Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 20, No. 4, pp. 769–779, 2024

SOME OPTIMALITY CONDITIONS FOR MINIMAX PROBLEM*

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The paper is dedicated to Professor Masao Fukushima's 75th birthday.‡

Abstract: Minimax problem has an important role in optimization, global optimization, game theory, and operations research. In [7], optimality conditions have been formulated for the maxmin problem. In a general case, since the maxmin and minimax values are not always equal, therefore, the optimality conditions for both problems might be different. The classical minimax theorem of von Neuman [21] deals with the equality conditions of maxmin and minimax values. In this paper, we derive new optimality conditions for the minimax problem based on Duvobizkii-Milyution theory (Duvobizkii and Milyuton in USSR Comput Math Math Phys 5:1-80, 1965).

Key words: *optimization, game theory, optimality condition, maxmin problem*

Mathematics Subject Classification: *49J35, 90C26, 90C27, 49K35*

1 Introduction

Classical minimax theory due to Von Neumann plays an important role in optimization and game theory. Minimax problems and techniques appear in different fields of research including game theory, optimization, and control theory. Many engineering and economics problems such as combinatorial optimization problems of scheduling, location, allocation, and packing as well as inventory problems are formulated as minimax applications. There

*This work was supported by the research grant (P2021-4193) of the National University of Mongolia.

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‡I have known Professor Masao Fukushima since 2001 when I first met him at Kyoto University during my JSPS fellowship period. I was invited two times by him to give talks at the seminars of his department. During 2005-2009, former my master's student Mend-Amar Majig from the National University of Mongolia did his Ph.D. under the supervision of Prof. M. Fukushima. We have also his joint research papers(M. Majig, B.Barsbold, R.Enkhbat and M.Fukushima, A Global Optimization Approach for Solving Non-Monotone Variational Inequality Problems, Optimization, Vol.58, No.7, pp.871-881, 2009; M.Mend-Amar, R.Enkhbat and M.Fukushima, Evolutionary Algorithm for Generalized Nash Equilibrium Problems, in book: "Optimization, Simulation, and Control",pp.97-106, Springer, 2013). In 2007, he was a plenary speaker at the second international conference on optimization and optimal control held in Ulaanbaatar, Mongolia. In 2017, he was invited by the School of Mathematics and Computer Sciences of the National University of Mongolia for our joint research. We thank very much Prof. M. Fukushima for his fruitful collaboration with his Mongolian colleagues and congratulate him on the occasion of his 75th birthday, and wish him all the success and happiness.

DOI: https://doi.org/10.61208/pjo-2024-006

are many works [16, 17, 26, 12, 13, 20, 10, 25] devoted to theory and algorithms of minimax and maxmin problems.

In recent years, many optimality conditions have developed for minimax problems. For instance, Truong et.al [2] proposed necessary optimality conditions in terms of upper or lower subdifferentials of both cost and constraint functions for minimax optimization problems. Linan and Yuan-Feng [30] are concerned with the study of optimality conditions for minimax optimization problems with an infinite number of constraints. Anulekha Dhara, Aparna Mehra [1] concerned with the second-order optimality conditions for minimax problems. Yu-Hong Dai, Liwei Zhang [3] provided both necessary optimality conditions and sufficient optimality conditions for the local minimax points of constrained minimax optimization problems.

Also, minimax problems find applications in areas such as machine learning, and signal processing, and it has been extensively studied in recent years in [18, 19, 24]. There are many generalizations of minimax theorems. Assume that X and Y are nonempty sets and $f : X \times Y \to \mathbb{R}$. A *minimax theorem* is a theorem which asserts that, under certain conditions,

$$\min_Y \max_X f = \max_X \min_Y f.$$

Theorem 1.1 ([20]). *Let X and Y be nonempty compact, convex subsets of Euclidean space, and f be continuous.*

Suppose that f is quasiconcave on X, that is to say,

$$\text{for all } y \in Y \text{ and } \lambda \in \mathbb{R}, L_f(\lambda, y) \text{ is a nonempty and convex}$$

and f is quasiconvex on Y, that is to say,

$$\text{for all } x \in X \text{ and } \lambda \in \mathbb{R}, L_f(x, \lambda) \text{ is a nonempty and convex.}$$

