower semicontinuous for all $u \in \mathcal{W}$. Hence, the function $\Xi_{\omega,f}(u, \cdot)$ is lower semicontinuous for all $u \in \mathcal{W}$. It follows from the condition $\mathfrak{H}(\mathcal{W})$ that $\mathcal{W}$ is a nonempty, closed and convex set. Thus, applying Lemma 2.7, the minimization problem (3.2) attains a unique minimum $v_{\omega,f}(u) \in \mathcal{W}$, for any $u \in \mathcal{W}$ and $\omega > 0$ fixed. □

The optimality condition for the minimization problem (3.2) are described as follows.

Lemma 3.4. *Suppose that all the assumptions of Lemma 3.3 hold. Then, for each $u \in \mathcal{W}$ and $\omega > 0$ fixed,*

$$\langle \mathcal{G}u - f + \omega(v_{\omega,f}(u) - u), v - v_{\omega,f}(u)\rangle_E + \mathcal{P}(\delta u, \delta v) - \mathcal{P}(\delta u, \delta v_{\omega,f}(u)) + \sum_{i=1}^{k} \Upsilon_i^0 (\gamma_i u; \gamma_i v - \gamma_i v_{\omega,f}(u)) \geq 0, \tag{3.3}$$

holds for all $v \in \mathcal{W}$, where $v_{\omega,f}(u)$ is a unique solution of the problem (3.2)*.*

Proof. For each $u \in \mathcal{W}$ and $\omega > 0$, let $v_{\omega,f}(u)$ be a unique solution of the problem (3.2). Hence, using the chain rule for generalized subgradient in [30, Proposition 3.35(ii) and Proposition 3.37(ii)], Lemma 3.2(ii) and the optimality condition for the problem (3.2) (see [14, Theorem 1.23]), one has

$$\begin{aligned} 0 &\in \partial_2 \Xi_{\omega,f}(u, v_{\omega,f}(u)) \\ &\subset \mathcal{G}u - f + \partial_2(\mathcal{P} \circ \delta)(\delta u, v_{\omega,f}(u)) \\ &\quad + \sum_{i=1}^{k} \partial_2(\varphi_i \circ \gamma_i)(u, v_{\omega,f}(u)) + \omega(v_{\omega,f}(u) - u) \\ &\subset \mathcal{G}u - f + \delta^* \partial_2 \mathcal{P}(\delta u, \delta v_{\omega,f}(u)) \\ &\quad + \sum_{i=1}^{k} \gamma_i^* \partial_2 \Upsilon_i^0(\gamma_i u; \gamma_i v_{\omega,f}(u) - \gamma_i u) + \omega(v_{\omega,f}(u) - u), \end{aligned}$$

where φ_i is defined by (3.1), $\delta^* \colon E_{\mathcal{P}}^* \to E^*$ and $\gamma_i^* \colon E_i^* \to E^*$ are the adjoint operators to δ and γ_i, respectively for all $i \in \{1, \ldots, k\}$. This implies that there exist $z \in \partial_2 \mathcal{P}(\delta u, \delta v_{\omega,f}(u))$ and $\zeta_i \in \partial_2 \varphi_i(\gamma_i u, \gamma_i v_{\omega,f}(u)) = \partial_2 \Upsilon_i^0(\gamma_i u; \gamma_i v_{\omega,f}(u) - \gamma_i u)$ such that

$$f - \mathcal{G}u - \omega(v_{\omega,f}(u) - u) = \delta^* z + \sum_{i=1}^{k} \gamma_i^* \zeta_i. \tag{3.4}$$

For each $i \in \{1, \ldots, k\}$, since δ^* and γ_i^* are adjoint operators to δ and γ_i, respectively, it follows from (3.4) that for all $v \in \mathcal{W}$,

$$\begin{aligned} &\langle -\mathcal{G}u + f - \omega(v_{\omega,f}(u) - u), v - v_{\omega,f}(u)\rangle_E \\ &\quad = \langle \delta^* z, v - v_{\omega,f}(u)\rangle_E + \sum_{i=1}^{k} \langle \gamma_i^* \zeta_i, v - v_{\omega,f}(u)\rangle_E \\ &\quad = \langle z, \delta v - \delta v_{\omega,f}(u)\rangle_E + \sum_{i=1}^{k} \langle \zeta_i, \gamma_i v - \gamma_i v_{\omega,f}(u)\rangle_E \\ &\quad \leq \mathcal{P}(\delta u, \delta v) - \mathcal{P}(\delta u, \delta v_{\omega,f}(u)) \\ &\qquad + \sum_{i=1}^{k} (\varphi_i(\gamma_i u, \gamma_i v) - \varphi_i(\gamma_i u, \gamma_i v_{\omega,f}(u))) \\ &\quad = \mathcal{P}(\delta u, \delta v) - \mathcal{P}(\delta u, \delta v_{\omega,f}(u)) \\ &\qquad + \sum_{i=1}^{k} \big(\Upsilon_i^0(\gamma_i u; \gamma_i v - \gamma_i u) - \Upsilon_i^0(\gamma_i u; \gamma_i v_{\omega,f}(u) - \gamma_i u)\big) \end{aligned}$$

$$\leq \mathcal{P}(\delta u, \delta v) - \mathcal{P}(\delta u, \delta v_{\omega,f}(u)) + \sum_{i=1}^{k} \Upsilon_i^0(\gamma_i u; \gamma_i v - \gamma_i v_{\omega,f}(u)),$$

that is,

$$\langle \mathcal{G}u - f + \omega(v_{\omega,f}(u) - u), v - v_{\omega,f}(u)\rangle_E + \mathcal{P}(\delta u, \delta v) - \mathcal{P}(\delta u, v_{\omega,f}(u)) + \sum_{i=1}^{k} \Upsilon_i^0(\gamma_i u; \gamma_i v - \gamma_i v_{\omega,f}(u)) \geq 0.$$

Thus, for each $u \in \mathcal{W}$, the inequality (3.3) holds for all $v \in \mathcal{W}$. □

Now, we consider the function $\mathcal{F}_{\omega,f} \colon \mathcal{W} \to \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{F}_{\omega,f}(u) &= \sup_{v\in\mathcal{W}} \{-\Xi_{\omega,f}(u,v)\} \\ &= -\inf_{v\in\mathcal{W}} \Xi_{\omega,f}(u,v) = -\Xi_{\omega,f}(u, v_{\omega,f}(u)).\end{aligned} \tag{3.5}$$

In what follows, the function $\mathcal{F}_{\omega,f}$ is called to be a regularized gap function of Problem 2.1. We shall assert that $\mathcal{F}_{\omega,f}$ is a gap function of Problem 2.1.

