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ON THE SPLIT COMMON FIXED POINT PROBLEM BY FINITE SYSTEMS OF EQUILIBRIA: AN ITERATIVE SCHEME WITH M- DEMICONTRACTIVE MAPPINGS

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Abstract: The purpose of this paper is to study the strong convergence of a general iterative scheme to find a common element of the set of solutions to a system of equilibrium problems. Our algorithm is inspired by those presented by Moudafi [22] and Takahashi [31, 32] on the standard equilibrium problem with the use of nonexpansive mappings. In our case we consider a system of equilibrium problems by introducing a new concept of M- demicontractive mappings which can be intermediately situated between nonexpansive operators and demicontractive ones. This allows us to exhibit a solution to a split common fixed point problem formulated as a system of equilibria. A numerical example written in MATLAB illustrates the convergence of our algorithm.

Key words: *split common fixed point, finite systems of equilibria, iterative scheme, strong convergence, M-demicontractive mappings, rate of convergence, MATLAB simulation, numerical example*

Mathematics Subject Classification: *47J20, 47J25, 47J26*

1 Introduction and Preliminaries

Let us agree to define our system of equilibrium problems. Let H be a real Hilbert space endowed with the usual inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|.\|$. Let $C_1, \dots, C_m$ be a sequence of nonempty, closed and convex subsets of H and let $(F_i)_{i=1,\dots,m}$ be a sequence of real-valued bifunctions such that F_i is defined on $C_i \times C_i$ for each $i = 1, 2 \dots, m$. The corresponding system of equilibrium problems we consider aims at finding a point x in $C = \bigcap_{i=1}^m C_i$ such that :

$$\text{(SEP)} \qquad \mathrm{F_i(x, y) \geq 0 \text{ for all } y \in C_i, \text{ and all } i = 1, \dots, m.} \tag{1.1}$$

In the case $m = 1$, the system (SEP) turns out to be the single equilibrium problem, which unifies several mathematical problems such as minimization problems, monotones inclusion problems, variational inequality problems, complementarity problems and fixed point problems, see [2, 6, 37] for example. The field of concrete applications of equilibrium problems and their related systems is very wide. Actually, broad classes of decision models arising from operation research, finance, transportation, network and structural analysis, elasticity and mathematical economics to many other engineering problems lead to the formulation in (1.1) for $m = 1$.

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The problem (1.1) extends the single formalism of equilibria to systems of such problems, covering in particular various forms of feasibility problems [1] and [15].

From the algorithmic viewpoint, several methods for solving scalar equilibrium problems have been developed extensively. The most popular approaches are the proximal point methods and the gradient ones. In the following, we review the general framework for the first one, the second kind of iterative methods is more expansive, we refer to [34] for more details. It is well known that the proximal-point method was firstly extended by Martinet [20] from optimization problems to classical variational inequalities. A further extension has been done by Rockafellar [26] to maximal monotone inclusions. The paper by Moudafi [22] was the first work that extended the proximal methods to equilibrium problems, wherein it was proved, as in the case of inclusions in [26], that when the solution set is nonempty, the sequence (x_k) of the iterative scheme defined by the associated proximal-point method is weakly convergent to a solution. Note that this convergence is not of a strong type in general, which has been underlined by Güler in [16] through an example in the space l^2 for the optimization case. Recently, more attention has been given to develop efficient and implementable numerical methods to solve equilibrium problems, see [34] for more details. Takahashi et al [31, 32] introduced a new iterative scheme based on the viscosity approximation method for finding the best approximation for a common element of the set of solutions of a system of an equilibrium problem and a fixed point problem with nonexpansive mappings for which they obtained strong convergence results, in this regard we also refer to [10, 19].

More recently, in [3], Buong proposed a regularization extragradient method for solving a system of equilibrium problems. Unfortunately, the main result [3, Thm 2.1] fails in the proof. Apparently, this reason has motivated the author of [3] to take back the same problem in [4] by considering a finite family of inverse strongly monotone operators.

The first goal of this paper is to adapt and investigate the Moudafi's proximal point method for solving the equilibrium system (1.1) in the Hilbert framework. We prove the strong convergence of the proposed iterative algorithm to a selected solution of the equilibrium system (1.1). The second goal of this work is to use our iterative algorithm proposed for equilibrium system (1.1), for solving the split common fixed point problem as defined by Censor and Segal in [5], by focusing on the relationship between the two problems. Finally, we give a numerical example for our main result in space of real numbers.

Recall that the main idea for the proximal point method is to use an adequate resolvent for equilibrium functions. If C is a convex set of H, $F : C \times C \to \mathbb{R}$, $r > 0$ and $x \in H$, the resolvent of F at x with parameter r, denoted by $J_r^F(x)$, is the set of solutions of the perturbed equilibrium problem : Find $z \in C$ such that

$$rF(z, y) + \langle y - z, z - x\rangle \geq 0 \;\; \forall y \in C.$$

We remark, see [10, Lemma 2.15], that if $F(x, y) = f(y) - f(x)$ for $x, y \in C$ and $C \subset \operatorname{dom} f$, then $J_r^F(x) = (I + r\partial f)^{-1}(x)$ where ∂f is the convex subdifferential of f. More generally, if $F(x, y) = \sup_{u \in Ax}\langle u, y - x\rangle$ where A is a maximal monotone operator and $C \subset \operatorname{int} \operatorname{dom} A$ then J_r^F is the Rockafellar's resolvent of the operator $A + N_C$. Here $\operatorname{dom} f$ is the domain of function f and N_C is the normal operator to convex set C defined for all $z \in C$, by $N_C(z) := \{w \in H : \langle z - u, w\rangle \geq 0, \quad \forall u \in H\}$.

In the sequel, the following usual conditions and lemmas will be used :

1. $F(x, x) = 0$ for each $x \in C$;
2. for each $x \in C$, $y \to F(x, y)$ is convex ;

3. for each $x \in C$, $y \to F(x, y)$ is lower semicontinuous;

4. F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for each $x, y \in C$;

5. $\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$ for any $x, y, z \in C$.

Lemma 1.1 (Minty's lemma [2])**.** *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ and consider the equilibrium problem (EP) : find $\overline{x} \in C$ such that $F(\overline{x}, y) \geq 0, \quad \forall y \in C$, and the associated dual problem (DEP) : find $\overline{x} \in C$ such that $F(y, \overline{x}) \leq 0, \ \forall y \in C$.*

(i) *If F satisfies (4), then each solution of (EP) is a solution to (DEP).*

(ii) *Conversely, if F satisfies (1), (2) and (5) then each solution of (DEP) is a solution to (EP).*

Remark 1.2. The following observation will be useful is the in the sequel:

(i) If a bifunction $F : C \times C \to \mathbb{R}$ satisfies [(1)-(5)] and the set of solutions of equilibrium problem (EP), say S, is not empty, then S is closed and convex. Indeed, let (x_n) be a sequence in S converging (strongly) to a point x. By (4), for each $n \geq 0$, x_n is also a solution to (DEP), then we have $\liminf_{n \longrightarrow \infty} F(y, x_n) \leq 0$. Now, use (3) and see that $F(y, x) \leq 0$, which means that x is a solution to (DEP). Under conditions (1), (2) and (5), x is a solution to (EP). Hence, the set S is closed. On the other hand, from the convexity of $y \to F(x, y)$ (condition (2)) and Minty's Lemma follows the convexity of S.

(ii) In view of the point (i) and the fact that the intersection of any closed and convex sets is closed and convex, whenever the bifunction $F_i : C_i \times C_i \to \mathbb{R}$ satisfies [(1)-(5)] for all $i \geq 1$ and the problem (1.1) is consistent, it can be readily seen that the set of solutions to (1.1) is closed and convex.

Lemma 1.3 ([8])**.** *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ satisfying [(1)-(4)]. Then the following are equivalent :*

(i) *F is maximal : $(x, u) \in C \times H$ and $F(x, y) \leq \langle u, x - y \rangle, \forall y \in C$ imply that $F(x, y) + \langle u, x - y \rangle \geq 0, \forall y \in C$;*

(ii) *for each $x \in H$ and $r > 0$, there exists a unique $z = J_r^F(x)$ such that*

$$rF(z, y) + \langle y - z, z - x \rangle \geq 0, \ \forall y \in C. \tag{1.2}$$

Let us mention that the maximality of F is ensured when assumptions (1), (2) and (5) are satisfied (see [6, Lemma 2.1]). Thus the associated resolvent mapping $J_r^F : H \to C$ is well defined and satisfies the following

Lemma 1.4 ([10])**.** *Let C be a nonempty closed convex subset of H and let F be a bi-function from $C \times C$ into $\mathbb{R}$ satisfying [(1)-(5)]. Then,*

(i) *J_r^F is firmly nonexpansive; i.e, for all $x, y \in H$*

$$\left\| J_r^F(x) - J_r^F(y) \right\|^2 \leq \langle J_r^F(x) - J_r^F(y), x - y \rangle;$$

(ii) $\overline{x} = J_r^F(\overline{x}) \Leftrightarrow F(\overline{x}, y) \geq 0, \forall y \in C.$

A mapping $T : C \longrightarrow H$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C.$$

If T is nonexpansive, then the set of all fixed points $FixT$, which is defined by

$$FixT = \{x \in C : x = Tx\}$$

is always closed and convex, see [17].

Definition 1.5. Suppose $T : C \longrightarrow C$ is nonexpansive and K a closed convex nonempty subset of C. We say that T is attracting w.r.t K if for every $x \in C \backslash K$, $z \in K$

$$\|Tx - z\| < \|x - z\|.$$

If $K = FixT$, we simply speak of attracting mappings.

