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NEW BOUNDS OF THE MINIMUM H-EIGENVALUE FOR SPARSE Z-TENSORS*

Xiuyun Feng, Gang Wang† and Yiju Wang

Abstract: Sparse tensors play a fundamental role in hypergraphs and stability of nonlinear systems. In this paper, we establish new bounds of the minimum H-eigenvalue for a Z-tensor by its majorization matrix's digraph and representation matrix's digraph. Numerical examples are proposed to verify that our conclusions are more accurate and require fewer calculations than existing results. Based on the lower bound estimations for the minimum H-eigenvalue, we provide some checkable sufficient conditions for the positive definiteness of Z-tensors.

Key words: *sparse tensors, Z-tensors, minimum H-eigenvalue, positive definiteness*

Mathematics Subject Classification: *15A18, 15A42*

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1 Introduction

Let $\mathbb{C}(\mathbb{R})$ be the set of complex (real) numbers and $\mathbb{C}^n(\mathbb{R}^n)$ be the set of n-dimensional complex (real) vectors. An m-order n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ is a multi-way array with entries

$$a_{i_1 i_2 \ldots i_m} \in \mathbb{C}, \quad i_k \in N = \{1, 2, \ldots, n\}, \quad k = 1, 2, \ldots, m.$$

Tensor $\mathcal{A}$ is called nonnegative (positive) if $a_{i_1 i_2 \ldots i_m} \geq 0 (a_{i_1 i_2 \ldots i_m} > 0)$.

Tensor is a higher-order extension of the matrix. Therefore, many ideas and associated properties for the matrix, such as determinant and eigenvalue theory, can be extended to higher order tensor by exploring its multilinear algebra properties [3, 17]. Particularly, the following definition of tensor eigenvalues were introduced in [11, 17].

Definition 1.1. Let $\mathcal{A}$ be an m-order n-dimensional tensor. (λ, x) is called an eigenpair of tensor $\mathcal{A}$ if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \tag{1.1}$$

where $(\mathcal{A}x^{m-1})_i = \sum\limits_{i_2,\ldots,i_m=1}^{n} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m}$ and $x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^T$. In this case, (λ, x) is called an H-eigenpair if they are both real.

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†Corresponding author

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Tensor eigenvalue problems have attracted a lot of researchers due to their wide applications in medical resonance imaging [1, 17, 18], higher-order Markov chains [14], positive definiteness of multivariate forms in automatical control [15]. In particular, some important properties of nonnegative tensors and M-tensors have been established in [3, 5, 12, 20, 21, 24]. For instance, sharp bounds for the minimum H-eigenvalue of nonsingular M-tensors have been proposed [8, 23, 25]. In fact, Z-tensors are the general form of M-tensors, which were investigated in tensor complementarity problems [7, 13]. So far, however, there has been little discussion about the bounds for the minimum H-eigenvalue of Z-tensors. An interesting problem arises: can the minimum H-eigenvalue of Z-tensors be estimated as the minimum H-eigenvalue of M-tensors? Recently, sparse tensor eigenvalue problems, which the number of non-zero elements in the tensor is far less than the number of zero elements, appear in hypergraphs [2, 19, 26] and stability of a nonlinear system [4, 12]. If we employ the existing methods [8, 23, 25] to estimate the bounds of the minimum H-eigenvalue with high-dimensional variables, the computational time is prohibitively long. Therefore, the sparsity of tensors encourages us to develop new methods for estimating the minimum H-eigenvalues.

Motivated and inspired by the above works, we investigate the relations between sparse tensors and their majorization matrix's digraph and representation matrix's digraph introduced by [6, 9, 16]. Based on their majorization matrix's digraph and representation matrix's digraph, we establish two tight the minimum H-eigenvalue for Z-tensors with reduced calculations. The obtained results not only improve the resluts of [8, 23, 25], but also widely apply to Z-tensors. Further, we propose several sufficient conditions to test positive definiteness of even-order real supersymmetric sparse tensors, as well as nonsingular M-tensors.

The remainder of the paper is organized as follows. In Section 2, we recall important definitions and preliminary results. In Section 3, we establish the bounds of the minimum H-eigenvalue for sparse Z-tensors, and show the respective advantages of two theorems. We apply the bounds of the minimum H-eigenvalue to check the positive definiteness of even-order real supersymmetric sparse tensors and nonsingular M-tensors in Section 4.

2 Preliminary

In this section, we introduce important definitions and related properties of the tensor analysis [3, 17].

Definition 2.1. Let $\mathcal{A}$ be an m-order n-dimensional tensor.

(i) Let $\sigma(\mathcal{A})$ be the set of all H-eigenvalues of $\mathcal{A}$. Then the minimum H-eigenvalue $\tau(\mathcal{A})$ and the H-spectral radius $\rho(\mathcal{A})$ of $\mathcal{A}$ are denoted by

$$\tau(\mathcal{A}) = \min\{\lambda : \lambda \in \sigma(\mathcal{A})\};\ \rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

(ii) $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $I \subset \{1, 2, \dots, n\}$ such that

$$a_{i_1 i_2 \dots i_m} = 0, \quad \forall i_1 \in I, i_2, \dots, i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then it is called irreducible.

(iii) $\mathcal{A}$ is called a Z-tensor if it can be written as $\mathcal{A} = c\mathcal{I} - \mathcal{B}$, where $c > 0$, $\mathcal{I}$ is a unit tensor with entries

$$\delta_{i_1 i_2 \dots i_m} = \begin{cases} 1, & \text{if } i_1 = i_2 = \cdots = i_m \\ 0, & \text{otherwise} \end{cases}$$

and $\mathcal{B}$ is a nonnegative tensor. If $c \geq \rho(\mathcal{B})$, then $\mathcal{A}$ is said to be an M-tensor. Further, $c > \rho(\mathcal{B})$, then $\mathcal{A}$ is said to be a nonsingular M-tensor.

The directed graph of a nonnegative matrix $A = (a_{ij})$ has as vertices the indices $\{1, \dots, n\}$, and there is an arc from vertex i to vertex j if $a_{ij} \neq 0$. Matrix A is irreducible, if and only if one can get from any vertex to any other vertex (perhaps in several steps) and is called a strongly connected graph [6, 9, 16].

Definition 2.2. Let $\mathcal{A}$ be an m-order n-dimensional nonnegative tensor.

(i) A nonnegative matrix $\mathring{\mathcal{A}} = (a_{ij})_{n\times n}$ is called the majorization associated to tensor $\mathcal{A}$, if the (i,j)-th element of $\mathring{\mathcal{A}}$ is defined to be $a_{ij\dots j}$ for any $i, j \in N$.

(ii) A nonnegative matrix $\mathcal{G}(\mathcal{A}) = (a_{ij})_{n\times n}$ is called the representation associated to the tensor $\mathcal{A}$, if the (i,j)-th element of $\mathcal{G}(\mathcal{A})$ is defined to be $\sum\limits_{j\in i_2,\dots,i_m} a_{ii_2\dots i_m}$.