Then

$$\min_Y \max_X f = \max_X \min_Y f,$$

where $L_f(\lambda, y) = \{x : x \in X, f(x, y) \geq \lambda\}$ *and* $L_f(x, \lambda) = \{y : y \in Y, f(x, y) \leq \lambda\}$

In 1941, Kakutani [15] analyzed von Neumann's proof from a viewpoint of the fixed point theorem. In 1952, Fan [9] generalized Theorem 1.1 to the case X and Y are compact, convex subsets of (infinite dimensional) locally convex spaces and the quasiconcave and quasiconvex conditions are somewhat relaxed, while Nikaido [22], using Brouwer's fixed-point theorem directly, generalized the same result to the case when X and Y are nonempty compact, convex subsets of (not necessary locally convex) topological vector spaces and f is only required to be separately continuous. Nikaido also showed in [23] that, if the quasiconcave and quasiconvex functions are replaced by concave and convex, then a proof of the minimax theorem can be proven by elementary calculus.

Theorem 1.2 ([29]). *Let X be a topological space, Y be a compact separable topological space, and $f : X \times Y \to \mathbb{R}$ be separately continuous.*

Suppose that, for all $x_0, x_1 \in X$, there exits a continuous map $h : [0, 1] \to X$ such that $h(0) = x_0$, $h(1) = x_1$ and, for all $y \in Y$ and $\lambda \in \mathbb{R}$,

$$\{t : t \in [0, 1], f(h(t), y) \geq \lambda\}$$

is connected in [0,1].

Suppose also that, for all nonempty finite subsets W of X and $\lambda \in \mathbb{R}$,

$$L_f(W,\lambda) = \{y : y \in Y, f(x,y) < \lambda\}$$

is connected in Y.

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

In 1953, Fan took the theory of minimax theorems out of the context of convex subsets of vector spaces when he established the following results.

Theorem 1.3 ([8]). *Let X be a nonempty set and Y be a nonempty compact topological space. Let $f : X \times Y \to \mathbb{R}$ be lower semicontinuous on Y.*

Suppose that f is concave like on X and convex like on Y, that is to say:

$$\text{for all } x_1, x_2 \in X \text{ and } \alpha \in [0,1], \text{ there exists } x_3 \in X \text{ such that}$$
$$f(x_3,\cdot) \geq \alpha f(x_1,\cdot) + (1-\alpha) f(x_2,\cdot) \text{ on } Y,$$

and

$$\text{for all } y_1, y_2 \in Y \text{ and } \beta \in [0,1], \text{ there exists } y_3 \in Y \text{ such that}$$
$$f(\cdot,y_3) \geq \beta f(\cdot,y_1) + (1-\beta) f(\cdot,y_2) \text{ on } X.$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

In 1972, Terkelsen proved first mixed minimax theorem.

Theorem 1.4 ([27]). *Let X be a nonempty set and Y be a nonempty compact topological space. Let $f : X \times Y \to \mathbb{R}$ be lower semicontinuous on Y.*

Suppose that for all $x_1, x_2 \in X$ and $\alpha \in [0,1]$, there exists $x_3 \in X$ such that

$$f(x_3,\cdot) \geq [f(x_1,\cdot) + f(x_2,\cdot)]/2 \text{ on } Y.$$

Suppose also that, for all nonempty finite subsets W of X and $\lambda \in \mathbb{R}$, $L_f(W,\lambda)$ is connected in Y. Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

In [4], the following discrete minimax problem has been considered.

$$\min_{x \in D} \psi(x) \tag{1.1}$$

where, $\psi(x) = \max\limits_{1 \leq i \leq m} h_i(x)$ and $h_i(x)$ are continuously differentiable functions on a convex closed set $D \subset \mathbb{R}^n$. The optimality conditions for this problem are given by the following assertion.

Theorem 1.5 ([4]). *If x^* is a solution to problem (1.1), then*

$$\inf_{x \in D} \max_{i \in I(x^*)} \langle \frac{\partial h_i(x^*)}{\partial x}, x - x^* \rangle = 0 \tag{1.2}$$

where, $I(x)=\{i\in\{1,2,\cdots,m\}\mid h_i(x)=\psi(x)\}$.