Lemma 1.6 ([1, Proposition 2.10])**.** *Suppose C is a closed convex nonempty set, $T_1, T_2, \dots, T_m : C \longrightarrow C$ are attracting and $\bigcap_{i=1}^m FixT_i$ is nonempty. Then*

$$Fix(T_m T_{m-1} \dots T_1) = \bigcap_{i=1}^m FixT_i$$

and $T_m T_{m-1} \dots T_1$ is attracting.

Corollary 1.7. *Under assumptions of Lemma 1.4 we claim that*

(i) *the mapping J_r^F is attracting;*

(ii) *$\overline{x}$ is a solution to (1.1) if, and only if, $\overline{x} = J_r^{F_i}(\overline{x})$ $\forall i = 1, \dots m$, which is equivalent to $\overline{x} = J_r^{F_m} J_r^{F_{m-1}} \dots J_r^{F_1}(\overline{x})$.*

Proof. (i) Let $x \notin Fix(J_r^F)$ and $y \in Fix(J_r^F)$, we have $F(y, J_r^F x) \geq 0$ and

$$rF(J_r^F x, y) + \langle J_r^F x - x, y - J_r^F x \rangle \geq 0.$$

By adding the two last inequalities and using monotonicity of F, we infer

$$\langle J_r^F x - x, y - J_r^F x \rangle \geq 0,$$

which leads to

$$\langle J_r^F x - y, y - x \rangle + \|y - x\|^2 - \|x - J_r^F x\|^2 \geq 0.$$

Now, since J_r^F is firmly nonexpansive and $x \neq J_r^F(x)$, we finally get

$$\|J_r^F x - y\|^2 \leq \|y - x\|^2 - \|x - J_r^F x\|^2 < \|y - x\|^2.$$

(ii) Observe that

$$\begin{aligned} \overline{x} \text{ is a solution to (1.1)} \quad &\Leftrightarrow \quad J_r^{F_i}(\overline{x}) = \overline{x} \; \forall i = 1, \dots, m \;\; \text{(see Lemma 1.4 (ii))} \\ &\Leftrightarrow \quad \overline{x} = J_r^{F_m} J_r^{F_{m-1}} \dots J_r^{F_1}(\overline{x}) \;\; \text{(see Lemma 1.6)}. \end{aligned}$$

□

Definition 1.8. A mapping T is said to be demiclosed if for any sequence (v^k) in H, the following implication holds: if $v^k \rightharpoonup v$ and $T(v^k) \longrightarrow w$, then $T(v) = w$, where " $\rightharpoonup$ " and " $\longrightarrow$ " stand for the weak convergence and strong convergence, respectively. If $v = 0$, the zero vector in H, then T is called demiclosed at zero.

Lemma 1.9 ([14, Theorem 10.3])**.** *Let H be a real Hilbert space, C a nonempty closed convex subset of H, and $T : C \longrightarrow H$ a nonexpansive mapping such that $FixT$ is nonempty. Then, the mapping $I - T$ is demiclosed on C, where I is the identity mapping, that is*

$$v^k \rightharpoonup v \text{ in } H \text{ and } (I-T)(v^k) \longrightarrow w$$

implies that $v \in C$ and $w = (I - T)v$.

Lemma 1.10 ([30])**.** *Let H be a real Hilbert space, let (α_k) be a sequence of real numbers such that $0 < \liminf_{k\to+\infty} \alpha_k \le \limsup_{k\to+\infty} \alpha_k < 1$ and let (x^k) and (y^k) be bounded sequences in H such that $x^{k+1} = \alpha_k x^k + (1 - \alpha_k)y^k$ and*

$$\limsup_{k\to 0} \left(\|y^{k+1} - y^k\| - \|x^{k+1} - x^k\| \to 0.\right.$$

Then $\|y^k - x^k\|\big) \longrightarrow 0$.

Lemma 1.11 ([36])**.** *Let (a^k) be a sequence of nonnegative real numbers such that $a_{k+1} \le (1 - t_k)a_k + \delta_k$ where $\sum_{k=1}^{+} \infty t_k = +\infty$ and $\limsup_{k\to+\infty} \frac{\delta_k}{t_k} \le 0$. Then $a_k \to 0$.*

2 Strong Convergence Theorem

We are now ready to introduce our iterative process :

Algorithm 2.1. From $x^0 \in C_m$, we generate a sequence (x^k) by

$$x^{k+1} = \alpha_k x^k + \beta_k g(x^k) + \gamma_k J_r^{F_m} J_r^{F_{m-1}} \dots J_r^{F_1}(x^k). \tag{2.1}$$

Theorem 2.2. *Let $C = \bigcap_{i=1}^{m} C_i$ be nonempty, and let F_i, for $i = 1, \dots, m$, be a family of bi-functions from $C_i \times C_i$ into $\mathbb{R}$ satisfying [(1)-(5)]. Suppose that the set S of solutions to (1.1) is nonempty and let $g : C_m \longrightarrow C_m$ be a $\delta -$ contaction, $\delta \in (0, 1)$. Assume that $\alpha_k, \beta_k, \gamma_k$ are positive scalars such that $\alpha_k + \beta_k + \gamma_k = 1$, $\beta_k \to 0$, $\sum \beta_k = +\infty$ and $(\alpha_k) \subset [c, d]$ for some $c, d \in (0, 1)$. Then, the sequence (x^k) strongly converges to a point $\overline{x} \in S$ which satisfies $\overline{x} = \Pi_S(g(\overline{x}))$.*

Proof. To simplify, we take $m = 2$.
Step 1: $\Pi_S \circ g : C_m \longrightarrow S$ is a contraction. Indeed,

$$\|\Pi_S \circ g(x) - \Pi_S \circ g(y)\| \le \|g(x) - g(y)\| \le \delta\|x - y\|,$$

where $\delta \in (0, 1)$. Then, there exists a unique $\overline{x}$ such that $\Pi_S \circ g(\overline{x}) = \overline{x}$.
Step 2: We first set $u^k = J_r^{F_1}(x^k)$. Since $\overline{x} \in S$, in view of Lemma 1.6, it follows that $\overline{x} = J_r^{F_1}(\overline{x})$. Thus

$$\|u^k - \overline{x}\| = \|J_r^{F_1}(x^k) - J_r^{F_1}(\overline{x})\| \le \|x^k - \overline{x}\|.$$

Set $M = \max\{\|x^0 - \overline{x}\|, \frac{1}{1-\delta}\|g(\overline{x}) - \overline{x}\|\}$ and suppose that $\|x^k - \overline{x}\| \leq M$ to see that

$$\begin{aligned} \|x^{k+1} - \overline{x}\| &= \|\alpha_k(x^k - \overline{x}) + \beta_k(g(x^k) - \overline{x}) + \gamma_k(J_r^{F_2}(u_k) - \overline{x})\| \\ &\leq \alpha_k\|x^k - \overline{x}\| + \beta_k\|g(x^k) - \overline{x}\| + \gamma_k\|u^k - \overline{x}\| \\ &\leq (\alpha_k + \gamma_k)\|x^k - \overline{x}\| + \beta_k(\delta\|x^k - \overline{x}\| + \|g(\overline{x}) - \overline{x}\|) \\ &\leq (1 - \beta_k + \delta\beta_k)\|x^k - \overline{x}\| + \beta_k\|g(\overline{x}) - \overline{x}\| \\ &= (1 - (1-\delta)\beta_k)\|x^k - \overline{x}\| + (1-\delta)\beta_k\frac{1}{1-\delta}\|g(\overline{x}) - \overline{x}\| \\ &\leq M. \end{aligned}$$

Hence, by induction, (x^k) and (u^k) are bounded.

Step 3: We shall show that $(x^{k+1} - x^k)$ and $(u^{k+1} - u^k)$ strongly converge to 0. Indeed, by definition of $u_k = J_r^{F_1}(x^k)$ and $u^{k+1} = J_r^{F_1}(x^{k+1})$, which are in C_1, and replacing respectively y by u^{k+1} and u^k in (1.3), we have

$$F_1(u^k, u^{k+1}) + \frac{1}{r}\langle u^{k+1} - u^k, u^k - x^k\rangle \geq 0$$

and

$$F_1(u^{k+1}, u^k) + \frac{1}{r}\langle u^k - u^{k+1}, u^{k+1} - x^{k+1}\rangle \geq 0.$$

By summing the last two inequalities and using monotonicity of F_1, we get

$$\begin{aligned} 0 &\leq \left\langle u^{k+1} - u^k, u^k - x^k\right\rangle + \left\langle u^k - u^{k+1}, u^{k+1} - x^{k+1}\right\rangle \\ &= \left\langle u^{k+1} - u^k, (u^k - x^k) - (u^{k+1} - x^{k+1})\right\rangle \\ &= -\|u^{k+1} - u^k\|^2 + \left\langle u^{k+1} - u^k, x^{k+1} - x^k\right\rangle, \end{aligned}$$

which implies $\|u^{k+1} - u^k\| \leq \|x^{k+1} - x^k\|$. Let us write $x^{k+1} = \alpha_k x^k + (1-\alpha_k)y^k$ with

$$y^k = \frac{\beta_k}{1-\alpha_k}g(x^k) + \frac{\gamma_k}{1-\alpha_k}J(x^k),$$

where $J = J_r^{F_2} \circ J_r^{F_1}$. Therefore, given that $\|g(x^k)\|$, $\|J(x^k)\|$ are upper-bounded by $K > 0$ and $\alpha_k + \beta_k + \gamma_k = 1$, it follows that

$$\begin{aligned} \|y^{k+1} - y^k\| - \|x^{k+1} - x^k\| &= \left\|\left(\frac{\beta_{k+1}}{1-\alpha_{k+1}}g(x^{k+1}) + \frac{\gamma_{k+1}}{1-\alpha_{k+1}}J(x^{k+1})\right)\right. \\ &\quad \left. - \left(\frac{\beta_k}{1-\alpha_k}g(x^k) + \frac{\gamma_k}{1-\alpha_k}J(x^k)\right)\right\| - \|x^{k+1} - x^k\| \\ &\leq \frac{\beta_{k+1}\delta}{1-\alpha_{k+1}}\|x^{k+1} - x^k\| + \left|\frac{\beta_{k+1}}{1-\alpha_{k+1}} - \frac{\beta_k}{1-\alpha_k}\right|\|g(x^k\| \\ &\quad + \frac{\gamma_{k+1}}{1-\alpha_{k+1}}\|x^{k+1} - x^k\| + \left|\frac{\gamma_{k+1}}{1-\alpha_{k+1}} - \frac{\gamma_k}{1-\alpha_k}\right|\|J(x^k\| \\ &\quad - \|x^{k+1} - x^k\| \\ &\leq \frac{\beta_{k+1}}{1-\alpha_{k+1}}(\delta + 1)\|x^{k+1} - x^k\| + 2K\left|\frac{\beta_{k+1}}{1-\alpha_{k+1}} - \frac{\beta_k}{1-\alpha_k}\right|. \end{aligned}$$