(iii) We associate with $\mathring{\mathcal{A}}$ digraphs as $\Gamma_{\mathring{\mathcal{A}}} = \Big(V(\mathring{\mathcal{A}}), E(\mathring{\mathcal{A}})\Big)$, where $V(\mathring{\mathcal{A}}) = \{1, \dots, n\}$ is the vertex set of $\Gamma_{\mathring{\mathcal{A}}}$, and $E(\mathring{\mathcal{A}}) = \{e_{ij} : e_{ij} = a_{ij\dots j} \neq 0, i \neq j\}$ is the arc set of $\Gamma_{\mathring{\mathcal{A}}}$, i.e., e_{ij} is the directed edge of $\Gamma_{\mathring{\mathcal{A}}}$.

(iv) We associate with $\mathcal{G}(\mathcal{A})$ digraphs as $\Gamma_{\mathcal{G}(\mathcal{A})} = (V(\mathcal{G}(\mathcal{A})), E(\mathcal{G}(\mathcal{A})))$, where $V(\mathcal{G}(\mathcal{A})) = \{1, \dots, n\}$ is the vertex set of $\Gamma_{\mathcal{G}(\mathcal{A})}$, and $E(\mathcal{G}(\mathcal{A})) = \{g_{ij} : g_{ij} = \sum\limits_{j\in\{i_2,\dots,i_m\}} a_{ii_2\dots i_m} \neq 0, i \neq j\}$ the arc set of $\Gamma_{\mathcal{G}(\mathcal{A}))}$, i.e., g_{ij} is the directed edge of $\Gamma_{\mathcal{G}(\mathcal{A})}$. Tensor $\mathcal{A}$ is called weakly irreducible if $\mathcal{G}(\mathcal{A})$ is irreducible.

From Theorem 2.3 of [16], if $\mathring{\mathcal{A}}$ is irreducible, then $\mathcal{A}$ is irreducible. Further, if $\mathcal{A}$ is irreducible, then $\mathcal{A}$ is weakly irreducible in [9]. When $\mathcal{A}$ is a general tensor, we use $|\mathcal{A}|$ to denote the nonnegative tensor composed of $\mathcal{A}$. In this paper, $|\mathring{\mathcal{A}}|$ and $\mathcal{G}(|\mathcal{A}|)$ denote the majorization matrix's digraph and representation matrix's digraph of general tensors, respectively. In what follows, we propose characterizations of eigenvector corresponding to the minimum H-eigenvalue of Z-tensors.

Lemma 2.3. *(i) (Theorem 1.3 of [3]) Let $\mathcal{A}$ be an m-order n-dimensional nonnegative tensor, then there exist $\lambda_0 \geq 0$ and a nonnegative vector $x_0 \neq 0$ such that*

$$\mathcal{A}x_0^{m-1} = \lambda_0 x_0^{[m-1]}.$$

(ii) (Theorem 4.1 of [6]) Let $\mathcal{A}$ be an m-order n-dimensional weakly irreducible nonnegative tensor, then there exists a unique x such that $(\rho(\mathcal{A}), x)$ is a positive eigenpair.

Lemma 2.4. *Let $\mathcal{Q}$ be an m-order n-dimensional Z-tensor. Then, there exists a nonnegative vector v such that*

$$\mathcal{Q}v^{m-1} = \tau(\mathcal{Q})v^{[m-1]}.$$

Further, if $\mathcal{Q}$ is weakly irreducible, there exists a positive vector v such that

$$\mathcal{Q}v^{m-1} = \tau(\mathcal{Q})v^{[m-1]}.$$

Proof. Since $\mathcal{Q}$ is a Z-tensor, there exists a nonnegative tensor $\mathcal{A}$ such that

$$\mathcal{Q} = \lambda\mathcal{I} - \mathcal{A} \text{ and } \rho(\mathcal{A}) = \lambda - \tau(\mathcal{Q}).$$

By (i) of Lemma 2.3, there exists a a nonnegative vector v such that

$$\mathcal{A}v^{m-1} = \rho(\mathcal{A})v^{[m-1]} = (\lambda - \tau(\mathcal{Q}))v^{[m-1]}.$$

Hence,

$$(\lambda\mathcal{I} - \mathcal{A})v^{m-1} = \tau(\mathcal{Q})v^{[m-1]},$$

which implies

$$\mathcal{Q}v^{m-1} = \tau(\mathcal{Q})v^{[m-1]}.$$

From (ii) of Lemma 2.3, we obtain further results. □

We end this section with important results of [8, 10, 22, 23, 25]. Given an m-order n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$, denote

$$\Delta_i = \{(i_2, \dots, i_m) : i_j = i \ for\ some\ j \in 2, \dots, m,\ where\ i, i_2, \dots, i_m \in N\},$$

$$\overline{\Delta}_i = \{(i_2, \dots, i_m) : i_j \neq i \ for\ any\ j \in 2, \dots, m,\ where\ i, i_2, \dots, i_m \in N\},$$

$$r_i(\mathcal{A}) = \sum_{\substack{i_2,\dots i_m \in N \\ \delta_{ii_2\dots i_m}=0}} |a_{ii_2\dots i_m}|, \quad r_i^{\Delta_i}(\mathcal{A}) = \sum_{\substack{(i_2,\dots,i_m)\in\Delta_i \\ \delta_{ii_2\dots i_m}=0}} |a_{ii_2\dots i_m}|,$$

$$r_i^{\overline{\Delta}_i}(\mathcal{A}) = \sum_{(i_2,\dots,i_m)\in\overline{\Delta}_i} |a_{ii_2\dots i_m}|, \quad r_i(\mathcal{A}) = r_i^{\Delta_i}(\mathcal{A}) + r_i^{\overline{\Delta}_i}(\mathcal{A}).$$

Lemma 2.5 (Theorem 2.1 of [8])**.** *Let $\mathcal{A}$ be an m-order n-dimensional irreducible M-tensor. Then*

$$\min_{i\in N} R_i(\mathcal{A}) \leq \tau(\mathcal{A}) \leq \max_{i\in N} R_i(\mathcal{A}),$$

where $R_i(\mathcal{A}) = \sum\limits_{i_2,\dots,i_m\in N} a_{ii_2\dots i_m}$.

Lemma 2.6 (Theorem 2.2 of [8])**.** *Let $\mathcal{A}$ be an m-order n-dimensional weakly irreducible nonsingular M-tensor. Then,*

$$\min_{\substack{i,j\in N \\ i\neq j}} \phi_{i,j}(\mathcal{A}) \leq \tau(\mathcal{A}) \leq \max_{\substack{i,j\in N \\ i\neq j}} \phi_{i,j}(\mathcal{A}),$$

where $\phi_{i,j}(\mathcal{A}) = \frac{1}{2}\left\{a_{i\dots i} + a_{j\dots j} - r_i^j(\mathcal{A}) - \sqrt{\left(a_{i\dots i} - a_{j\dots j} - r_i^j(\mathcal{A})\right)^2 - 4a_{ij\dots j}r_j(\mathcal{A})}\right\}$.

Lemma 2.7 (Theorem 2.1 of [25])**.** *Let $\mathcal{A}$ be an m-order n-dimensional irreducible M-tensor. Then,*

$$\min_{\substack{i,j\in N \\ i\neq j}} \widetilde{\Psi}_{i,j}(\mathcal{A}) \leq \tau(\mathcal{A}) \leq \max_{\substack{i,j\in N \\ i\neq j}} \widetilde{\Psi}_{i,j}(\mathcal{A}),$$

where $\widetilde{\Psi}_{i,j}(\mathcal{A}) = \frac{1}{2}\left\{a_{i\dots i} + a_{j\dots j} - r_i^{\Delta_i}(\mathcal{A}) - \sqrt{\left(a_{i\dots i} - a_{j\dots j} - r_i^{\Delta_i}(\mathcal{A})\right)^2 + 4r_i^{\overline{\Delta}_i}(\mathcal{A})r_j(\mathcal{A})}\right\}$.