Theorem 3.5. *Suppose that the hypotheses $\mathfrak{H}(\mathcal{P})(a)$, $\mathfrak{H}(\Upsilon)(b)$, $\mathfrak{H}(\mathcal{W})$, $\mathfrak{H}(\mathcal{K})$ and $\mathfrak{H}(f)$ hold, and $\delta \in \mathcal{L}(E, E_{\mathcal{P}})$, $\gamma_i \in \mathcal{L}(E, E_{\Upsilon_i})$ for all $i \in \{1, ..., k\}$. Then, for any $\omega > 0$, the function $\mathcal{F}_{\omega,f}$ is a gap function for Problem 2.1.*

Proof. (a) For all $u \in \mathcal{W}$, we have

$$\begin{aligned}\mathcal{F}_{\omega,f}(u) &= \sup_{v\in\mathcal{W}} \{-\Xi_{\omega,f}(u,v)\} \\ &\geq -\Xi_{\omega,f}(u,u) \\ &= \langle f - \mathcal{G}u, u - u\rangle_E - \mathcal{P}(\delta u, \delta u) + \mathcal{P}(\delta u, \delta u) \\ &\quad - \sum_{i=1}^{k} \Upsilon_i^0\left(\gamma_i u; \gamma_i u - \gamma_i u\right) - \frac{\omega}{2}\left\|u - u\right\|_E^2 \\ &= -\sum_{i=1}^{k} \Upsilon_i^0\left(\gamma_i u; \mathbf{0}_{E_i}\right) = 0.\end{aligned}$$

(b) Suppose that u^* is a solution of Problem 2.1. From (3.5), we have

$$\begin{aligned}\mathcal{F}_{\omega,f}(u^*) &= \sup_{v\in\mathcal{W}} \{-\Xi_{\omega,f}(u^*,v)\} \\ &= -\Xi_{\omega,f}(u^*, v_{\omega,f}(u^*)).\end{aligned} \tag{3.6}$$

Moreover, since u^* is a solution of Problem 2.1, we obtain

$$\langle \mathcal{G}u^* - f, v_{\omega,f}(u^*) - u^*\rangle_E + \mathcal{P}(\delta u^*, \delta v_{\omega,f}(u^*)) - \mathcal{P}(\delta u^*, \delta u^*) + \sum_{i=1}^{k} \Upsilon_i^0\left(\gamma_i u^*; \gamma_i v_{\omega,f}(u^*) - \gamma_i u^*\right) \geq 0. \tag{3.7}$$

It follows from the result of Lemma 3.4 that

$$\langle \mathcal{G}u^* - f + \omega(v_{\omega,f}(u^*) - u^*), u^* - v_{\omega,f}(u^*)\rangle_E$$

$$+ \mathcal{P}(\delta u^*, \delta u^*) - \mathcal{P}(\delta u^*, \delta v_{\omega,f}(u^*))$$
$$+ \sum_{i=1}^{k} \Upsilon_i^0 \left(\gamma_i u^*; \gamma_i u^* - \gamma_i v_{\omega,f}(u^*)\right) \geq 0. \tag{3.8}$$

Combining (3.7) and (3.8), thanks to the assumption $\mathfrak{H}(\Upsilon)$(b), we get

$$-\omega \|v_{\omega,f}(u^*) - u^*\|_E^2 \geq 0.$$

This implies that

$$\|v_{\omega,f}(u^*) - u^*\|_E^2 \leq 0,$$

and so $u^* = v_{\omega,f}(u^*)$. Therefore, it follows from (3.6) that $\mathcal{F}_{\omega,f}(u^*) = 0$.

Conversely, for any $x^* \in \mathcal{W}$, we assume that $\mathcal{F}_{\omega,f}(u^*) = 0$. This implies $-\Xi_{\omega,f}(u^*, v) \leq 0$ for all $v \in P$, i.e., $\Xi_{\omega,f}(u^*, v) \geq 0$ for all $v \in P$. Since $\Xi_{\omega,f}(u^*, u^*) = 0$, u^* solves the following convex minimization problem

$$\min_{v \in \mathcal{W}} \Xi_{\omega,f}(u^*, v).$$

Using the optimality condition for this problem, we have $0 \in \partial_2 \Xi_{\omega,f}(u^*, u^*)$. From similar arguments to those used in the proof of Lemma 3.4 with fixed first argument of the function $\Xi_{\omega,f}$, we obtain that for each $v \in \mathcal{W}$,

$$f - \mathcal{G}u^* = \delta^* z^* + \sum_{i=1}^{k} \gamma_i^* \zeta_i^*,$$

where $z^* \in \partial_2 \mathcal{P}(\delta u^*, \delta u^*)$ and $\zeta_i^* \in \partial_2 \varphi_i(\gamma_i u^*; \gamma_i u^*)$ for all $i \in \{1, \ldots, k\}$. Then, for all $v \in \mathcal{W}$,

$$\begin{aligned}
&\langle -\mathcal{G}u^* + f, v - u^* \rangle_E \\
&= \langle \delta^* z^*, v - u^* \rangle_E + \sum_{i=1}^{k} \langle \gamma_i^* \zeta_i^*, v - u^* \rangle_E \\
&= \langle z^*, \delta v - \delta u^* \rangle_E + \sum_{i=1}^{k} \langle \zeta_i^*, \gamma_i v - \gamma_i u^* \rangle_E \\
&\leq \mathcal{P}(\delta u^*, \delta v) - \mathcal{P}(\delta u^*, \delta u^*) + \sum_{i=1}^{k} \left(\varphi_i(\gamma_i u^*; \gamma_i v) - \varphi_i(\gamma_i u^*; \gamma_i u^*)\right) \\
&= \mathcal{P}(\delta u^*, \delta v) - \mathcal{P}(\delta u^*, \delta u^*) + \sum_{i=1}^{k} \left(\Upsilon_i^0(\gamma_i u^*; \gamma_i v - \gamma_i u^*) - \Upsilon_i^0(\gamma_i u^*; \mathbf{0}_{E_i})\right) \\
&= \mathcal{P}(\delta u^*, \delta v) - \mathcal{P}(\delta u^*, \delta u^*) + \sum_{i=1}^{k} \Upsilon_i^0(\gamma_i u^*; \gamma_i v - \gamma_i u^*),
\end{aligned}$$

that is,

$$\langle \mathcal{G}u^*, v - u^* \rangle_E + \mathcal{P}(\delta u^*, \delta v) - \mathcal{P}(\delta u^*, \delta u^*)$$
$$+ \sum_{i=1}^{k} \Upsilon_i^0(\gamma_i u^*; \gamma_i v - \gamma_i u^*) \geq \langle f, v - u^* \rangle_E$$

which implies that u^* is a solution of Problem 2.1. Thus, $\mathcal{F}_{\omega,f}$ is a gap function for Problem 2.1. □

Next, we will establish the D-gap function for Problem 2.1 by using the regularized gap functions of the Fukushima type given above. To this end, let the regularized gap function $\mathcal{F}_{\omega,f}$ be defined by (3.5). Now, we will consider the function $\mathfrak{D}^f_{\omega,\rho}\colon \mathcal{W} \to \mathbb{R}$ defined by

$$\mathfrak{D}^f_{\omega,\rho}(u) = \mathcal{F}_{\omega,f}(u) - \mathcal{F}_{\rho,f}(u) \tag{3.9}$$

where $\rho > \omega > 0$. Then, we obtain the following property of the function $\mathfrak{D}^f_{\omega,\rho}$.

