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OPTIMALITY CONDITIONS FOR DIRECTIONAL EFFICIENCY IN SET-VALUED OPTIMIZATION

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Abstract: The aim of our paper is to establish optimality conditions on dual spaces for constrained and unconstrained set-valued optimization problems with respect to some new notions of directional Pareto efficiency. The methods we employ are different for each of the two cases. For the constrained problem we use a fuzzy extremal principle, and for the unconstrained one we use the incompatibility between openness and minimality to prove that the limit of a sequence of directional minima for a perturbation of the objective set-valued map F is a critical point for F.

Key words: *set-valued optimization, directional efficiency, openness, sum stability*

Mathematics Subject Classification: *4C60, 46G05, 90C46*

1 Introduction and Preliminaries

Optimization using set-valued maps has proved to be very useful in many research fields such as duality principles in vector optimization, inverse problems for partial differential equations and variational inequalities, image processing and reconstruction, game theory, viability theory and many others (see, for example, [2], [17] and [27]). On the other hand, problems from location theory and mathematical programming where some directions are considered more important than others motivate the study of directional efficiency. Regularity for set-valued maps, efficiency, calculus with generalized differentiation objects and other properties concerning the main objects in variational analysis have been intensively studied in the light of directional phenomena in many research papers in the last years (to cite a few of them, see [1], [3], [4], [6], [9], [15], [24], [25] and the references therein). In this paper we continue the investigation started in [6] and [11] concerning a notion of directional Pareto efficiency with respect to one and two sets of directions in view of obtaining necessary optimality conditions for set-valued optimization problems on dual spaces. The methods we propose are based on the extremal principle and some sensitivity issues. Using the extremal principle to derive optimality conditions in constrained optimization is a natural idea that comes from the fact that the extremal principle is a variational analog of the convex separation principle in nonconvex settings and has the advantage that it is a straightforward method (see [12], [20] and [22]). The second part of our study concerns the case of an unconstrained optimization problem approached from the perspective of sensitivity, which we conclude with the proof that the limit of a sequence of directional Pareto minima for an Aubin-type perturbation of the objective set-valued map F is a critical point for F. This is in

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line with a popular topic in optimization that is the description of asymptotic behavior of families of minimization problems. See [10], [14] and [19] and the references therein for more details and results on the subject. One direction of research is to be able to characterize the solution (in various acceptances: approximate minima, Geoffrion/Borwein/Benson minima, weak minima) of an optimization problem as the limit of a sequence of minimizers for a family of approximating problems of the initial one, with the natural amendment that the approximating solutions are more easily obtained. On the other hand, proving that the limit of a sequence of approximating solutions is a critical point for a certain problem is especially useful in numerical analysis where algorithms search for critical points rather than actual solutions.

The paper is structured as follows. After a first introductory section where we give the notations that are used throughout and the reference problem and notions of efficiency, the two main sections follow the two main purposes of the paper, that is optimality conditions for constrained and unconstrained set-valued optimization problems. In Section 2, we deal with the constrained case and we use a fuzzy extremal principle in order to arrive at optimality conditions. The method is pretty straightforward and it starts from the facts that a minimum point for a constrained optimization problem is an extremal point of a system in which the constraints and the objective map of the problem are involved and then, that the Approximate Extremal Principle holds in any Asplund space. Hence, the section consists of only one main result. In Section 3 we prove that the limit of a sequence of directional Pareto minima for a perturbation of the objective set-valued map F with directional Lipschitz-like set-valued maps is a critical point for F, in the Fermat generalized sense using the normal (Mordukhovich) coderivative. In order to do that, some auxiliary results are needed, that are also interesting for their own sake. More precisely, we introduce a weaker notion of directional openness (see [10] and [11]) with respect to two sets of directions in the input and output space of the objective map and also with respect to the ordering cone in the output space; with respect to this notion, we prove that the sum between a directionally open and a directionally Aubin-continuous set-valued map is still directionally open, both in a local and in a global version.

Throughout this paper X and Y denote real Banach spaces, unless otherwise stated. We denote by $B(x,r)$ the open ball centered in $x \in X$ with radius $r > 0$, by B_X the unit ball in X and by S_X the unit sphere of X. The notations $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{cone} A$ stand for the topological interior, the topological closure and the conical hull respectively of a nonempty set $A \subset X$.

If $f : X \to \mathbb{R} \cup \{\infty\}$ is a function with values on the extended real line, then the domain of f is the set $\operatorname{dom} f = \{x \in X \mid f(x) < \infty\}$.

For a set-valued map $F : X \rightrightarrows Y$, the domain and the graph are the sets $\operatorname{Dom} F = \{x \in X \mid F(x) \neq \emptyset\}$ and $\operatorname{Gr} F = \{(x,y) \in X \times Y \mid y \in F(x)\}$, respectively. Given $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, we say that F is Lipschitz-like around $(\overline{x}, \overline{y})$ with modulus $l > 0$ if there exists a neighborhood U of $\overline{x}$ such that

$$F(x) \subset F(u) + l\|x - u\| \operatorname{cl} B_X, \ \forall x, u \in U.$$

By the symbol X^* we denote the topological dual of X, while w^* stands for the weak* topology on X^*. If $x^* \in X^*$ and $x \in X$, then by $\langle x^*, x\rangle$ we understand the value of the functional x^* applied to the element x.

We recall next some of the generalized differential constructions developed by Mordukhovich and his collaborators (see [21]) on the dual spaces that we use throughout this paper. Let $\Omega \subset X$ be a nonempty subset of X, $\varepsilon \geq 0$ and $\overline{x} \in \Omega$. The set of $\varepsilon-$normals to

Ω at $\overline{x}$ is

$$\widehat{N}_{\varepsilon}(\Omega,\overline{x}) := \left\{ x^* \in X^* \mid \limsup_{x \xrightarrow{\Omega} \overline{x}} \frac{\langle x^*, x-\overline{x}\rangle}{\|x-\overline{x}\|} \leq \varepsilon \right\},$$

where the notation $x \xrightarrow{\Omega} \overline{x}$ means that $x \in \Omega$ and $x \to \overline{x}$. If $\varepsilon = 0$, the elements in the right-hand side of the expression above are called Fréchet (or regular) normals and their collection is the Fréchet normal cone to Ω at $\overline{x}$, denoted by $\widehat{N}(\Omega,\overline{x})$.

The basic (or limiting, or Mordukhovich) normal cone to Ω at $\overline{x}$ is

$$N(\Omega,\overline{x}) := \{x^* \in X^* \mid \exists \varepsilon_n \xrightarrow{(0,\infty)} 0, x_n \xrightarrow{\Omega} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}_{\varepsilon_n}(\Omega, x_n), \forall n \in \mathbb{N}\},$$

where the notation $x_n^* \xrightarrow{w^*} x^*$ means convergence in the w^*-topology.

If X is an Asplund space (i.e., every convex continuous function $f : U \to \mathbb{R}$ defined on an open convex subset U of X is Fréchet differentiable on a dense subset of U) and Ω is closed around $\overline{x}$ (i.e., there is a neighborhood U of $\overline{x}$ such that $\Omega \cap \operatorname{cl} U$ is closed), the formula for the basic normal cone looks as follows:

$$N(\Omega,\overline{x}) = \{x^* \in X^* \mid \exists x_n \xrightarrow{\Omega} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}(\Omega, x_n), \forall n \in \mathbb{N}\}.$$

For a set-valued map $F : X \rightrightarrows Y$ between the Banach spaces X and Y, the Fréchet coderivative of F at $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ is the set-valued map $\widehat{D}^*F(\overline{x},\overline{y}) : Y^* \rightrightarrows X^*$ given by

$$\widehat{D}^*F(\overline{x},\overline{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in \widehat{N}(\operatorname{Gr} F, (\overline{x},\overline{y}))\},$$

and the normal coderivative of F at $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ is the set-valued map $D^*F(\overline{x},\overline{y}) : Y^* \rightrightarrows X^*$ given by

$$D^*F(\overline{x},\overline{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N(\operatorname{Gr} F, (\overline{x},\overline{y}))\}.$$

For a function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ finite at $\overline{x} \in X$, the basic (or limiting, or Mordukhovich) subdifferential of f at $\overline{x}$ is the set

$$\partial f(\overline{x}) := \{x^* \in X^* \mid (x^*, -1) \in N(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\},$$

where $\operatorname{epi} f := \{(x,y) \in X \times \mathbb{R} \mid y \geq f(x)\}$.