3 Bounds of the Minimum H-Eigenvalue of a Sparse Z-Tensor

In this section, we establish new bounds of the minimum H-eigenvalue for a sparse Z-tensor via its majorization matrix's digraph and representation matrix's digraph, which can reduce calculations and improve the results in [10, 25].

Theorem 3.1. *Let $\mathcal{A}$ be an m-order n-dimensional Z-tensor with $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i) = \{i : \exists\ j \in N$ such that $g_{ij} \in E(\mathcal{G}(|\mathcal{A}|))\} \neq \emptyset$. Then*

$$\min_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}} \Psi_{i,j}(\mathcal{A}) \leq \tau(\mathcal{A}) \leq \max_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}\bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A}),$$

where

$$\Psi_{i,j}(\mathcal{A}) = \frac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} - r_i^{\Delta_i}(\mathcal{A}) - \sqrt{\left(a_{i\ldots i} - a_{j\ldots j} - r_i^{\Delta_i}(\mathcal{A})\right)^2 + 4r_i^{\overline{\Delta}_i}(\mathcal{A})r_j(\mathcal{A})}\right\}.$$

Proof. Since $\mathcal{A}$ is a Z-tensor, there exists a nonnegative eigenvector $x = (x_1, \ldots, x_n)^\top$ corresponding to $\tau(\mathcal{A})$ such that

$$\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}. \tag{3.1}$$

Without loss of generality, we assume $x_{t_1} \geq x_{t_2} \geq \ldots x_{t_n} \geq 0$ from Lemma 2.4.

(i) We first show $\min\limits_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}} \Psi_{i,j}(\mathcal{A}) \leq \tau(\mathcal{A})$. Since $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(t_1) \neq \emptyset$, we set $x_{t_1} \geq x_{t_s} = \max\{x_{t_i} : \sum\limits_{(i_2,\ldots,i_m)\in\Delta_{t_1}} |a_{t_1 i_2\ldots i_m}| \neq 0\}$, which means $g_{t_1 t_s} \in \Gamma_{\mathcal{G}(|\mathcal{A}|)}$. In view of the t_1-th equation of (3.1), we deduce

$$\begin{aligned}
(a_{t_1\ldots t_1} - \tau(\mathcal{A}))x_{t_1}^{m-1} &= \sum_{\substack{(i_2,\ldots,i_m)\in\Delta_{t_1}\\ \delta_{t_1 i_2\ldots i_m}=0}} |a_{t_1 i_2\ldots i_m}|x_{i_2}\ldots x_{i_m} + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}_{t_1}} |a_{t_1 i_2\ldots i_m}|x_{i_2}\ldots x_{i_m}\\
&\leq \sum_{\substack{(i_2,\ldots,i_m)\in\Delta_{t_1}\\ \delta_{t_1 i_2\ldots i_m}=0}} |a_{t_1 i_2\ldots i_m}|x_{t_1}^{m-1} + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}_{t_1}} |a_{t_1 i_2\ldots i_m}|x_{t_s}^{m-1}\\
&= r_{t_1}^{\Delta_{t_1}}(\mathcal{A})x_{t_1}^{m-1} + r_{t_1}^{\overline{\Delta}_{t_1}}(\mathcal{A})x_{t_s}^{m-1},
\end{aligned}$$

equivalently,

$$\left(a_{t_1\ldots t_1} - \tau(\mathcal{A}) - r_{t_1}^{\Delta_{t_1}}(\mathcal{A})\right)x_{t_1}^{m-1} \leq r_{t_1}^{\overline{\Delta}_{t_1}}(\mathcal{A})x_{t_s}^{m-1}. \tag{3.2}$$

We now break up the argument into two cases.

Case 1: $x_{t_s} > 0$. It follows from (3.1) and $i = t_s$ that

$$(a_{t_s\ldots t_s} - \tau(\mathcal{A}))x_{t_s}^{m-1} = \sum_{\delta_{t_s i_2\ldots i_m}=0} |a_{t_s i_2\ldots i_m}|x_{i_2}\ldots x_{i_m} \leq r_{t_s}(\mathcal{A})x_{t_1}^{m-1}. \tag{3.3}$$

Multiplying inequalities (3.2) and (3.3) gives

$$\left(a_{t_1\ldots t_1} - \tau(\mathcal{A}) - r_{t_1}^{\Delta_{t_1}}(\mathcal{A})\right)(a_{t_s\ldots t_s} - \tau(\mathcal{A}))x_{t_1}^{m-1}x_{t_s}^{m-1} \leq r_{t_1}^{\overline{\Delta}_{t_1}}(\mathcal{A})r_{t_s}(\mathcal{A})x_{t_1}^{m-1}x_{t_s}^{m-1}.$$

From $x_{t_1}^{m-1}x_{t_s}^{m-1} > 0$, it holds that

$$\left(a_{t_1\ldots t_1} - \tau(\mathcal{A}) - r_{t_1}^{\Delta_{t_1}}(\mathcal{A})\right)(a_{t_s\ldots t_s} - \tau(\mathcal{A})) \leq r_{t_1}^{\overline{\Delta}_{t_1}}(\mathcal{A})r_{t_s}(\mathcal{A}), \tag{3.4}$$

that is,

$$\tau(\mathcal{A})^2 - \left(a_{t_1\ldots t_1} + a_{t_s\ldots t_s} - r_{t_1}^{\Delta_{t_1}}(\mathcal{A})\right)\tau(\mathcal{A}) + a_{t_s\ldots t_s}\left(a_{t_1\ldots t_1} - r_{t_1}^{\Delta_{t_1}}(\mathcal{A})\right) - r_{t_1}^{\overline{\Delta}_{t_1}}(\mathcal{A})r_{t_s}(\mathcal{A}) \leq 0.$$

Solving for $\tau(\mathcal{A})$, we obtain

$$\tau(\mathcal{A}) \geq \frac{1}{2}\left\{a_{t_1\ldots t_1} + a_{t_s\ldots t_s} - r_{t_1}^{\Delta_{t_1}}(\mathcal{A}) - \sqrt{\left(a_{t_1\ldots t_1} - a_{t_s\ldots t_s} - r_{t_1}^{\Delta_{t_1}}(\mathcal{A})\right)^2 + 4r_{t_1}^{\overline{\Delta}_{t_1}}(\mathcal{A})r_{t_s}(\mathcal{A})}\right\},$$

which implies

$$\min_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}} \Psi_{i,j}(\mathcal{A}) \le \Psi_{t_1,t_s}(\mathcal{A}) \le \tau(\mathcal{A}).$$

Case 2: $x_{t_s} = 0$. Then, $a_{t_1\ldots t_1} - \tau(\mathcal{A}) - r_{t_1}^{\Delta_{t_1}}(\mathcal{A}) \le 0$ and it is obvious that satisfy (3.4).

(ii) We prove $\tau(\mathcal{A}) \le \max_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}\bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A})$. We break up the argument into two cases.

