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# POSITIVE DEFINITENESS OF SIXTH-ORDER PAIRED SYMMETRIC CAUCHY TENSORS

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**Abstract:** In this paper, we consider a sixth-order paired symmetric Cauchy tensor and its generating vectors. First, we investigate the conditions for positive definiteness and positive semidefiniteness of the sixth-order paired symmetric Cauchy tensor. Necessary and sufficient conditions for its positive definiteness are provided based on the structural characteristics of the tensor. Subsequently, we apply the concept of the M-eigenvalue to the sixth-order paired symmetric Cauchy tensor, and further discuss related properties. We also provide three M-eigenvalue inclusion intervals for the sixth-order paired symmetric Cauchy tensor, which give three upper bounds for its M-spectral radius.

**Key words:** *paired symmetric tensor, Cauchy tensor, positive definiteness, M-eigenvalue*

**Mathematics Subject Classification:** *15A06, 15A18, 15A69, 65H17*

## 1 Introduction

As we know, tensors of order three or higher are called higher-order tensors in mathematics. In recent years, increasing attention has been paid to these higher-order tensors, especially to higher-order structural tensors [4, 26, 29, 30, 35], such as the $M$ tensor, $B$ tensor, $P$ tensor, Positive tensor, Hilbert tensor, and Cauchy tensor. Moreover, there exists a special kind of tensor, which we call the paired symmetric tensor [15].

For any positive integers $m$ and $n$, we denote an arbitrary $2m$-th order $n$ dimensional tensor with $\mathcal{B} = (b_{p_1q_1p_2q_2\cdots p_mq_m})$. Clearly, we can divide its indices into $m$ adjacent blocks $\{p_1q_1\}, \cdots, \{p_mq_m\}$. $\mathcal{B}$ is called a $2m$-th order $n$ dimensional paired symmetric tensor if its entries possess minor symmetries:

$$b_{p_1q_1p_2q_2\cdots p_mq_m} = b_{q_1p_1p_2q_2\cdots p_mq_m} = b_{p_1q_1q_2p_2\cdots p_mq_m} = \cdots = b_{p_1q_1p_2q_2\cdots q_mp_m}.$$

In other words, the entries of $\mathcal{B}$ are invariant under any permutation of indices in every block $\{p_iq_i\}$ for $i \in [m]$. We call $\mathcal{B}$ a $2m$-th order $n$ dimensional strongly paired symmetric tensor if its entries possess major symmetries:

$$b_{p_1q_1p_2q_2\cdots p_mq_m} = b_{p_2q_2p_1q_1\cdots p_mq_m} = b_{p_3q_3p_2q_2\cdots p_mq_m} = \cdots = b_{p_mq_mp_2q_2\cdots p_1q_1},$$

that is, the entries of $\mathcal{B}$ are invariant under any permutation of every block. Clearly, both paired symmetric tensor and strong paired symmetric tensor are partially symmetric tensors.

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At present, a typical example of paired symmetric tensor is elastic tensor [9, 11]. The elastic constants of the elastic tensor possess minor and major symmetries. There have been numerous studies on the fourth-order elastic tensors [10, 13, 17, 18, 24, 28, 33, 36]. The positive definiteness of elastic tensors and higher-order elastic tensors is called strong ellipticity. The strong ellipticity property of equilibrium equations is tremendously significant in elasticity theory and has been thoroughly investigated [6, 7, 9, 10, 11, 12, 13, 16, 27, 31, 32, 36]. Owing to its application to higher-order elastic tensors and other domains, the positive definiteness of elastic tensors has been an essential topic in discussing the properties of paired symmetric tensors. However, little research has been done on sixth-order paired symmetric tensors.

According to references [9, 13, 15, 8, 20, 21] and other pertinent sources, research on sixth-order paired symmetric tensors is situated at the intersection of advanced mathematical analysis and applied physics. Theoretical studies have demonstrated that sixth-order paired symmetric tensors exhibit unique symmetries akin to fourth-order elastic tensors, enabling the extension of fundamental properties and determinacy conditions from the fourth to the sixth order. The practical significance of these tensors lies in their application to model the mechanical behavior of transversely isotropic elastic materials. In contrast to fourth-order models, sixth-order tensors offer a more refined representation, intricately linking each component to the six components of stress and strain. This heightened modeling capability is crucial for comprehending and predicting the intricate responses of materials in engineering applications. Furthermore, the eigenvalues of sixth-order paired symmetric tensors are closely linked to their positive definiteness, providing a valuable avenue for determining physical properties. This relationship facilitates the exploration of the tensors' positive definiteness, offering insights into the behavior of materials under different conditions. The broader context of the study involves the extensive application of high-order tensors in various scientific and engineering domains. As computational capabilities improve, the exploration of higher-order tensors becomes increasingly feasible, opening new possibilities for sophisticated modeling and analysis. In essence, research on sixth-order paired symmetric tensors contributes to both theoretical intricacies and practical challenges, enriching tensor analysis and its applications in materials science and engineering.

Motivated by the Cauchy tensor [2, 5, 25] and sixth-order paired symmetric tensor [15], we discuss a sixth-order Cauchy tensor that satisfies the paired symmetry property. While previous studies on paired symmetric tensors focused largely on fourth-order elastic tensors, we expand the scope to higher-order tensors by investigating sixth-order paired symmetric Cauchy tensors. Because sixth-order Cauchy tensors possess symmetries similar to those of fourth-order elastic tensors, we extend conditions for the positive definiteness of elastic tensors to characterize the properties of the sixth-order paired symmetric Cauchy tensor constructed here. Furthermore, the relationship we establish between the positive semidefiniteness of these sixth-order Cauchy tensors and the monotonicity of their associated polynomials could facilitate future investigations into analogous higher-order symmetric configurations.

On the other hand, higher-order tensor eigenvalues proposed by Lim [19] and Qi [22], have been an active area of research in recent years [1, 14, 19, 22, 23, 25]. In [10, 24], the authors introduced the concept of M-eigenvalue for a fourth-order partially symmetric tensor. In [24], the authors discussed strong ellipticity condition via M-eigenvalues. In [18], the authors gave sufficient conditions for the M-positive definiteness of partially symmetric tensors. In [3, 12, 17], the authors studied M-eigenvalue inclusion intervals for a fourth-order partially symmetric tensor. In [34], Wang, Qi and Zhang propose a practical approach to determine the largest M-eigenvalue of a fourth-order partially symmetric tensor. In [15], the concept of M-eigenvalue was further discussed. The authors extended this concept and demonstrated the relationship between the M-eigenvalues of sixth-order tensors and their positive definiteness. While most research on tensor eigenvalues has focused primarily on fourth-order tensors, this work discusses the M-eigenvalue problem of a higher-order symmetric tensor. We apply the concept of the M-eigenvalue to the sixth-order Cauchy tensor, and further discuss related properties. We give three M-eigenvalue inclusion intervals for a sixth-order Cauchy tensor, which provide upper bound for the M-spectral radius. Additionally, we discussed the inclusion relationships between the intervals through two numerical examples.

## 2 Preliminaries

In this paper, vectors are denoted by lowercase boldface letters, such as $\mathbf{x}$, $\mathbf{y}$, and tensors are denoted by calligraphic capitals, such as $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. For any $\mathbf{x} = (x_1, x_2, \cdots, x_n)^\top$, $\mathbf{x} \geq (>)0$ signifies $x_i \geq (>)0$ for all $i \in [n]$, where $[n] = \{1, 2, \cdots, n\}$ for any positive integer $n$. For any $\mathbf{x} = (x_1, x_2, \cdots, x_n)^\top$, $\mathbf{y} = (y_1, y_2, \cdots, y_n)^\top$, $\mathbf{x} \geq \mathbf{y}$ ($\mathbf{x} \leq \mathbf{y}$) means $x_i \geq y_i$ ($x_i \leq y_i$ ) for all $i \in [n]$. Let $\mathbb{R}$ represent the set of reals, $\mathbb{R}^n$ signify the $n$ dimensional real Euclidean space, and $\mathbb{R}^n_+$ denote the set of $n$ dimensional nonnegative vectors. We use $\mathbb{T}_{m,n}$ to signify the set of $m$th-order $n$ dimensional real tensors.