Lemma 3.6. *Suppose that the hypotheses of Theorem 3.5 hold. Then, for any $\rho > \omega > 0$, we have*

$$\left\|u - v_{\rho,f}(u)\right\|_E^2 \leq \frac{2}{\rho - \omega}\mathfrak{D}^f_{\omega,\rho}(u) \leq \left\|u - v_{\omega,f}(u)\right\|_E^2, \tag{3.10}$$

for all $u \in \mathcal{W}$, where

$$v_{\omega,f}(u) = \arg\min_{v\in\mathcal{W}} \Xi_{\omega,f}(u,v) \text{ and } v_{\rho,f}(u) = \arg\min_{v\in\mathcal{W}} \Xi_{\rho,f}(u,v).$$

Proof. By the definitions of the gap functions $\mathcal{F}_{\omega,f}, \mathcal{F}_{\rho,f}$ and the function $\mathfrak{D}^f_{\omega,\rho}$, we see that

$$\begin{aligned}\mathfrak{D}^f_{\omega,\rho}(u) &= \sup_{v\in\mathcal{W}}\{-\Xi_{\omega,f}(u,v)\} - \sup_{v\in\mathcal{W}}\{-\Xi_{\rho,f}(u,v)\} \\ &\leq -\Xi_{\omega,f}(u, v_{\omega,f}(u)) + \Xi_{\rho,f}(u, v_{\omega,f}(u)) \\ &= \frac{\rho-\omega}{2}\left\|u - v_{\omega,f}(u)\right\|_E^2.\end{aligned}$$

Thus, the right-hand-side inequality in (3.10) holds. Similarly, we obtain the left-hand-side inequality in (3.10). □

Theorem 3.7. *Suppose that the hypotheses of Theorem 3.5 hold. Then, for any $\rho > \omega > 0$, the function $\mathfrak{D}^f_{\omega,\rho}$ defined by (3.9) is a gap function for Problem 2.1.*

Proof. (a) It is clearly follows from (3.10) that $\mathfrak{D}^f_{\omega,\rho}(u) \geq 0$, for all $u \in \mathcal{W}$.

(b) Suppose that u^* is a solution of Problem 2.1. It follows from Theorem 3.5 that $\mathcal{F}_{\omega,f}(u^*) = \mathcal{F}_{\rho,f}(u^*) = 0$ and so $\mathfrak{D}^f_{\omega,\rho}(u^*) = 0$.

Conversely, for any $u^* \in \mathcal{W}$ such that $\mathfrak{D}^f_{\omega,\rho}(u^*) = 0$. From (3.10), we have $u^* = v_{\rho,f}(u^*)$. Applying Lemma 3.4 with $u = u^*$ and $\omega = \rho$, we have,

$$\begin{aligned}&\langle \mathcal{G}u^* - f, v - u^*\rangle_E + \mathcal{P}(\delta u^*, \delta v) - \mathcal{P}(\delta u^*, \delta u^*) \\ &\qquad + \sum_{i=1}^k \Upsilon_i^0(\gamma_i u^*; \gamma_i v - \gamma_i u^*) \geq 0,\end{aligned}$$

for all $v \in \mathcal{W}$, which implies that u^* is a solution of Problem 2.1. Thus, $\mathfrak{D}^f_{\omega,\rho}$ is a gap function of Problem 2.1. □

Remark 3.8. (i) As discussed in the introduction, no work has been established on D-gap functions for variational-hemivariational inequalities. As a result, our Theorem 3.7 is new.

(ii) Furthermore, using a formulation of the optimality condition in Lemma 3.4, the method of proof in Theorem 3.5 for the regularized gap function $\mathcal{F}_{\omega,f}$ considered to investigate the D-gap function $\mathfrak{D}^f_{\omega,\rho}$ for EVHIs is different from the corresponding results on regularized gap functions in [8, 15].

4 Global Error Bounds

In this section, we construct some global error bounds for Problem 2.1 given by the regularized gap function $\mathcal{F}_{\omega,f}$ and the D-gap function $\mathfrak{D}^f_{\omega,\rho}$ considered in Section 3.

Lemma 4.1. *Let $u^* \in \mathcal{W}$ be the unique solution to Problem 2.1. Assume that the hypotheses $\mathfrak{H}(\mathcal{G})$, $\mathfrak{H}(\mathcal{P})$, $\mathfrak{H}(\Upsilon)$, $\mathfrak{H}(\mathcal{W})$, $\mathfrak{H}(\mathcal{K})$, $\mathfrak{H}(\delta)$, $\mathfrak{H}(\gamma)$, $\mathfrak{H}(f)$ and $\mathfrak{H}(const.)$ hold. Then, for each $u \in \mathcal{W}$, we have*

$$\|u-u^*\|_E \le \frac{l_{\mathcal{G}} + \rho + \sum_{i=1}^k L_{\Upsilon_i} c^2_{\Upsilon_i}}{m_{\mathcal{G}} - \alpha_{\mathcal{P}} c^2_{\mathcal{P}} - \sum_{i=1}^k L_{\Upsilon_i} c^2_{\Upsilon_i}} \|u - v_{\rho,f}(u)\|_E. \tag{4.1}$$

Proof. For each $u \in \mathcal{W}$, since $v_{\rho,f}(u) \in \mathcal{W}$ and $u^* \in \mathcal{W}$ is a solution of Problem 2.1,

$$\begin{aligned}\langle \mathcal{G}u^* - f, v_{\rho,f}(u) - u^*\rangle_E + \mathcal{P}(\delta u^*, \delta v_{\rho,f}(u)) - \mathcal{P}(\delta u^*, \delta u^*)\\ + \sum_{i=1}^k \Upsilon_i^0(\gamma_i u^*; \gamma_i v_{\rho,f}(u) - \gamma_i u^*) \ge 0.\end{aligned} \tag{4.2}$$

Moreover, we add (3.3) with $\omega = \rho, v = u^*$ and obtain

$$\begin{aligned}\langle \mathcal{G}u - f + \rho(v_{\rho,f}(u) - u), u^* - v_{\rho,f}(u)\rangle_E + \mathcal{P}(\delta u, \delta u^*) - \mathcal{P}(\delta u, \delta v_{\rho,f}(u))\\ + \sum_{i=1}^k \Upsilon_i^0\left(\gamma_i u; \gamma_i u^* - \gamma_i v_{\rho,f}(u)\right) \ge 0.\end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3) yields

$$\begin{aligned}0 \le\ & \langle \mathcal{G}u^* - \mathcal{G}u, v_{\rho,f}(u) - u^*\rangle_E\\ & + \mathcal{P}(\delta u^*, \delta v_{\rho,f}(u)) - \mathcal{P}(\delta u^*, \delta u^*) + \mathcal{P}(\delta u, \delta u^*) - \mathcal{P}(\delta u, \delta v_{\rho,f}(u))\\ & + \sum_{i=1}^k \left[\Upsilon_i^0(\gamma_i u^*; \gamma_i v_{\rho,f}(u) - \gamma_i u^*) + \Upsilon_i^0\left(\gamma_i u; \gamma_i u^* - \gamma_i v_{\rho,f}(u)\right)\right]\\ & + \rho \left\langle v_{\rho,f}(u) - u, u^* - v_{\rho,f}(u)\right\rangle_E .\end{aligned} \tag{4.4}$$