Case 1: $\mathcal{G}(|\mathcal{A}|)$ is irreducible. Then, $\mathcal{A}$ is weakly irreducible. From Lemma 2.4, there exists a positive vector $x = (x_1,\ldots,x_n)^\top$ such that

$$0 < x_{t_n} \le x_{t_r} = \min\{x_{t_j} : \sum_{(i_2,\ldots,i_m)\in\Delta_{t_n}} |a_{t_n i_2\ldots i_m}| \ne 0, j\in N\}.$$

Following the similar arguments to the proof of (i), we have

$$\left(a_{t_n\ldots t_n} - \tau(\mathcal{A}) - r_{t_n}^{\Delta_{t_n}}(\mathcal{A})\right) x_{t_n}^{m-1} \ge r_{t_n}^{\overline{\Delta}_{t_n}}(\mathcal{A}) x_{t_r}^{m-1} \ge 0 \tag{3.5}$$

and

$$(a_{t_r\ldots t_r} - \tau(\mathcal{A}))x_{t_r}^{m-1} = \sum_{\delta_{t_r i_2\ldots i_m=0}} |a_{t_r i_2\ldots i_m}| x_{i_2}\ldots x_{i_m} \ge r_{t_r}(\mathcal{A}) x_{t_n}^{m-1} \ge 0. \tag{3.6}$$

Multiplying (3.5) with (3.6) and utilizing $x_{t_r} \ge x_{t_n} > 0$ yield

$$\left(a_{t_n\ldots t_n} - \tau(\mathcal{A}) - r_{t_n}^{\Delta_{t_n}}(\mathcal{A})\right)(a_{t_r\ldots t_r} - \tau(\mathcal{A})) \ge r_{t_n}^{\overline{\Delta}_{t_n}}(\mathcal{A}) r_{t_r}(\mathcal{A}).$$

Solving for $\tau(\mathcal{A})$, one has

$$\tau(\mathcal{A}) \le \frac{1}{2}\left\{a_{t_n\ldots t_n} + a_{t_r\ldots t_r} - r_{t_n}^{\Delta_{t_n}}(\mathcal{A}) - \sqrt{\left(a_{t_n\ldots t_n} - a_{t_r\ldots t_r} - r_{t_n}^{\Delta_{t_n}}(\mathcal{A})\right)^2 + 4 r_{t_n}{}^{\overline{\Delta}_{t_n}}(\mathcal{A}) r_{t_r}(\mathcal{A})}\right\},$$

which shows

$$\tau(\mathcal{A}) \le \Psi_{t_n,t_r}(\mathcal{A}) \le \max_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}} \Psi_{i,j}(\mathcal{A}).$$

Case 2: $\mathcal{G}(|\mathcal{A}|)$ is reducible. For any $\epsilon > 0$, set

$$|\mathcal{A}(\epsilon)| = |\mathcal{A}| + \Phi(\epsilon) \quad and \quad \Phi(\epsilon) = (\theta_{i_1\ldots i_m}),$$

where

$$\theta_{i_1\ldots i_m} = \begin{cases} \theta_{ij\ldots j} = \epsilon, & if j-i = 1, 1-n, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{G}(|\mathcal{A}(\epsilon)|)$ is irreducible. Following the similar proof of Case 1, we obtain

$$\tau(\mathcal{A}(\epsilon)) \le \max_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}(\epsilon)|)}} \Psi_{i,j}(\mathcal{A}(\epsilon)).$$

Letting $\epsilon \to 0$, we obtain

$$\tau(\mathcal{A}) \le \max_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}\bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A}).$$

Thus, the desired result holds. □

When the condition $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i) \ne \emptyset$ is replaced by weak irreducibility of $\mathcal{A}$, we obtain tight conclusions.

Corollary 3.2. *Let $\mathcal{A}$ be an m-order n-dimensional weakly irreducible Z-tensor. Then,*

$$\min_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}} \Psi_{i,j}(\mathcal{A}) \le \tau(\mathcal{A}) \le \max_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}} \Psi_{i,j}(\mathcal{A}).$$

Remark 3.3. Compared with Theorem 2.1 of [25] under irreducibility of $\mathcal{A}$, we weaken the condition into weak irreducibility. Further, the results of Corollary 3.2 have minor computations yet tight bounds, i.e.,

$$\min_{\substack{i,j\in N\\ i\neq j}} \widetilde{\Psi}_{i,j}(\mathcal{A}) \le \min_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}} \Psi_{i,j}(\mathcal{A}) \le \tau(\mathcal{A}) \le \max_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}} \Psi_{i,j}(\mathcal{A}) \le \max_{\substack{i,j\in N\\ i\neq j}} \widetilde{\Psi}_{i,j}(\mathcal{A}).$$

Indeed, compared with weak irreducibility of $\mathcal{A}$, $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i) \neq \emptyset$ is a condition to verify and meet, which can achieve exact results.

When the condition $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i) \neq \emptyset$ is removed, we obtain general results.

Corollary 3.4. *Let $\mathcal{A}$ be an m-order n-dimensional Z-tensor. Then,*

$$\min_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}\bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A}) \le \tau(\mathcal{A}) \le \max_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}\bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A}),$$

where

$$\Psi_{i,j}(\mathcal{A}) = \frac{1}{2}\left\{a_{i\dots i} + a_{j\dots j} - r_i^{\Delta_i}(\mathcal{A}) - \sqrt{\left(a_{i\dots i} - a_{j\dots j} - r_i^{\Delta_i}(\mathcal{A})\right)^2 + 4r_i^{\overline{\Delta}_i}(\mathcal{A})r_j(\mathcal{A})}\right\}.$$

Proof. When $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i) \neq \emptyset$, by Theorem 3.1, the results hold. We only prove $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i) = \emptyset$. For any $\epsilon > 0$, set $|\mathcal{A}(\epsilon)| = |\mathcal{A}| + \Phi(\epsilon)$. Then, $|\mathcal{A}(\epsilon)|$ is weakly irreducible. Using Corollary 3.2, we have

$$\min_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}} \Psi_{i,j}(\mathcal{A}(\epsilon)) \le \tau(\mathcal{A}(\epsilon)) \le \max_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}} \Psi_{i,j}(\mathcal{A}(\epsilon)).$$

Letting $\epsilon \to 0$, we obtain

$$\min_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}\bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A}) \le \tau(\mathcal{A}) \le \max_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}\bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A}).$$

□

Remark 3.5. Without imposing any condition on Z-tensors, we obtain the bounds of the minimum H-eigenvalue $\tau(\mathcal{A})$ in Corollary 3.4. Compared with Theorem 2.1 of [25], the result of Corollary 3.4 has wide applications and good numerical results.

Now, we arrive at the following bounds for Z-tensors by their majorization matrix's digraph.