In [7], the following maxmin problem has been examined from a viewpoint of optimality conditions.

$$\max_{x\in D}\tilde{\psi}(x) \tag{1.3}$$

$$D=\{x\in Q\mid f_i(x)\le 0,\ i=1,2,...,m\} \tag{1.4}$$

where, $\tilde{\psi}(x)=\min\limits_{y\in A}f(x,y)$, Q is a convex in $\mathbb{R}^n$, $intQ\neq\emptyset$, $f_i:\mathbb{R}^n\to\mathbb{R},\ i=1,...,m$ are differentiable functions, $f(x,y)$ and $\frac{\partial f(x,y)}{\partial x}$ are continuous on $D\times A$, A is compact in $\mathbb{R}^s$.

Theorem 1.6 ([7])**.** *If x^0 is a solution to problem (1.3)-(1.4), then there exist numbers $\alpha_1,\ \alpha_2,...,\alpha_{n+1},\ \lambda_1,\lambda_2,...\lambda_m$ and points $y^i\in Y(x^0)$, $i=1,2,...(n+1)$ such that*

$$\begin{cases}\lambda_i\ge 0,\ \lambda_i f_i(x^0)=0,\ i=1,2,...,m\\ \alpha_i\ge 0,\ i=1,2,...,n+1\\ \langle\sum\limits_{i=1}^{n+1}\alpha_i\frac{\partial f(x^0,y^i)}{\partial x}+\sum\limits_{i=1}^{m}\lambda_i\frac{\partial f_i(x^0)}{\partial x},x-x^0\rangle\ge 0,\forall x\in Q\end{cases} \tag{1.5}$$

where,

$$Y(x^0)=\{y\in A\mid f(x^0,y)=\tilde{\psi}(x^0)\}.$$

In 2001, I. Tseveendorj in [26] considered the following type of discrete maxmin problems called piecewise convex maximization problem.

Definition 1 ([26])**.** A function $\varphi:\mathbb{R}^n\to\mathbb{R}$ is called a piecewise convex function if it can be decomposed into:

$$\varphi(x)=\min_{1\le j\le m}f_j(x)$$

where $f_j:\mathbb{R}^n\to\mathbb{R}$ is convex for all $j=1,2,\cdots,m$.

Definition 2 ([26])**.** A problem

$$\max_{x\in D}\varphi(x) \qquad\qquad (PCMP)$$

is called a piecewise convex maximization problem, if $\varphi(x)$ is a piecewise convex function.

Piecewise convex maximization problem (PCMP) has been studied also in [5] and has many applications [20].

Proposition 1.7 ([26])**.** *If $z\in D$ is global maximizer of (PCMP) then for all $k\in I(z)=\{i\in\{1,2,\cdots,m\}\mid f_i(z)=\varphi(z)\}$*

$$\partial f_k(y)\bigcap N(D_k(z),y)\neq\mathbf{0}$$

$$\forall\ y\ \ s.t.\ f_k(y)=\varphi(z),$$

where

$$D_k(z)=D\cap\{x\mid f_j(x)>\varphi(z),\ \forall j=1,2,\cdots,m\setminus k\},$$

$$\partial f(y)=\{y^*\in\mathbb{R}\mid f(x)-f(y)\ge\langle y^*,x-y\rangle,\forall x\in\mathbb{R}^n\},$$

and

$$N(D,c)=\{c\in\mathbb{R}^n\mid\langle c,x-c\rangle\le 0,\ \forall x\in D\}.$$

$\boxed{2}$ Optimality Conditions for Minimax Problem

We consider the following minimax problem:

$$\min_{x \in D} \max_{y \in A} f(x, y) \tag{2.1}$$

where $f(x, y)$ and $\frac{\partial f(x,y)}{\partial x}$ are continuous on $D \times A$, A is compact in $\mathbb{R}^s$, D is convex and compact in $\mathbb{R}^n$.

According to the minimax Theorem [6] there exists a value v:

$$v = \min_{x \in D} \max_{y \in A} f(x, y). \tag{2.2}$$

We introduce the function

$$\varphi(x) = \max_{y \in A} f(x, y).$$

Definition 3. A point $x^0 \in D$ is called a local minimax solution if it is a local minimizer of the function $\varphi(x)$.

Assume that

$$\frac{\partial f(x^0, y)}{\partial x} \neq \mathbf{0} \subset \mathbb{R}^n, \ \forall y \in A.$$

Introduce the set

$$Y(x^0) = \{y \in A \mid f(x^0, y) = \varphi(x^0)\}. \tag{2.3}$$

It is clear that $Y(x^0)$ is compact.