The positive dual cone of a cone $Q \subset Y$ is defined by

$$Q^+ := \{y^* \in Y^* \mid \langle y^*, y\rangle \geq 0,\ \forall y \in Q\}.$$

Throughout this paper $K \subset Y$ represents a proper (i.e., $K \neq Y$ and $K \neq \{0\}$) convex cone that induces a partial order relation on the space Y; when additional assumptions on it will be needed, they will be stated explicitly. We have in view the following set-valued optimization problem with geometric constraints

$$(P) \quad \text{minimize } F(x), \text{ subject to } x \in \Omega,$$

where $F : X \rightrightarrows Y$ is a set-valued map and $\Omega \subset X$ is a nonempty set.

Let $L \subset S_X$ and $M \subset S_Y$ be two nonempty closed sets; we recall from [6] and [11] the next directional minimality concepts related to (P).

Definition 1.1. (i) One says that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (\Omega \times Y)$ is a local directional Pareto minimum point for F on Ω with respect to L and M if there exists a neighborhood U of $\overline{x}$ such that

$$[F(U \cap \Omega \cap (\overline{x} + \operatorname{cone} L)) \cap (\overline{y} - \operatorname{cone} M) - \overline{y}] \cap -K \subset K. \tag{1.1}$$

(ii) If $\operatorname{int} K \neq \emptyset$, one says that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (\Omega \times Y)$ is a local weak directional Pareto minimum point for F on Ω with respect to L and M if there exists a neighborhood U of $\overline{x}$ such that

$$[F(U \cap \Omega \cap (\overline{x} + \operatorname{cone} L)) \cap (\overline{y} - \operatorname{cone} M) - \overline{y}] \cap - \operatorname{int} K = \emptyset.$$

(iii) One says that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (\Omega \times Y)$ is a local directional Pareto minimum point for F on Ω with respect to L if there exists a neighborhood U of $\overline{x}$ such that

$$[F(U \cap \Omega \cap (\overline{x} + \operatorname{cone} L)) - \overline{y}] \cap -K \subset K. \tag{1.2}$$

Notice that (1.1) is equivalent to

$$[F(U \cap \Omega \cap (\overline{x} + \operatorname{cone} L)) - \overline{y}] \cap (-\operatorname{cone} M) \cap -K \subset K,$$

and so, if $K \subset \operatorname{cone} M$, the relation reduces to (1.2). That is, if $K \subset \operatorname{cone} M$, the set of directions M plays no role in the efficiency notion from (i) and the latter reduces to the notion in (iii).

Of course, in the unconstrained case, $\Omega = X$ in the definition above. The optimality conditions that we establish next are with respect to this notions of efficiency, (ii) and (iii) more precisely.

2 Optimality Conditions Based on the Extremal Principle

In this section we obtain some optimality conditions for (P) using the extremal principle developed by Kruger and Mordukhovich (see [18] and [20]). Being a variational analog of the convex separation principle in nonconvex settings, the extremal principle is naturally a very useful tool in deriving optimality conditions in constrained optimization and has been thus used in various papers (see [12],[13]). For the reader's convenience, we state here the definitions of the extremal system and the fuzzy form of the extremal principle that we use in our result. Roughly speaking, a common point of sets is locally extremal if these sets can be locally pushed apart by linear small perturbations in such a way that the resulting sets have empty intersection.

Definition 2.1. Let Ω_1, $\Omega_2 \subset X$ and let $\overline{x} \in \Omega_1 \cap \Omega_2$.

(i) The system $\{\Omega_1, \Omega_2, \overline{x}\}$ is called extremal if there are sequences $(a_{1n}), (a_{2n}) \xrightarrow{n\to\infty} 0$ and U a neighborhood of $\overline{x}$ such that

$$(\Omega_1 - a_{1n}) \cap (\Omega_2 - a_{2n}) \cap U = \emptyset \ \forall n.$$

(ii) It is said that the extremal system $\{\Omega_1, \Omega_2, \overline{x}\}$ satisfies the Approximate Extremal Principle if for every $\varepsilon > 0$ there are $x_1 \in \Omega_1 \cap (\overline{x} + \varepsilon B_X)$, $x_2 \in \Omega_2 \cap (\overline{x} + \varepsilon B_X)$ and $x^* \in (\widehat{N}(\Omega_1, x_1) + \varepsilon B_{X^*}) \cap (-\widehat{N}(\Omega_2, x_2) + \varepsilon B_{X^*})$ such that $\|x^*\| = \frac{1}{2}$.

Remark 2.2. According to Theorem 2.20 from [21], the Approximate Extremal Principle holds in any Asplund space; that is, every extremal system in an Asplund space satisfies the Approximate Extremal Principle. For more details on the extremal principle, its history and its extensions we refer the reader to [5], [22] and [23].

We recall next the definitions of two commonly used qualification conditions both for calculus and optimality conditions that we will also use as hypotheses in the main result of the section.

Definition 2.3. (i) Let Ω_1 and Ω_2 be two nonempty closed subsets of the space X. One says that Ω_1 and Ω_2 are allied at $\overline{x} \in \Omega_1 \cap \Omega_2$ if for every $(x_{in}) \xrightarrow{\Omega_i} \overline{x}$, $x_{in}^* \in \widehat{N}(\Omega_i, x_{in})$, $i = 1, 2$, the relation $x_{1n}^* + x_{2n}^* \to 0$ implies that $x_{1n}^* \to 0$ and $x_{2n}^* \to 0$.

(ii) A subset $\Omega \subset X$ is called sequentially normally compact (SNC) at $\overline{x} \in \Omega$ if for every sequences $(\varepsilon_n, x_n, x_n^*) \subset [0, \infty) \times \Omega \times X^*$ satisfying

$$\varepsilon_n \to 0,\ x_n \to \overline{x},\ x_n^* \in \widehat{N}_{\varepsilon_n}(\Omega, x_n) \,\text{and}\, x_n^* \xrightarrow{w^*} 0,$$

it follows that $\|x_n^*\| \to 0$ for $n \to \infty$.

The result we obtain in the sequel is a direct extension of Theorem 3.7 from [6] for the case of directional Pareto efficiency with respect to two sets of directions.