Theorem 3.6. *Let $\mathcal{A}$ be an m-order n-dimensional Z-tensor with $\Gamma_{|\mathring{\mathcal{A}}|}(i) = \{i : \exists\ j \in N$ such that $e_{ij} \in E(|\mathring{\mathcal{A}}|)\} \neq \emptyset$. Then,*

$$\min_{e_{ij}\in\Gamma_{|\mathring{\mathcal{A}}|}} \kappa_{i,j}(\mathcal{A}) \le \tau(\mathcal{A}) \le \max_{e_{ij}\in\Gamma_{|\mathring{\mathcal{A}}|}\bigcup j-i=1,1-n} \kappa_{i,j}(\mathcal{A}),$$

where

$$\kappa_{i,j}(\mathcal{A}) = \frac{1}{2}\left\{a_{i\dots i} + a_{j\dots j} - r_i^{'}(\mathcal{A}) - \sqrt{(a_{i\dots i} - a_{j\dots j} - r_i^{'}(\mathcal{A}))^2 + 4\widetilde{r}_i(\mathcal{A})r_j(\mathcal{A})}\right\},$$

$$\widetilde{r}_i(\mathcal{A}) = \sum_{\substack{\delta_{i_2\dots i_m=1}\\ \delta_{ii_2\dots i_m=0}}} |a_{ii_2\dots i_m}| \quad \textit{and} \quad r_i^{'}(\mathcal{A}) = r_i(\mathcal{A}) - \widetilde{r}_i(\mathcal{A}).$$

Proof. Let $(\tau(\mathcal{A}), x)$ be an H-eigenpair of $\mathcal{A}$. Without loss of generality, we assume that $x_{t_1} \geq x_{t_2} \geq \cdots \geq x_{t_n} \geq 0$ from Lemma 2.4.

(i) Since $\Gamma_{|\mathring{\mathcal{A}}|}(i) \neq \emptyset$, there exists $j \neq t_1$ with $a_{t_1 j \ldots j} \neq 0$. Assume

$$a_{t_1 t_l \ldots t_l} = 0,\ l = 2, 3, \ldots, s-1,\ a_{t_1 t_s \ldots t_s} \neq 0\ (2 \leq s \leq n),$$

which implies $e_{t_1 t_s} \in \Gamma_{|\mathring{\mathcal{A}}|}$. In view of the t_1-th equation of (3.1), we deduce

$$\begin{aligned}(a_{t_1 \ldots t_1} - \tau(\mathcal{A}))x_{t_1}^{m-1} &= \sum_{\delta_{i_2 \ldots i_m = 0}} |a_{t_1 i_2 \ldots i_m}| x_{i_2} \ldots x_{i_m} + \sum_{\substack{\delta_{i_2 \ldots i_m = 1} \\ \delta_{t_1 i_2 \ldots i_m = 0}}} |a_{t_1 i_2 \ldots i_m}| x_{i_2} \ldots x_{i_m} \\ &\leq \sum_{\delta_{i_2 \ldots i_m = 0}} |a_{t_1 i_2 \ldots i_m}| x_{t_1}^{m-1} + \sum_{\substack{\delta_{i_2 \ldots i_m = 1} \\ \delta_{t_1 i_2 \ldots i_m = 0}}} |a_{t_1 i_2 \ldots i_m}| x_{t_s}^{m-1} \\ &= r'_{t_1}(\mathcal{A}) x_{t_1}^{m-1} + \widetilde{r}_{t_1}(\mathcal{A}) x_{t_s}^{m-1},\end{aligned}$$

equivalently,

$$\left(a_{t_1 \ldots t_1} - \tau(\mathcal{A}) - r'_{t_1}(\mathcal{A})\right) x_{t_1}^{m-1} \leq \widetilde{r}_{t_1}(\mathcal{A}) x_{t_s}^{m-1}. \tag{3.7}$$

Next, we break up the argument into two cases.

Case 1: $x_{t_s} > 0$. It follows from (3.1) and $i = t_s$ that

$$(a_{t_s \ldots t_s} - \tau(\mathcal{A}))x_{t_s}^{m-1} \leq r_{t_s}(\mathcal{A}) x_{t_1}^{m-1}. \tag{3.8}$$

Multiplying (3.7) and (3.8) gives

$$\left(a_{t_1 \ldots t_1} - \tau(\mathcal{A}) - r'_{t_1}(\mathcal{A})\right)(a_{t_s \ldots t_s} - \tau(\mathcal{A})) x_{t_1}^{m-1} x_{t_s}^{m-1} \leq \widetilde{r}_{t_1}(\mathcal{A}) r_{t_s}(\mathcal{A}) x_{t_1}^{m-1} x_{t_s}^{m-1}.$$

From $x_{t_1}^{m-1} x_{t_s}^{m-1} > 0$, it holds that

$$\left(a_{t_1 \ldots t_1} - \tau(\mathcal{A}) - r'_{t_1}(\mathcal{A})\right)(a_{t_s \ldots t_s} - \tau(\mathcal{A})) \leq \widetilde{r}_{t_1}(\mathcal{A}) r_{t_s}(\mathcal{A}). \tag{3.9}$$

Solving for $\tau(\mathcal{A})$, we deduce

$$\begin{aligned}\tau(\mathcal{A}) &\geq \tfrac{1}{2}\left\{a_{t_1 \ldots t_1} + a_{t_s \ldots t_s} - r'_{t_1}(\mathcal{A}) - \sqrt{(a_{t_1 \ldots t_1} - a_{t_s \ldots t_s} - r'_{t_1}(\mathcal{A}))^2 + 4\widetilde{r}_{t_1}(\mathcal{A}) r_{t_s}(\mathcal{A})}\right\} \\ &\geq \min_{e_{ij} \in \Gamma_{|\mathring{\mathcal{A}}|}} \kappa_{i,j}(\mathcal{A}).\end{aligned}$$

Case 2: $x_{t_s} = 0$. Then, $a_{t_1 \ldots t_1} - \tau(\mathcal{A}) - r'_{t_1}(\mathcal{A}) \leq 0$. Clearly, $\tau(\mathcal{A}) \geq a_{t_1 \ldots t_1} - r'_{t_1}(\mathcal{A})$ and (3.9) holds.

(ii) We prove $\tau(\mathcal{A}) \leq \max\limits_{e_{ij} \in \Gamma_{|\mathring{\mathcal{A}}|} \bigcup j-i=1,1-n} \kappa_{i,j}(\mathcal{A})$. We break up the argument into two cases.

Case 1: $|\mathring{\mathcal{A}}|$ is irreducible. Then, $\mathcal{A}$ is irreducible and weakly irreducible. According to Lemma 2.4, without loss of generality, there exists an H-eigenpair $(\tau(\mathcal{A}), x)$ with $x_{t_1} \geq x_{t_2} \geq \cdots \geq x_{t_n} > 0$. By $\Gamma_{|\mathring{\mathcal{A}}|}(t_n) \neq \emptyset$, there exists $j \neq t_n$ with $a_{t_n j \ldots j} \neq 0$. Suppose

$$a_{t_n t_l \ldots t_l} = 0,\ l = n-1, n-2, \ldots, n-r,\ a_{t_n t_r \ldots t_r} \neq 0\ (2 \leq r \leq n),$$

which implies $e_{t_n t_r} \in \Gamma_{|\mathring{\mathcal{A}}|}$.