Then, as $\beta_k \longrightarrow 0$ and $0 < \liminf \alpha_k \leq \limsup \alpha_k < 1$, it results that

$$\limsup_{k\to 0}(\|y^{k+1} - y^k\| - \|x^{k+1} - x^k\|) \leq 0.$$

This allows us to Lemma 1.10 to get $\|y^k - x^k\| \longrightarrow 0$. Thus

$$\|x^{k+1} - x^k\| = \|\alpha_k x^k + (1-\alpha_k)y^k - x^k\| = (1-\alpha_k)\|y^k - x^k\| \longrightarrow 0.$$

In addition, since $\|u^{k+1} - u^k\| = \|J_r^{F_1}(x^{k+1}) - J_r^{F_1}(x^k)\| \leq \|x^{k+1} - x^k\|$, we also have $\|u^{k+1} - u^k\| \longrightarrow 0$.

Step 4: Now, we will prove that the sequence $(x^k - u^k)$ strongly converges to 0. To do that observe first that

$$\begin{aligned}\|u^k - \overline{x}\|^2 &= \|J_r^{F_1}(x^k) - J_r^{F_1}(\overline{x})\|^2 \\ &\leq \langle J_r^{F_1}(x^k) - J_r^{F_1}, x^k - \overline{x}\rangle \quad \text{(by Lemma 1.4, (ii))} \\ &= \langle u^k - \overline{x}, x^k - \overline{x}\rangle \\ &= \frac{1}{2}(\|u^k - \overline{x}\|^2 + \|x^k - \overline{x}\|^2 - \|u^k - x^k\|^2).\end{aligned}$$

Hence

$$\|u^k - \overline{x}\|^2 \leq \|x^k - \overline{x}\|^2 - \|x^k - u^k\|^2,$$

which implies that

$$\begin{aligned}\|x^{k+1} - \overline{x}\|^2 &\leq \beta_k\|g(x^k) - \overline{x}\|^2 + \alpha_k\|x^k - \overline{x}\|^2 + \gamma_k\|J_r^{F_2}(u^k) - \overline{x}\|^2 \\ &\leq \beta_k\|g(x^k) - \overline{x}\|^2 + \alpha_k\|x^k - \overline{x}\|^2 + \gamma_k\|u^k - \overline{x}\|^2 \\ &\leq \beta_k\|g(x^k) - \overline{x}\|^2 + (\alpha_k + \gamma_k)\|x^k - \overline{x}\|^2 - \gamma_k\|x^k - u^k\|^2.\end{aligned}$$

Accordingly, we obtain

$$\begin{aligned}\gamma_k\|x^k - u^k\|^2 &\leq \beta_k\|g(x^k) - \overline{x}\|^2 + (1-\beta_k)\|x^k - \overline{x}\|^2 - \|x^{k+1} - \overline{x}\|^2 \\ &\leq \beta_k\|g(x^k) - \overline{x}\|^2 + \|x^k - \overline{x}\|^2 - \|x^{k+1} - \overline{x}\|^2 \\ &\leq \beta_k\|g(x^k) - \overline{x}\|^2 + (\|x^k - \overline{x}\| - \|x^{k+1} - \overline{x}\|)(\|x^k - \overline{x}\| + \|x^{k+1} - \overline{x}\|) \\ &\leq \beta_k\|g(x^k) - \overline{x}\|^2 + \|x^{k+1} - x^k\|(\|x^{k+1} - \overline{x}\| + \|x^k - \overline{x}\|).\end{aligned}$$

Since $(\beta_k) \longrightarrow 0$ and $(\alpha_k) \subset [c,d]$ for $c, d \in (0,1)$, then the sequence (γ_k) with $\gamma_k = 1 - \alpha_k - \beta_k$ doesn't converge to 0, so using Step 3, we deduce that $x^k - u^k \longrightarrow 0$.

Step 5: Let us show that

$$\limsup_{k\longrightarrow+\infty}\langle g(\overline{x}) - \overline{x}, x^k - \overline{x}\rangle \leq 0. \tag{2.2}$$

In view of the boundedness of (x^k), there exists a subsequence (x^{k_i}) which weakly converges to some $w \in H$ such that

$$\limsup_{k\to+\infty}\langle g(\overline{x}) - \overline{x}, x^k - \overline{x}\rangle = \lim_{i\to+\infty}\langle g(\overline{x}) - \overline{x}, x^{k_i} - \overline{x}\rangle.$$

Moreover, as $(x^k - u^k)$ strongly converges to 0, we deduce also that (u^{k_i}) weakly converges to w.

We will prove that $w \in S$. Taking into account $u^k = J_r^{F_1}(x^k)$ and (1.3) we obtain from the monotonicity of F_1 the following

$$\langle y - u^k, u^k - x^k\rangle \geq rF_1(y, u^k) \quad \forall y \in C_1. \tag{2.3}$$

$F_1(y,.)$ being convex and lower semicontinuous (lsc), it is weakly lsc, so when i goes to $+\infty$ in (2.3), we get

$$rF_1(y,w) \leq \liminf_{i\to+\infty} rF_1(y,u^{k_i}) \leq \liminf_{i\to+\infty} \langle y-u^{k_i}, u^{k_i}-x^{k_i}\rangle. \tag{2.4}$$

Coming back to the fact that (u^{k_i}) weakly converges to w and (x^k-u^k) strongly converges to 0, we deduce that

$$\limsup_i \langle y-u^{k_i}, u^{k_i}-x^{k_i}\rangle = 0.$$

In addition, by (2.4), we also have

$$F_1(y,w) \leq 0 \quad \forall y \in C_1,$$

which means that w is a dual solution of $EP(F_1)$. Consequently, due to Minty's Lemma 1.1 (ii), we are able to conclude that $w \in EP(F_1)$.

Next, we show that $w \in EP(F_2)$. To this end, with $\alpha_k+\beta_k+\gamma_k = 1$ we can write

$$\begin{aligned} x^{k+1} - J_r^{F_2}(u^k) &= \alpha_k x^k + \beta_k g(x^k) + \gamma_k J_r^{F_2}(u^k) - J_r^{F_2}(u^k) \\ &= \alpha_k(x^k - J_r^{F_2}(u^k)) + \beta_k(g(x^k) - J_r^{F_2}(u^k)). \end{aligned}$$

Then

$$\begin{aligned} \|J_r^{F_2}(u^{k+1}) - x^{k+1}\| &\leq \|J_r^{F_2}(u^{k+1}) - J_r^{F_2}(u^k)\| + \|J_r^{F_2}(u^k) - x^{k+1}\| \\ &\leq \|u^{k+1}-u^k\| + \alpha_k\|x^k - J_r^{F_2}(u^k)\| + \beta_k\|g(x^k) - J_r^{F_2}(u^k)\| \\ &\leq (1-(\beta_k+\gamma_k))\|x^k - J_r^{F_2}(u^k)\| + (\|u^{k+1}-u^k\| \\ &\quad + \beta_k\|g(x^k) - J_r^{F_2}(u^k)\|). \end{aligned}$$

Now, let us set $a_k = \|x^k - J_r^{F_2}(u^k)\|$, $\delta_k = \|u^{k+1}-u^k\| + \beta_k\|g(x^k) - J_r^{F_2}(u^k)\|$ and $t_k = \beta_k+\gamma_k$. Since $\gamma_k > 0$ and $\gamma_k \nrightarrow 0$, then $\sum_{k=0}^{+\infty}\gamma_k = +\infty$. Furthermore, as $\delta_k \to 0$, we also have $\limsup_k \frac{\delta_k}{\gamma_k} \leq 0$. Hence, by virtue of Lemma 1.11, we obtain $\lim_{k\to+\infty}\|J_r^{F_2}(u^k) - x^k\| = 0$, which combined with

$$\|J_r^{F_2}(u^k) - u^k\| \leq \|J_r^{F_2}(u^k) - x^k\| + \|x^k - u^k\|$$

leads to $\lim_{k\to+\infty}\|J_r^{F_2}(u^k) - u^k\| = 0$.

Therefore, by taking $T = J_r^{F_2}$ in Lemma 1.9 and using the fact that $u^{k_i} \xrightarrow{w} w$ and $(I - J_r^{F_2})(u^{k_i}) \xrightarrow{s} 0$, we see that $0 = (I - J_r^{F_2})(w)$, i.e., $J_r^{F_2}(w) = w$ and then $w \in EP(F_2)$.