**Definition 2.1** ([15])**.** For any $\mathcal{B} = (b_{ijklpq}) \in \mathbb{T}_{6,3}$, if its entries satisfy

$$b_{ijklpq} = b_{jiklpq} = b_{ijlkpq} = b_{ijklqp} \quad \forall\, i, j, k, l, p, q \in [3], \tag{2.1}$$

we call $\mathcal{B}$ a paired symmetric tensor. Furthermore, we call the paired symmetric tensor $\mathcal{B} \in \mathbb{T}_{6,3}$ sixth-order elasticity tensor [11] if its entries satisfy

$$b_{ijklpq} = b_{klijpq} = b_{ijpqkl} \quad \forall\, i, j, k, l, p, q \in [3]. \tag{2.2}$$

For any tensor $\mathcal{B} = (b_{ijklpq}) \in \mathbb{T}_{6,3}$, the corresponding homogeneous polynomial can be defined as

$$\mathcal{B}\mathbf{x}^2\mathbf{y}^2\mathbf{z}^2 = \sum_{i,j,k,l,p,q=1}^{3} b_{ijklpq} x_i x_j y_k y_l z_p z_q \quad \forall\, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3. \tag{2.3}$$

**Definition 2.2** ([25])**.** Suppose that a real tensor $\mathcal{C} = (c_{j_1 \cdots j_m})$ is defined by

$$c_{j_1 \cdots j_m} = \frac{1}{c_{j_1} + c_{j_2} + \cdots + c_{j_m}} \quad j_k \in [n],\ k \in [m],$$

where vector $\mathbf{c} = (c_1, c_2, \cdots, c_n)^\top \in \mathbb{R}^n$. Then, we call $\mathcal{C}$ an $m$th-order $n$ dimensional symmetric Cauchy tensor, the vector $\mathbf{c} \in \mathbb{R}^n$ is called the generating vector of $\mathcal{C}$.

From the definition of the Cauchy tensor [25] and fourth-order Cauchy tensor [2], the sixth-order Cauchy tensor can be represented as follows,

**Definition 2.3.** Suppose that a real tensor $\mathcal{C} = (c_{ijklpq})$ is defined by

$$c_{ijklpq} = \frac{1}{a_i + a_j + b_k + b_l + c_p + c_q} \quad i, j \in [m],\ k, l \in [n],\ p, q \in [s],$$

where the vector $\mathbf{a} = (a_1, a_2, \cdots, a_m)^\top \in \mathbb{R}^m$, $\mathbf{b} = (b_1, b_2, \cdots, b_n)^\top \in \mathbb{R}^n$ and $\mathbf{c} = (c_1, c_2, \cdots, c_s)^\top \in \mathbb{R}^s$. Then, we say $\mathcal{C}$ a sixth-order Cauchy tensor, and the vectors $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{c} \in \mathbb{R}^s$ are called the generating vectors of $\mathcal{C}$.

Obviously, the generating vectors $\mathbf{a} \in \mathbb{R}^m, \mathbf{b} \in \mathbb{R}^n$ and $\mathbf{c} \in \mathbb{R}^s$ should satisfy

$$a_i + a_j + b_k + b_l + c_p + c_q \neq 0, \quad i, j \in [m],\ k, l \in [n],\ p, q \in [s].$$

The entries of this sixth-order Cauchy tensor satisfy

$$c_{ijklpq} = c_{jiklpq} = c_{ijlkpq} = c_{ijklqp} = \frac{1}{a_i + a_j + b_k + b_l + c_p + c_q} \tag{2.4}$$

and

$$c_{ijklpq} = c_{klijpq} = c_{ijpqkl} = \frac{1}{a_i + a_j + b_k + b_l + c_p + c_q}, \tag{2.5}$$

where $i, j \in [m],\ k, l \in [n],\ p, q \in [s]$. Thus, it is also a strong paired symmetric tensor.

Furthermore, if $\mathbf{a} = \mathbf{b} = \mathbf{c}$ and $m = n = s$, the sixth-order paired symmetric Cauchy tensor reduces to the sixth-order symmetric Cauchy tensor. In the rest of this paper, we always consider the sixth-order real paired symmetric Cauchy tensor. Therefore, we call them sixth-order Cauchy tensor for simplicity.

## 3 Positive Definiteness of Sixth-Order Paired Symmetric Cauchy Tensor

In this section, we consider the positive definiteness of the sixth-order paired symmetric Cauchy tensor. We propose several necessary and sufficient conditions under which the concerned tensors are positive definite.

For any Cauchy tensor $\mathcal{C} = (c_{ijklpq})$, the corresponding homogeneous polynomial is defined by

$$\begin{aligned} f(\mathbf{x}, \mathbf{y}, \mathbf{z}) =& \mathcal{C}\mathbf{x}^2\mathbf{y}^2\mathbf{z}^2 \\ =& \sum_{i,j\in[m],k,l\in[n],p,q\in[s]} c_{ijklpq} x_i x_j y_k y_l z_p z_q \\ =& \sum_{i,j\in[m],k,l\in[n],p,q\in[s]} \frac{x_i x_j y_k y_l z_p z_q}{a_i + a_j + b_k + b_l + c_p + c_q}, \\ & \forall\, \mathbf{x} \in \mathbb{R}^m,\ \mathbf{y} \in \mathbb{R}^n \text{ and } \mathbf{z} \in \mathbb{R}^s. \end{aligned} \tag{3.1}$$

**Definition 3.1** ([25])**.** For any vectors $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{z} \in \mathbb{R}^s$, the tensor $\mathcal{C}$ is called positive semidefinite if $f(\mathbf{x}, \mathbf{y}, \mathbf{z}) \geq 0$. For any vectors $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{x} \neq 0$, $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} \neq 0$ and $\mathbf{z} \in \mathbb{R}^s$, $\mathbf{z} \neq 0$, the tensor $\mathcal{C}$ is called positive definite if $f(\mathbf{x}, \mathbf{y}, \mathbf{z}) > 0$. Similarly, for any vectors $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{z} \in \mathbb{R}^s$, the tensor $\mathcal{C}$ is called negative semidefinite if $f(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq 0$. For any vectors $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{x} \neq 0$, $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} \neq 0$ and $\mathbf{z} \in \mathbb{R}^s$, $\mathbf{z} \neq 0$, the tensor $\mathcal{C}$ is called negative definite if $f(\mathbf{x}, \mathbf{y}, \mathbf{z}) < 0$.

First, we give a necessary and sufficient condition under which the concerned tensor is positive semidefinite.

**Theorem 3.2.** *Let vectors $\boldsymbol{a} \in \mathbb{R}^m$, $\boldsymbol{b} \in \mathbb{R}^n$ and $\mathbf{c} \in \mathbb{R}^s$ be the vectors that generate the sixth-order Cauchy tensor $\mathcal{C}$. Then, the tensor $\mathcal{C}$ is positive semidefinite if and only if $a_i + b_k + c_p > 0$ for all $i \in [m]$, $k \in [n]$, $p \in [s]$.*

*Proof.* First, we assume that $a_i + b_k + c_p > 0$ for all $i \in [m]$, $k \in [n]$, $p \in [s]$. For any $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$,

$\mathbf{z} \in \mathbb{R}^s$, we have

$$\begin{aligned}
f(\mathbf{x},\mathbf{y},\mathbf{z}) =& \mathcal{C}\mathbf{x}^2\mathbf{y}^2\mathbf{z}^2 \\
=& \sum_{i,j\in[m],k,l\in[n],p,q\in[s]} c_{ijklpq} x_i x_j y_k y_l z_p z_q \\
=& \sum_{i,j\in[m],k,l\in[n],p,q\in[s]} \frac{x_i x_j y_k y_l z_p z_q}{a_i + a_j + b_k + b_l + c_p + c_q} \\
=& \sum_{i,j\in[m],k,l\in[n],p,q\in[s]} \int_0^1 t^{a_i+a_j+b_k+b_l+c_p+c_q-1} x_i x_j y_k y_l z_p z_q dt \\
=& \int_0^1 (\sum_{i\in[m]} t^{a_i-\frac{1}{6}} x_i)^2 (\sum_{k\in[n]} t^{b_k-\frac{1}{6}} y_k)^2 (\sum_{p\in[s]} t^{c_p-\frac{1}{6}} z_p)^2 dt \\
\geq& 0,
\end{aligned}$$

which means that the tensor $\mathcal{C}$ is positive semidefinite.