Since $\mathcal{G}$ is Lipschitz continuous with the constant $l_{\mathcal{G}}$ and the condition $\mathfrak{H}(\mathcal{G})$(b) holds, we have

$$\begin{aligned}& \langle \mathcal{G}u^* - \mathcal{G}u, v_{\rho,f}(u) - u^*\rangle_E\\ & = \langle \mathcal{G}u^* - \mathcal{G}u, v_{\rho,f}(u) - u\rangle_E - \langle \mathcal{G}u^* - \mathcal{G}u, u^* - u\rangle_E\\ & \le l_{\mathcal{G}}\|u-u^*\|_E \|u - v_{\rho,f}(u)\|_E - m_{\mathcal{G}}\|u-u^*\|_E^2.\end{aligned} \tag{4.5}$$

Moreover, we also obtain

$$\begin{aligned}& \rho \left\langle v_{\rho,f}(u) - u, u^* - v_{\rho,f}(u)\right\rangle_E\\ & = \rho \left\langle v_{\rho,f}(u) - u, u^* - u\right\rangle_E + \rho \left\langle v_{\rho,f}(u) - u, u - v_{\rho,f}(u)\right\rangle_E\\ & \le \rho\|v_{\rho,f}(u) - u\|_E \|u^* - u\|_E - \rho\|v_{\rho,f}(u) - u\|_E^2\\ & \le \rho\|v_{\rho,f}(u) - u\|_E \|u^* - u\|_E.\end{aligned} \tag{4.6}$$

Using the conditions $\mathfrak{H}(\mathcal{P})$(b) and $\mathfrak{H}(\delta)$ lead to

$$\mathcal{P}(\delta u^*, \delta v_{\rho,f}(u)) - \mathcal{P}(\delta u^*, \delta u^*) + \mathcal{P}(\delta u, \delta u^*) - \mathcal{P}(\delta u, \delta v_{\rho,f}(u))$$

$$\le \alpha_{\mathcal{P}} \|\delta u^* - \delta u\|_{E_{\mathcal{P}}} \|\delta v_{\omega,f}(u) - \delta u^*\|_{E_{\mathcal{P}}}$$
$$\le \alpha_{\mathcal{P}} c_{\mathcal{P}}^2 \|u^* - u\|_E^2 + \alpha_{\mathcal{P}} c_{\mathcal{P}}^2 \|u^* - u\|_E \|u - v_{\rho,f}(u)\|_E. \tag{4.7}$$

For each $i \in \{1, \ldots, k\}$, by the conditions $\mathfrak{H}(\Upsilon)$(b) and $\mathfrak{H}(\gamma)$, we have

$$\Upsilon_i^0(\gamma_i u^*; \gamma_i v_{\rho,f}(u) - \gamma_i u^*) + \Upsilon_i^0(\gamma_i u; \gamma_i u^* - \gamma_i v_{\rho,f}(u))$$
$$\le L_{\Upsilon_i} \|\gamma_i u^* - \gamma_i u\|_{E_{\Upsilon_i}} \|\gamma_i v_{\rho,f}(u) - \gamma_i u^*\|_{E_{\Upsilon_i}}$$
$$\le L_{\Upsilon_i} c_{\Upsilon_i}^2 \|u^* - u\|_E^2 + L_{\Upsilon_i} c_{\Upsilon_i}^2 \|u^* - u\|_E \|u - v_{\rho,f}(u)\|_E. \tag{4.8}$$

From (4.4)–(4.8), we have

$$\left(m_{\mathcal{G}} - \alpha_{\mathcal{P}} c_{\mathcal{P}}^2 - \sum_{i=1}^k L_{\Upsilon_i} c_{\Upsilon_i}^2 \right) \|u - u^*\|_E^2$$
$$\le \left(l_{\mathcal{G}} + \rho + \sum_{i=1}^k L_{\Upsilon_i} c_{\Upsilon_i}^2 \right) \|u - u^*\|_E \|u - v_{\rho,f}(u)\|_E.$$

This implies that

$$\|u - u^*\|_E \le \frac{l_{\mathcal{G}} + \rho + \sum_{i=1}^k L_{\Upsilon_i} c_{\Upsilon_i}^2}{m_{\mathcal{G}} - \alpha_{\mathcal{P}} c_{\mathcal{P}}^2 - \sum_{i=1}^k L_{\Upsilon_i} c_{\Upsilon_i}^2} \|u - v_{\rho,f}(u)\|_E.$$

Thus, the inequality (4.1) holds. □

From Lemma 4.1, we get the following global error bound for Problem 2.1 by using the regularized gap function of Fukushima type $\mathcal{F}_{\omega,f}$.

Theorem 4.2. *Let $u^* \in \mathcal{W}$ be the unique solution to Problem 2.1. Assume that the hypotheses of Lemma 4.1 hold. Then, for each $u \in \mathcal{W}$, we can get the following global error bound by the gap function $\mathcal{F}_{\omega,f}$ for Problem 2.1:*

$$\|u - u^*\|_E \le \frac{l_{\mathcal{G}} + \omega + \sum_{i=1}^k L_{\Upsilon_i} c_{\Upsilon_i}^2}{m_{\mathcal{G}} - \alpha_{\mathcal{P}} c_{\mathcal{P}}^2 - \sum_{i=1}^k L_{\Upsilon_i} c_{\Upsilon_i}^2} \sqrt{\frac{2}{\omega} \mathcal{F}_{\omega,f}(u)}. \tag{4.9}$$

Proof. For any $u \in \mathcal{W}$, taking $v = u$ in (3.3), we have

$$\langle \mathcal{G}u - f + \omega(v_{\omega,f}(u) - u), u - v_{\omega,f}(u) \rangle_E + \mathcal{P}(\delta u, \delta u) - \mathcal{P}(\delta u, v_{\omega,f}(u))$$
$$+ \sum_{i=1}^k \Upsilon_i^0(\gamma_i u; \gamma_i u - \gamma_i v_{\omega,f}(u)) \ge 0.$$

Equivalently,

$$-\langle \mathcal{G}u - f, v_{\omega,f}(u) - u \rangle_E - \mathcal{P}(\delta u, v_{\omega,f}(u)) + \mathcal{P}(\delta u, \delta u)$$
$$+ \sum_{i=1}^k \Upsilon_i^0(\gamma_i u; \gamma_i u - \gamma_i v_{\omega,f}(u)) - \frac{\omega}{2} \|u - v_{\omega,f}(u)\|_E^2$$
$$\ge \frac{\omega}{2} \|u - v_{\omega,f}(u)\|_E^2,$$

which implies that

$$-\Xi_{\omega,f}(u, v_{\omega,f}(u)) \geq \frac{\omega}{2}\|u - v_{\omega,f}(u)\|_E^2. \tag{4.10}$$

It follows from (3.5) and (4.10) that

$$\|u - v_{\omega,f}(u)\|_E \leq \sqrt{\frac{2}{\omega}\mathcal{F}_{\omega,f}(u)}. \tag{4.11}$$

From taking $\rho = \omega$ in (4.1) and (4.11), we obtain

$$\|u - u^*\|_E \leq \frac{l_{\mathcal{G}} + \omega + \sum_{i=1}^k L_{\Upsilon_i} c_{\Upsilon_i}^2}{m_{\mathcal{G}} - \alpha_{\mathcal{P}} c_{\mathcal{P}}^2 - \sum_{i=1}^k L_{\Upsilon_i} c_{\Upsilon_i}^2} \sqrt{\frac{2}{\omega}\mathcal{F}_{\omega,f}(u)}.$$

Thus, the inequality (4.9) holds. □

Without using the Lipschitz continuity of $\mathcal{G}$, we can also provide an error bound for Problem 2.1.