Following the similar arguments to the proof of (3.7) and (3.8), we deduce

$$\left(a_{t_n \ldots t_n} - \tau(\mathcal{A}) - r'_{t_n}(\mathcal{A})\right) x_{t_n}^{m-1} \geq \widetilde{r}_{t_n}(\mathcal{A}) x_{t_r}^{m-1} \geq 0 \tag{3.10}$$

and

$$(a_{t_r\ldots t_r}-\tau(\mathcal{A}))x_{t_r}^{m-1}\geq r_{t_r}(\mathcal{A})x_{t_n}^{m-1}\geq 0. \tag{3.11}$$

Multiplying (3.10) with (3.11) and utilizing $x_{t_r}\geq x_{t_n}>0$ yield

$$\left(a_{t_n\ldots t_n}-\tau(\mathcal{A})-r'_{t_n}(\mathcal{A})\right)(a_{t_r\ldots t_r}-\tau(\mathcal{A}))\geq \widetilde{r}_{t_n}(\mathcal{A})r_{t_r}(\mathcal{A}).$$

Solving for $\tau(\mathcal{A})$, one shows

$$\tau(\mathcal{A})\leq \kappa_{t_n,t_r}(\mathcal{A})\leq \max_{e_{ij}\in\Gamma_{|\mathring{\mathcal{A}}|}}\kappa_{i,j}(\mathcal{A}).$$

Case 2: $|\mathring{\mathcal{A}}|$ is reducible. For any $\epsilon>0$, set

$$|\mathcal{A}(\epsilon)|=|\mathcal{A}|+\Phi(\epsilon)\quad and\quad \Phi(\epsilon)=(\theta_{i_1\ldots i_m}).$$

Thus, $|\mathring{\mathcal{A}}(\epsilon)|$ is irreducible. Following the similar proof of Case 1, we deduce

$$\tau(\mathcal{A}(\epsilon))\leq \max_{e_{ij}\in\Gamma_{|\mathring{\mathcal{A}}(\epsilon)|}}\kappa_{i,j}(\mathcal{A}(\epsilon)).$$

Letting $\epsilon\to 0$, we obtain

$$\tau(\mathcal{A})\leq \max_{e_{ij}\in\Gamma_{|\mathring{\mathcal{A}}|}\bigcup j-i=1,1-n}\kappa_{i,j}(\mathcal{A}).$$

Combining Cases 1 and 2, we obtain the desired results. □

Corollary 3.7. *Let $\mathcal{A}$ be an m-order n-dimensional irreducible Z-tensor. Then,*

$$\min_{e_{ij}\in\Gamma_{|\mathring{\mathcal{A}}|}}\kappa_{i,j}(\mathcal{A})\leq\tau(\mathcal{A})\leq\max_{e_{ij}\in\Gamma_{|\mathring{\mathcal{A}}|}}\kappa_{i,j}(\mathcal{A}).$$

Corollary 3.8. *Let $\mathcal{A}$ be an m-order n-dimensional Z-tensor. Then,*

$$\min_{e_{ij}\in\Gamma_{|\mathring{\mathcal{A}}|}\bigcup j-i=1,1-n}\kappa_{i,j}(\mathcal{A})\leq\tau(\mathcal{A})\leq\max_{e_{ij}\in\Gamma_{|\mathring{\mathcal{A}}|}\bigcup j-i=1,1-n}\kappa_{i,j}(\mathcal{A}).$$

Remark 3.9. Compared with Theorem 4.5 of [23], the result of Corollary 3.8 has minor computations yet tight bounds without irreducibility.

In what follows, we test the efficiency of the obtained results.

Example 3.10. Let $\mathcal{A}$ be a 3-order 4-dimensional Z-tensor with non-zero elements defined as follows:

$$a_{ijk}=\begin{cases} a_{111}=8; a_{112}=-1; a_{122}=-2; a_{123}=-1; a_{133}=-2;\\ a_{222}=10; a_{232}=-1; a_{233}=-1; a_{243}=-1;\\ a_{333}=15; a_{334}=-1; a_{343}=-1; a_{344}=-4;\\ a_{421}=-6; a_{422}=-1; a_{424}=-1; a_{444}=12.\end{cases}$$

We compute $\tau(\mathcal{A})=7.063$ and identify

$$\mathcal{G}(|\mathcal{A}|)=\begin{pmatrix} 11 & 4 & 3 & 0\\ 0 & 13 & 3 & 1\\ 0 & 0 & 13 & 6\\ 6 & 8 & 0 & 11\end{pmatrix}\quad and\quad |\mathring{\mathcal{A}}|=\begin{pmatrix} 10 & 2 & 2 & 0\\ 0 & 12 & 1 & 0\\ 0 & 0 & 11 & 4\\ 0 & 1 & 0 & 12\end{pmatrix}.$$

By Theorem 2.1 of [8], one has

$$2\leq\tau(\mathcal{A})\leq 9.$$

From Theorem 2.2 of [8], we have

$\phi_{1,2}(\mathcal{A}) = 3.127$	$\phi_{1,3}(\mathcal{A}) = 3.000$	$\phi_{1,4}(\mathcal{A}) = 2.000$
$\phi_{2,1}(\mathcal{A}) = 7.000$	$\phi_{2,3}(\mathcal{A}) = 7.228$	$\phi_{2,4}(\mathcal{A}) = 7.000$
$\phi_{3,1}(\mathcal{A}) = 8.000$	$\phi_{3,2}(\mathcal{A}) = 9.000$	$\phi_{3,4}(\mathcal{A}) = 6.821$
$\phi_{4,1}(\mathcal{A}) = 4.000$	$\phi_{4,2}(\mathcal{A}) = 4.459$	$\phi_{4,3}(\mathcal{A}) = 4.000$

Therefore,

$$\min_{i,j\in\{1,2,3,4\},i\neq j} \phi_{i,j}(\mathcal{A}) = 2 \leq \tau(\mathcal{A}) \leq 9 = \max_{i,j\in\{1,2,3,4\},i\neq j} \phi_{i,j}(\mathcal{A}).$$

From Theorem 2.1 of [25], following the similar computations of $\phi_{i,j}(\mathcal{A})$, we obtain

$$\min_{i,j\in\{1,2,3,4\},i\neq j} \widetilde{\Psi}_{i,j}(\mathcal{A}) = 2.699 \leq \tau(\mathcal{A}) \leq 7.725 = \max_{i,j\in\{1,2,3,4\},i\neq j} \widetilde{\Psi}_{i,j}(\mathcal{A}).$$

It follows from Theorem 4.5 of [23] that

$$2.597 \leq \tau(\mathcal{A}) \leq 7.725.$$

From matrix $\mathcal{G}(|\mathcal{A}|)$, we know that $\mathcal{A}$ is weakly irreducible. It follows from Theorem 3.1 that

$\Psi_{1,2}(\mathcal{A}) = 4.346$	$\Psi_{1,3}(\mathcal{A}) = 4.218$	$\Psi_{2,3}(\mathcal{A}) = 7.417$	$\Psi_{2,4}(\mathcal{A}) = 6.228$
$\Psi_{3,4}(\mathcal{A}) = 6.821$	$\Psi_{4,1}(\mathcal{A}) = 2.848$	$\Psi_{4,2}(\mathcal{A}) = 5.890$	

i.e.,

$$\min_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}} \Psi_{i,j}(\mathcal{A}) = 2.848 \leq \tau(\mathcal{A}) \leq 7.417 = \max_{g_{ij}\in\Gamma_{\mathcal{G}(|\mathcal{A}|)}\bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A}).$$

By matrix $\mathring{\mathcal{A}}$, for any $i \in \{1,2,3,4\}$, we observe $\Gamma_{|\mathring{\mathcal{A}}|}(i) \neq \emptyset$. Recalling Theorem 3.6, we compute