Next, let us assume the sixth-order Cauchy tensor $\mathcal{C}$ is positive semidefinite. Taking $\mathbf{x} = \mathbf{e}_i \in \mathbb{R}^m$, $\mathbf{y} = \mathbf{e}_k \in \mathbb{R}^n$, $\mathbf{z} = \mathbf{e}_p \in \mathbb{R}^s$, then,

$$f(\mathbf{e}_i, \mathbf{e}_k, \mathbf{e}_p) = \mathcal{C}\mathbf{e}_i^2\mathbf{e}_k^2\mathbf{e}_p^2 = \frac{1}{2(a_i + b_k + c_p)} \geq 0, \quad i \in [m],\ k \in [n],\ p \in [s].$$

where $\mathbf{e}_i$, $\mathbf{e}_k$ and $\mathbf{e}_p$ are the $i$th, $k$th and $p$th coordinate vectors, respectively. Obviously, we have $a_i + b_k + c_p > 0$ for all $i \in [m],\ k \in [n],\ p \in [s]$. □

Next, we give a necessary and sufficient condition for the sixth-order Cauchy tensor $\mathcal{C}$ to be positive definite.

**Theorem 3.3.** *Let vectors $\boldsymbol{a} \in \mathbb{R}^m$, $\boldsymbol{b} \in \mathbb{R}^n$ and $\boldsymbol{c} \in \mathbb{R}^s$ be the vectors that generate the sixth-order Cauchy tensor $\mathcal{C}$. Then, the tensor $\mathcal{C}$ is positive definite if and only if $a_i + b_k + c_p > 0$ for all $i \in [m]$, $k \in [n]$, $p \in [s]$, and the elements of generating vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are mutually distinct.*

*Proof.* First, suppose that the tensor $\mathcal{C}$ is positive definite. Thereby, the tensor $\mathcal{C}$ is positive semidefinite. By Theorem 3.2, $a_i + b_k + c_p > 0$ for all $i \in [m]$, $k \in [n]$, $p \in [s]$. Without loss of generality, we assume that two elements of the vector $\mathbf{a}$ are equal, and it can be assumed that $a_1 = a_2 = \tilde{a}$.

Let $\mathbf{x} = (1, -1, 0, \cdots, 0)^\top \in \mathbb{R}^m$, $\mathbf{y} = (1, 0, 0, \cdots, 0)^\top \in \mathbb{R}^n$, and $\mathbf{z} = (0, 1, 0, \cdots, 0)^\top \in \mathbb{R}^s$. We have:

$$\begin{aligned}
& \mathcal{C}\mathbf{x}^2\mathbf{y}^2\mathbf{z}^2 \\
=& \sum_{i,j\in[m],k,l\in[n],p,q\in[s]} c_{ijklpq} x_i x_j y_k y_l z_p z_q \\
=& \sum_{i,j\in[m],k,l\in[n],p,q\in[s]} \frac{x_i x_j y_k y_l z_p z_q}{a_i + a_j + b_k + b_l + c_p + c_q} \\
=& \frac{1}{2(\tilde{a} + b_1 + c_2)} (x_1^2 y_1^2 z_2^2 + x_1 x_2 y_1^2 z_2^2 + x_1 x_2 y_1^2 z_2^2 + x_2^2 y_1^2 z_2^2) \\
=& \frac{1}{2(\tilde{a} + b_1 + c_2)} [1 + (-1) + (-1) + 1] \\
=& 0.
\end{aligned}$$

This is in contradiction with the assumption that the tensor $\mathcal{C}$ is positive definite. Thus, the elements of generating vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are mutually distinct.

Next, we assume $a_i + b_k + c_p > 0$ for all $i \in [m]$, $k \in [n]$, $p \in [s]$, and the elements of generating vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are mutually distinct. From Theorem 3.2, the sixth-order Cauchy tensor $\mathcal{C}$ is positive semidefinite. Assume that there exists nonzero vectors $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$, and $\mathbf{z} \in \mathbb{R}^s$ such that

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathcal{C}\mathbf{x}^2\mathbf{y}^2\mathbf{z}^2 = 0$$

From the proof of Theorem 3.2, we can obtain:

$$\int_0^1 (\sum_{i\in[m]} t^{a_i-\frac{1}{6}} x_i)^2 (\sum_{k\in[n]} t^{b_k-\frac{1}{6}} y_k)^2 (\sum_{p\in[s]} t^{c_p-\frac{1}{6}} z_p)^2 dt = 0,$$

which implies

$$\sum_{i\in[m]} t^{a_i-\frac{1}{6}} x_i = 0 \quad t \in (0,1],$$

or

$$\sum_{k\in[n]} t^{b_k-\frac{1}{6}} y_k = 0 \quad t \in (0,1],$$

or

$$\sum_{p\in[s]} t^{c_p-\frac{1}{6}} z_p = 0 \quad t \in (0,1].$$

Without losing generality, we assume that:

$$\sum_{i\in[m]} t^{a_i-\frac{1}{6}} x_i = 0 \quad t \in (0,1],$$

that is,

$$t^{a_1-\frac{1}{6}} x_1 + t^{a_2-\frac{1}{6}} x_2 + \cdots + t^{a_m-\frac{1}{6}} x_m \equiv 0, \quad t \in (0,1].$$

Thus,

$$x_1 + t^{a_2-a_1} x_2 + \cdots + t^{a_m-a_1} x_m \equiv 0, \quad t \in (0,1].$$

Due to continuity and the condition that all components of $\mathbf{a}$ are mutually distinct, it follows readily that $x_1 = 0$. Then, we have

$$x_2 + t^{a_3-a_2} x_3 + \cdots + t^{a_m-a_2} x_m \equiv 0, \quad t \in (0,1],$$

which implies $x_2 = 0$.

By repeating this process, we can gradually obtain:

$$x_1 = x_2 = \cdots = x_m = 0,$$

this contradicts $\mathbf{x} \neq 0$. Hence, the sixth-order Cauchy tensor $\mathcal{C}$ is positive definite. Thus, the conclusion is established, and the proof is complete. $\square$

Next, we show the relationship between the positive definite of the sixth-order Cauchy tensor and the monotonicity of a homogeneous polynomial with respect to the proposed Cauchy tensor. First, we express the definition of the monotonicity of a homogeneous polynomial with respect to the sixth-order Cauchy tensor.

**Definition 3.4.** For any $\mathbf{x}$, $\hat{\mathbf{x}} \in \mathbb{R}^m$, $\mathbf{y}$, $\hat{\mathbf{y}} \in \mathbb{R}^n$, and $\mathbf{z}$, $\hat{\mathbf{z}} \in \mathbb{R}^s$, $f(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is called monotonically increasing (or monotonically decreasing) if $f(\mathbf{x}, \mathbf{y}, \mathbf{z}) \geq f(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ when $\mathbf{x} \geq \hat{\mathbf{x}}$, $\mathbf{y} \geq \hat{\mathbf{y}}$, and $\mathbf{z} \geq \hat{\mathbf{z}}$ (or $\mathbf{x} \leq \hat{\mathbf{x}}$, $\mathbf{y} \leq \hat{\mathbf{y}}$, and $\mathbf{z} \leq \hat{\mathbf{z}}$). $f(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is called strictly monotonically increasing (or strictly monotonically decreasing) if $f(\mathbf{x}, \mathbf{y}, \mathbf{z}) > f(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ when $\mathbf{x} \geq \hat{\mathbf{x}}$, $\mathbf{x} \neq \hat{\mathbf{x}}$, $\mathbf{y} \geq \hat{\mathbf{y}}$, $\mathbf{y} \neq \hat{\mathbf{y}}$ and $\mathbf{z} \geq \hat{\mathbf{z}}$, $\mathbf{z} \neq \hat{\mathbf{z}}$ (or $\mathbf{x} \leq \hat{\mathbf{x}}$, $\mathbf{x} \neq \hat{\mathbf{x}}$, $\mathbf{y} \leq \hat{\mathbf{y}}$, $\mathbf{y} \neq \hat{\mathbf{y}}$ and $\mathbf{z} \leq \hat{\mathbf{z}}$, $\mathbf{z} \neq \hat{\mathbf{z}}$).