Theorem 4.3. *Let $u^* \in \mathcal{W}$ be the unique solution to Problem 2.1. Assume that the hypotheses $\mathfrak{H}(\mathcal{G})(b)$, $\mathfrak{H}(\mathcal{P})$, $\mathfrak{H}(\Upsilon)$, $\mathfrak{H}(\mathcal{W})$, $\mathfrak{H}(\mathcal{K})$, $\mathfrak{H}(\delta)$, $\mathfrak{H}(\gamma)$ and $\mathfrak{H}(f)$ hold. Then, for each $\omega > 0$, $u \in \mathcal{W}$, for any $\omega > 0$ satisfying*

$$m_{\mathcal{G}} - \alpha_{\mathcal{P}} c_{\mathcal{P}}^2 - \sum_{i=1}^k L_{\Upsilon_i} c_{\Upsilon_i}^2 - \frac{\omega}{2} > 0,$$

one has

$$\|u - u^*\|_E \leq \frac{1}{\sqrt{m_{\mathcal{G}} - \alpha_{\mathcal{P}} c_{\mathcal{P}}^2 - \sum_{i=1}^k L_{\Upsilon_i} c_{\Upsilon_i}^2 - \frac{\omega}{2}}} \sqrt{\mathcal{F}_{\omega,f}(u)}. \tag{4.12}$$

Proof. Let $u^* \in \mathcal{W}$ be the unique solution to Problem 2.1. Fix an arbitrary $u \in \mathcal{W}$, it follows from the definition of $\mathcal{F}_{\omega,f}$ that

$$\begin{aligned}
\mathcal{F}_{\omega,f}(u) &= \sup_{v \in \mathcal{W}} \{-\Xi_{\omega,f}(u, v)\} \\
&\geq -\Xi_{\omega,f}(u, u^*) \\
&= \langle f - \mathcal{G}u, u^* - u\rangle_E + \mathcal{P}(\delta u, \delta u) - \mathcal{P}(\delta u, \delta u^*) \\
&\quad - \sum_{i=1}^k \Upsilon_i^0(\gamma_i u; \gamma_i u^* - \gamma_i u) - \frac{\omega}{2}\big\|u - u^*\big\|_E^2.
\end{aligned} \tag{4.13}$$

Since u^* is a solution to Problem 2.1, we have

$$\begin{aligned}
\langle \mathcal{G}u^* - f, u - u^*\rangle_E &+ \mathcal{P}(\delta u^*, \delta u) - \mathcal{P}(\delta u^*, \delta u^*) \\
&+ \sum_{i=1}^k \Upsilon_i^0(\gamma_i u^*; \gamma_i u - \gamma_i u^*) \geq 0.
\end{aligned} \tag{4.14}$$

The condition $\mathfrak{H}(\mathcal{G})$(c) implies that

$$\langle f - \mathcal{G}u, u^* - u\rangle_E - \langle \mathcal{G}u^* - f, u - u^*\rangle_E$$

$$
\begin{aligned}
&= \langle \mathcal{G}u^* - \mathcal{G}u, u^* - u\rangle_E \\
&\geq m_{\mathcal{G}}\|u - u^*\|_E^2. \qquad (4.15)
\end{aligned}
$$

It follows from the conditions $\mathfrak{H}(\mathcal{P})$(b), $\mathfrak{H}(\Upsilon)$(b), $\mathfrak{H}(\delta)$ and $\mathfrak{H}(\gamma)$ that

$$
\begin{aligned}
&- \left[\mathcal{P}(\delta u, \delta u^*) - \mathcal{P}(\delta u, \delta u) + \mathcal{P}(\delta u^*, \delta u^*) - \mathcal{P}(\delta u^*, \delta)\right] \\
&- \sum_{i=1}^{k} \left[\Upsilon_i^0(\gamma_i u; \gamma_i u^* - \gamma_i u) + \Upsilon_i^0(\gamma_i u^*; \gamma_i u - \gamma_i u^*)\right] \\
&\geq -\alpha_{\mathcal{P}}\|\delta u^* - \delta u\|_{E_{\mathcal{P}}}^2 - \sum_{i=1}^{k} L_{\Upsilon_i}\|\gamma_i u^* - \gamma_i u\|_{E_{\Upsilon_i}}^2 \\
&\geq -\left(\alpha_{\mathcal{P}} c_{\mathcal{P}}^2 + \sum_{i=1}^{k} L_{\Upsilon_i} c_{\Upsilon_i}^2\right)\|u - u^*\|_E^2. \qquad (4.16)
\end{aligned}
$$

Having in mind relations (4.14)–(4.16), it follows that

$$
\begin{aligned}
\langle f - \mathcal{G}u, u^* - u\rangle_E + \mathcal{P}(\delta u, \delta u) - \mathcal{P}(\delta u, \delta u^*) - \sum_{i=1}^{k} \Upsilon_i^0(\gamma_i u; \gamma_i u^* - \gamma_i u) \\
\geq \left(m_{\mathcal{G}} - \alpha_{\mathcal{P}} c_{\mathcal{P}}^2 - \sum_{i=1}^{k} L_{\Upsilon_i} c_{\Upsilon_i}^2\right)\|u - u^*\|_E^2. \qquad (4.17)
\end{aligned}
$$

Combining (4.13) and (4.17), we have

$$
\left(m_{\mathcal{G}} - \alpha_{\mathcal{P}} c_{\mathcal{P}}^2 - \sum_{i=1}^{k} L_{\Upsilon_i} c_{\Upsilon_i}^2 - \frac{\omega}{2}\right)\left\|u - u^*\right\|_E^2 \leq \mathcal{F}_{\omega,f}(u).
$$

Then, the desired inequality (4.12) follows. □

We conclude this section with the global error bounds for Problem 2.1 associated with the D-gap function.