Theorem 2.4. *Let $F : X \rightrightarrows Y$ be a set-valued map between the Asplund spaces X and Y, $K \subset Y$ a closed convex cone with nonempty interior, $L \subset S_X$ and $M \subset S_Y$ two closed sets with* cone$M$ *convex and $(\overline{x}, \overline{y}) \in \operatorname{Gr}F$. Suppose that F is Lipschitz-like around $(\overline{x}, \overline{y})$, Ω and $(\overline{x} + \operatorname{cone}L)$ are allied at $\overline{x}$, $K \cap \operatorname{cone}M$ is (SNC) at 0, $\operatorname{int}K \cap \operatorname{cone}M \neq \emptyset$ and $(\overline{x}, \overline{y})$ is a local weak directional Pareto minimum point for F on Ω with respect to the sets of directions L and M.*

Then there exists $y^* \in (K \cap \operatorname{cone}M)^+ \setminus \{0\}$ such that

$$0 \in D^*F(\overline{x}, \overline{y})(y^*) + N(\Omega, \overline{x}) + N(\operatorname{cone}L, 0).$$

Proof. Let us define

$$\Omega_1 := [\Omega \cap (\overline{x} + \operatorname{cone}L)] \times [(\overline{y} - K) \cap (\overline{y} - \operatorname{cone}M)] \text{ and } \Omega_2 := \operatorname{Gr}F.$$

Choose $U = U_{\overline{x}} \times Y$, where $U_{\overline{x}}$ is the neighborhood from the definition of the minimality of $(\overline{x}, \overline{y})$, and $\overline{k} \in \operatorname{int}K \cap \operatorname{cone}M$. We claim that

$$\Omega_1 \cap \left[\Omega_2 + \left(0, \frac{\overline{k}}{n}\right)\right] \cap U = \emptyset,$$

for all n. If we suppose otherwise, we get (x, y) such that

$$\begin{aligned} & x \in \Omega \cap (\overline{x} + \operatorname{cone}L) \cap U_{\overline{x}}, \\ & y \in (\overline{y} - K) \cap (\overline{y} - \operatorname{cone}M), \\ & y \in F(x) + \frac{\overline{k}}{n}, \end{aligned} \tag{2.1}$$

which yields that $y - \frac{\overline{k}}{n} \in F(x)$ and also that $y - \frac{\overline{k}}{n} \in \overline{y} - \operatorname{cone}M - \frac{\overline{k}}{n} \subset \overline{y} - \operatorname{cone}M - \operatorname{cone}M \subset \overline{y} - \operatorname{cone}M$. Moreover, $y - \frac{\overline{k}}{n} - \overline{y} \in -\operatorname{int}K - K \subset -\operatorname{int}K$ and thus we obtain

$$y - \frac{\overline{k}}{n} - \overline{y} \in [F(\Omega \cap (\overline{x} + \operatorname{cone}L) \cap U_{\overline{x}}) \cap (\overline{y} - \operatorname{cone}M) - \overline{y}] \cap (-\operatorname{int}K),$$

which is a contradiction to the minimality of $(\overline{x},\overline{y})$.

Therefore, the system $\{\Omega_1,\Omega_2,(\overline{x},\overline{y})\}$ is extremal and we can apply the Approximate Extremal Principle in order to get, for all $n\in\mathbb{N}$, some sequences $(x_{1n},y_{1n})\in\Omega_1\cap\big[(\overline{x},\overline{y})+\frac{1}{n}B_{X\times Y}\big]$, $(x_{2n},y_{2n})\in\Omega_2\cap\big(\overline{x},\overline{y})+\frac{1}{n}B_{X\times Y}\big]$ and $(x_n^*,y_n^*)\in\big[\widehat{N}(\Omega_1,(x_{1n},y_{1n}))+\frac{1}{n}B_{X^*\times Y^*}\big]\cap\big[-\widehat{N}(\Omega_2,(x_{2n},y_{2n}))+\frac{1}{n}B_{X^*\times Y^*}\big]$ such that $\|(x_n^*,y_n^*)\|=\frac{1}{2}$. Further, we deduce the existence of other sequences $(u_{1n}^*,v_{1n}^*)\in\widehat{N}(\Omega_1,(x_{1n},y_{1n}))$, $(u_{2n}^*,v_{2n}^*)\in\widehat{N}(\Omega_2,(x_{2n},y_{2n}))$ and $(a_{in}^*,b_{in}^*)\in B_{X^*\times Y^*}$, $i=1,2$ such that

$$\begin{aligned}(x_n^*,y_n^*)&=(u_{1n}^*,v_{1n}^*)+\frac{1}{n}(a_{1n}^*,b_{1n}^*)\\(x_n^*,y_n^*)&=-(u_{2n}^*,v_{2n}^*)+\frac{1}{n}(a_{2n}^*,b_{2n}^*).\end{aligned}\tag{2.2}$$

It is easy to see that the sequences (x_n^*) and (y_n^*) are bounded and thus w^*-convergent on some subsequences to some elements x^* and y^* respectively. Hence, by passing to the limit in $(-x_n^*,-y_n^*)\in\widehat{N}(\Omega_2,(x_{2n},y_{2n}))-\frac{1}{n}B_{X^*\times Y^*}$, we get that $(-x^*,-y^*)\in N(\mathrm{Gr}F,(\overline{x},\overline{y}))$, which is equivalent to $-x^*\in D^*F(\overline{x},\overline{y})(y^*)$ and then to $0\in x^*+D^*F(\overline{x},\overline{y})(y^*)$.

Let us now justify that y^* is nonzero. Suppose, on the contrary, that $y^*=0$. There is a sequence $z_n^*\in B_{Y^*}$ such that $y_n^*-\frac{1}{n}z_n^*\in\widehat{N}((\overline{y}-K)\cap(\overline{y}-\mathrm{cone}M),y_{1n})$. Since $y_n^*-\frac{1}{n}z_n^*\xrightarrow{w^*}0$ and $y_{1n}\to\overline{y}$, from the SNC property of $K\cap\mathrm{cone}M$ at 0 follows that $y_n^*-\frac{1}{n}z_n^*\to 0$. Also, there are some sequences $(r_n^*,s_n^*)\in\frac{1}{n}B_{X^*\times Y^*}$ such that $(-x_n^*+r_n^*,-y_n^*+s_n^*)\in\widehat{N}(\mathrm{Gr}F,(x_{2n},y_{2n}))$, or, using the definition of the coderivative, $-x_n^*+r_n^*\in\widehat{D}^*F(x_{2n},y_{2n})(y_n^*-s_n^*)$. Now, using the Lipschitz property of F around $(\overline{x},\overline{y})$ and the fact that $y_n^*-s_n^*\to 0$, Theorem 1.43 from [21] ensures that we also have $-x_n^*+r_n^*\to 0$. From that we immediately see that $x_n^*\to 0$ and this yields a contradiction since $\|(x_n^*,y_n^*)\|=\frac{1}{2}$. Therefore, $y^*\neq 0$.

From the property of alliedness of Ω and $(\overline{x}+\mathrm{cone}L)$ at $\overline{x}$, it follows that there are some elements $a_n\in\Omega\cap B\left(x_{1n},\frac{1}{n}\right)$, $l_n\in(\overline{x}+\mathrm{cone}L)\cap B\left(x_{1n},\frac{1}{n}\right)$ such that

$$x_n^*\in\widehat{N}(\Omega\cap(\overline{x}+\mathrm{cone}L),x_{1n})+\frac{1}{n}B_{X^*}\subset\widehat{N}(\Omega,a_n)+\widehat{N}(\overline{x}+\mathrm{cone}L,l_n)+\frac{2}{n}B_{X^*}.$$

Hence, there are $a_n^*\in\widehat{N}(\Omega,a_n)$ and $l_n^*\in\widehat{N}(\overline{x}+\mathrm{cone}L,l_n)$ such that

$$a_n^*+l_n^*-x_n^*\to 0.$$

Next we prove that these two sequences cannot be both unbounded. Suppose, on the contrary, that they are. Then, for every n, there is k_n sufficiently large such that

$$n<\min\{\|a_{k_n}^*\|,\|l_{k_n}^*\|\}.\tag{2.3}$$

For simplicity, we denote the subsequences $(a_{k_n}^*)$ and $(l_{k_n}^*)$ by (a_n^*) and (l_n^*) respectively. We have that $\frac{1}{n}a_n^*\in\widehat{N}(\Omega,a_n)$ and $\frac{1}{n}l_n^*\in\widehat{N}(\overline{x}+\mathrm{cone}L,l_n)$, and from

$$\frac{1}{n}\|a_n^*+l_n^*\|\leq\frac{1}{n}\|a_n^*+l_n^*-x_n^*\|+\frac{1}{n}\|x_n^*\|$$

we obtain that $\frac{1}{n}(a_n^*+l_n^*)\to 0$. Using again the alliedness of Ω and $(\overline{x}+\mathrm{cone}L)$ at $\overline{x}$, it follows that $\frac{1}{n}a_n^*\to 0$ and $\frac{1}{n}l_n^*\to 0$, but this is a contradiction to the relation (2.3). Therefore, the sequences (a_n^*) and (l_n^*) are bounded and thus w^*-convergent on some subsequences to some elements a^* and l^* respectively. Finally, we get by passing to the limit that $x^* = a^* + l^* \in N(\Omega, \overline{x}) + N(\overline{x} + \text{cone}L, \overline{x}) = N(\Omega, \overline{x}) + N(\text{cone}L, 0)$.