Finally, since $w \in S$, the characterization of projection $\overline{x} = \Pi_S(g(\overline{x}))$ gives us $\langle g(\overline{x}) - \overline{x}, w - \overline{x}\rangle \leq 0$, which implies that

$$\begin{aligned} \limsup_k \langle g(\overline{x}) - \overline{x}, x^k - \overline{x}\rangle &= \lim_i \langle g(\overline{x}) - \overline{x}, x^{k_i} - \overline{x}\rangle \\ &= \langle g(\overline{x}) - \overline{x}, w - \overline{x}\rangle \\ &\leq 0. \end{aligned}$$

Step 6: Having in mind that $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$, we can write

$$
\begin{aligned}
\|x^{k+1}-\overline{x}\|^2 &= \|\alpha_k(x^k-\overline{x})+\gamma_k(Jx^k-\overline{x})+\beta_k(g(x^k)-\overline{x})\|^2\\
&\le \|\alpha_k(x^k-\overline{x})+\gamma_k(Jx^k-\overline{x})\|^2+2\beta_k\langle g(x^k)-\overline{x},x^{k+1}-\overline{x}\rangle\\
&\le (\alpha_k\|(x^k-\overline{x})\|+\gamma_k\|Jx^k-\overline{x}\|)^2+2\beta_k\langle g(x^k)-g(\overline{x}),x^{k+1}-\overline{x}\rangle\\
&\quad +2\beta_k\langle g(\overline{x})-\overline{x},x^{k+1}-\overline{x}\rangle\\
&\le (\alpha_k+\gamma_k)^2\|x^k-\overline{x}\|^2+\beta_k\delta[\|x^k-\overline{x}\|^2+\|x^{k+1}-\overline{x}\|^2]\\
&\quad +2\beta_k\langle g(\overline{x})-\overline{x},x^{k+1}-\overline{x}\rangle\\
&\le (1-\beta_k)^2\|x^k-\overline{x}\|^2+\beta_k\delta\|x^k-\overline{x}\|^2+\beta_k\delta\|x^{k+1}-\overline{x}\|^2\\
&\quad +2\beta_k\langle g(\overline{x})-\overline{x},x^{k+1}-\overline{x}\rangle.
\end{aligned}
$$

Let us set $1-t_k=\frac{(1-\beta_k)^2+\beta_k\delta}{1-\delta\beta^k}$ and $\delta_k=\frac{2\beta_k}{1-\delta\beta_k}\langle g(\overline{x})-\overline{x},x^{k+1}-\overline{x}\rangle$. Of course we have

$$
\begin{aligned}
\limsup_k \frac{\delta_k}{t_k} &= \limsup_k\left[\frac{2\beta_k}{1-\delta\beta_k}\langle g(\overline{x})-\overline{x},x^{k+1}-\overline{x}\rangle\times\frac{1-\delta\beta_k}{(2-\beta_k-2\delta)\beta_k}\right]\\
&= \limsup_k\frac{2}{2-\beta_k-2\delta}\langle g(\overline{x})-\overline{x},x^{k+1}-\overline{x}\rangle\\
&\le 0.
\end{aligned}
$$

The last inequality comes from (2) and the fact that $\lim_k\frac{2}{2-\beta_k-2\delta}=\frac{1}{1-\delta}$. On the other hand, remark that $\beta_k\ge 0$ implies $1-\delta\beta_k\le 1$ and that $\beta_k\longrightarrow 0$ ensures the existence of some constant $K>0$ such that for every $k\ge K$, $\beta_k\le 1-\delta$. In this way, observe that $(2-\beta_k-2\delta)\ge(1-\delta)$ and then $t_k\ge(1-\delta)\beta_k$ for $k\ge K$. Accordingly,

$$\sum_k t_k\ge(1-\delta)\sum_k\beta_k=+\infty.$$

Then, with $a_k=\|x^k-\overline{x}\|$, the conditions of Lemma (1.11) are satisfied. This permits to conclude that (x^k) strongly converges to $\overline{x}=\Pi_S(g(\overline{x}))$. □

Remark 2.3. Let us emphasize that:

(i) The theorem above is not impacted by the order of the inequalities of the system (1.1), we can commute them freely in the formulation of the problem.

(ii) In [27], the author has developed an iterative scheme strongly convergent to a common solution of variational inequalities and systems of equilibrium problems and fixed points of families and semigroups of nonexpansive mappings. Our algorithm (2.1) is a special case of the one in [27, Theorem 3.1] from which we can derive our result in Theorem 2.2, but for completeness, we have included our independent proof to let also our paper be self-contained for the reader convenience.

3 Application to the Split Common Fixed Point Problem

Let H_1 and H_2 be two real Hilbert spaces and let $A:H_1\longrightarrow H_2$ be a bounded linear operator. Given non-linear operators $(U_i)_{i=1}^m:H_1\longrightarrow H_1$ and $(T_i)_{i=1}^m:H_2\longrightarrow H_2$ where $m\ge 1$ is an integer. The split common fixed point problem (SCFPP) consists at finding a point $\overline{x}$ satisfying the system:

$$\overline{x}\in\bigcap_{i=1}^m Fix\,U_i\quad such\quad that\quad A\overline{x}\in\bigcap_{j=1}^m Fix\,T_j. \tag{3.1}$$

We denote by Γ the set of solutions to the problem (3.1).
In particular, if $m = 1$ then the problem (3.1) reduces to finding a point $\overline{x}$ with the following property

$$\overline{x} \in Fix\,U \quad such \quad that \quad A\overline{x} \in Fix\,T. \tag{3.2}$$

(3.2) is usually called the two set split common fixed point problem. We denote by Γ_0 the solution set of problem (3.2).

From the chronological viewpoint, the problem (3.1) was firstly introduced by Censor and Segal in [5] in Euclidean spaces. Later, Moudafi [24] initiated the algorithmic treatment for solving the two split common fixed point problem for the class of demicontractive operators in Hilbert space by the use of Féjer-monotonicity sequences. Thereafter, Wang and Xu [35] proposed a cyclic iterative algorithm for solving (SCFPP) for directed operators. In recent years, many authors have made several efforts to develop implementable iterative methods for solving these problems, in this sense we quote here for example [5], [33], [7], [24], [18].

In the present paper our goal is to develop an iterative algorithm for solving problems (3.2) and (3.1) by formulating theses problems as a system the equilibrium problems purposed in Section 2. Actually, for given bifunctions $F_i : C_i \times C_i \to \mathbb{R}$, we observe that $\overline{x}$ is a solution to (1.1) if and only if $\overline{x}$ is a fixed point of the mapping $J_r^{F_m} J_r^{F_{m-1}} \dots J_r^{F_1}$ (see Corollary 1.7). This means that

$$\overline{x} \in \bigcap_{i=i}^{m} Fix J_r^{F_i}.$$

Therefore, the problem (1.1) is equivalent to (3.1) with $H_1 = H_2 = H$, $A = id_H$ and $U_i = T_i = J_r^{F_i}$ for $1 \leq i \leq m$. Conversely, given two Hilbert spaces H_1 and H_2, with inner product denoted respectively by $\langle \cdot, \cdot \rangle_{H_1}$ and $\langle \cdot, \cdot \rangle_{H_2}$, two operators $U : H_1 \longrightarrow H_1$, $T : H_2 \longrightarrow H_2$ with nonempty intersection of fixed point sets $Fix\,U$ and $Fix\,T$, and a bounded linear operator $A : H_1 \longrightarrow H_2$, a decisive step in our treatment is based on the fact that problem (3.1) can be converted into the problem (1.1) for a suitable class of operators.

Let us recall first some important concepts of operators which are usually involved in iterative methods for the split common fixed point.

Definition 3.1. The mapping $K : H \longrightarrow H$ is said to be β-demicontractive if there exists $\beta \in \mathbb{R}$ such that

$$\|Kx - q\|^2 \leq \|x - q\|^2 + \beta \|Kx - x\|^2 \quad \forall x \in H, q \in Fix\,K, \tag{3.3}$$

which is equivalent to

$$\langle x - Kx, x - q \rangle \geq \frac{1 - \beta}{2} \|Kx - x\|^2 \quad \forall x \in H, q \in Fix\,K.$$

Usually, the constant β is supposed to be in the interval $[0, 1)$, in this case the mapping K is called demicontractive mapping. For negative values of β the class of β-demicontractive mappings is a subject to a great restriction, such a class (with negative value of β) was considered in [1] under the name of strongly attracting maps. In particular, a mapping K which satisfies (3.3) with $\beta = -1$ is called pseudo-contractive. Notice that the class of β-demicontractive mappings properly includes the class of quasi-nonexpansive mappings in the case when the constant β is supposed to be a positive number. Note also that a mapping

K satisfying (3.3) with $\beta = 1$ is usually called hemi-contractive and it was considered by some authors in connection with the strong convergence of the implicit Mann-type iteration (see [25] and [28] for example).

To start the treatment of this section let us assume that the operators U and T are demicontractive operators with constant $\beta \in [0,1)$ and $\mu \in [0,1)$ respectively, set $\tilde{C_1} = Fix\,U$, $\tilde{C_2} = A^{-1}(Fix\,T)$ and consider two closed and convex subsets in H_1, C_1 and C_2 such that $\tilde{C_1} \subset C_1$ and $\tilde{C_2} \subset C_2$. We further introduce the real-valued bifunctions $F_U : C_1 \times C_1 \longrightarrow \mathbb{R}$ and $F_{(A,T)} : C_2 \times C_2 \longrightarrow \mathbb{R}$ defined respectively by

$$\begin{cases} F_U(x,y) = \langle x - Ux, y - x\rangle_{H_1} & \forall x, y \in C_1, \\ F_{(A,T)}(x,y) = \langle Ax - T(Ax), A(y-x)\rangle_{H_2} & \forall x, y \in C_2. \end{cases} \tag{3.4}$$