Theorem 4.4. *Let $u^* \in \mathcal{W}$ be the unique solution to Problem 2.1. Assume that the hypotheses of Lemma 4.1 hold. Then, for each $u \in \mathcal{W}$, we can get the following global error bound by $\mathfrak{D}_{\omega,\rho}^f$ for Problem 2.1:*

$$
\|u - u^*\|_E \leq \frac{l_{\mathcal{G}} + \rho + \sum_{i=1}^{k} L_{\Upsilon_i} c_{\Upsilon_i}^2}{m_{\mathcal{G}} - \alpha_{\mathcal{P}} c_{\mathcal{P}}^2 - \sum_{i=1}^{k} L_{\Upsilon_i} c_{\Upsilon_i}^2}\sqrt{\frac{2}{\rho - \omega}\mathfrak{D}_{\omega,\rho}^f(u)}. \qquad (4.18)
$$

Proof. The inequality (4.18) is a consequence of (3.10) and (4.1). □

Remark 4.5. (i) By Remark 3.8 (i), the error bound for Problem 2.1 in Theorem 4.4 with respect to the D-gap function $\mathfrak{D}_{\omega,\rho}^f$ is new.

(ii) On the other hand, the new error bounds in Theorem 4.2 and Theorem 4.3 via the regularized gap function $\mathcal{F}_{\omega,f}$ extend to the corresponding results in [8,15]. Furthermore, Theorem 4.2 and Theorem 4.3 also extend to the error bound studied in Proposition 3.4 of [39] for strongly monotone variational inequalities.

5 Application to Contact Mechanics

The contact model will be described in this section, together with its variational formulation, which demonstrates the applicability of the abstract results presented in the previous sections. The physical setting and notation are as follows.

An elastic body occupies an open, connected and bounded set Ω in $\mathbb{R}^l$ ($l = 2, 3$) with Lipschitz continuous boundary Γ divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 with meas$(\Gamma_1) > 0$. The body is fixed on Γ_1 and in contact on Γ_3 with a foundation. Moreover, it is in equilibrium under the action of a surface traction of density $\boldsymbol{f}_2$ on Γ_2 and a volume force of density $\boldsymbol{f}_0$ in Ω.

Let $\mathbb{S}^l$ be the space of second order symmetric tensors on $\mathbb{R}^l$. Denote by $\boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^l$ and $\boldsymbol{v} = (v_i) \in \mathbb{R}^l$, where $i, j \in \{1, ..., l\}$. Let $\boldsymbol{\nu} = (\nu_i)$ be the unit outward normal vector on the boundary Γ and $\boldsymbol{x} = (x_i)$ for a generic point in $\Omega \cup \Gamma$. Unless stated otherwise, denote $\mathbf{0}$ by the zero element of $\mathbb{R}^l$ and $\mathbb{S}^l$, and the summation convention over repeated indices is used. The inner products and the Euclidean norms on $\mathbb{R}^l$ and $\mathbb{S}^l$ are given by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \qquad \|\boldsymbol{u}\| = (\boldsymbol{u} \cdot \boldsymbol{u})^{\frac{1}{2}}, \quad \text{for all } \boldsymbol{u} = (u_i), \boldsymbol{v} = (v_i) \in \mathbb{R}^l;$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}}, \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}) \in \mathbb{S}^l, \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^l.$$

For a vector field $\boldsymbol{v}$, $v_\nu := \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau := \boldsymbol{v} - v_\nu \boldsymbol{\nu}$ denote the normal and tangential components of $\boldsymbol{v}$ on Γ. Also, the normal and tangential components of the stress field $\boldsymbol{\sigma}$ on the boundary are denoted by $\sigma_\nu := (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau := \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$. For the stress and strain fields, we shall use the Hilbert space $V = L^2(\Omega; \mathbb{S}^l)$ with the canonical inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_V := \int_\Gamma \sigma_{ij}(\boldsymbol{x})\tau_{ij}(\boldsymbol{x})dx, \quad \boldsymbol{\sigma}, \boldsymbol{\tau} \in V$$

and the associated norm $\|\cdot\|_V$. The function space for the displacement field is defined by

$$E := \{\boldsymbol{v} = (v_i) \in H^1(\Omega; \mathbb{R}^l) \mid \boldsymbol{v} = \mathbf{0} \text{ a.e on } \Gamma_1\}.$$

It follows from an application of Korn's inequality and meas$(\Gamma_1) > 0$ that E is real Hilbert space endowed with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_E := \int_\Gamma \boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{\varepsilon}(\boldsymbol{v})dx, \quad \boldsymbol{u}, \boldsymbol{v} \in E,$$

and the associated norm $\|\cdot\|_E$, where $\boldsymbol{\varepsilon}$ represents the deformation operator defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{v_{i,j} + v_{j,i}}{2} \quad \forall \boldsymbol{v} \in V.$$

We shall use Div to denote the divergence operator given by

$$\operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j}) = \left(\frac{\partial \sigma_{ij}}{\partial x_j}\right)$$

and the same symbol $\boldsymbol{v}$ for the trace of a function $\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^l)$ on Γ. By the Sobolev trace theorem, we have

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3;\mathbb{R}^l)} \le \|\delta\|\|\boldsymbol{v}\|_E, \quad \forall \boldsymbol{v} \in E,$$

where $\|\delta\|$ is the norm of the trace operator $\delta \colon E \to L^2(\Gamma_3; \mathbb{R}^l)$. With the aforementioned discussions, we revisit the following formulation of contact problems considered in [9,11,12].

Problem 5.1. Find a displacement field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^l$ and a stress field $\boldsymbol{\sigma}\colon \Omega \to \mathbb{S}^l$ such that

$$\boldsymbol{\sigma} = \mathcal{M}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{5.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{5.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{5.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \tag{5.4}$$

$$\begin{cases} u_\nu \le g, \quad \sigma_\nu + \zeta_\nu \le 0, \\ (u_\nu - g)(\sigma_\nu + \zeta_\nu) = 0, \quad \zeta_\nu \in \partial h_\nu(u_\nu), \end{cases} \quad \text{on } \Gamma_3, \tag{5.5}$$

$$\|\boldsymbol{\sigma}_\tau\| \le \mathcal{N}_b(u_\nu), \ -\boldsymbol{\sigma}_\tau = \mathcal{N}_b(u_\nu)\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \ \text{ if } \boldsymbol{u}_\tau \ne \boldsymbol{0}, \quad \text{on } \Gamma_3. \tag{5.6}$$

The elastic constitutive law is described in (5.1), where $\mathcal{M}\colon \Omega \times \mathbb{S}^l \to \mathbb{S}^l$ denotes the elasticity operator and satisfies the following conditions:

$$\begin{cases} \text{(a) there exists } L_{\mathcal{M}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^l, \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \qquad \|\mathcal{M}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{M}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{M}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|; \\ \text{(b) } \mathcal{M}(\cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^l \\ \qquad \text{with } \mathcal{M}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0} \text{ for a.e. } \boldsymbol{x} \in \Omega; \\ \text{(c) there exists } m_{\mathcal{M}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^l, \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \qquad (\mathcal{M}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{M}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{M}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2. \end{cases} \tag{5.7}$$