Moreover, passing to the limit in $y_n^* \in \widehat{N}((\overline{y} - K) \cap (\overline{y} - \text{cone}M), y_{1n})$, we get that $y^* \in N(\overline{y} - K \cap \text{cone}M, \overline{y}) = N(-K \cap \text{cone}M, 0) = (K \cap \text{cone}M)^+$, and the proof is completed. □

A simple example to prove the validity of this result is given next.

Example 2.5. Let $F : [0, \infty) \rightrightarrows [0, \infty)$, $F(x) = [x, x + 1]$, $L = M = \{1\}$, $K = [0, \infty)$, $\Omega = [0, 1]$ and $(\overline{x}, \overline{y}) = (0, 0)$. It is easy to check that all hypotheses are fulfilled for these choices and in the end, the verification of the conclusion comes to finding an element $y^* \in (K \cap \text{cone}\, M)^+ = (0, \infty)$ such that

$$(0, -y^*) \in N(\text{Gr}\, F, (0, 0)).$$

Or, due to the fact that $\text{Gr}\, F$ is a convex set, this further reduces to finding a $y^* \in (0, \infty)$ such that

$$y^* \cdot y \geq 0$$

for all $x \in [0, \infty)$ and all $y \in [x, x + 1]$, which is obviously true.

Remark 2.6. (i) Notice that in the case when $M = S_Y$, the result above is similar to Theorem 3.7 from [6], in the cited paper the case of strong minimality being dealt. Moreover, if F is a single-valued map and both L and M are the whole unit spheres in X and Y respectively, that is, $(\overline{x}, \overline{y})$ is a classical Pareto minimum point for $f := F$, then one recovers the usual optimality condition that $0 \in \partial(v^* \circ f)(\overline{x}) + N(\Omega, \overline{x})$ for some nonzero positive multiplier $v^* \in K^+ \setminus \{0\}$.

(ii) In [13] the author obtains optimality conditions associated with the classical weak Pareto minimum concept for a constrained set-valued optimization problem with geometric constraints using directly the extremal principle. The main idea is that a minimum point for a map generates an extremal system formed by the graph of the map and the constraints set; then the conditions in the extremal principle are turned into optimality conditions for a real-valued map written using the support function of the objective map. Our result is a Fermat-type necessary condition written for the objective map where the hypotheses of generalized compactness properties and alliedness of some of the sets involved in the structure of the problem and the regularity of the objective map were essential.

3 Optimality Conditions Based on Openness

We recall from [9] a notion of directional linear openness of a set-valued map with respect to two sets of directions. Let $F : X \rightrightarrows Y$ be a set-valued map, $L \subset S_X$, $M \subset S_Y$, $\alpha > 0$ and $(\overline{x}, \overline{y}) \in \text{Gr}F$.

Definition 3.1. We call F $\alpha-$directionally linearly open at $(\overline{x}, \overline{y})$ with respect to L and M if there exists $\varepsilon > 0$ such that for all $\rho \in (0, \varepsilon)$ it holds that

$$B(\overline{y}, \alpha\rho) \cap (\overline{y} - \text{cone}\, M) \subset F(B(\overline{x}, \rho) \cap (\overline{x} + \text{cone}\, L)).$$

As it happens in the classical, non-directional case, this notion is equivalent to two other properties: F being $\alpha-$directionally metrically regular at $(\overline{x}, \overline{y})$ with respect to L and M

and F^{-1} being $\alpha^{-1}-$directionally Aubin continuous at $(\overline{y}, \overline{x})$ with respect to M and L (see [9, Proposition 2.4] and [26, Theorem 1.5]).

We introduce next a weaker notion of directional openness of F with respect to the sets of directions L and M and, additionally, to the ordering cone K of the output space Y. Besides α, we consider another strictly positive constant $\beta > 0$.

Definition 3.2. We call F $(\alpha, \beta)-$directionally open at $(\overline{x}, \overline{y})$ with respect to the sets of directions (L, M) and the cone K if there exists $\varepsilon > 0$ such that for all $\rho \in (0, \varepsilon)$ the following relation holds:

$$B(\overline{y}, \alpha\rho) \cap (\overline{y} - \operatorname{cone} M) \subset F(B(\overline{x}, \rho) \cap (\overline{x} + \operatorname{cone} L)) + K \cap B(0, \beta\rho).$$

The aim of this section is twofold. First we prove a stability result that shows that the sum between two set-valued maps having each one of the properties of directional openness with respect to (L, M) and K and the directional Aubin continuity with respect to S_X and M at every point of their graphs, respectively, is still a directional open set-valued map with respect to (L, M) and K at every point of its graph as well. This stability we prove in the global case and in a local version also. Our second aim is to derive optimality conditions for the problem (P) in the unconstrained case when $\Omega = X$.

The main technical tools we use in the next result are a directional minimal time function and a form of the well-known Ekeland variational principle. We recall from [8] the definition and the main properties we need for the minimal time function: if $\Omega \subset X$ and $M \subset S_X$ are two nonempty sets, then the directional minimal time function with respect to M, $T_M(\cdot, \Omega) : X \to [0, \infty]$, is given by

$$T_M(x, \Omega) := \inf\{t \geq 0 \mid \exists u \in M \ : \ x + tu \in \Omega\} \text{ for all } x \in X.$$

In the particular case when the target set Ω is a singleton, we write $T_M(x, \{u\}) =: T_M(x, u)$. It is easy to check that $T_M(x, u)$ is $+\infty$ if $u - x \notin \operatorname{cone} M$ and $\|u - x\|$ if $u - x \in \operatorname{cone} M$. If one of the sets M and Ω is closed, and the other one is compact, then $T_M(\cdot, \Omega)$ is lower semicontinuous (see [8, Proposition 2.4]).

The first result in this section concerns the preservation of directional openness with respect to the directions (L, M) and K at directional pseudo-Lipschitz perturbation in the global case. For similar results see [7], [10], [28] and [29]. In fact, our result represents a directional version of [10, Theorem 4.1].

Theorem 3.3. *Let $F, G : X \rightrightarrows Y$ be two set-valued maps with* $\operatorname{Gr} F$ *and* $\operatorname{Gr} G$ *locally closed. Let $K \subset Y$ be a closed and convex cone and $L \subset S_X$, $M \subset S_Y$ be two closed sets of directions with* $\operatorname{cone} M$ *convex. Suppose that* $\operatorname{Dom}(F+G)$ *is nonempty and let α, β, γ be strictly positive constants with $\beta < \alpha$. If F is $(\alpha, \gamma)-$directionally open with respect to (L, M) and K at every point of its graph and G is $\beta-$directionally Aubin continuous at every point of its graph with respect to S_X and M, then $F + G$ is $\left(2^{-1}(\alpha - \beta), \gamma\right)-$directionally open with respect to (L, M) and K at every point of its graph.*

Proof. Let $(x_0, w_0) \in \operatorname{Gr}(F + G)$ be arbitrary, so there are $y_0 \in F(x_0)$ and $z_0 \in G(x_0)$ such that $w_0 = y_0 + z_0$.