Of course, if the problem (3.2) admits a solution $\overline{x}$ then $\overline{x} \in \tilde{C_1} \cap \tilde{C_2} \subset C_1 \cap C_2$, which means that $U(\overline{x}) = \overline{x}$ and $T(A\overline{x}) = A\overline{x}$. Hence, $F_U(\overline{x}, y) = 0$ for each $y \in C_1$ and $F_{(A,T)}(\overline{x}, y) = 0$ for each $y \in C_2$ and then $\overline{x}$ solve the problem (1.1) in the case $m = 2$, $F_1 = F_U$ and $F_2 = F_{(A,T)}$. Conversely, if we suppose that the system of equilibrium problem (1.1) admits a solution for the bifunctions F_U and $F_{(A,T)}$, then there exists a common element $\overline{x} \in C_1 \cap C_2$ such that

$$\begin{cases} \langle \overline{x} - U\overline{x}, y - \overline{x}\rangle_{H_1} \geq 0 & \forall y \in C_1 \\ \langle A\overline{x} - T(A\overline{x}), A(y - \overline{x})\rangle_{H_2} \geq 0 & \forall y \in C_2. \end{cases} \tag{3.5}$$

Given that $\tilde{C_1} \subset C_1$ and $\tilde{C_2} \subset C_2$ then the system (3.5) implies

$$\begin{cases} \langle \overline{x} - U\overline{x}, y - \overline{x}\rangle_{H_1} \geq 0 & \forall y \in \tilde{C_1} \\ \langle A\overline{x} - T(A\overline{x}), A(y - \overline{x})\rangle_{H_2} \geq 0 & \forall y \in \tilde{C_2}. \end{cases} \tag{3.6}$$

Using the fact that the operator U is demicontractive and $\tilde{C_1} = Fix\,U$ we deduce that

$$\langle \overline{x} - U\overline{x}, \overline{x} - y\rangle_{H_1} \geq \frac{1-\beta}{2}\|\overline{x} - U\overline{x}\|^2 \geq 0 \quad \forall y \in \tilde{C_1}. \tag{3.7}$$

If we compare the last inequality with the first inequality of the system (3.6) we see that

$$\langle \overline{x} - U\overline{x}, \overline{x} - y\rangle_{H_1} = 0 \quad \forall y \in \tilde{C_1}. \tag{3.8}$$

Then, combining (3.7) and (3.8) we obtain $\|\overline{x} - U\overline{x}\| = 0$. The same technique can be used to prove that $T(A\overline{x}) = A\overline{x}$. Indeed, the operator T being demicontractive and $\tilde{C_2} = A^{-1}(Fix\,T)$, for a fixed point y in $\tilde{C_2}$, one has $Ay \in Fix\,T$ and moreover

$$\langle A\overline{x} - T(A\overline{x}), A\overline{x} - Ay\rangle_{H_2} \geq \frac{1-\mu}{2}\|A\overline{x} - T(A\overline{x})\|^2 \geq 0. \tag{3.9}$$

From last inequality and the second inequality of system (3.6) as well as the linearity of operator A it follows that

$$\langle A\overline{x} - T(A\overline{x}), A\overline{x} - Ay\rangle_{H_2} = 0 \quad \forall y \in \tilde{C_2}.$$

Therefore, $\|A\overline{x} - T(A\overline{x})\| = 0$. Consequently, $A\overline{x}$ is a fixed point of T, hence $\overline{x}$ solve the problem (3.2). As a conclusion, the problem (3.2) is equivalent to the system of equilibria (1.1), with $m = 2$, whenever the operators T and U are demicontractives. By the argument of induction, the problem (3.1) is easily shown to be equivalent to (1.1) for any $m \geq 1$.

Next we aim at applying the iterative scheme developed in section 2 to solve the two problems (3.1) and (3.2). But before this, let us observe that the bifunctions $F_U, F_{(A,T)}$ are not monotone in general if the operators U and T are only demicontractives. To see this fact, let us consider for example: Take $U : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $U(x) = x^2$. Clearly, 0 is the unique fixed point of U. On the other hand, for any $x \in \mathbb{R}$,

$$|U(x) - 0|^2 = |U(x)|^2 = |x|^2 \leq |x|^2 + \beta|U(x) - x|^2,$$

for any $\beta \geq 0$. Thus U is β-demicontractive for any $\beta \geq 0$. But if we take $F_U(x, y) = \langle x - U(x), y - x\rangle$ for each $x, y \in C_1$, wherein C_1 is a closed and convex subset of $\mathbb{R}$ containing 0, it is a simple matter to check that F_U is not monotone (for instance the monotonicity condition fails for $x = 1$ and $y = 2$).
For this reason linked to the monotonicity property (on F_U and $F_{(A,T)}$), we suggest the following alternative, which can be regarded as a concept leading to a subclass of demicontractive operators.

Definition 3.2. The mapping $K : H \longrightarrow H$ is said to be M-demicontractive (or β-M-demicontractive) if there exists $\beta \in \mathbb{R}$ such that

$$\begin{cases} \|Kx - y\|^2 \leq \|x - y\|^2 + \beta\|Kx - x\|^2 \quad \forall(x, y) \in H \times Fix\, K. \\ \langle Kx - Ky, x - y\rangle \leq \|x - y\|^2 \quad \forall(x, y) \in H \times (H \setminus Fix\, K). \end{cases} \tag{3.10}$$

Remark 3.3. It is clear that any M-demicontractive operator is demicontractive but the inverse is not true. To see that let us take $K : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $K(x) = x^2$ for every $x \in \mathbb{R}$. Then K is demi-contractive while the second inequality of (3.10) is not satisfied for $x = 1$ and $y = 2$.

Example 3.4. As an example of a demicontractive mapping we take $K : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $K(x) = -\frac{4}{5}x$ for every $x \in \mathbb{R}$. Then (3.10) is satisfied. In fact, 0 is clearly the unique fixed point of K. Furthermore, for any $x \in \mathbb{R}$,

$$|K(x) - 0|^2 = |K(x)|^2 = \frac{16}{25}|x|^2 \leq |x|^2 \leq |x|^2 + \beta|K(x) - x|^2,$$

for any $\beta \geq 0$. Thus, K is demicontractive. In addition, for any $(x, y) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$, we have

$$(K(x) - K(y))(x - y) = -\frac{4}{5}(x - y)^2 \leq (x - y)^2.$$

Thus, K is M-demicontractive. We observe that this example can be generalized to any non-expansive operator. Precisely, if K is non-expansive operator, for any $x, y \in H$, we have

$$\|K(x) - K(y)\| \leq \|x - y\|.$$

Hence,

$$\|K(x) - K(y)\|\|x - y\| \leq \|x - y\|^2.$$

This implies that

$$\langle K(x) - K(y), x - y\rangle \leq \|x - y\|^2,$$

and then the second inequality of (3.10) is satisfied. Moreover, it is well known that any nonexpansive operator is demicontractive and satisfies the first inequality of (3.10). Accordingly, the set of nonexpansive operators is a subclass of M-demicontractive operators.
A further interesting question that arises in this context is whether the mapping $I - K$ disposes at the demi-closedness property at 0 for a given M-demicontractive operator $K : H \longrightarrow H$. The answer is negative as shows the following counterexample:

Example 3.5. Let us consider the mapping $K : \mathbb{R} \longrightarrow \mathbb{R}$ defined by:

$$\begin{cases} K(x) := \dfrac{2x}{x+1}, & x \in]1, \infty[; \\ K(x) := 0, & x \in]-\infty, 1]. \end{cases}$$

Then $I - K$ is not demi-closed at point 0. Actually, the sequence $y_n := \frac{n+1}{n}$ converges to 1 and when $n \longrightarrow \infty$, $y_n - K(y_n) \longrightarrow 0$ (because $K(y_n) = \frac{2y_n}{y_n+1} = \frac{2(n+1)}{2n+1}$) while $1 - K(1) = 1 \neq 0$. However, K is M-demicontracive as the inequalities of (3.10) can be checked as follows. Firstly, observe that K has a unique fixed point 0 and

$$|Kx - 0|^2 \leq |x|^2 + \frac{3}{4}|Kx - x|^2 \quad \forall x \in \mathbb{R}.$$

Furthermore, a direct calculus gives us

$$(K(x) - K(y))(x - y) \leq |x - y|^2, \quad \forall x \in \mathbb{R}, \ \forall y \in \mathbb{R} \setminus \{0\}.$$

Hence, K is M-demicontractive with the coefficient $\beta = \frac{3}{4}$.

Lemma 3.6. *Let $U : H_1 \longrightarrow H_1$, $T : H_2 \longrightarrow H_2$ be M-demicontractive operators with some constants $\beta \in [0,1)$ and $\mu \in [0,1)$ respectively, with $Fix\,U$, $Fix\,T$ are non empty sets, and let $A : H_1 \longrightarrow H_2$ be a bounded linear operator. For any closed and convex subsets of H_1, C_1 and C_2, such that $C_1 \supseteq Fix\,U$ and $C_2 \supseteq A^{-1}(Fix\,T)$, the bifunctions F_U and $F_{(A,T)}$ defined in (3.4) are monotones.*

Proof. Let $(x, y) \in C_1 \times C_1$. There are two cases to treat:

First case: $y \in Fix\,U$.

From the first inequality of (3.10) it follows that

$$\|Ux - y\|^2 \leq \|x - y\|^2 + \beta\|Ux - x\|^2,$$

which is equivalent to

$$\langle x - Ux, x - y\rangle_{H_1} \geq \frac{1-\beta}{2}\|x - Ux\|^2.$$

Now, using the definition of F_U and the fact that $\beta \in [0,1)$, we infer

$$F_U(x, y) \leq \frac{\beta - 1}{2}\|x - U(x)\|^2 \leq 0.$$

On the other hand, we have $F_U(y, x) = \langle y - Uy, x - y\rangle_{H_1} = 0$ (since y is a fixed point of U), then, in this case, $F_U(x, y) + F_U(y, x) \leq 0$.

Second case: $y \in C_1 \setminus Fix\,U$.

By the definition of M-demicontractive operators, the second inequality of (3.10) holds and we have

$$\begin{aligned} F_U(x, y) + F_U(y, x) &= \langle x - Ux, y - x\rangle_{H_1} + \langle y - Uy, x - y\rangle_{H_1} \\ &= \langle Ux - Uy, x - y\rangle_{H_1} - \|x - y\|^2 \\ &\leq 0. \end{aligned}$$

In both of the two cases, we have the inequality $F_U(x,y)+F_U(y,x) \leq 0$, hence the bifunction F_U is monotone.