Equation (5.2) represents the equilibrium equation and the classical displacement-traction boundary conditions are described equations (5.3) and (5.4), where $\boldsymbol{f}_0$ and $\boldsymbol{f}_2$ are assumed to satisfy

$$\boldsymbol{f}_0 \in L^2(\Omega; \mathbb{R}^l), \quad \boldsymbol{f}_2 \in L^2(\Gamma_2; \mathbb{R}^l). \tag{5.8}$$

We also define $\boldsymbol{f} \in V^*$ by the relation

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle_V = (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^l)} + (\boldsymbol{f}_2, \boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^l)} \quad \forall \boldsymbol{v} \in V. \tag{5.9}$$

The contact condition formulated on the surface Γ_3 is represented in (5.5), where $g\colon \Gamma_3 \to \mathbb{R}$ describes the thickness of the elastic layer. Assume that

$$g \in L^2(\Gamma_3), \quad g(\boldsymbol{x}) \ge 0 \quad \text{a.e. on } \Gamma_3. \tag{5.10}$$

Moreover, we define an admissible set K in E as follows:

$$K = \{\boldsymbol{v} \in E \mid v_\nu \le g \text{ on } \Gamma_3\}.$$

For the potential function $h_\nu\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$, we assume

$$\begin{cases} \text{(a) } h_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there} \\ \quad \text{exists } \widetilde{e} \in L^2(\Gamma_3) \text{ such that } h_\nu(\cdot, \widetilde{e}(\cdot)) \in L^1(\Gamma_3). \\ \text{(b) } h_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) there exist } \bar{c}_0, \bar{c}_1 \ge 0 \text{ such that} \\ \qquad |\partial h_\nu(\boldsymbol{x}, r)| \le \bar{c}_0 + \bar{c}_1|r| \text{ for all } r \in \mathbb{R} \text{ and a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(d) there exists } L_{h_\nu} \ge 0 \text{ such that} \\ \qquad h_\nu^0(\boldsymbol{x}, s_1; r_2 - r_1) + h_\nu^0(\boldsymbol{x}, s_2; r_1 - r_2) \\ \qquad\qquad \le L_{h_\nu}|s_1 - s_2||r_1 - r_2|, \\ \quad \forall r_1, r_2, s_1, s_2 \in \mathbb{R} \text{ and a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{5.11}$$

The condition (5.6) represents a version of Coulomb's law of dry friction, where $\mathcal{N}_b\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}_+$ denotes the friction bound which may depend on the normal displacement u_ν, and we assume

$$\begin{cases} \text{(a) } \mathcal{N}_b(\cdot,r) \text{ is measurable on } \Gamma_3 \text{ for all } r\in\mathbb{R}. \\ \text{(b) } \mathcal{N}_b(\boldsymbol{x},r)=0 \text{ for all } r\le 0, \\ \quad \mathcal{N}_b(\boldsymbol{x},r)\ge 0 \text{ for all } r\ge 0, \text{ a.e. } \boldsymbol{x}\in\Gamma_3; \\ \text{(c) } (\mathcal{N}_b(\boldsymbol{x},r_1)-\mathcal{N}_b(\boldsymbol{x},r_2))(r_1-r_2)\ge 0, \\ \quad \forall r_1,r_2\in\mathbb{R} \text{ and a.e. } \boldsymbol{x}\in\Gamma_3; \\ \text{(d) there exists } L_{\mathcal{N}_b}>0 \text{ such that} \\ \qquad |\mathcal{N}_b(\boldsymbol{x},r_1)-\mathcal{N}_b(\boldsymbol{x},r_2)|\le L_{\mathcal{N}_b}|r_1-r_2| \\ \quad \forall r_1,r_2\in\mathbb{R} \text{ and a.e. } \boldsymbol{x}\in\Gamma_3. \end{cases} \tag{5.12}$$

We refer to [12, 30, 37] for more information and mechanical interpretation of static contact models with elastic materials. The variational formulation of the contact problem 5.1 is in the following form:

Problem 5.2. Find a displacement field $\boldsymbol{u}\in K$ such that

$$\begin{aligned}(\mathcal{M}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}))_V &+ \int_{\Gamma_3}\mathcal{N}_b(u_\nu)\cdot(\|\boldsymbol{v}_\tau\|-\|\boldsymbol{u}_\tau\|)ds \\ &+\int_{\Gamma_3}h_\nu^0(u_\nu;v_\nu-u_\nu)ds\ge\int_\Omega\boldsymbol{f}_0\cdot(\boldsymbol{v}-\boldsymbol{u})dx+\int_{\Gamma_2}\boldsymbol{f}_2\cdot(\boldsymbol{v}-\boldsymbol{u})d\Gamma\end{aligned}$$

for all $\boldsymbol{v}\in K$.

To apply the results presented in the previous sections on Problem 5.2, we let $k=1$, $\mathcal{W}=K$, $E_{\mathcal{P}}=L^2(\Gamma_3;\mathbb{R}^l)$ with δ the trace operator from E to $E_{\mathcal{P}}$, $E_\Upsilon=E_{\Upsilon_1}=L^2(\Gamma_3;\mathbb{R})$ with $\gamma\boldsymbol{v}=\gamma_1\boldsymbol{v}=v_\nu$ for $\boldsymbol{v}\in E$, and we define

$$\begin{aligned}&\mathcal{G}\colon E\to E^*,\quad \langle\mathcal{G}\boldsymbol{u},\boldsymbol{v}\rangle_E=(\mathcal{M}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_V \quad \text{for } \boldsymbol{u},\boldsymbol{v}\in E,\\ &\mathcal{P}\colon L^2(\Gamma_3;\mathbb{R}^l)\times L^2(\Gamma_3;\mathbb{R}^l)\to\mathbb{R},\\ &\qquad \mathcal{P}(\delta\boldsymbol{u},\delta\boldsymbol{v})=\int_{\Gamma_3}\mathcal{N}_b(u_\nu)\|\boldsymbol{v}_\tau\|ds \quad \text{for } \boldsymbol{u},\boldsymbol{v}\in E,\\ &\Upsilon\colon L^2(\Gamma_3;\mathbb{R})\to\mathbb{R},\quad \Upsilon(\gamma\boldsymbol{v})=\int_{\Gamma_3}h_\nu(v_\nu)ds \quad \text{for } \boldsymbol{u},\boldsymbol{v}\in E,\\ &f=\boldsymbol{f}\in V^*,\quad \langle\boldsymbol{f},\boldsymbol{v}\rangle_E=\int_\Omega\boldsymbol{f}_0\cdot\boldsymbol{v}dx+\int_{\Gamma_2}\boldsymbol{f}_2\cdot\boldsymbol{v}d\Gamma \text{ for } \boldsymbol{u},\boldsymbol{v}\in E.\end{aligned}$$

It is easily seen that all conditions of Theorem 2.10 are satisfied with $m_{\mathcal{G}}=m_{\mathcal{M}}$, $l_{\mathcal{G}}=L_{\mathcal{M}}$, $\alpha_{\mathcal{P}}=L_{\mathcal{N}_b}$ and $L_\Upsilon=L_{\Upsilon_1}=L_{h_\nu}$.