Let us define the set $A := \{(x, y, z, k) \mid y \in F(x), z \in G(x), k \in K\}$. Since $\operatorname{Gr} F$ and $\operatorname{Gr} G$ are locally closed and K is closed, it follows that A is locally closed, so there exists $\rho > 0$ such that $A \cap \operatorname{cl} W$ is closed, where

$$W := B(x_0, \rho) \times B(y_0, (\alpha + \gamma)\rho) \times B(z_0, \beta\rho) \times B(0, \gamma\rho).$$

Let $u_0 \in B(w_0, 2^{-1}(\alpha-\beta)\rho) \cap (w_0 - \mathrm{cone}M)$. Our aim is to prove that $u_0 \in (F+G)(B(x_0,\rho) \cap (x_0 + \mathrm{cone}L)) + K \cap B(0,\gamma\rho)$.

Suppose τ is in $(0,1)$ such that $\|u_0 - w_0\| < 2^{-1}\tau(\alpha-\beta)\rho$. Let us define the function $f : A \cap \mathrm{cl}W \subset X \times (Y \times Y \times Y) \to \mathbb{R} \cup \{\infty\}$, by

$$f(x,y,z,k) := T_{-M}(y+z+k; \{u_0\}) := T_{-M}(y+z+k; u_0),$$

where on the product space $Y \times Y \times Y$ we consider the norm

$$\|(y,z,k)\|_0 := \max\{(\alpha+\gamma)^{-1}\|y\|, \beta^{-1}\|z\|, \gamma^{-1}\|k\|\}.$$

It is easy to see that f is positive, and since M is closed and $\{u_0\}$ is compact, f is also lower semicontinuous. Also, since $u_0 - (y_0 + z_0) = u_0 - w_0 \in -\mathrm{cone}M$, the point $(x_0, y_0, z_0, 0) \in \mathrm{dom}f$, and so we can apply the modified variational principle of Ekeland from [11] to get, for all $\varepsilon > 0$, $(x_\varepsilon, y_\varepsilon, z_\varepsilon, k_\varepsilon) \in A \cap \mathrm{cl}W$ such that

$$f(x_\varepsilon, y_\varepsilon, z_\varepsilon, k_\varepsilon) \le f(x_0, y_0, z_0, 0) - \varepsilon(T_{-L}(x_\varepsilon; x_0) + \|(y_\varepsilon, z_\varepsilon, k_\varepsilon) - (y_0, z_0, 0)\|_0) \tag{3.1}$$

and, for all $(x,y,z,k) \in A \cap \mathrm{cl}W$,

$$f(x_\varepsilon, y_\varepsilon, z_\varepsilon, k_\varepsilon) \le f(x,y,z,k) + \varepsilon(T_{-L}(x; x_\varepsilon) + \|(y_\varepsilon, z_\varepsilon, k_\varepsilon) - (y,z,k)\|_0). \tag{3.2}$$

From (3.1) we get that

$$T_{-M}(y_\varepsilon + z_\varepsilon + k_\varepsilon; u_0) \le \|u_0 - w_0\| - \varepsilon(T_{-L}(x_\varepsilon; x_0) + \|(y_\varepsilon - y_0, z_\varepsilon - z_0, k_\varepsilon)\|_0),$$

which further implies, because the left-hand side is positive, that $T_{-L}(x_\varepsilon; x_0) < \infty$ and $T_{-M}(y_\varepsilon + z_\varepsilon + k_\varepsilon; u_0) < \infty$, which are equivalent to

$$x_\varepsilon \in x_0 + \mathrm{cone}L \text{ and } u_0 - (y_\varepsilon + z_\varepsilon + k_\varepsilon) \in -\mathrm{cone}M. \tag{3.3}$$

Hence, relation (3.1) becomes

$$\|u_0 - (y_\varepsilon + z_\varepsilon + k_\varepsilon)\| \le \|u_0 - w_0\| - \varepsilon(\|x_\varepsilon - x_0\| + \|(y_\varepsilon - y_0, z_\varepsilon - z_0, k_\varepsilon)\|_0),$$

which implies that

$$\begin{aligned} &\varepsilon\|x_\varepsilon - x_0\| \le \|u_0 - w_0\| \\ &\varepsilon(\alpha+\gamma)^{-1}\|y_\varepsilon - y_0\| \le \|u_0 - w_0\| \\ &\varepsilon\beta^{-1}\|z_\varepsilon - z_0\| \le \|u_0 - w_0\| \\ &\varepsilon\gamma^{-1}\|k_\varepsilon\| \le \|u_0 - w_0\|. \end{aligned} \tag{3.4}$$

By choosing $\varepsilon = 2^{-1}\tau(\alpha-\beta)$, the previous relations yield that $(x_\varepsilon, y_\varepsilon, z_\varepsilon, k_\varepsilon) \in W$.

Now, if $u_0 = y_\varepsilon + z_\varepsilon + k_\varepsilon$, then it would be true that $u_0 \in (F+G)(B(x_0,\rho) \cap (x_0 + \mathrm{cone}L)) + K \cap B(0,\gamma\rho)$, and the proof it would be done.

We will now prove that in fact, it is not possible that $u_0 \neq y_\varepsilon + z_\varepsilon + k_\varepsilon$. Suppose, by means of contradiction, that $u_0 \neq y_\varepsilon + z_\varepsilon + k_\varepsilon$. Then we can correctly define

$$v := \frac{C}{\|u_0 - y_\varepsilon + z_\varepsilon + k_\varepsilon\|}(u_0 - (y_\varepsilon + z_\varepsilon + k_\varepsilon)) \in -\mathrm{cone}M,$$

where C is an arbitrary constant strictly smaller than α.

Let $t > 0$ be small enough such that

$$y_\varepsilon + tv \in B(y_\varepsilon, \alpha t) \cap (y_\varepsilon - \mathrm{cone} M) \subset F(B(x_\varepsilon, t) \cap (x_\varepsilon + \mathrm{cone} L)) + K \cap B(0, \gamma t),$$

by the $(\alpha, \gamma)-$directional openness of F at $(x_\varepsilon, y_\varepsilon)$ with respect to (L, M) and K. We add k_ε to the previous inclusion to get that

$$\begin{aligned} y_\varepsilon + tv + k_\varepsilon &\in F(B(x_\varepsilon, t) \cap (x_\varepsilon + \mathrm{cone} L)) + K \cap B(0, \gamma t) \\ +k_\varepsilon &\subset F(B(x_\varepsilon, t) \cap (x_\varepsilon + \mathrm{cone} L)) + K \cap B(k_\varepsilon, \gamma t). \end{aligned}$$

Hence there exist $a \in B(x_\varepsilon, t) \cap (x_\varepsilon + \mathrm{cone} L)$ and $k \in K \cap B(k_\varepsilon, \gamma t)$ such that

$$y_\varepsilon + tv + k_\varepsilon - k \in F(a). \tag{3.5}$$

Furthermore, we can choose $p \in \mathrm{cone} L$ and $q \in Y$ with $\|p\| < 1$ and $\|q\| < 1$ such that

$$a = x_\varepsilon + tp \text{ and } k = k_\varepsilon + \gamma tq. \tag{3.6}$$

It follows from (3.5) that $y_\varepsilon + tv - \gamma tq \in F(x_\varepsilon + tp)$.