Now, we shall prove that $F_{(A,T)}$ is monotone. At first recall that

$$C_2 \supseteq \tilde{C_2} = A^{-1}(Fix\, T).$$

Take any $x, y \in C_2$. There are again two cases to discuss.

First case: $y \in \tilde{C_2}$, in this case, $Ay \in Fix\, T$, so given that T is M-demicontractive with constant $\mu \in [0,1)$, we have

$$\langle z - Tz, z - q\rangle_{H_2} \geq \frac{1-\mu}{2}\|z - Tz\|^2 \quad (z,q) \in (H_2 \times Fix\, T). \tag{3.11}$$

By making the choice $z = Ax$ and $q = Ay$ in (3.11) we obtain

$$\langle Ax - T(Ax), Ax - Ay\rangle_{H_2} \geq \frac{1-\mu}{2}\|Ax - T(Ax)\|^2.$$

Thus

$$F_{(A,T)}(x,y) \leq \frac{\mu-1}{2}\|Ax - T(Ax)\|^2 \leq 0.$$

As $Ay \in FixT$, we can write

$$F_{(A,T)}(y,x) = \langle Ay - T(Ay), Ax - Ay\rangle_{H_2} = 0,$$

which implies that

$$F_{(A,T)}(x,y) + F_{(A,T)}(y,x) \leq 0.$$

Second case: $y \in C_2 \setminus \tilde{C_2}$. In this case, we have $Ay \notin Fix\, T$, so let us write the second inequality in the definition of M-demicontractivity of T as follows :

$$\langle Tz - Tz', z - z'\rangle_{H_2} \leq \|z - z'\|^2 \quad \forall (z,z') \in (H_2 \times (H_2 \setminus FixT)). \tag{3.12}$$

Then by injecting the values $z = Ax$ and $z' = Ay$ in (3.12) we get

$$\langle T(Ax) - T(Ay), Ax - Ay\rangle_{H_2} \leq \|Ax - Ay\|^2.$$

Of course, the last inequality is equivalent to $F_{(A,T)}(x,y) + F_{(A,T)}(y,x) \leq 0$, which means that the bifunction $F_{(A,T)}$ is monotone. □

Next, we introduce the concept of F-upper hemicontinuity mapping which can be viewed as the split counterpart of the standard upper hemicontinuity of equilibrium bifunctions associated to the operators.

Definition 3.7 ([21, Definition 2.2])**.** Let C be any subset of a real Hilbert space H. A mapping $K : C \longrightarrow H$ is said to be F-upper hemicontinuous (lower hemicontinuous) if for all $v \in C$, the function $u \longmapsto \langle K(u), u - v\rangle$ is weakly upper semicontinuous (lower semicontinuous) on C.

This kind of semi-continuity, introduced firstly by Fan (See [13],[12]), is well known in the framework of the existence theory of variational inequalities. Note that in the literature, the F-lower hemicontinuity is called F-hemicontinuity. We mention that Ernest and Théra [11] proved the following result: If K is linear and continuous operator then the F-lower hemicontinuity of K is equivalent to the following property

$$\text{if} \quad x_k \rightharpoonup 0 \quad \text{then} \quad \liminf_k \langle K(x_k), x_k\rangle \geq 0. \tag{3.13}$$

Example 3.8. Let us consider any linear and continuous mapping $K : H \longrightarrow H$ such that the implication (3.13) is not satisfied. Then K is not F-hemicontinuous, whereas $I - K$ is demiclosed at 0. Conversely, we present the following example:

Fix $a > 1$ and consider the function defined on $[0, 1]$ by

$$K(x) = \begin{cases} \frac{-1}{x} + a, & \text{si } x \in]\frac{1}{2a}, 1], \\ -a & \text{si } x \in]0, \frac{1}{2a}], \\ a & \text{si } x = 0. \end{cases}$$

$K(x)$ is F-hemicontinuous on $[0, 1]$ but obviously $I - K$ is not demiclosed at 0. Clearly, the sequence $y_n := \frac{1}{2an}, n \geq 1$ converges to 0 and $y_n - K(y_n) \longrightarrow a$ (when $n \longrightarrow \infty$) while $(I - K)(0) = -a \neq a$.

Lemma 3.9. *Let $U : H_1 \longrightarrow H_1$, $T : H_2 \longrightarrow H_2$ be F-upper hemicontinuous mappings, with $Fix\,U$, $Fix\,T$ are non empty sets, and let $A : H_1 \longrightarrow H_2$ be a bounded linear operator. For any closed and convex subsets of H_1, C_1 and C_2, such that $C_1 \supseteq Fix\,U$ and $C_2 \supseteq A^{-1}(Fix\,T)$, the bifunctions F_U and $F_{(A,T)}$ defined in (3.4) verify the condition (5).*

Proof. Let x, y, z be fixed points in C_1, take $x_t := (1 - t)x + tz$ an arbitrary element in segment $[z, x]$, then x_t converges to x if $t \longrightarrow 0^+$ and we have

$$\begin{aligned} \limsup_{t\downarrow 0^+} F_U(x_t, y) &= \limsup_{t\downarrow 0^+} \langle x_t - U(x_t), y - x_t\rangle \\ &\leq \limsup_{t\downarrow 0^+} \langle x_t, y - x_t\rangle + \limsup_{t\downarrow 0^+} \langle U(x_t), x_t - y\rangle \\ &\leq \langle x, y - x\rangle + \langle U(x), x - y\rangle. \quad \text{(by F-hemicontinuity of } U). \\ &= F_U(x, y). \end{aligned}$$

Then the condition (5) is satisfied by bifunction F_U. A similar argument permits to conclude that the condition (5) is also satisfied by bifunction $F_{(A,T)}$ under F-hemicontinuity condition on T. ◻

Now, we are in position to give our convergence result relative to the split common fixed point problem (3.2).

Algorithm 3.10. From $x^0 \in C_2$, we generate a sequence (x^k) by

$$x^{k+1} = \alpha_k x^k + \beta_k g(x^k) + \gamma_k J_r^{F_{(A,T)}} J_r^{F_U}(x^k), \tag{3.14}$$

where $J_r^{F_U}$ and $J_r^{F_{(A,T)}}$ are the resolvent of the special bifunctions F_U and $F_{(A,T)}$ defined by the expression (3.4) and $g : C_2 \longrightarrow C_2$ an arbitrary δ -contraction, $\delta \in (0, 1)$, while $\alpha_k, \beta_k, \gamma_k$ are positive scalars such that $\alpha_k + \beta_k + \gamma_k = 1$, $\beta_k \to 0$, $\sum \beta_k = +\infty$ and $(\alpha_k) \subset [c, d]$ for some $c, d \in (0, 1)$.

Theorem 3.11. *Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator. Let $U : H_1 \longrightarrow H_1$ and $T : H_2 \longrightarrow H_2$ be F-hemicontinuous and M-demicontractive operators with constant $\beta \in [0, 1)$ and $\mu \in [0, 1)$ respectively such that $Fix\,U \neq \emptyset$ and $Fix\,T \neq \emptyset$. If the problem (3.2) is consistent (i.e., Γ_0 is nonempty) then the sequence (x^k) generated by the Algorithm (3.14) strongly converges to a point $\overline{x} \in \Gamma_0$.*

Proof. Recall that the bifunction F_U is given by

$$F_U(x,y) = \langle x - Ux, y - x \rangle_{H_1},$$

for each x, y in the closed and convex subset C_1 of H_1 such that $C_1 \supseteq Fix\, U$ while the other one $F_{(A,T)}$ is defined by

$$F_{(A,T)}(x,y) = \langle Ax - T(Ax), A(y-x) \rangle_{H_2},$$

for each x, y in the closed and convex subset C_2 of H_1 such that $C_2 \supseteq A^{-1}(Fix\, T)$. Then, based on the above discussion we see that the problems (3.2) and (1.1) are equivalent i.e., $S_0 = \Gamma_0$, with S_0 is the set of solutions of system of equilibrium problems defined by the bifunctions F_U and $F_{(A,T)}$. By using Remark 1.2, we see that Γ_0 is a closed and convex set. Hence, we have,

$$\begin{aligned} \|\Pi_{\Gamma_0} \circ g(x) - \Pi_{\Gamma_0} \circ g(y)\| &= \|\Pi_{S_0} \circ g(x) - \Pi_{S_0} \circ g(y)\|, \\ &\leq \|g(x) - g(y)\|, \\ &\leq \delta \|x - y\|. \end{aligned}$$

Therefore, $\Pi_{\Gamma_0} \circ g : C_2 \longrightarrow \Gamma_0$ is a contraction and there exists a unique point $\overline{x}$ such that $\Pi_{\Gamma_0} \circ g(\overline{x}) = \overline{x}$. Furthermore, the bifunctions $F_U : C_1 \times C_1 \longrightarrow \mathbb{R}$ and $F_(A,T) : C_2 \times C_2 \longrightarrow \mathbb{R}$ verify the conditions [(1)-(5)]. Precisely, For F_U, it is clear that $F_U(x,x) = 0$ for each $x \in C_1$. Since the inner product is linear and continuous then the conditions (2), (3) trivially hold. The monotonicity of F_U is guaranteed by Lemma 3.6 under M-demicontractivity of the operator U, and the condition (5) is ensured by Lemma 3.9 under F-hemicontinuity of mapping U. For $F_{(A,T)}$, the operator A being linear we have $A(0) = 0$, hence $F_{(A,T)}(x,x) = 0$ for each $x \in C_2$. Moreover, using the continuity and again the linearity of A, the conditions (2), (3) are automatically satisfied. The monotonicity of $F_{(A,T)}$ is a consequence of Lemma 3.6 under M-demicontractivity of operator T, and the condition (5) comes Lemma 3.9 under F-hemicontinuity of T.