$\kappa_{1,2}(\mathcal{A}) = 4.000$	$\kappa_{1,3}(\mathcal{A}) = 3.848$	$\kappa_{2,3}(\mathcal{A}) = 7.228$
$\kappa_{3,4}(\mathcal{A}) = 6.821$	$\kappa_{4,1}(\mathcal{A}) = 3.628$	$\kappa_{4,2}(\mathcal{A}) = 4.459$

i.e.,

$$\min_{e_{ij}\in\Gamma_{|\mathring{\mathcal{A}}|}} \kappa_{i,j}(\mathcal{A}) = 3.848 \leq \tau(\mathcal{A}) \leq 7.228 = \max_{e_{ij}\in\Gamma_{|\mathring{\mathcal{A}}|}\bigcup j-i=1,1-n} \kappa_{i,j}(\mathcal{A}).$$

It is easy to see that the bounds in Theorems 3.1 and 3.6 are sharper than those of Theorems 2.1 and 2.2 of [8], Theorem 2.1 of [25] and Theorem 4.5 of [23]. In general, Theorems 3.1 and 3.6 have their own advantages. In the above example, the result of Theorem 3.6 is sharper than that of Theorem 3.1. The following example reveals the reverse result.

Example 3.11. Let $\mathcal{A}$ be an 3-order 4-dimensional Z-tensor with non-zero elements defined as follows:

$$a_{ijk} = \begin{cases} a_{111} = 6; a_{121} = -1; a_{122} = -2; a_{132} = -1; \\ a_{212} = -2; a_{213} = -1; a_{222} = 8; a_{233} = -2; \\ a_{313} = -1; a_{331} = -1; a_{333} = 8; a_{344} = -1; \\ a_{421} = -1; a_{422} = -1; a_{424} = -2; a_{444} = 11. \end{cases}$$

We can compute $\tau(\mathcal{A}) = 4.601$ and obtain

$$\mathcal{G}(|\mathcal{A}|) = \begin{pmatrix} 7 & 4 & 1 & 0 \\ 3 & 10 & 3 & 0 \\ 2 & 0 & 10 & 1 \\ 1 & 4 & 0 & 13 \end{pmatrix} \quad and \quad |\mathring{\mathcal{A}}| = \begin{pmatrix} 6 & 2 & 0 & 0 \\ 0 & 8 & 2 & 0 \\ 0 & 0 & 8 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

From matrix $\mathcal{G}(|\mathcal{A}|)$, for any $i \in \{1,2,3,4\}$, we know $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i) \neq \emptyset$. In view of Theorem 3.1, we compute

$\Psi_{1,2}(\mathcal{A}) = 2.347$	$\Psi_{1,3}(\mathcal{A}) = 3.146$	$\Psi_{2,1}(\mathcal{A}) = 2.536$
$\Psi_{2,3}(\mathcal{A}) = 3.838$	$\Psi_{3,1}(\mathcal{A}) = 4.000$	$\Psi_{3,4}(\mathcal{A}) = 5.298$
$\Psi_{4,1}(\mathcal{A}) = 4.298$	$\Psi_{4,2}(\mathcal{A}) = 5.298$	

and obtain

$$\min_{g_{ij} \in \Gamma_{\mathcal{G}(|\mathcal{A}|)}} \Psi_{i,j}(\mathcal{A}) = 2.347 \leq \tau(\mathcal{A}) \leq 5.298 = \max_{g_{ij} \in \Gamma_{\mathcal{G}(|\mathcal{A}|)} \bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A}).$$

By matrix $|\mathring{\mathcal{A}}|$, for any $i \in \{1,2,3,4\}$, we observe $\Gamma_{|\mathring{\mathcal{A}}|}(i) \neq \emptyset$. It follows from Theorem 3.6 that

$\kappa_{1,2}(\mathcal{A}) = 2.258$	$\kappa_{2,3}(\mathcal{A}) = 3.628$
$\kappa_{3,4}(\mathcal{A}) = 5.298$	$\kappa_{4,2}(\mathcal{A}) = 5.764$

which shows

$$\min_{e_{ij} \in \Gamma_{|\mathring{\mathcal{A}}|}} \kappa_{i,j}(\mathcal{A}) = 2.258 \leq \tau(\mathcal{A}) \leq 5.764 = \max_{e_{ij} \in \Gamma_{|\mathring{\mathcal{A}}|} \bigcup j-i=1,1-n} \kappa_{i,j}(\mathcal{A}).$$

In what follows, we introduce "sunflower" k-uniform hypergraph given in [2] to show our results.

Example 3.12. A 4-uniform hypergraph $\mathcal{H}$ with ten vertices $V = \{1,2,\ldots,10\}$ and three edges $E = \{e_1 = \{1,2,3,4\}, e_2 = \{1,5,6,7\}, e_3 = \{1,8,9,10\}\}$.

We obtain its degrees are $d_1 = 3, d_i = 1$ for $i = 2,\ldots,10$, and $\Delta = \max_{i \in [n]} d_i = 3$. Indeed, Laplacian tensor $\mathcal{L}(\mathcal{H})$ is sparse and can be characterized with non-zero elements as follows:

$$\mathcal{L}(\mathcal{H})_{i_1,\ldots,i_k} = \begin{cases} a_{1111} = 3, a_{2222} \cdots = a_{9999} = a_{10101010} = 1 \\ a_{1234} = a_{2341} = a_{3412} = a_{4123} = -\frac{1}{6} \\ a_{1567} = a_{5671} = a_{6715} = a_{7156} = -\frac{1}{6} \\ a_{18910} = a_{89101} = a_{91018} = a_{10189} = -\frac{1}{6}, \end{cases}$$

where a_{ijkl} is supersymmetric. We identify

$$\mathcal{G}(|\mathcal{L}(\mathcal{H})|) = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Certainly, $\mathcal{G}(|\mathcal{L}(\mathcal{H})|)$ is irreducible and $\mathcal{L}(\mathcal{H})$ is weakly irreducible. Therefore, it follows from Theorem 3.1 that

$$0 \leq \tau(\mathcal{L}(\mathcal{H})) \leq 0,$$

which implies that $\tau(\mathcal{A}) = 0$ and Laplacian tensor $\mathcal{L}(\mathcal{H})$ is positive semidefinite.

It is noted that $a_{ijjj} = 0$ for all $x, j \in [10], i \neq j$. Thus, Theorem 3.6 does not apply to determining the bounds of the minimum H-eigenvalues. We apply Corollary 3.8 to finding its bounds as follows:

$$0 \leq \tau(\mathcal{L}(\mathcal{H})) \leq 0,$$

which shows that $\tau(\mathcal{L}(\mathcal{H})) = 0$ and Laplacian tensor $\mathcal{L}(\mathcal{H})$ is positive semidefinite.

4 Checking the Positive Definiteness of a Sparse Z-Tensor

As we know, the positive definiteness of a sparse Z-tensor with even order if and only if its minimum H-eigenvalue is positive [18]. Zhang *et al.* [24] show that a Z-tensor is a nonsingular M-tensor if and only if the minimum H-eigenvalue is positive. In this section, we establish the following adequate conditions for checking a nonsingular M-tensor and positive definiteness via Theorems 3.1 and 3.6, Corollaries 3.4 and 3.8.