We make use now of the fact that G is $\beta-$directionally Aubin continuous at $(x_\varepsilon, z_\varepsilon)$ with respect to S_X and M, or, equivalently, G^{-1} is $\beta^{-1}-$directionally linearly open at $(z_\varepsilon, x_\varepsilon)$ with respect to M and S_X. That is, for $t > 0$ sufficiently small, it holds that

$$B(x_\varepsilon, t) \cap (x_\varepsilon - \mathrm{cone}\, S_X) \subset G^{-1}(B(z_\varepsilon, \beta t) \cap (z_\varepsilon + \mathrm{cone}\, M)).$$

Thus, there is $f \in B(z_\varepsilon, \beta t)$ such that $f \in G(a)$ for which we can write $f = z_\varepsilon + \beta tr$ by choosing $r \in \mathrm{cone} M$ with $\|r\| < 1$.

The use of these expressions of the constants a, f and k is that we obtained the points

$$(x_\varepsilon + tp, y_\varepsilon + tv - \gamma tq, z_\varepsilon + \beta tr, k_\varepsilon + \gamma tq)$$

to be both in A and in W since they represent small perturbations of $(x_\varepsilon, y_\varepsilon, z_\varepsilon, k_\varepsilon) \in W$ and W was defined as an open set. Hence we can use (3.2) for these points and write:

$$\begin{aligned} f(x_\varepsilon, y_\varepsilon, z_\varepsilon, k_\varepsilon) \le\ & f(x_\varepsilon + tp, y_\varepsilon + tv - \gamma tq, z_\varepsilon + \beta tr, k_\varepsilon + \gamma tq) \\ & + \varepsilon(T_{-L}(x_\varepsilon + tp; x_\varepsilon) + \|(tv - \gamma tq, \beta tr, \gamma tq)\|_0). \end{aligned} \tag{3.7}$$

We have $f(x_\varepsilon, y_\varepsilon, z_\varepsilon, k_\varepsilon) = T_{-M}(y_\varepsilon + z_\varepsilon + k_\varepsilon; u_0) = \|u_0 - (y_\varepsilon + z_\varepsilon + k_\varepsilon)\|$, $T_{-L}(x_\varepsilon + tp; x_\varepsilon) = t\|p\|$ because $x_\varepsilon - (x_\varepsilon + tp) = -tp \in -\mathrm{cone} L$, and

$$\begin{aligned} f(x_\varepsilon + tp, y_\varepsilon + tv - \gamma tq, z_\varepsilon + \beta tr, k_\varepsilon + \gamma tq) &= T_{-M}(y_\varepsilon + z_\varepsilon + k_\varepsilon + tv + \beta tr; u_0) \\ &= \|u_0 - (y_\varepsilon + z_\varepsilon + k_\varepsilon + tv + \beta tr)\| \end{aligned} \tag{3.8}$$

since $-\beta tr \in -\mathrm{cone} M$, $u_0 - (y_\varepsilon + z_\varepsilon + k_\varepsilon + tv) = \left(1 - \frac{tC}{\|u_0 - (y_\varepsilon + z_\varepsilon + k_\varepsilon)\|}\right)(u_0 - y_\varepsilon - z_\varepsilon - k_\varepsilon) \in -\mathrm{cone}\, M$ for t small enough and thus $u_0 - (y_\varepsilon + z_\varepsilon + k_\varepsilon + tv + \beta tr) \in -\mathrm{cone}\, M - \mathrm{cone}\, M \subset -\mathrm{cone}\, M$ because $\mathrm{cone} M$ is convex.

Thus,

$$f(x_\varepsilon + tp, y_\varepsilon + tv - \gamma tq, z_\varepsilon + \beta tr, k_\varepsilon + \gamma tq) \le \beta t\|r\| + \|u_0 - (y_\varepsilon + z_\varepsilon + k_\varepsilon + tv)\|$$

and (3.7) becomes

$$\|u_0 - (y_\varepsilon + z_\varepsilon + k_\varepsilon)\| \leq \beta t\|r\| + \|u_0 - (y_\varepsilon + z_\varepsilon + k_\varepsilon) - tv\| + \varepsilon(t\|p\| + t\|(v - \gamma q, \beta r, \gamma q)\|_0).$$

By the definition of v, we have

$$\|u_0 - (y_\varepsilon + z_\varepsilon + k_\varepsilon) - tv\| = |\|u_0 - (y_\varepsilon + z_\varepsilon + k_\varepsilon)\| - tC|.$$

By choosing, as before, t small enough such that $\|u_0 - (y_\varepsilon + z_\varepsilon + k_\varepsilon)\| \geq tC$, we get that

$$0 \leq \beta t\|r\| - tC + \varepsilon(t\|p\| + t\|(v - \gamma q, \beta r, \gamma q)\|_0),$$

and so

$$0 \leq \beta - C + 2^{-1}\tau(\alpha - \beta)(1 + \max\{(\alpha + \gamma)^{-1}\|v - \gamma q\|, \beta^{-1}\beta\|r\|, \gamma^{-1}\gamma\|q\|\}).$$

That is, since $\|v - \gamma q\| \leq C + \gamma < \alpha + \gamma$,

$$C - \beta \leq 2^{-1}\tau(\alpha - \beta)(1 + 1).$$

By passing to the limit with $C \to \alpha$, we get that

$$\alpha - \beta \leq \tau(\alpha - \beta),$$

which yields the contradiction that $\tau \geq 1$.

Therefore the only possibility for u_0 is to be the sum $y_\varepsilon + z_\varepsilon + k_\varepsilon$, and this completes the proof. □

We are trying in the sequel to obtain a local version of this stability result. Note that by the $(\alpha, \beta)-$directional openness of F with respect to (L, M) and K *around* $(\overline{x}, \overline{y}) \in \operatorname{Gr}F$ we understand that there exist $\varepsilon > 0$ and some neighbours U and V for $\overline{x}$ and $\overline{y}$ respectively such that for all $(x, y) \in (U \times V) \cap \operatorname{Gr} F$ and $\rho \in (0, \varepsilon)$ one has

$$B(y, \alpha\rho) \cap (y - \operatorname{cone}M) \subset F(B(x, \rho) \cap (x + \operatorname{cone}L)) + K \cap B(0, \beta\rho).$$

Similarly for the directional Aubin continuity of G with respect to S_X and M *around* $(\overline{x}, \overline{z})$. We prove the following intermediate result.

Theorem 3.4. *Let $F, G : X \rightrightarrows Y$ be two set-valued maps with $\operatorname{Gr}F$ and $\operatorname{Gr}G$ locally closed. Let $K \subset Y$ be a closed and convex cone and $L \subset S_X$, $M \subset S_Y$ be two closed sets of directions with $\operatorname{cone}M$ convex. Suppose that $\operatorname{Dom}(F + G)$ is nonempty and α, β, γ are strictly positive constants with $\beta < \alpha$. If F is $(\alpha, \gamma)-$directionally open with respect to (L, M) and K around $(\overline{x}, \overline{y}) \in \operatorname{Gr}F$ and G is $\beta-$directionally Aubin continuous around $(\overline{x}, \overline{z}) \in \operatorname{Gr}G$ with respect to S_X and M, then there exists $\varepsilon > 0$ such that for all $(x, y, z) \in B(\overline{x}, \varepsilon) \times B(\overline{y}, \varepsilon) \times B(\overline{z}, \varepsilon)$ with $y \in F(x)$ and $z \in G(x)$, and for all $\rho \in (0, \varepsilon)$, one has*

$$B(y + z, 2^{-1}(\alpha - \beta)\rho) \cap (y + z - \operatorname{cone}M) \subset (F + G)(B(x, \rho) \cap (x + \operatorname{cone}L)) + K \cap B(0, \gamma\rho).$$