Now, if the problem (3.2) admits a solution $\overline{x}$, then $\overline{x}$ is also a solution to the system of equilibrium points (1.1) with $F_1 = F_U$ and $F_2 = F_{(A,T)}$. It remains only to take $m = 2$ in the Algorithm (2.1) and replace $J_r^{F_1}$ and $J_r^{F_2}$ by $J_r^{F_U}$ and $J_r^{F_{(A,T)}}$ respectively to derive from Theorem 1.1 that the sequence (x^k) generated by the Algorithm (3.14) strongly converges to a point $\overline{x} \in \Gamma_0$. □

In this paragraph, we obtain a further result for the problem (3.1) with a family of M-demicontractive mappings. Let $(U_i)_{i=1}^m : H_1 \longrightarrow H_1$ and $(T_i)_{i=1}^m : H_2 \longrightarrow H_2$ be M-demicontractive mappings. Assume that $\bigcap_{i=1}^m Fix\, U_i \neq \emptyset$ and $\bigcap_{i=1}^m Fix\, T_i \neq \emptyset$. Observe that these sets are convex and closed subsets (since the mappings T_i and U_i are M-demicontractive) and F-hemicontinuous. Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator. For any $i \in \{1, \ldots, m\}$, set $C_{U_i} = FixU_i$, $C_{(A,T_i)} = A^{-1}(FixT_i)$ and choose any closed and convex subset of H_1 noted D_i, such that $D_i \supseteq C_{U_i} \cup C_{(A,T_i)}$. By Γ we denote the set of solutions to the problem (3.1) and assume that Γ is not empty, which implies that the set D_i is also nonempty for any $i \in \{1, \ldots, m\}$. Also observing that $D_i \supset C_{U_i}$ and $D_i \supset C_{(A,T_i)}$. For each $i \in \{1, \ldots, m\}$, let us introduce the bifunctions

$$F_{U_i} : D_i \times D_i \longrightarrow \mathbb{R}, \quad F_{U_i}(x,y) = \langle x - U_i x, y - x \rangle_{H_1}, \tag{3.15}$$

and

$$F_{(A,T_i)} : D_i \times D_i \longrightarrow \mathbb{R}, \quad F_{(A,T_i)}(x,y) = \langle Ax - T_i(Ax), A(y-x) \rangle_{H_2}. \tag{3.16}$$

Observing from now that, if $\overline{x}$ solves (3.1) then $F_{U_i}(\overline{x}, y) = F_{(A,T_i)}(\overline{x}, y) = 0$, for each $y \in D_i$ and each $i \in \{1 \dots m\}$. In the following we consider for each $i \in \{1 \dots m\}$ and for some $\lambda \in (0, 1)$ the bifunction

$$G_i^\lambda : D_i \times D_i \longrightarrow \mathbb{R},\ G_i^\lambda(x, y) = (1 - \lambda)F_{U_i}(x, y) + \lambda F_{(A,T_i)}(x, y). \tag{3.17}$$

With the well-known result below of Lemma 3.12 we are able to prove that $\{G_i^\lambda, 1 \leq i \leq m\}$ satisfy our conditions [(1)-(5)]. Besides, it will give us the link between the common solution of $EP(F_{U_i}, D_i)$ and $EP(F_{(A,T_i)}, D_i)$ and the solution to $EP(G_i^\lambda, D_i)$.

Lemma 3.12. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i \in \{1 \dots k\}$ let $F_i : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying conditions [(1)-(5)]. Then*

(i) $\sum_{i=1}^{k} \lambda_i F_i$ *satisfies the conditions [(1)-(5)], where $\lambda_i \in (0.1)$ for every $i \in \{1 \dots k\}$ and* $\sum_{i=1}^{k} \lambda_i = 1$.

(ii) *If $\bigcap_{i=1}^{k} S(F_i, C) \neq \emptyset$ then $S(\sum_{i=1}^{k} \lambda_i F_i, C) = \bigcap_{i=1}^{k} S(F_i, C)$, where $S(F_i, C)$ is the set solution of equilibrium problem $EP(F_i, C)$.*

Proof. For (i) see [18]. For (ii) see [29]. □

By this moment we present our convergence result relatively to the split common fixed point problem (3.1).

Algorithm 3.13. From $x^0 \in C_m$, we generate a sequence (x^k) by

$$x^{k+1} = \alpha_k x^k + \beta_k g(x^k) + \gamma_k J_r^{G_m^\lambda} J_r^{G_{m-1}^\lambda} \dots J_r^{G_1^\lambda}(x^k). \tag{3.18}$$

where, $J_r^{G_i^\lambda}$ is the resolvent of the special bifunction G_i^λ defined by the expression (3.17), and $g : C_m \longrightarrow C_m$ an arbitrary δ -contraction, $\delta \in (0, 1)$, while $\alpha_k, \beta_k, \gamma_k$ are positive scalars such that $\alpha_k + \beta_k + \gamma_k = 1$, $\beta_k \to 0$, $\sum \beta_k = +\infty$ and $(\alpha_k) \subset [c, d]$ for some $c, d \in (0, 1)$.

Theorem 3.14. *Let $(U_i)_{i=1}^m : H_1 \longrightarrow H_1$ and $(T_i)_{i=1}^m : H_2 \longrightarrow H_2$ be F-hemicontinuous and M-demicontractive mappings with nonempty intersection of fixed point sets $\bigcap_{i=1}^{m} Fix\, U_i$ and $\bigcap_{i=1}^{m} Fix\, T_i$. Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator. If the solution set Γ is nonempty, i.e., the problem (3.1) is consistent then the sequence (x^k) generated by the Algorithm (3.18) strongly converges to a point $\overline{x} \in \Gamma$.*

Proof. Let us first recall the construction of the sequence of bifunctions $\{G_i^\lambda, i = 1, \dots, m\}$ used to convert the problem (3.1) into a system of equilibrium problems. The first point is the use the assumption Γ is not empty. Let $\overline{x} \in \Gamma$. This is equivalent to

$$\overline{x} \in \left(\bigcap_{i=1}^{m} Fix\, U_i \right) \bigcap A^{-1} \left(\bigcap_{i=1}^{m} Fix\, T_i \right). \tag{3.19}$$

Now, for each $i \in \{1, \ldots, m\}$, take any closed and convex subset D_i of H_1 such that

$$D_i \supset Fix\, U_i \cup A^{-1}(Fix\, T_i). \tag{3.20}$$

Clearly, from (3.19) it follows that $\overline{x} \in \bigcap_{i=1}^m D_i$. For a given value of $i \in \{1, \ldots, m\}$, we construct two bifunctions F_{U_i} and $F_{(A,T_i)}$ over the domain $D_i \times D_i$, which are defined respectively by relations (3.15) and (3.16). Then, take the convex combination of F_{U_i} and $F_{(A,T_i)}$. Let us denote by G_i^λ this combination for some $\lambda \in (0, 1)$, so G_i^λ is defined by relation (3.17). Let S be the set of solutions of the system of equilibrium problems defined by $\{G_i^\lambda, i = 1, \ldots, m\}$.i.e.,

$$S := \left\{ \overline{x} \in \bigcap_{i=1}^{m} D_i \quad \text{such that} \quad G_i^\lambda(\overline{x}, y) \geq 0, \quad \forall y \in D_i \right\}.$$

Now, observe that Lemma 3.6 and Lemma 3.9 ensure that the bifunctions F_{U_i} and $F_{(A,T_i)}$ satisfy the conditions [(1)-(5)] for all $i \in \{1, \ldots, m\}$. Consequently, by the use of the point (i) of Lemma 3.12, the bifunction G_i^λ satisfies the conditions [(1)-(5)] for all $i \in \{1, \ldots, m\}$. At this stage, thanks to these conditions on G_i^λ, we are able to claim that $\Gamma = S$. Indeed, if $\overline{x} \in \Gamma$, then it is clear that $\overline{x} \in \bigcap_{i=1}^m D_i$ and for any $\{1 \leq i \leq m\}$, $G_i(\overline{x}, y) = 0$ for all $y \in D_i$, hence $\overline{x} \in S$. Conversely, if $\overline{x} \in S$, then by the point (ii) of Lemma 3.12, for any $\{1 \leq i \leq m\}$, we have

$$\begin{cases} F_{U_i}(\overline{x}, y) & \forall y \in D_i, \\ F_{(A,T_i)}(\overline{x}, y) & \forall y \in D_i. \end{cases} \tag{3.21}$$

According to the case of two set split common fixed point discussed above, for any arbitrary $\{1 \leq i \leq m\}$, we conclude that $\overline{x} \in Fix\, U_i$ and $A\overline{x} \in Fix\, T_i$, which means that $\overline{x} \in \Gamma$. Use now Remark 1.2 and see that $\Gamma = S$ is a closed and convex subset of H_1. Thus, in view of the projection mapping on this subset we infer

$$\begin{aligned} \|\Pi_\Gamma \circ g(x) - \Pi_\Gamma \circ g(y)\| &= \|\Pi_S \circ g(x) - \Pi_S \circ g(y)\|, \\ &\leq \|g(x) - g(y)\|, \\ &\leq \delta \|x - y\|. \end{aligned}$$

Therefore, $\Pi_\Gamma \circ g : C_m \longrightarrow \Gamma$ is a contraction. This implies that there exists a unique point $\overline{x}$ such that $\Pi_\Gamma \circ g(\overline{x}) = \overline{x}$. The strong convergence of the sequence (x_k) to $\overline{x}$ is immediate from Theorem 2.2. □

4 A Numerical Example:

In this section, we give a numerical example to support our main Theorem.