Theorem 2.1 ([4])**.** *The function $\varphi(x)$ is continuous on $\mathbb{R}^n$ and differentiable in any directions $h \in \mathbb{R}^n$ with $\|h\| = 1$ at a point $x \in \mathbb{R}^n$ and*

$$\frac{\partial \varphi(x)}{\partial x} = \min_{y \in Y(x)} \langle \frac{\partial f(x, y)}{\partial x}, h \rangle.$$

Definition 4. A direction $h \in \mathbb{R}^n$ is said to be a descent direction of the function $\varphi(x)$ at a point x^0, if there exist a neighborhood $V(h)$ of the vector h and a number $\varepsilon_0 > 0$ such that

$$\varphi(x^0) > \varphi(x^0 + \varepsilon \bar{h}), \ \forall \bar{h} \in V(h), \ \forall \varepsilon \in (0, \varepsilon_0).$$

Denote by K the set of descent directions of the function $\varphi(x)$ at a point x^0.

Lemma 2.2. *The set K of descent directions is an open cone.*

Proof. Let h be a descent direction of the function $\varphi(x)$ at a point x^0. Since the function $\varphi(x)$ is continuous, then by the definition of descent directions, there exist scalars $\delta > 0$ and $\varepsilon_0 > 0$ such that

$$\varphi(x^0) > \varphi(x^0 + \varepsilon \bar{h}) \tag{2.4}$$

hold for all $\bar{h}$ and ε satisfying

$$\|h - \bar{h}\| < \delta, \quad \varepsilon \in (0, \varepsilon_0).$$

Now we show that λh directions for all $\lambda > 0$ are also descent. Indeed, if we take $\bar{\delta} = \lambda \delta$ and $\bar{\varepsilon} = \frac{\varepsilon_0}{\lambda}$ then (2.4) holds for hold for all $\bar{h}$ and ε such that

$$\|\lambda h - \lambda \bar{h}\| < \bar{\delta}, \quad \varepsilon \in (0, \bar{\varepsilon}).$$

$\square$

Lemma 2.3. *A set*

$$K_0 = \left\{ h \in \mathbb{R}^n \mid \max_{y \in Y(x^0)} \langle \frac{\partial f(x^0, y)}{\partial x}, h \rangle < 0 \right\} \tag{2.5}$$

is an open convex subcone of the cone of the descent directions of the function $\varphi(x)$ at a point x^0.

Proof. By construction, it is clear that K_0 is an open and convex cone. Indeed, since the function

$$g(h) = \max_{y \in Y(x^0)} \langle \frac{\partial f(x^0, y)}{\partial x}, h \rangle$$

is convex, then K_0 is convex.

Let us show $K_0 \subset K$. Assume that $h \in K_0$. Due to continuity of $g(h)$, there exists a neighborhood $V(h)$ of h such that

$$g(\bar{h}) = \max_{y \in Y(x^0)} \langle \frac{\partial f(x^0, y)}{\partial x}, \bar{h} \rangle < 0, \ \forall \bar{h} \in V(h).$$

The function $\varphi(x)$ is differentiable in directions [5], and it follows that

$$\varphi(x^0 + \varepsilon \bar{h}) - \varphi(x^0) = \varepsilon \max_{y \in Y(x^0)} \langle \frac{\partial f(x^0, y)}{\partial x}, \bar{h} \rangle + o(\varepsilon).$$

It is clear that for a sufficiently small ε $(0 \leq \varepsilon \leq \varepsilon_0)$, we have

$$\varphi(x^0 + \varepsilon \bar{h}) < \varphi(x^0), \ \forall \bar{h} \in V(h), \ \forall \varepsilon \in (0, \varepsilon_0).$$

It is means that $h \in K$ and $K_0 \subset K$.

The proof is complete. □

Now we introduce the set

$$M(x^0) = \left\{ \frac{\partial f(x^0, y)}{\partial x} \mid y \in Y(x^0) \right\}. \tag{2.6}$$

It is clear that $M(x^0)$ is compact.

Denote by $N(x^0)$ the convex hull of $M(x^0)$, i.e.,

$$N(x^0) = conv M(x^0).$$

By definition of the convex hull and theorem of Caratheodory [14], we have

$$conv M(x^0) = \left\{ \sum_{i=1}^{n+1} \alpha_i v_i \mid \sum_{i=1}^{n+1} \alpha_i = 1, \ \alpha_i \geq 0, \ v_i \in M(x^0), \ i = 1, 2, ..., n+1 \right\} \tag{2.7}$$

It can be checked that $N(x^0)$ is compact.