**Theorem 3.5.** *Let $\boldsymbol{a} \in \mathbb{R}^m$, $\boldsymbol{b} \in \mathbb{R}^n$ and $\boldsymbol{c} \in \mathbb{R}^s$ be the vectors that generate the sixth-order Cauchy tensor $\mathcal{C}$. Then, the sixth-order Cauchy $\mathcal{C}$ is positive semidefinite if and only if the homogeneous polynomial $f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is monotonically increasing in $\mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^s_+$.*

*Proof.* First, when the tensor $\mathcal{C}$ is positive semidefinite, suppose $\mathbf{x}$, $\hat{\mathbf{x}} \in \mathbb{R}^m_+$, $\mathbf{y}$, $\hat{\mathbf{y}} \in \mathbb{R}^n_+$, $\mathbf{z}$, $\hat{\mathbf{z}} \in \mathbb{R}^s_+$, and $\mathbf{x} \geq \hat{\mathbf{x}}$, $\mathbf{y} \geq \hat{\mathbf{y}}$, $\mathbf{z} \geq \hat{\mathbf{z}}$. It can be obtained from Theorem 3.2 that $a_i + b_k + c_p > 0$ for all $i \in [m]$, $k \in [n]$, $p \in [s]$. Furthermore,

$$\begin{aligned}
& f(\mathbf{x}, \mathbf{y}, \mathbf{z}) - f(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \\
=& \mathcal{C}\mathbf{x}^2\mathbf{y}^2\mathbf{z}^2 - \mathcal{C}\hat{\mathbf{x}}^2\hat{\mathbf{y}}^2\hat{\mathbf{z}}^2 \\
=& \sum_{i,j\in[m],k,l\in[n],p,q\in[s]} \frac{1}{a_i + a_j + b_k + b_l + c_p + c_q}(x_i x_j y_k y_l z_p z_q - \hat{x}_i \hat{x}_j \hat{y}_k \hat{y}_l \hat{z}_p \hat{z}_q) \\
\geq& 0,
\end{aligned}$$

which implies that $f(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is monotonically increasing in $\mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^s_+$.

On the other hand, if the homogeneous polynomial $f(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is monotonically increasing in $\mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^s_+$. Let $\mathbf{x} = \mathbf{e}_i \in \mathbb{R}^m_+$, $\hat{\mathbf{x}} = 0 \in \mathbb{R}^m_+$, $\mathbf{y} = \mathbf{e}_k \in \mathbb{R}^n_+$, $\hat{\mathbf{y}} = 0 \in \mathbb{R}^n_+$, and $\mathbf{z} = \mathbf{e}_p \in \mathbb{R}^s_+$, $\hat{\mathbf{z}} = 0 \in \mathbb{R}^s_+$, we have

$$\frac{1}{2(a_i + b_k + c_p)} = \mathcal{C}\mathbf{xxyyzz} = f(\mathbf{x}, \mathbf{y}, \mathbf{z}) \geq f(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}) = \mathcal{C}\hat{\mathbf{x}}\hat{\mathbf{x}}\hat{\mathbf{y}}\hat{\mathbf{y}}\hat{\mathbf{z}}\hat{\mathbf{z}} = 0,$$

which implies that $a_i + b_k + c_p > 0$ for all $i \in [m]$, $k \in [n]$, $p \in [s]$. By Theorem 3.2, the tensor $\mathcal{C}$ is positive semidefinite and the proof is complete. $\square$

**Theorem 3.6.** *Let $\boldsymbol{a} \in \mathbb{R}^m$, $\boldsymbol{b} \in \mathbb{R}^n$ and $\boldsymbol{c} \in \mathbb{R}^s$ be the vectors that generate the sixth-order Cauchy tensor $\mathcal{C}$. If the tensor $\mathcal{C}$ is positive definite, then the homogeneous polynomial $f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is strictly monotonically increasing in $\mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^s_+$.*

*Proof.* Suppose the tensor $\mathcal{C}$ is positive definite. From Theorem 3.3, we have $a_i + b_k + c_p > 0$ for all $i \in [m]$, $k \in [n]$, $p \in [s]$. For any $\mathbf{x} \geq \hat{\mathbf{x}}$, $\mathbf{x} \neq \hat{\mathbf{x}}$, $\mathbf{x} \in \mathbb{R}^m_+$, $\mathbf{y} \geq \hat{\mathbf{y}}$, $\mathbf{y} \neq \hat{\mathbf{y}}$, $\mathbf{y} \in \mathbb{R}^n_+$ and $\mathbf{z} \geq \hat{\mathbf{z}}$, $\mathbf{z} \neq \hat{\mathbf{z}}$, $\mathbf{z} \in \mathbb{R}^s_+$, there exist indexes $i_0 \in [m]$, $k_0 \in [n]$, $p_0 \in [s]$ such that $x_{i_0} > \hat{x}_{i_0} \geq 0$, $y_{k_0} > \hat{y}_{k_0} \geq 0$ and $z_{p_0} > \hat{z}_{p_0} \geq 0$.

Then,

$$\begin{aligned}
&f(\mathbf{x},\mathbf{y},\mathbf{z})-f(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})\\
=&\mathcal{C}\mathbf{x}^2\mathbf{y}^2\mathbf{z}^2-\mathcal{C}\hat{\mathbf{x}}^2\hat{\mathbf{y}}^2\hat{\mathbf{z}}^2\\
=&\sum_{\substack{i,j\in[m],k,l\in[n],p,q\in[s]\\(i,j,k,l,p,q)\neq(i_0,i_0,k_0,k_0,p_0,p_0)}} c_{ijklpq}(x_ix_jy_ky_lz_pz_q-\hat{x}_i\hat{x}_j\hat{y}_k\hat{y}_l\hat{z}_p\hat{z}_q)\\
&+c_{i_0i_0k_0k_0p_0p_0}(x_{i_0}^2y_{k_0}^2z_{p_0}^2-\hat{x}_{i_0}^2\hat{y}_{k_0}^2\hat{z}_{p_0}^2)\\
=&\sum_{\substack{i,j\in[m],k,l\in[n],p,q\in[s]\\(i,j,k,l,p,q)\neq(i_0,i_0,k_0,k_0,p_0,p_0)}} \frac{1}{a_i+a_j+b_k+b_l+c_p+c_q}(x_ix_jy_ky_lz_pz_q-\hat{x}_i\hat{x}_j\hat{y}_k\hat{y}_l\hat{z}_p\hat{z}_q)\\
&+\frac{1}{2(a_{i_0}+b_{k_0}+c_{p_0})}(x_{i_0}^2y_{k_0}^2z_{p_0}^2-\hat{x}_{i_0}^2\hat{y}_{k_0}^2\hat{z}_{p_0}^2)\\
>&0.
\end{aligned}$$

That is, the homogeneous polynomial $f(\mathbf{x},\mathbf{y},\mathbf{z})$ is strictly monotonically increasing in $\mathbb{R}^m_+\times\mathbb{R}^n_+\times\mathbb{R}^s_+$. □

Next, we propose an example to show the strictly monotonically increasing property of the polynomial $f(\mathbf{x},\mathbf{y},\mathbf{z})$ is not a sufficient condition for the positive definiteness of the sixth-order Cauchy tensor.

**Example 3.7.** Let $\mathcal{A}$ be a sixth-order Cauchy tensor with generating vectors $\mathbf{a}=(3,3,3)^\top$, $\mathbf{b}=(6,6,6,6)^\top$ and $\mathbf{c}=(9,9,9,9)^\top$. Then, we have

$$a_{ijklpq}=\frac{1}{2(3+6+9)}=\frac{1}{36},\quad i,j\in[3],k,l,p,q\in[4]$$

and the homogeneous polynomial

$$f(\mathbf{x},\mathbf{y},\mathbf{z})=\mathcal{A}\mathbf{x}^2\mathbf{y}^2\mathbf{z}^2=\frac{1}{36}\sum_{i,j\in[3],k,l\in[4],p,q\in[4]}x_ix_jy_ky_lz_pz_q.$$

It can be readily verified that the polynomial $f(\mathbf{x},\mathbf{y},\mathbf{z})$ is strictly monotonically increasing in $\mathbb{R}^3_+\times\mathbb{R}^4_+\times\mathbb{R}^4_+$. However, according to Theorem 3.3, the tensor $\mathcal{A}$ does not qualify as positive definite.