Let $\lambda_{1,E}>0$ and $\lambda_{1\nu,E}>0$ be the smallest eigenvalues of the eigenvalue problem

$$\boldsymbol{u}\in E,\quad \int_\Omega\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx=\lambda\int_{\Gamma_3}\boldsymbol{u}\cdot\boldsymbol{v}\,d\Gamma\quad\forall\boldsymbol{v}\in E,$$

and the eigenvalue problem

$$\boldsymbol{u}\in E,\quad \int_\Omega\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx=\lambda\int_{\Gamma_3}u_\nu v_\nu d\Gamma\quad\forall\boldsymbol{v}\in E,$$

respectively. Then we may take

$$c_{\mathcal{P}} = \lambda_{1,E}^{-1/2}, \quad c_{\Upsilon} = \lambda_{1\nu,E}^{-1/2}.$$

Using Theorem 2.10, we can conclude that Problem 5.2 admits a solution. Furthermore, the smallness condition

$$L_{\mathcal{N}_b}\lambda_{1,E}^{-1} + L_{h_\nu}\lambda_{1\nu,E}^{-1} < m_{\mathcal{M}} \tag{5.13}$$

guarantees that Problem 5.2 is uniquely solvable (cf. [9, 11, 12]).

Next, for any parameter $\omega > 0$, we introduce the function $\widehat{\mathcal{F}}_{\omega,\boldsymbol{f}_{0,2}} : K \to \mathbb{R}$ defined by

$$\begin{aligned}\widehat{\mathcal{F}}_{\omega,\boldsymbol{f}_{0,2}}(\boldsymbol{u}) = \sup_{\boldsymbol{v}\in K}\Bigg(&(\mathcal{M}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{v}))_V + \int_{\Gamma_3} \mathcal{N}_b(u_\nu)\cdot(\|\boldsymbol{u}_\tau\| - \|\boldsymbol{v}_\tau\|)ds \\ &- \int_{\Gamma_3} h_\nu^0(u_\nu; v_\nu - u_\nu)ds + \int_\Omega \boldsymbol{f}_0\cdot(\boldsymbol{u}-\boldsymbol{v})dx \\ &+ \int_{\Gamma_2} \boldsymbol{f}_2\cdot(\boldsymbol{u}-\boldsymbol{v})d\Gamma - \frac{\omega}{2}\|\boldsymbol{u}-\boldsymbol{v}\|_E^2\Bigg).\end{aligned} \tag{5.14}$$

Applying Theorem 3.5, Theorem 3.7, Theorem 4.2, Theorem 4.3 and Theorem 4.4, we directly obtain the following error estimates with $l_{\mathcal{G}} = L_{\mathcal{M}}$.

Theorem 5.1. *Let $\boldsymbol{u}^* \in K$ be the unique solution to Problem 5.2. Under the hypotheses* (5.7)–(5.13)*, the following hold.*

(i) *For each $\omega > 0$, $\boldsymbol{f}_0 \in L^2(\Omega;\mathbb{R}^l)$ and $\boldsymbol{f}_2 \in L^2(\Gamma_2;\mathbb{R}^l)$, $\widehat{\mathcal{F}}_{\omega,\boldsymbol{f}_{0,2}}$ defined by* (5.14)*, is a regularized gap function for Problem 5.2.*

(ii) *If $\omega > 0$ then, for each $\boldsymbol{u} \in K$, it holds*

$$\|\boldsymbol{u}-\boldsymbol{u}^*\|_E \le \frac{L_{\mathcal{M}} + \omega + L_{h_\nu}\lambda_{1\nu,E}^{-1}}{m_{\mathcal{M}} - L_{\mathcal{N}_b}\lambda_{1,E}^{-1} - L_{h_\nu}\lambda_{1\nu,E}^{-1}}\sqrt{\frac{2}{\omega}\widehat{\mathcal{F}}_{\omega,\boldsymbol{f}_{0,2}}(\boldsymbol{u})}. \tag{5.15}$$

(iii) *If $\omega > 0$ satisfying*

$$m_{\mathcal{M}} - L_{\mathcal{N}_b}\lambda_{1,E}^{-1} - L_{h_\nu}\lambda_{1\nu,E}^{-1} - \frac{\omega}{2} > 0,$$

then, for each $\boldsymbol{u} \in K$, it also holds

$$\|\boldsymbol{u}-\boldsymbol{u}^*\|_E \le \frac{1}{\sqrt{m_{\mathcal{M}} - L_{\mathcal{N}_b}\lambda_{1,E}^{-1} - L_{h_\nu}\lambda_{1\nu,E}^{-1} - \frac{\omega}{2}}}\sqrt{\widehat{\mathcal{F}}_{\omega,\boldsymbol{f}_{0,2}}(\boldsymbol{u})}. \tag{5.16}$$

Theorem 5.2. *Let $\boldsymbol{u}^* \in K$ be the unique solution to Problem 5.2. Under the hypotheses* (5.7)–(5.13)*, the following hold.*

(i) *For any $\rho > \omega > 0$, $\boldsymbol{f}_0 \in L^2(\Omega;\mathbb{R}^l)$ and $\boldsymbol{f}_2 \in L^2(\Gamma_2;\mathbb{R}^l)$, the function $\widehat{\mathfrak{D}}_{\omega,\rho}^{\boldsymbol{f}_{0,2}} : K \to \mathbb{R}$ defined by*

$$\widehat{\mathfrak{D}}_{\omega,\rho}^{\boldsymbol{f}_{0,2}}(\boldsymbol{u}) = \widehat{\mathcal{F}}_{\omega,\boldsymbol{f}_{0,2}}(\boldsymbol{u}) - \widehat{\mathcal{F}}_{\rho,\boldsymbol{f}_{0,2}}(\boldsymbol{u})$$

is the D-gap function for Problem 5.2.

(ii) *If* $\omega > \rho > 0$ *then, for each* $\boldsymbol{u} \in K$*, it holds*

$$\|\boldsymbol{u} - \boldsymbol{u}^*\|_E \leq \frac{L_{\mathcal{M}} + \rho + L_{h_\nu}\lambda_{1\nu,E}^{-1}}{m_{\mathcal{M}} - L_{\mathcal{N}_b}\lambda_{1,E}^{-1} - L_{h_\nu}\lambda_{1\nu,E}^{-1}} \sqrt{\frac{2}{\rho - \omega}\widehat{\mathfrak{D}}_{\omega,\rho}^{\boldsymbol{f}_{0,2}}(\boldsymbol{u})}. \tag{5.17}$$

Remark 5.3. Theorem 5.1 and Theorem 5.2 give the upper bounds of the distance between an arbitrary displacement field in the admissible set and the unique solution of the contact problem. Computing the upper bounds in (5.15)–(5.17) is based on the regularized gap function $\widehat{\mathcal{F}}_{\omega,\boldsymbol{f}_{0,2}}$ and the D-gap function $\widehat{\mathfrak{D}}_{\omega,\rho}^{\boldsymbol{f}_{0,2}}(\boldsymbol{u})$ with depending on the data of the such contact problem.

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