Lemma 2.4.

$$\max_{v \in M(x^0)} \langle v, h \rangle = \max_{v \in N(x^0)} \langle v, h \rangle, \ \forall h \in \mathbb{R}^n.$$

The proof is obvious.

Now we can present the cone K_0 as

$$K_0 = \left\{h \in \mathbb{R}^n \mid \langle v, h\rangle < 0,\ \forall v \in N(x^0)\right\} \tag{2.8}$$

Consider the cone $\hat{K}$ generated by the set $N(x^0)$:

$$\begin{gathered}\hat{K} = \{\lambda v \mid \lambda \geq 0,\ v \in N(x^0)\} = \\ \left\{\sum_{i=1}^{n+1} \alpha_i v_i \mid \alpha_i \geq 0,\ v_i \in M(x^0),\ i = 1, 2, ..., n+1\right\}\end{gathered} \tag{2.9}$$

Introduce the conjugate cone K_0^* to K_0:

$$K_0^* = \{v \in \mathbb{R}^n \mid \langle v, x\rangle \leq 0,\ \forall x \in K_0\}. \tag{2.10}$$

Lemma 2.5.

$$K_0^* = \hat{K}.$$

Proof. It is obvious that $\hat{K} \subseteq K_0^*$. Now let us show the inverse inclusion $K_0^* \subseteq \hat{K}$. On the contrary, assume that $\bar{v} \in K_0^*$ but $\bar{v} \notin \hat{K}$. Clearly, $\hat{K}$ is a convex and closed cone. Then due to the separation theorem, there exists a linear functional $c \in \mathbb{R}^n$ and a scalar γ strictly separating $\hat{K}$ from $\bar{v}$:

$$\langle c, \bar{v}\rangle > \gamma \geq \langle c, v\rangle,\ \forall x \in \hat{K}.$$

Since the set $\hat{K}$ is convex cone , it can be shown that the above inequality holds only for $\gamma = 0$. Hence, we have

$$\langle c, \bar{v}\rangle > 0 \geq \langle c, v\rangle,\ \forall x \in \hat{K}.$$

Since $\langle c, \bar{v}\rangle > 0$ then there exists a neighborhood $V(c)$ of c such that

$$\langle c, h\rangle > 0,\ \forall h \in V(c).$$

Taking into account that $convM(x^0) \subseteq \hat{K}$, we have

$$c \in closure(K_0) = \{h \in \mathbb{R} \mid \langle v, h\rangle \leq 0, \forall v \in convM(x^0)\}. \tag{2.11}$$

By (2.11), we have that there exists $\tilde{h} \in V(c) \cap K_0$ such that $\langle\tilde{h}, \bar{v}\rangle > 0$. Consequently, we have $\bar{v} \notin K_0^*$ which contradicts $\bar{v} \in K_0^*$. This is proves the lemma. □

Now we consider the following minimax problem and formulate optimality conditions. Problem (2.1) can be written as:

$$\min_{x \in D} \varphi(x), \tag{2.12}$$

$$D = \{x \in Q \mid g_i(x) \leq 0,\ i = 1, 2, ..., m\} \tag{2.13}$$

where, $\varphi(x) = \max_{y \in A} f(x, y)$, Q is a convex set in $\mathbb{R}^n$, $intQ \neq \emptyset$, $g_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2 \dots, m$ are differentiable functions.

The optimality conditions of the problem (2.12)-(2.13) is given by the following theorem.

Theorem 2.6. *If x^0 is a minimax solution to problem (2.12)-(2.13), then there exist numbers $\alpha_1,\ \alpha_2 \dots, \alpha_{n+1},\ \lambda_1, \lambda_2, ...\lambda_m$ with $\sum_{i=1}^{n+1} \alpha_i^2 + \sum_{i=1}^{m} \lambda_i^2 \neq 0$ and points $y^i \in Y(x^0)$, $i = 1, 2, ...n+1$ such that*

$$\left\{\begin{array}{l}\langle \sum_{i=1}^{n+1} \alpha_i \frac{\partial f(x^0, y^i)}{\partial x} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x^0)}{\partial x}, x - x^0\rangle \leq 0, \forall x \in Q \\ \lambda_i \geq 0,\ \lambda_i g_i(x^0) = 0,\ i = 1, 2 \dots, m \\ \alpha_i \geq 0,\ i = 1, 2 \dots, n+1.\end{array}\right. \tag{2.14}$$

Proof. By Lemmas 2.3 and 2.5, we construct cone K_0 of descent directions for the function $\varphi(x) = \min_{y \in A} f(x, y)$ at the point x^0 and corresponding conjugate cone K_0^*.