Example 4.1. Let $H_1 = \mathbb{R}^2$, $H_2 = \mathbb{R}$. Define the mappings $U : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T : \mathbb{R} \longrightarrow \mathbb{R}$ and $A : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $U(x_1, x_2) = (-x_1, x_2)$, $T(x) = -\frac{4}{5}x$ and $A(x_1, x_2) = x_1 + x_2$ for all x, x_1, x_2 in $\mathbb{R}$. Clearly, $Fix\, U = \{0\} \times \mathbb{R}$, $Fix\, T = \{0\}$, A is a bounded and linear operator. In addition, T is M-demicontractive. Let us show that U is M-demicontractive. Take two points $x = (x_1, x_2)$ and $q = (0, q_2)$ in $\mathbb{R}^2$ and $Fix\, U$ respectively. Then

$$\begin{aligned} \|Ux - q\|^2 &= \|(-x_1, x_2) - (0, q_2)\|^2 \\ &= |x_1|^2 + |x_2 - q_2|^2 \\ &= \|x - q\|^2 \\ &\leq \|x - q\|^2 + \beta \|Ux - q\|^2, \quad \text{for some } \beta \in [0, 1). \end{aligned}$$

Then the first inequality in (3.10) is satisfied. For the second inequality of (3.10), for any points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R} \setminus \{0\} \times \mathbb{R}$ we write

$$\begin{aligned}\langle Ux - Uy, x - y\rangle_{\mathbb{R}^2} &= \langle(-x_1, x_2) - (-y_1, y_2), (x_1 - y_1, x_2 - y_2)\rangle_{\mathbb{R}^2}\\ &= -(x_1 - y_1)^2 + (x_2 - y_2)^2\\ &\leq \|(x_1, x_2) - (y_1, y_2)\|^2.\end{aligned}$$

This shows that U is M-demicontractive mapping for any $\beta \in [0, 1)$. Now, our problem (3.2) is equivalent to finding $(\overline{x_1}, \overline{x_2}) \in (0, \mathbb{R})$ such that

$$\overline{x_1} + \overline{x_2} = 0.$$

In other terms

$$\overline{x_1} = \overline{x_2} = 0.$$

Hence, $(0, 0)$ is the unique solution to the problem (3.2).

Let C_1 be any closed and convex subset of $\mathbb{R}^2$ such that $\{0\} \times \mathbb{R} \subset C_1$. For example take $C_1 = \mathbb{R} \times \mathbb{R}$. We further introduce the bifunction $F_U : C_1 \times C_1 \longrightarrow \mathbb{R}$ defined for any $\big((x_1, x_2), (y_1, y_2)\big) \in C_1 \times C_1$ by

$$F_U\big((x_1, x_2), (y_1, y_2)\big) := \langle(x_1, x_2) - U(x_1, x_2), (y_1, y_2) - (x_1, x_2)\rangle_{\mathbb{R}^2} = 2x_1(y_1 - x_1).$$

It is clear that the conditions (1), (2), (3) and (4) are satisfied. For the monotonicity of F_U, observe that

$$F_U\big((x_1, x_2), (y_1, y_2)\big) + F_U\big((y_1, y_2), (x_1, x_2)\big) = -2(y_1 - x_1)^2 \leq 0.$$

Let C_2 be a closed and convex subset of $\mathbb{R}^2$ such that $\tilde{C_2} := A^{-1}\{0\} \subset C_2$ (note that $A^{-1}\{0\} = \{(x, -x), x \in \mathbb{R}\}$), for example we my choose $C_2 = \mathbb{R} \times \mathbb{R}$. For any $x = (x_1, x_2), y = (y_1, y_2) \in C_2 \times C_2$, we define $F_{(A,T)} : C_2 \times C_2 \longrightarrow \mathbb{R}$ by

$$F_{(A,T)}(x, y) := \big(Ax - T(Ax)\big).A(y - x) = \frac{9}{5}(x_1 + x_2).\big((y_1 - x_1) + (y_2 - x_2)\big).$$

It is clear that the condition (1), (2), (3) and (4) are satisfied. The monotonicity of $F_{(A,T)}$ is also fulfilled thanks to the following easy observation

$$\begin{aligned}F_{(A,T)}\big((x_1, x_2), (y_1, y_2)\big) + F_{(A,T)}\big((y_1, y_2), (x_1, x_2)\big) &= -\frac{9}{5}\big((x_1 + x_2) - (y_1 + y_2)\big)^2\\ &\leq 0.\end{aligned}$$

So the bifunction $F_{(A,T)}$ verifies all the conditions (1)-(5).

On the other hand, $\overline{x} := (0, 0) \in C_1 \cap C_2$ and it is a common solution for the system of equilibrium problems defined by two bifunctions F_U and $F_{(A,T)}$. Accordingly, the Algorithm (3.14) applied to this example converges to a solution $\overline{x}$ to the corresponding split common fixed point problem.

We turn now our attention to the case when $r = 1$ (in the definition of the resolvent of the underlying bifunction) and $u_k := J_1^{F_U}(x_k)$, for every $k \geq 0$ to obtain $u_k \in C_1$ and

$$\langle y - u_k, 2u_k - U(u_k) - x_k\rangle_{\mathbb{R}^2} \geq 0 \quad \forall y \in C_1. \tag{4.1}$$

In particular, for $y = -u_k + U(u_k) + x_k$ in (4.1) we get $-\|2u_k - U(u_k) - x_k\|^2 = 0$, which means that $2u_k - U(u_k) - x_k = 0$. Of course, $u_k \in C_1 = \mathbb{R}^2$. Then

$$\boxed{u_k = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} x_k.} \tag{4.2}$$

Let us put $v_k := J_1^{F_{(A,T)}} J_1^{F_U}(x_k) = J_1^{F_{(A,T)}}(u_k)$ for every $k \geq 0$, so $v_k \in C_2$ and

$$F_{(A,T)}(v_k, y) + \langle y - v_k, v_k - u_k \rangle_{\mathbb{R}^2} \geq 0 \quad \forall y \in C_2. \tag{4.3}$$

We shall transform the expression (4.3) by writing $F_{(A,T)}(v_k, y)$ as a scalar product in $\mathbb{R}^2$. To do that let us introduce the following notation:
$v_k := \begin{pmatrix} v_{k,1} \\ v_{k,2} \end{pmatrix}$, $\lambda_k := \frac{9}{5}(v_{k,1} + v_{k,2})$, $\theta_k := \lambda_k . \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Accordingly, write $F_{(A,T)}(v_k, y)$ as the following

$$F_{(A,T)}(v_k, y) = \langle y - v_k, \theta_k \rangle_{\mathbb{R}^2}$$

and see that (4.3) is equivalent to

$$\langle y - v_k, v_k - u_k + \theta_k \rangle_{\mathbb{R}^2} \geq 0 \quad \forall y \in C_2. \tag{4.4}$$

In particular, for $y = u_k - \theta_k$ in (4.4) we get $-\|v_k - u_k + \theta_k\|^2 = 0$. By elementary calculus we obtain

$$\boxed{v_k = \begin{pmatrix} \frac{14}{23} & \frac{-9}{23} \\ \frac{-9}{23} & \frac{14}{23} \end{pmatrix} u_k.} \tag{4.5}$$

Hence, combine (4.2) with (4.5) to get that

$$\boxed{v_k = M x_k,} \tag{4.6}$$

where

$$M = \begin{pmatrix} \frac{14}{69} & \frac{-9}{23} \\ \frac{-3}{23} & \frac{14}{23} \end{pmatrix}.$$

In our algorithm (3.14) let us inject $\alpha_k := \frac{k+1}{2k+1}$, $\beta_k := \frac{1}{100k}$, $\gamma_k := \frac{100k^2-2k-1}{200k^2+100k}$ and $g : C_2 \longrightarrow C_2$ defined by $g(x, y) := \frac{1}{2}(x, y)$ to obtain that $x_1 \in C_2$ and for every $k \geq 1$

$$\boxed{x_{k+1} = \frac{200k^2 + 202k + 1}{400k^2 + 200k} x_k + \frac{100k^2 - 2k - 1}{200k^2 + 100k} M x_k}. \tag{4.7}$$

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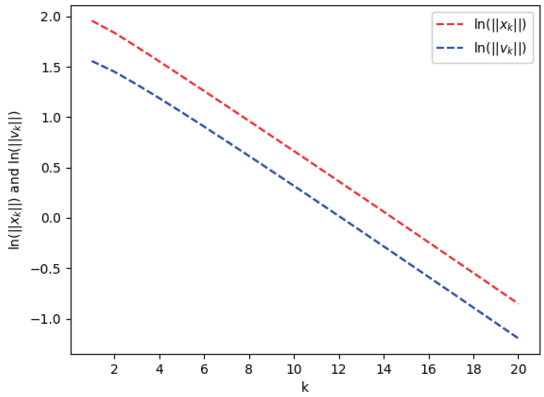

Figure 1: Convergence of sequences $\ln(\|x_k\|)$ and $\ln(\|v_k\|)$ to 0, with $x_0 = (-5, 5)$.

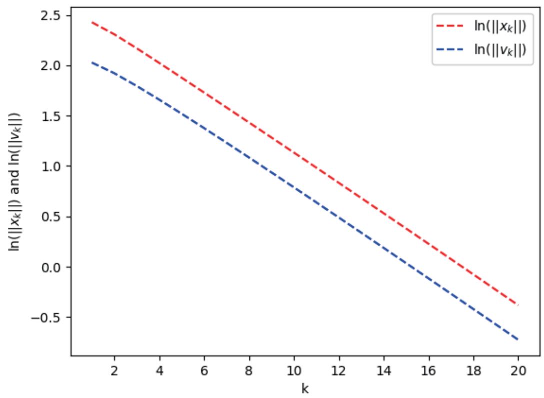

Figure 2: Convergence of sequences $\ln(\|x_k\|)$ and $\ln(\|v_k\|)$ to 0, with $x_0 = (-8, 8)$.

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