Proof. Due to the assumptions made, there is $\delta > 0$ such that the following assertions hold: $\operatorname{Gr}F \cap [\operatorname{cl}(B(\overline{x}, \delta)) \times \operatorname{cl}(B(\overline{y}, \delta))]$ and $\operatorname{Gr}G \cap [\operatorname{cl}(B(\overline{x}, \delta)) \times \operatorname{cl}(B(\overline{z}), \delta))]$ are closed, for any $(x, y) \in \operatorname{Gr}F \cap [B(\overline{x}, \delta) \times B(\overline{y}, \delta)]$ and any $t \in (0, \delta)$,

$$B(y, \alpha t) \cap (y - \operatorname{cone}M) \subset F(B(x, t) \cap (x + \operatorname{cone}L)) + K \cap B(0, \gamma t),$$

and finally, for every $(x, z) \in \mathrm{Gr}G \cap [B(\overline{x}, \delta) \times B(\overline{z}, \delta)]$ and all $t \in (0, \delta)$, one has

$$B(x, t) \cap (x - \operatorname{cone} S_X) \subset G^{-1}(B(z, \beta t) \cap (z + \operatorname{cone} M)).$$

Let us now choose $\varepsilon = \frac{\delta}{2}$ and fix some $(x, y, z) \in B(\overline{x}, \varepsilon) \times B(\overline{y}, \varepsilon) \times B(\overline{z}, \varepsilon)$ with $y \in F(x)$ and $z \in G(x)$, and also $\rho \in (0, \varepsilon)$. The proof of the fact that the inclusion

$$B(y + z, 2^{-1}(\alpha - \beta)\rho) \cap (y + z - \mathrm{cone}M) \subset (F + G)(B(x, \rho) \cap (x + \mathrm{cone}L)) + K \cap B(0, \gamma\rho)$$

holds is basically the proof of the previous theorem, so we skip the technical details. It is sufficient to notice that, in our present context with the notation from the cited theorem, it is still true that F is $(\alpha, \gamma)-$directionally open with respect to (L, M) and K at $(x_\varepsilon, y_\varepsilon)$ and G^{-1} is $\beta^{-1}-$directionally linearly open at $(z_\varepsilon, x_\varepsilon)$ with respect to M and S_X since

$$(x_\varepsilon, y_\varepsilon) \in \mathrm{Gr}F \cap [B(\overline{x}, \rho) \times B(\overline{y}, \rho)] \subset \mathrm{Gr}F \cap [B(\overline{x}, \delta) \times B(\overline{y}, \delta)]$$

and

$$(x_\varepsilon, z_\varepsilon) \in \mathrm{Gr}G \cap [B(\overline{x}, \rho) \times B(\overline{z}, \rho)] \subset \mathrm{Gr}G \cap [B(\overline{x}, \delta) \times B(\overline{z}, \delta)].$$

Thus the proof is completed. □

Further, to get from this result to the local stability of the sum of F and G we use the following notion of sum-stability of the pair (F, G).

Definition 3.5. Let $F, G : X \rightrightarrows Y$ be two set-valued maps and $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Y$. The pair (F, G) is called locally sum-stable around $(\overline{x}, \overline{y}, \overline{z})$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in B(\overline{x}, \delta)$ and every $w \in (F + G)(x) \cap B(\overline{y} + \overline{z}, \delta)$ there exist $y \in F(x) \cap B(\overline{y}, \varepsilon)$ and $z \in G(x) \cap B(\overline{z}, \varepsilon)$ such that $w = y + z$.

Corollary 3.6. *Let $F, G : X \rightrightarrows Y$ be two set-valued maps with $\mathrm{Gr}F$ and $\mathrm{Gr}G$ locally closed. Let $K \subset Y$ be a closed and convex cone and $L \subset S_X$, $M \subset S_Y$ two closed sets of directions with $\mathrm{cone}M$ convex. Suppose that $\mathrm{Dom}(F + G)$ is nonempty and α, β, γ are strictly positive constants with $\beta < \alpha$. If F is $(\alpha, \gamma)-$directionally open with respect to (L, M) and K around $(\overline{x}, \overline{y}) \in \mathrm{Gr}F$ and G is $\beta-$directionally Aubin continuous around $(\overline{x}, \overline{z}) \in \mathrm{Gr}G$ with respect to S_X and M and the pair (F, G) is locally sum-stable around $(\overline{x}, \overline{y}, \overline{z})$, then $F + G$ is $(2^{-1}(\alpha - \beta), \gamma)-$directionally open with respect to (L, M) and K around $(\overline{x}, \overline{y} + \overline{z})$.*

Proof. From the previous result we know that there exists $\varepsilon > 0$ such that for all $(x, y, z) \in B(\overline{x}, \varepsilon) \times B(\overline{y}, \varepsilon) \times B(\overline{z}, \varepsilon)$ with $y \in F(x)$, $z \in G(x)$ and for all $\rho \in (0, \varepsilon)$, one has

$$B(y+z, 2^{-1}(\alpha-\beta)\rho) \cap (y+z-\operatorname{cone} M) \subset (F+G)(B(x, \rho) \cap (x+\operatorname{cone} L)) + K \cap B(0, \gamma\rho). \tag{3.9}$$

Let ε be the one from above and $\delta > 0$ the one handed by the locally sum-stability property of (F, G). We can assume, without loss of generality, that $\delta < \varepsilon$. We want to prove that for all $(x, w) \in \mathrm{Gr}(F + G) \cap (B(\overline{x}, \delta) \times B(\overline{y} + \overline{z}, \delta))$ and for all $\rho \in (0, \varepsilon)$, one has

$$B(w, 2^{-1}(\alpha - \beta)\rho) \cap (w - \operatorname{cone} M) \subset (F + G)(B(x, \rho) \cap (x + \operatorname{cone} L)) + K \cap B(0, \gamma\rho).$$

This is true from (3.9) because, again from the sum-stability of (F, G), we can write $w = y + z$ with $y \in F(x) \cap B(\overline{y}, \varepsilon)$ and $z \in G(x) \cap B(\overline{z}, \varepsilon)$. The proof is complete. □

For the second aim of this section, that is, obtaining optimality conditions for the unconstrained (P), we recall some sufficient conditions for the directional openness of a set-valued map that one can find in the proof of Theorem 3.10 from [11]. We formulate this result here, for the coherence of our presentation.

Theorem 3.7. *Let X and Y be finite dimensional spaces, $\emptyset \neq L \subset S_X$ and $\emptyset \neq M \subset S_Y$ be closed sets such that* $\operatorname{cone} L$ *and* $\operatorname{cone} M$ *are convex, K be a closed convex cone in Y and let $F : X \rightrightarrows Y$ be a set-valued map with $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Suppose that the following are satisfied:*

(i) $\operatorname{Gr} F$ *is closed;*

(ii) *there exist $c > 0,\ r > 0$ such that for all $y^* \in K^+$ and $u \in M$ with $\langle y^*, u\rangle = 1$ and for every $z^* \in B(0_Y, 2c)$, $(x, y) \in \operatorname{Gr} F \cap (B(\overline{x}, r) \times B(\overline{y}, r))$ and $x^* \in \widehat{D}^* F(x, y)(y^* + z^*)$, there is $w \in L$ such that*
$$-\langle x^*, w\rangle \geq c\|y^* + z^*\|.$$

Then there exists $\varepsilon > 0$ such that for all $a \in (0, c)$, for all $\rho \in (0, \varepsilon)$ and for all $(x, y) \in \operatorname{Gr} F \cap \big(B(\overline{x}, 2^{-1}r) \times B(\overline{y}, 2^{-1}r)\big)$,
$$B(y, \rho a) \cap (y - \operatorname{cone} M) \subset F(B(x, \rho) \cap (x + \operatorname{cone} L)) + K. \tag{3.10}$$

Remark 3.8. Making a slight modification in the proof mentioned above, namely, with the therein notation, applying the Ekeland variational principle on the domain $\operatorname{Gr} F \times (K \cap \operatorname{cl} B(0, a\rho))$, one can obtain the next form of the openness of F: there exists $\varepsilon > 0$ such that for all $a \in (0, c)$, for all $\rho \in (0, \varepsilon)$ and for all $(x, y) \in \operatorname{Gr} F \cap \big(B(\overline{x}, 2^{-1}r) \times B(\overline{y}, 2^{-1}r)\big)$,
$$B(y, \rho a) \cap (y - \operatorname{cone} M) \subset F(B(x, \rho) \cap (x + \operatorname{cone} L)) + K \cap B(0, (a + 1)\rho). \tag{3.11}$$

The optimality conditions we give next are based on the opposition between openness and minimality, but the hypotheses do not concern a minimal point for the objective set-valued map F, but a sequence of minimal points for some perturbations of F. As such, we prove that the limit of such a sequence is a critical point for F.