## 4 M-Eigenvalue and M-Eigenvalue Inclusion Intervals

In this section, we apply the concept of M-eigenvalue for the sixth-order three dimensional paired symmetric tensor introduced in Huang and Qi [15] to the sixth-order Cauchy tensor $\mathcal{C}$ with generating vectors $\mathbf{a}\in\mathbb{R}^m,\mathbf{b}\in\mathbb{R}^n$ and $\mathbf{c}\in\mathbb{R}^s$. Then we discuss several related properties.

For any sixth-order Cauchy tensor $\mathcal{C}$, the corresponding homogeneous polynomial is given in (3.1). For any $\mathbf{x}\in\mathbb{R}^m$, $\mathbf{y}\in\mathbb{R}^n$ and $\mathbf{z}\in\mathbb{R}^s$, We can represent $\mathcal{C}\mathbf{x}\mathbf{y}\mathbf{y}\mathbf{z}\mathbf{z}$, $\mathcal{C}\mathbf{x}\mathbf{x}\mathbf{y}\mathbf{z}\mathbf{z}$, $\mathcal{C}\mathbf{x}\mathbf{x}\mathbf{y}\mathbf{y}\mathbf{z}$

as follows

$$\begin{aligned}(\mathcal{C}\mathbf{xyyzz})_i &:= \sum_{j\in[m],k,l\in[n],p,q\in[s]} c_{ijklpq}x_jy_ky_lz_pz_q, \quad \forall i\in[m],\\ (\mathcal{C}\mathbf{xxyzz})_k &:= \sum_{i,j\in[m],l\in[n],p,q\in[s]} c_{ijklpq}x_ix_jy_lz_pz_q, \quad \forall k\in[n],\\ (\mathcal{C}\mathbf{xxyyz})_p &:= \sum_{i,j\in[m],k,l\in[n],q\in[s]} c_{ijklpq}x_ix_jy_ky_lz_q, \quad \forall p\in[s].\end{aligned} \tag{4.1}$$

Obviously, for any $\mathbf{x}\in\mathbb{R}^m$, $\mathbf{y}\in\mathbb{R}^n$ and $\mathbf{z}\in\mathbb{R}^s$, we have

$$\langle \mathbf{x}, \mathcal{C}\mathbf{xyyzz}\rangle = \mathcal{C}\mathbf{xxyyzz}, \quad \langle \mathbf{y}, \mathcal{C}\mathbf{xxyzz}\rangle = \mathcal{C}\mathbf{xxyyzz}, \quad \langle \mathbf{z}, \mathcal{C}\mathbf{xxyyz}\rangle = \mathcal{C}\mathbf{xxyyzz}.$$

**Definition 4.1** ([15])**.** For any sixth-order Cauchy tensor $\mathcal{C}$, if there exist $\mathbf{x}\in\mathbb{R}^m$, $\mathbf{y}\in\mathbb{R}^n$ and $\mathbf{z}\in\mathbb{R}^s$ and $\lambda\in\mathbb{R}$ such that

$$\begin{cases}\mathcal{C}\mathbf{xyyzz}=\lambda\mathbf{x},\\ \mathcal{C}\mathbf{xxyzz}=\lambda\mathbf{y},\\ \mathcal{C}\mathbf{xxyyz}=\lambda\mathbf{z},\\ \mathbf{x}^\top\mathbf{x}=1,\\ \mathbf{y}^\top\mathbf{y}=1,\\ \mathbf{z}^\top\mathbf{z}=1,\end{cases} \tag{4.2}$$

where $(\mathcal{C}\mathbf{xyyzz})_i$, $(\mathcal{C}\mathbf{xxyzz})_k$, and $(\mathcal{C}\mathbf{xxyyz})_p$ are defined in (4.1). Then, we call $\lambda$ an M-eigenvalue of $\mathcal{C}$, and $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are the eigenvectors of $\mathcal{C}$ associated with $\lambda$.

According to Theorem 3.2 in reference [15], we know that for any sixth-order Cauchy tensor $\mathcal{C}$ generated by the vectors $\mathbf{a}\in\mathbb{R}^m$, $\mathbf{b}\in\mathbb{R}^n$ and $\mathbf{c}\in\mathbb{R}^s$, its M-eigenvalue always exists. Moreover, if $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are the eigenvectors of $\mathcal{C}$ associated with the M-eigenvalue $\lambda$, then, $\lambda=\mathcal{C}\mathbf{xxyyzz}$.

Furthermore, we also know that a sixth-order Cauchy tensor $\mathcal{C}$ with generating vectors $\mathbf{a}\in\mathbb{R}^m$, $\mathbf{b}\in\mathbb{R}^n$ and $\mathbf{c}\in\mathbb{R}^s$ is positive definite if and only if the smallest M-eigenvalue of $\mathcal{C}$ is positive.

**Theorem 4.2.** *Let $\mathcal{C}$ be a sixth-order Cauchy tensor with generating vectors $\boldsymbol{a}\in\mathbb{R}^m$, $\boldsymbol{b}\in\mathbb{R}^n$ and $\mathbf{c}\in\mathbb{R}^s$, and the elements of $\boldsymbol{a}$, $\boldsymbol{b}$, $\mathbf{c}$ are mutually distinct. If $\lambda$ is an M-eigenvalue of the tensor $\mathcal{C}$ with eigenvectors $\boldsymbol{x}\in\mathbb{R}^m_+\backslash\{0\}$, $\boldsymbol{y}\in\mathbb{R}^n_+\backslash\{0\}$, $\boldsymbol{z}\in\mathbb{R}^s_+\backslash\{0\}$, then, $\lambda\neq 0$.*

*Proof.* We employ a proof by contradiction by assuming that the tensor $\mathcal{C}$ has an M-eigenvalue $\lambda=0$ with eigenvector $\mathbf{z}$. Since $\mathbf{z}\geq 0$ and $\mathbf{z}\neq 0$, there exists at least one nonzero element $z_d>0$.

Combining (4.1) and (4.2), it follows that

$$\begin{aligned}\lambda z_d = 0 =&(\mathcal{C}\mathbf{xxyyz})_d\\ =&\sum_{i,j\in[m],k,l\in[n],p\in[s]} c_{ijklpd}x_ix_jy_ky_lz_p\\ =&\sum_{i,j\in[m],k,l\in[n],p\in[s]} \frac{x_ix_jy_ky_lz_p}{a_i+a_j+b_k+b_l+c_p+c_d}\\ =&\sum_{i,j\in[m],k,l\in[n],p\in[s]} \int_0^1 t^{a_i+a_j+b_k+b_l+c_p+c_d-1}x_ix_jy_ky_lz_p dt\\ =&\int_0^1 (\sum_{i\in[m]} t^{a_i-\frac{1}{6}}x_i)^2(\sum_{k\in[n]} t^{b_k-\frac{1}{6}}y_k)^2(\sum_{p\in[s]} t^{c_p-\frac{1}{6}}z_p)t^{c_d-\frac{1}{6}}dt.\end{aligned}$$

It can be concluded that

$$(\sum_{i\in[m]} t^{a_i-\frac{1}{6}}x_i)^2(\sum_{k\in[n]} t^{b_k-\frac{1}{6}}y_k)^2(\sum_{p\in[s]} t^{c_p-\frac{1}{6}}z_p)=0, \quad t\in(0,1].$$

Thus,

$$\begin{aligned} &\sum_{i\in[m]} t^{a_i-\frac{1}{6}}x_i=0,\ t\in(0,1],\\ or\ &\sum_{k\in[n]} t^{b_k-\frac{1}{6}}y_k=0,\ t\in(0,1],\\ or\ &\sum_{p\in[s]} t^{c_p-\frac{1}{6}}z_p=0,\ t\in(0,1]. \end{aligned}$$

Without losing generality, we assume that $\sum\limits_{p\in[s]} t^{c_p-\frac{1}{6}}z_p=0,\ t\in(0,1]$, that is,

$$t^{c_1-\frac{1}{6}}z_1+t^{c_2-\frac{1}{6}}z_2+\cdots+t^{c_s-\frac{1}{6}}z_s=0, \quad t\in(0,1].$$

Hence,

$$z_1+t^{c_2-c_1}z_2+\cdots+t^{c_s-c_1}z_s=0, \quad t\in(0,1].$$

By the continuity and the condition that all components of $\mathbf{c}$ are mutually distinct, it follows that $z_1=0$. Then we have

$$z_2+t^{c_3-c_2}z_3+\cdots+t^{c_s-c_2}z_s=0, \quad t\in(0,1],$$

which implies $z_2=0$.