Theorem 4.1. *Let $\mathcal{A}$ be an even m-order n-dimensional supersymmetric Z-tensor with $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i) \neq \emptyset$. If all $(i,j) \in \{(k,l) : g_{kl} \in \Gamma_{\mathcal{G}(|\mathcal{A}|)}, l \neq k\}$ and $a_{i\ldots i} > 0, i \in N$ such that*

$$\min_{g_{ij} \in \Gamma_{\mathcal{G}(|\mathcal{A}|)}} \Psi_{i,j}(\mathcal{A}) > 0, \tag{4.1}$$

then $\mathcal{A}$ is positive definite and a nonsingular M-tensor.

Proof. It follows from Theorem 3.1 and (4.1) that

$$\tau(\mathcal{A}) \geq \min_{g_{ij} \in \Gamma_{\mathcal{G}(|\mathcal{A}|)}} \Psi_{i,j}(\mathcal{A}) > 0,$$

which implies that $\mathcal{A}$ is positive definite and a nonsingular M-tensor. □

Corollary 4.2. *Let $\mathcal{A}$ be an even m-order n-dimensional supersymmetric Z-tensor. If*

$$\min_{g_{ij} \in \Gamma_{\mathcal{G}(|\mathcal{A}|)} \bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A}) > 0,$$

then $\mathcal{A}$ is positive definite and a nonsingular M-tensor.

Theorem 4.3. *Let $\mathcal{A}$ be an even m-order n-dimensional supersymmetric Z-tensor with $\Gamma_{|\mathring{\mathcal{A}}|}(i) \neq \emptyset$. If all $(i,j) \in \{(k,l) : e_{kl} \in \Gamma_{|\mathring{\mathcal{A}}|}, l \neq k\}$ and $a_{i\ldots i} > 0, i \in N$ such that*

$$\min_{e_{ij} \in \Gamma_{|\mathring{\mathcal{A}}|}} \kappa_{i,j}(\mathcal{A}) > 0,$$

then $\mathcal{A}$ is positive definite and a nonsingular M-tensor.

Proof. Following the similar arguments to the proof of Theorem 4.1, we obtain the desired results □

Corollary 4.4. *Let $\mathcal{A}$ be an even m-order n-dimensional supersymmetric Z-tensor. If*

$$\min_{g_{ij} \in \Gamma_{|\mathring{\mathcal{A}}|} \bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A}) > 0,$$

then $\mathcal{A}$ is positive definite and a nonsingular M-tensor.

In the following, we demonstrate the application of a Z-tensor in the stability of a nonlinear system. It is worthy that Deng [4] established the stability of a nonlinear system based on Lyapunov stability theorem.

Lemma 4.5. *(Theorem 3.4 of [4]) For the nonlinear system* $\sum: \dot{x} = \mathcal{A}_2x + \mathcal{A}_4x^3 + \cdots + \mathcal{A}_{2k}x^{2k-1}$*, if* $-\mathcal{A}_t$ *is positive definite* ($t = 2, 4, \ldots, 2k$)*, then the equilibrium point of* $\sum$ *is asymptotically stable.*

The following example shows that Theorem 4.1 and Corollary 4.4 are more effective in verifying the positive definiteness and the stability of a nonlinear system than existing conclusions.

Example 4.6. A nonlinear system is characterized as follow:

$$\sum : \begin{cases} \dot{x}_1 = -4.1x_1 + x_2 + x_3 + x_4 - 4x_1^3 + x_2^3 + \frac{x_1x_3^2}{2} \\ \dot{x}_2 = x_1 - 4.1x_2 + x_3 + x_4 - 4x_2^3 + 3x_1x_2^2 \\ \dot{x}_3 = x_1 + x_2 - 4.1x_3 + x_4 - x_3^3 + x_4^3 + \frac{x_1^2x_3}{2} \\ \dot{x}_4 = x_1 + x_2 + x_3 - 4.1x_4 - 7x_4^3 + 3x_3x_4^2. \end{cases}$$

Then $\sum$ can be written as $\dot{x} = \mathcal{A}_2x + \mathcal{A}_4x^3$, where $x = (x_1, x_2, x_3, x_4)^\top$,

$$\mathcal{A}_2 = \begin{pmatrix} -4.1 & 1 & 1 & 1 \\ 1 & -4.1 & 1 & 1 \\ 1 & 1 & -4.1 & 1 \\ 1 & 1 & 1 & -4.1 \end{pmatrix}$$

and $-\mathcal{A}_4$ is a 4-order 4-dimensional symmetric Z-tensor with non-zero elements defined as follows:

$$-a_{ijkl} = \begin{cases} a_{1111} = 4; a_{2222} = 4; a_{3333} = 1; a_{4444} = 7; \\ a_{1222} = a_{2221} = a_{2212} = a_{2122} = -1; a_{3444} = a_{4443} = a_{4434} = a_{4344} = -1; \\ a_{1331} = a_{3311} = a_{3113} = a_{1133} = -\frac{1}{4}. \end{cases}$$

Clearly, $-\mathcal{A}_2$ is positive definite. We verify

$$\mathcal{G}(|\mathcal{A}_4|) = \begin{pmatrix} \frac{9}{2} & 1 & \frac{1}{2} & 0 \\ 3 & 7 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} & 1 \\ 0 & 0 & 3 & 10 \end{pmatrix} \quad \text{and} \quad |\mathring{\mathcal{A}}_4| = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix}.$$

By computations, the bounds of the minimum H-eigenvalue in the different literatures are shown:

Table 1: The bounds of the minimum H-eigenvalue in different literatures

References	Bounds
Theorem 2.2 of [8]	$-0.2 \le \tau(\mathcal{A}_4) \le 4$
Theorem 4.5 of [23]	$-0.2 \le \tau(\mathcal{A}_4) \le 4$
Theorem 2.1 of [25]	$-0.2 \le \tau(\mathcal{A}_4) \le 4$
Theorem 3.1	$0.068 \le \tau(\mathcal{A}_4) \le 2$
Corollary 3.8	$0.068 \le \tau(\mathcal{A}_4) \le 4$

From Table 1, we can deduce that $-\mathcal{A}_4$ is positive definite and a nonsingular M-tensor by Theorem 4.1 and Corollary 4.4. Therefore, the equilibrium point of $\sum$ is asymptotically stable. However, Theorem 2.1 of [8], Theorem 4.5 of [23] and Theorem 2.1 of [25] cannot identify the positiveness of $-\mathcal{A}_4$ and cannot verify the asymptotical stability of $\sum$.

$\boxed{5}$ Conclusions

In this paper, we introduced majorization matrix's digraph and representation matrix's digraph of a sparse tensor. We established the improved bounds of the minimum H-eigenvalue of a sparse Z-tensor by the information of $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i)$ and $\Gamma_{|\mathring{\mathcal{A}}|}(i)$. Meanwhile, two sufficient conditions were proposed to check the positive definiteness of an even-order real supersymmetric tensor, as well as a nonsingular M-tensor. Further studies can be considered to develop certain algorithms for computing the minimum H-eigenvalue of sparse Z-tensors.

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Xiuyun Feng
School of Management Science, Qufu Normal University
Rizhao Shandong, 276800, China
E-mail address: fengxy9895@163.com

Gang Wang
School of Management Science, Qufu Normal University
Rizhao Shandong, 276800, China
E-mail address: wgglj1977@163.com

Yiju Wang
School of Management Science, Qufu Normal University
Rizhao Shandong, 276800, China
E-mail address: wyijumail@163.com