Let K_i be cones of feasible directions at a point x^0 for the following sets:

$$Q_i = \{x \in \mathbb{R}^n |\ g_i(x) \leq 0\},\ i = 1, 2 \ldots, m.$$

Also, K_i^* are the corresponding conjugate cones to $K_i,\ i = 1, 2 \ldots, m$. According to [6], K_i and K_i^* are constructed as follows:

$$\begin{aligned} &K_i = \{h \in \mathbb{R}^n \mid \langle g_i'(x^0), h\rangle < 0\}, \\ &K_i^* = \{\lambda_i \tfrac{\partial g_i(x^0)}{\partial x} \mid \lambda_i \geq 0\}, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{2.15}$$

Let K_Q be cone of feasible directions at the point x^0 for the set Q.

K_Q and K_Q^* can be constructed in the following way:

$$\begin{aligned} &K_Q = \{h \in \mathbb{R}^n |\ h = \alpha(x - x^0),\ x \in intQ,\ \alpha > 0\}, \\ &K_Q^* = \{v \in \mathbb{R}^n |\ \langle v, x - x^0\rangle \leq 0,\ \forall x \in Q\}. \end{aligned}$$

According to the Duvobizkii-Milyution theorem [6], if a point x^0 is a minimax solution to problem (2.12)-(2.13), then

$$\left(\bigcap_{i=0}^{m} K_i\right) \bigcap K_Q = \emptyset. \tag{2.16}$$

Since all cones of descent and feasible directions are convex, then due to the separation theorem [6], there exists $v_i^* \in K_i^*,\ i = 0, \ldots, m;\ v_Q^* \in K_Q^*$ such that $\sum_{i=1}^{m} (v_i^*)^2 + (v_Q^*)^2 \neq 0$ and

$$v_0^* + v_1^* + \cdots + v_m^* + v_Q^* = 0. \tag{2.17}$$

Consequently, the last equality implies that there exist numbers $\alpha_1,\ \alpha_2 \ldots, \alpha_{n+1}, \lambda_1, \lambda_2, ...\lambda_m$ with $\sum_{i=1}^{n+1} \alpha_i^2 + \sum_{i=1}^{m} \lambda_i^2 \neq 0$ and points $y^i \in Y(x^0)$, $i = 1, 2, ...n + 1$ such that

$$\begin{cases} \langle \sum_{i=1}^{n+1} \alpha_i \frac{\partial f(x^0, y^i)}{\partial x} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x^0)}{\partial x}, x - x^0\rangle \leq 0, \forall x \in intQ \\ \lambda_i \geq 0,\ \lambda_i g_i(x^0) = 0,\ i = 1, 2 \ldots, m \\ \alpha_i \geq 0,\ i = 1, 2 \ldots, n + 1. \end{cases} \tag{2.18}$$

Due to continuity of scalar product function in (2.18), we have

$$\langle \sum_{i=1}^{n+1} \alpha_i \frac{\partial f(x^0, y^i)}{\partial x} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x^0)}{\partial x}, x - x^0\rangle \leq 0, \forall x \in cl(intQ).$$

Since $intQ \neq \emptyset$, then by the theorem in [14], it implies that

$$cl(intQ) = cl(Q).$$

Thus,

$$\langle \sum_{i=1}^{n+1} \alpha_i \frac{\partial f(x^0, y^i)}{\partial x} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x^0)}{\partial x}, x - x^0\rangle \leq 0, \forall x \in cl(Q). \tag{2.19}$$

Now taking into account (2.19), we obtain

$$\langle \sum_{i=1}^{n+1} \alpha_i \frac{\partial f(x^0, y^i)}{\partial x} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x^0)}{\partial x}, x - x^0 \rangle \leq 0, \forall x \in Q,$$

which completes the proof. □

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Manuscript received 20 July 2023
revised 30 December 2023
accepted for publication 17 January 2024

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