By repeating the above process, we have

$$z_1=z_2=\cdots=z_s=0,$$

which is a contradiction with $\mathbf{z}\neq 0$. Hence, the sixth-order Cauchy tensor $\mathcal{C}$ has no zero M-eigenvalue. Similarly, if $\mathbf{x}\geq 0$, $\mathbf{x}\neq 0$ or $\mathbf{y}\geq 0$, $\mathbf{y}\neq 0$, we can also obtain that the sixth-order Cauchy tensor $\mathcal{C}$ has no zero M-eigenvalue, and the desired result holds. □

Next, we shall present a theorem concerning the M-eigenvalue inclusion interval of the sixth-order Cauchy tensor. Denote $\sigma(\mathcal{C})$ as the spectrum of tensor $\mathcal{C}$, which contains all M-eigenvalues of tensor $\mathcal{C}$. The spectral radius $\rho(\mathcal{C})$ of tensor $\mathcal{C}$ is defined as

$$\rho(\mathcal{C})=\{max\mid\lambda\mid:\lambda\in\sigma(\mathcal{C})\}.$$

**Theorem 4.3.** *Suppose $\mathcal{C}=(c_{ijklpq})$ is a sixth-order Cauchy tensor with generating vectors $\boldsymbol{a}\in\mathbb{R}^m$, $\boldsymbol{b}\in\mathbb{R}^n$ and $\boldsymbol{c}\in\mathbb{R}^s$. If $\lambda$ is an M-eigenvalue of $\mathcal{C}$, then,*

$$\begin{aligned}\lambda\in\sigma(\mathcal{C})\subseteq&\Phi(\mathcal{C})\\ =&\{\boldsymbol{z}\in C:|\boldsymbol{z}|\leq min\{\max_{i\in[m]}\{R_i(\mathcal{C})\},\max_{k\in[n]}\{S_k(\mathcal{C})\},\max_{p\in[s]}\{T_p(\mathcal{C})\}\},\end{aligned}$$

*where*

$$\begin{aligned} R_i(\mathcal{C})&=\sum_{j\in[m]}\sum_{k,l\in[n]}\sum_{p,q\in[s]}|\frac{1}{a_i+a_j+b_k+b_l+c_p+c_q}|,\\ S_k(\mathcal{C})&=\sum_{i,j\in[m]}\sum_{l\in[n]}\sum_{p,q\in[s]}|\frac{1}{a_i+a_j+b_k+b_l+c_p+c_q}|,\\ T_p(\mathcal{C})&=\sum_{i,j\in[m]}\sum_{k,l\in[n]}\sum_{q\in[s]}|\frac{1}{a_i+a_j+b_k+b_l+c_p+c_q}|. \end{aligned}$$

*Proof.* Let $\lambda$ be an M-eigenvalue of tensor $\mathcal{C}$ with eigenvectors $\mathbf{x} \in \mathbb{R}^m\backslash\{0\}$, $\mathbf{y} \in \mathbb{R}^n\backslash\{0\}$, $\mathbf{z} \in \mathbb{R}^s\backslash\{0\}$. Since $\mathbf{x}^\top\mathbf{x} = 1$, $\mathbf{y}^\top\mathbf{y} = 1$, $\mathbf{z}^\top\mathbf{z} = 1$, there exists index $u \in [m], v \in [n], w \in [s]$, such that

$$\begin{aligned} 0 < |x_u| &= \max_{i\in[m]}\{|x_i|\} \leq 1, \\ 0 < |y_v| &= \max_{k\in[n]}\{|y_k|\} \leq 1, \\ 0 < |z_w| &= \max_{p\in[s]}\{|z_p|\} \leq 1. \end{aligned} \tag{4.3}$$

From (4.1) and (4.2), we can obtain

$$\begin{aligned} \lambda x_u &= (\mathcal{C}\mathbf{xyyzz})_u \\ &= \sum_{j\in[m]}\sum_{k,l\in[n]}\sum_{p,q\in[s]} \frac{x_j y_k y_l z_p z_q}{a_u + a_j + b_k + b_l + c_p + c_q}. \end{aligned} \tag{4.4}$$

Taking absolute values on both sides of (4.4), using inequality (4.3) above, we obtain

$$\begin{aligned} |\lambda||x_u| &= |(\mathcal{C}\mathbf{xyyzz})_u| \\ &\leq \sum_{j\in[m]}\sum_{k,l\in[n]}\sum_{p,q\in[s]} |\frac{x_u}{a_u + a_j + b_k + b_l + c_p + c_q}|. \end{aligned} \tag{4.5}$$

Thus,

$$|\lambda| \leq \sum_{j\in[m]}\sum_{k,l\in[n]}\sum_{p,q\in[s]} |\frac{1}{a_u + a_j + b_k + b_l + c_p + c_q}| = R_u(\mathcal{C}) \leq \max_{i\in[m]}\{R_i(\mathcal{C})\}. \tag{4.6}$$

In the same way, the $v$th equation of

$$\lambda\mathbf{y} = \mathcal{C}\mathbf{xxyzz}$$

is

$$\begin{aligned} \lambda y_v &= (\mathcal{C}\mathbf{xxyzz})_v \\ &= \sum_{i,j\in[m]}\sum_{l\in[n]}\sum_{p,q\in[s]} \frac{x_i x_j y_l z_p z_q}{a_i + a_j + b_v + b_l + c_p + c_q}. \end{aligned} \tag{4.7}$$

Taking absolute values on both sides of (4.7), using inequality (4.3) above, we can obtain

$$\begin{aligned} |\lambda||y_v| &= |(\mathcal{C}\mathbf{xxyzz})_v| \\ &\leq \sum_{i,j\in[m]}\sum_{l\in[n]}\sum_{p,q\in[s]} |\frac{y_v}{a_i + a_j + b_v + b_l + c_p + c_q}|. \end{aligned} \tag{4.8}$$

Thus,

$$|\lambda| \leq \sum_{i,j\in[m]}\sum_{l\in[n]}\sum_{p,q\in[s]} |\frac{1}{a_i + a_j + b_v + b_l + c_p + c_q}| = S_v(\mathcal{C}) \leq \max_{k\in[n]}\{S_k(\mathcal{C})\}. \tag{4.9}$$

The $w$th equation of

$$\lambda\mathbf{z} = \mathcal{C}\mathbf{xxyyz}$$

is

$$\lambda z_w = (\mathcal{C}\mathbf{xxyyz})_w = \sum_{i,j\in[m]}\sum_{k,l\in[n]}\sum_{q\in[s]}\frac{x_ix_jy_ky_lz_q}{a_i+a_j+b_k+b_l+c_w+c_q}. \tag{4.10}$$

Taking absolute values on both sides of (4.10), using inequality (4.3) above, we obtain

$$|\lambda||z_w| = |(\mathcal{C}\mathbf{xxyyz})_w| \leq \sum_{i,j\in[m]}\sum_{k,l\in[n]}\sum_{q\in[s]}|\frac{z_w}{a_i+a_j+b_k+b_l+c_w+c_q}|. \tag{4.11}$$

Thus,

$$|\lambda| \leq \sum_{i,j\in[m]}\sum_{k,l\in[n]}\sum_{q\in[s]}|\frac{1}{a_i+a_j+b_k+b_l+c_w+c_q}| = T_w(\mathcal{C}) \leq \max_{p\in[s]}\{T_p(\mathcal{C})\}. \tag{4.12}$$

Combined with (4.6), (4.9) and (4.12), we have

$$|\lambda| \leq min\{\max_{i\in[m]}\{R_i(\mathcal{C})\}, \max_{k\in[n]}\{S_k(\mathcal{C})\}, \max_{p\in[s]}\{T_p(\mathcal{C})\}\}.$$

Thus, $\lambda \in \Phi(\mathcal{C})$, and the desired result holds. □

**Theorem 4.4.** *Suppose $\mathcal{C} = (c_{ijklpq})$ is a sixth-order Cauchy tensor with generating vectors $\boldsymbol{a} \in \mathbb{R}^m$, $\boldsymbol{b} \in \mathbb{R}^n$ and $\boldsymbol{c} \in \mathbb{R}^s$. If $\lambda$ is an M-eigenvalue of $\mathcal{C}$, then,*

$$\lambda \in \sigma(\mathcal{C}) \subseteq \Psi(\mathcal{C}) = U(\mathcal{C})\bigcap V(\mathcal{C})\bigcap W(\mathcal{C}),$$