Theorem 3.9. *Let X and Y be finite dimensional spaces and $F : X \rightrightarrows Y$ be a set-valued map with a sequence of perturbating maps $G_n : X \rightrightarrows Y$, $n \in \mathbb{N}$. Let $L \subset S_X$ be a closed set with* $\operatorname{cone} L$ *convex; let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ be the limit of a sequence of Pareto directional minima $(x_n, y_n) \in \operatorname{Gr}(F + G_n)$ for $F + G_n$ with respect to the set of directions L. Assume that*

(i) *$K \subset Y$ is a closed convex cone and let $u \in S_Y \cap (K \setminus -K)$;*

(ii) $\operatorname{Gr} F$ *is closed and* $\operatorname{Gr} G_n$ *is locally closed at every point close to $(\overline{x}, 0)$;*

(iii) *F is Lipschitz-like around $(\overline{x}, \overline{y})$ and for all n there is $\beta_n > 0$ such that G_n is β_n-directionally Aubin continuous with respect to S_X and $\{u\}$ around every point from its graph close to $(\overline{x}, 0)$; $\beta_n \to 0$;*

(iv) *for all n, the pair (F, G_n) is locally sum-stable around $(\overline{x}, \overline{y}, 0)$.*

Then, there are $x^* \in X^*$, $y^* \in K^+$ with $\langle x^*, l\rangle \geq 0$ for all $l \in L$ and $\langle y^*, u\rangle = 1$ such that
$$x^* \in D^* F(\overline{x}, \overline{y})(y^*). \tag{3.12}$$

Proof. The idea of the proof is to show that the openness condition from (3.11) can not hold and from that to deduce that condition (ii) of Theorem 3.7 is not satisfied and then, by negation, to obtain the optimality conditions. For that, suppose by means of contradiction that there is $\alpha > 0$ such that F is $(\alpha, \alpha + 1)-$directionally open with respect to $(L, \{u\})$ and K around $(\overline{x}, \overline{y})$. The sum-stability hypothesis ensures that for all n there are $u_n \in F(x_n)$ and $v_n \in G_n(x_n)$ such that $y_n = u_n + v_n$, $u_n \to \overline{x}$ and $v_n \to 0$.

Thus, since $(x_n, u_n, v_n) \to (\overline{x}, \overline{y}, 0)$ and $\beta_n \to 0$, we get the following facts valid for all n large enough: $\beta_n < \alpha$, $\operatorname{Gr} G_n$ is locally closed at (x_n, v_n), F is $(\alpha, \alpha+1)-$directionally open with respect to $(L, \{u\})$ and K around (x_n, u_n), G_n is β_n-directionally Aubin continuous around (x_n, v_n) with respect to S_X and $\{u\}$ and finally, the pair (F, G_n) is locally sum-stable around (x_n, u_n, v_n). These facts ensure, according to Corollary 3.6, that $F + G_n$ is $(2^{-1}(\alpha - \beta_n), \alpha+1)-$directionally open with respect to $(L, \{u\})$ and K around (x_n, u_n, v_n). That is, there is $\varepsilon_1 > 0$ such that for all $\rho \in (0, \varepsilon_1)$, one has in particular that

$$B(y_n, 2^{-1}(\alpha-\beta_n)\rho) \cap (y_n - \operatorname{cone}\{u\}) \subset (F+G_n)(B(x_n, \rho) \cap (x_n + \operatorname{cone} L)) + K \cap B(0, (\alpha+1)\rho). \tag{3.13}$$

By Proposition 3.7 from [6], in the assumption of the minimality of the points (x_n, y_n), one gets that there is $\varepsilon_2 > 0$ such that for all $r > 0$ it holds that

$$B(y_n, r) \cap (y_n - \operatorname{cone}\{u\}) \not\subset (F + G_n)(B(x_n, \varepsilon_2) \cap (x_n + \operatorname{cone} L)) + K.$$

But this is clearly a contradiction to relation (3.13) since, if $\varepsilon_1 > \varepsilon_2$ we can choose $r = 2^{-1}(\alpha - \beta_n)\varepsilon_2$, and if $\varepsilon_1 < \varepsilon_2$ we can choose $r = 2^{-1}(\alpha - \beta_n)\rho$ for arbitrary $\rho \in (0, \varepsilon_1)$ and write that

$$\begin{aligned} B(y_n, 2^{-1}(\alpha - \beta_n)\rho) \cap (y_n - \operatorname{cone}\{u\}) &\subset (F + G_n)(B(x_n, \rho) \cap (x_n + \operatorname{cone} L)) + K \\ &\subset (F + G_n)(B(x_n, \varepsilon_2) \cap (x_n + \operatorname{cone} L)) + K. \end{aligned}$$

Therefore, condition (ii) from Theorem 3.7 does not hold for L and $M := \{u\}$ and this means that for all n there are sequences $(x_n, y_n) \xrightarrow{\operatorname{Gr} F} (\overline{x}, \overline{y})$, $y_n^* \in K^+$ with $\langle y_n^*, u\rangle = 1$, $z_n^* \in B(0, 2n^{-1}) \subset Y^*$ and $x_n^* \in X^*$ such that

$$\begin{aligned} &x_n^* \in \widehat{D}^* F(x_n, y_n)(y_n^* + z_n^*) \\ &-\langle x_n^*, l\rangle < \frac{1}{n}\|y_n^* + z_n^*\|, \ \forall l \in L. \end{aligned} \tag{3.14}$$

The sequence (y_n^*) is bounded according to [16, Lemma 2.2.17], having that $u \in \operatorname{int} K$. From the Lipschitz property of F around $(\overline{x}, \overline{y})$ together with [21, Theorem 1.43] and the fact that z_n^* is also clearly bounded, we can conclude that (x_n^*) is also bounded. Therefore, on some subsequences, we have that $x_n^* \to x^* \in X^*$, $y_n^* \to y^* \in K^+$ and $z_n^* \to 0_{Y^*}$. Passing now to the limit in (3.14) we get that

$$\begin{aligned} &x^* \in D^* F(\overline{x}, \overline{y})(y^*), \text{ with } \langle y^*, u\rangle = 1, \\ &\langle x^*, l\rangle \geq 0, \forall l \in L, \end{aligned}$$

which is exactly the conclusion. □

Remark 3.10. This result can be seen as a generalization of Proposition 3.7 from [6], in the sense that we prove that not only a set-valued map can not be directionally open at a directional minimum point, but that it can not be open at the limit of a sequence of directional minimum points for some Lipschitz perturbations of it. Using the sufficient openness conditions from Remark 3.11, we arrive at the optimality conditions.

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