*where*

$$\begin{aligned}
U(\mathcal{C}) &= \bigcup_{i\in[m]}(\bigcap_{j\in[m],j\neq i} U_{i,j}(\mathcal{C})),\\
U_{i,j}(\mathcal{C}) &= \{\boldsymbol{z}\in C : (|\boldsymbol{z}| - (R_i(\mathcal{C}) - R_i^j(\mathcal{C})))|\boldsymbol{z}| \leq R_i^j(\mathcal{C})R_j(\mathcal{C})\},\\
R_i(\mathcal{C}) &= \sum_{j\in[m],k,l\in[n],p,q\in[s]}|\frac{1}{a_i+a_j+b_k+b_l+c_p+c_q}|,\\
R_i^j(\mathcal{C}) &= \sum_{k,l\in[n],p,q\in[s]}|\frac{1}{a_i+a_j+b_k+b_l+c_p+c_q}|.
\end{aligned}$$

$$\begin{aligned}
V(\mathcal{C}) &= \bigcup_{k\in[n]}(\bigcap_{l\in[n],l\neq k} V_{k,l}(\mathcal{C})),\\
V_{k,l}(\mathcal{C}) &= \{\boldsymbol{z}\in C : (|\boldsymbol{z}| - (S_k(\mathcal{C}) - S_k^l(\mathcal{C})))|\boldsymbol{z}| \leq S_k^l(\mathcal{C})S_l(\mathcal{C})\},\\
S_k(\mathcal{C}) &= \sum_{i,j\in[m],l\in[n],p,q\in[s]}|\frac{1}{a_i+a_j+b_k+b_l+c_p+c_q}|,\\
S_k^l(\mathcal{C}) &= \sum_{i,j\in[m],p,q\in[s]}|\frac{1}{a_i+a_j+b_k+b_l+c_p+c_q}|.
\end{aligned}$$

$$
\begin{aligned}
W(\mathcal{C}) &= \bigcup_{p\in[s]} \Big( \bigcap_{q\in[s],q\neq p} W_{p,q}(\mathcal{C}) \Big),\\
W_{p,q}(\mathcal{C}) &= \{ \boldsymbol{z} \in C : (|\boldsymbol{z}| - (T_p(\mathcal{C}) - T_p^q(\mathcal{C})))|\boldsymbol{z}| \leq T_p^q(\mathcal{C}) T_q(\mathcal{C}) \},\\
T_p(\mathcal{C}) &= \sum_{i,j\in[m],k,l\in[n],q\in[s]} \Big| \frac{1}{a_i + a_j + b_k + b_l + c_p + c_q} \Big|,\\
T_p^q(\mathcal{C}) &= \sum_{i,j\in[m],k,l\in[n]} \Big| \frac{1}{a_i + a_j + b_k + b_l + c_p + c_q} \Big|.
\end{aligned}
$$

*Proof.* Let $\lambda$ be an M-eigenvalue of tensor $\mathcal{C}$ with eigenvectors $\mathbf{x} \in \mathbb{R}^m \backslash \{0\}$, $\mathbf{y} \in \mathbb{R}^n \backslash \{0\}$, $\mathbf{z} \in \mathbb{R}^s \backslash \{0\}$. Since $\mathbf{x}^\top \mathbf{x} = 1$, there exists an index $u \in [m]$ such that

$$
|x_u| = \max_{i\in[m]} \{|x_i|\} > 0.
$$

Therefore,

$$
\begin{aligned}
\lambda x_u =& (\mathcal{C}\mathbf{xyyzz})_u\\
=& \sum_{j\in[m],k,l\in[n],p,q\in[s]} \frac{x_j y_k y_l z_p z_q}{a_u + a_j + b_k + b_l + c_p + c_q}\\
=& \sum_{j\in[m],j\neq t,k,l\in[n],p,q\in[s]} \frac{x_j y_k y_l z_p z_q}{a_u + a_j + b_k + b_l + c_p + c_q}\\
&+ \sum_{k,l\in[n],p,q\in[s]} \frac{x_t y_k y_l z_p z_q}{a_u + a_t + b_k + b_l + c_p + c_q},
\end{aligned}
$$

where $t \in [m]$, $t \neq u$, which is equivalent to

$$
\begin{aligned}
|\lambda| \leq& \sum_{j\in[m],j\neq t,k,l\in[n],p,q\in[s]} \Big| \frac{1}{a_u + a_j + b_k + b_l + c_p + c_q} \Big|\\
&+ \sum_{k,l\in[n],p,q\in[s]} \Big| \frac{1}{a_u + a_t + b_k + b_l + c_p + c_q} \Big| \frac{|x_t|}{|x_u|}.
\end{aligned} \tag{4.13}
$$

If $|x_t| = 0$, then,

$$
|\lambda| - \sum_{j\in[m],j\neq t,k,l\in[n],p,q\in[s]} \Big| \frac{1}{a_u + a_j + b_k + b_l + c_p + c_q} \Big| \leq 0,
$$

which implies $\lambda \in U_{u,t}(\mathcal{C}) \subseteq U(\mathcal{C})$.

If $|x_t| > 0$, we have

$$
\lambda x_t = (\mathcal{C}\mathbf{xyyzz})_t = \sum_{j\in[m],k,l\in[n],p,q\in[s]} \frac{x_j y_k y_l z_p z_q}{a_t + a_j + b_k + b_l + c_p + c_q},
$$

then,

$$
\begin{aligned}
|\lambda| \leq& \sum_{j\in[m],k,l\in[n],p,q\in[s]} \Big| \frac{1}{a_t + a_j + b_k + b_l + c_p + c_q} \Big| \Big| \frac{x_j}{x_t} \Big|\\
\leq& \sum_{j\in[m],k,l\in[n],p,q\in[s]} \Big| \frac{1}{a_t + a_j + b_k + b_l + c_p + c_q} \Big| \Big| \frac{x_u}{x_t} \Big|.
\end{aligned} \tag{4.14}
$$

Multiplying (4.13) and (4.14) yields

$$\begin{aligned}&(|\lambda| - \sum_{j\in[m],j\neq t,k,l\in[n],p,q\in[s]} |\frac{1}{a_u+a_j+b_k+b_l+c_p+c_q}|)|\lambda| \\ \leq &\sum_{k,l\in[n],p,q\in[s]} |\frac{1}{a_u+a_t+b_k+b_l+c_p+c_q}| \sum_{j\in[m],k,l\in[n],p,q\in[s]} |\frac{1}{a_t+a_j+b_k+b_l+c_p+c_q}|.\end{aligned}$$

Consequently,

$$(|\lambda| - (R_u(\mathcal{C}) - R_u^t(\mathcal{C})))|\lambda| \leq R_u^t(\mathcal{C})R_t(\mathcal{C}),$$

which implies $\lambda \in U_{u,t}(\mathcal{C})$. Due to the arbitrariness of $t$, it follows that $\lambda \in \bigcap_{j\in[m],j\neq i} U_{i,j}(\mathcal{C})$, and hence $\lambda \in \bigcup_{i\in[m]} (\bigcap_{j\in[m],j\neq i} U_{i,j}(\mathcal{C}))$.

Similar to above, we can obtain

$$\lambda \in \bigcup_{k\in[n]} (\bigcap_{l\in[n],l\neq k} V_{k,l}(\mathcal{C})) \ \ and \ \ \lambda \in \bigcup_{p\in[s]} (\bigcap_{q\in[s],q\neq p} W_{p,q}(\mathcal{C})).$$

Thus, $\lambda \in \Psi(\mathcal{C}) = U(\mathcal{C}) \bigcap V(\mathcal{C}) \bigcap W(\mathcal{C})$, and the desired result holds. □

The following conclusion will show the relationship between $\sigma(\mathcal{C})$, $\Phi(\mathcal{C})$ and $\Psi(\mathcal{C})$.

**Theorem 4.5.** *Let $\mathcal{C}$ be defined as in Theorem 4.3 and Theorem 4.4. Then,*

$$\sigma(\mathcal{C}) \subseteq \Psi(\mathcal{C}) \subseteq \Phi(\mathcal{C}).$$

*Proof.* By Theorem 4.3 and Theorem 4.4, we only need to prove $\Psi(\mathcal{C}) \subseteq \Phi(\mathcal{C})$. Without loss of generality, for any $\lambda \in \Psi(\mathcal{C})$, there exists an index $u \in [m]$, such that $\lambda \in U_{u,t}(\mathcal{C})$, for all $t \neq u$. Then,

$$(|\lambda| - (R_u(\mathcal{C}) - R_u^t(\mathcal{C})))|\lambda| \leq R_u^t(\mathcal{C})R_t(\mathcal{C}).$$

Obviously, according to the conditions described in Theorem 4.4, $R_u^t(\mathcal{C}) > 0$, $R_t(\mathcal{C}) > 0$ and $R_u^t(\mathcal{C})R_t(\mathcal{C}) > 0$. Thus,

$$\frac{|\lambda| - (R_u(\mathcal{C}) - R_u^t(\mathcal{C}))}{R_u^t(\mathcal{C})} \frac{|\lambda|}{R_t(\mathcal{C})} \leq 1,$$

which implies

$$\frac{|\lambda| - (R_u(\mathcal{C}) - R_u^t(\mathcal{C}))}{R_u^t(\mathcal{C})} \leq 1,$$

or

$$\frac{|\lambda|}{R_t(\mathcal{C})} \leq 1.$$

Therefore, we can obtain $|\lambda| \leq R_u(\mathcal{C})$ or $|\lambda| \leq R_t(\mathcal{C})$. Thus, $\lambda \in \Phi(\mathcal{C})$. As a result, $\sigma(\mathcal{C}) \subseteq \Psi(\mathcal{C}) \subseteq \Phi(\mathcal{C})$. The proof is complete. □

Next, we validate the inclusion relationships between the intervals through two numerical examples.

**Example 4.6.** We consider a sixth-order paired symmetric Cauchy tensor with generating vectors $\mathbf{a} = (1, 6, 9)^\top$, $\mathbf{b} = (2, 9, 15)^\top$, and $\mathbf{c} = (16, 6, 9)^\top$. According to Theorem 4.3, we can compute the range of $\lambda$ by designing the corresponding program, $\lambda \in \sigma(\mathcal{C}) \subseteq \Phi(\mathcal{C}) = \{\mathbf{z} : |\mathbf{z}| \leq 5.79315445223860\}$. According to Theorem 4.4, by designing the corresponding computational program, we can determine that the range of $\lambda$ is $\lambda \in \sigma(\mathcal{C}) \subseteq \Psi(\mathcal{C}) = U(\mathcal{C}) \bigcap V(\mathcal{C}) \bigcap W(\mathcal{C}) = (-1.67611448534124, 5.79315445223860)$. Obviously, $\Psi(\mathcal{C}) \subseteq \Phi(\mathcal{C})$.

**Example 4.7.** We consider a sixth-order paired symmetric Cauchy tensor with generating vectors $\mathbf{a} = (3, 8, 29)^\top$, $\mathbf{b} = (2, 9, 15, 26)^\top$, and $\mathbf{c} = (16, 6, 9, 35)^\top$. According to Theorem 4.3, we can compute the range of $\lambda$ by designing the corresponding program, $\lambda \in \sigma(\mathcal{C}) \subseteq \Phi(\mathcal{C}) = \{\mathbf{z} : |\mathbf{z}| \leq 8.54677573831066\}$. According to Theorem 4.4, by designing the corresponding computational program, we can determine that the range of $\lambda$ is $\lambda \in \sigma(\mathcal{C}) \subseteq \Psi(\mathcal{C}) = U(\mathcal{C}) \bigcap V(\mathcal{C}) \bigcap W(\mathcal{C}) = (-1.61376831328198, 8.54677573831066)$. Obviously, $\Psi(\mathcal{C}) \subseteq \Phi(\mathcal{C})$.

Below, based on the polynomial structural characteristics of the sixth-order Cauchy tensor $\mathcal{C}$, we also obtained a bound for its M-eigenvalue.

**Theorem 4.8.** *Suppose $\mathcal{C} = (c_{ijklpq})$ is a sixth-order Cauchy tensor with generating vectors $\boldsymbol{a} \in \mathbb{R}^m$, $\boldsymbol{b} \in \mathbb{R}^n$ and $\boldsymbol{c} \in \mathbb{R}^s$. If $\lambda$ is an M-eigenvalue of $\mathcal{C}$, then*

$$|\lambda| \leq mnsL(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}),$$

*where*

$$L(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = \max\{\frac{1}{|a_i + a_j + b_k + b_l + c_p + c_q|}\}.$$

*Proof.* Let $\lambda$ be an M-eigenvalue of the tensor $\mathcal{C}$ with eigenvectors $\mathbf{x} \in \mathbb{R}^m \backslash \{0\}$, $\mathbf{y} \in \mathbb{R}^n \backslash \{0\}$, $\mathbf{z} \in \mathbb{R}^s \backslash \{0\}$, then

$$\begin{aligned}
|\lambda| =& |\mathcal{C}\mathbf{xxyyzz}| \\
=& |\sum_{i,j=1}^{m} \sum_{k,l=1}^{n} \sum_{p,q=1}^{s} \frac{x_i x_j y_k y_l z_p z_q}{a_i + a_j + b_k + b_l + c_p + c_q}| \\
\leq& \sum_{i,j=1}^{m} \sum_{k,l=1}^{n} \sum_{p,q=1}^{s} \frac{|x_i||x_j||y_k||y_l||z_p||z_q|}{|a_i + a_j + b_k + b_l + c_p + c_q|} \\
\leq& L(\mathbf{a}, \mathbf{b}, \mathbf{c}) \sum_{i,j=1}^{m} \sum_{k,l=1}^{n} \sum_{p,q=1}^{s} |x_i||x_j||y_k||y_l||z_p||z_q| \\
=& L(\mathbf{a}, \mathbf{b}, \mathbf{c}) \|\mathbf{x}\|_1^2 \|\mathbf{y}\|_1^2 \|\mathbf{z}\|_1^2 \\
\leq& mnsL(\mathbf{a}, \mathbf{b}, \mathbf{c}),
\end{aligned}$$

where

$$L(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \max\{\frac{1}{|a_i + a_j + b_k + b_l + c_p + c_q|}\}.$$

□

Next, we use the method of Theorem 4.8 to calculate Example 4.6 and Example 4.7.

In Example 4.6, the generating vectors of the tensor $\mathcal{C}$ are $\mathbf{a} = (1, 6, 9)^\top$, $\mathbf{b} = (2, 9, 15)^\top$, and $\mathbf{c} = (16, 6, 9)^\top$. Then $L(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{1}{18}$. Hence, $|\lambda| \leq \frac{3\times3\times3}{18} = 1.5$.

In Example 4.7, the generating vectors of the tensor $\mathcal{C}$ are $\mathbf{a} = (3, 8, 29)^\top$, $\mathbf{b} = (2, 9, 15, 26)^\top$, and $\mathbf{c} = (16, 6, 9, 35)^\top$. Then $L(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{1}{22}$. Hence, $|\lambda| \leq \frac{3\times4\times4}{22} \approx 2.18$.

Clearly, from the comparison of the numerical examples in Theorem 4.8 and Theorem 4.4, the obtained eigenvalue intervals do not always represent absolute inclusion relations. However, the interval length obtained using Theorem 4.8 is slightly smaller than the one obtained using Theorem 4.4, and the computation process of Theorem 4.8 is simpler.

## $\boxed{5}$ Conclusions

In this paper, we consider a sixth-order Cauchy tensor. First, according to the structural characteristics of the sixth-order Cauchy tensor, some conditions for judging the positive definiteness of the sixth-order Cauchy tensor are given. In other words, we obtain the strong elliptic condition of the sixth-order Cauchy tensor. Moreover, we show the relationship between the positive definiteness of the sixth-order Cauchy tensor and the monotonicity of a homogeneous polynomial with respect to the proposed Cauchy tensor. In addition, we apply the concept of the M-eigenvalue to the sixth-order paired symmetric Cauchy tensor, and further discuss related properties. We give three M-eigenvalue inclusion intervals for the sixth-order paired symmetric Cauchy tensor, and the inclusion relations between them are also discussed.

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