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SEQUENTIAL HENIG PROPER OPTIMALITY CONDITIONS FOR MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS VIA SEQUENTIAL PROPER SUBDIFFERENTIAL CALCULUS

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Abstract: In this paper, in the absence of any constraint qualifications, we develop sequential necessary and sufficient optimality conditions for a constrained multiobjective fractional programming problem characterizing a Henig proper efficient solution in terms of the ϵ-subdifferentials and the subdifferentials of the functions. This is achieved by employing a sequential Henig subdifferential calculus rule of the sums of m ($m \geq 2$) proper convex vector valued mappings with a composition of two convex vector valued mappings. In order to present an example illustrating Our results, we establish the classical optimality conditions under Moreau-Rockafellar qualification condition. Our results are presented in the setting of reflexive Banach space in order to avoid the use of nets.

Key words: *sequential optimality conditions, Henig subdifferential, multiobjective fractional programming problem, Henig proper efficient solution*

Mathematics Subject Classification: *90C32, 90C46*

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1 Introduction

Fractional multiobjective programming has many applications such management science, operational research, economics and information theory (see [13, 14]). However, to obtain optimality conditions for a multiobjective fractional problem which is not generally convex, we often formulate an equivalent vector convex problem by using a parametric approach, but such vector convex program requires a regularity condition as generalized Slater constraint qualification which is not an easy task to verify and may fail to hold neither for finite or infinite dimensional convex program. To raise this drawbacks, many contributions have been focused on the characterizations of sequential optimality condition for a vector convex problem or a fractional optimization problem which avoid a constraint qualification (see [5, 6, 7, 11, 8, 15, 16]).

Upon solving a vector optimization problem, the set of efficient solutions is so big that it has poor properties and that it may contain anomalous or nondesirable points. In order to further refine the concept of an efficient solution, some authors introduced different kinds of properly efficient solution, such as Geoffrion properly efficient solution, Benson properly efficient solution, Henig properly efficient solution and super efficient solution. In different

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settings, Guerraggio et al. [3] gave some relations among properly efficient solutions. We know that Henig properly efficient solution not only has many desirable properties, but also has much weaker existence conditions than other properly efficient solutions.

The recent contributions in [10, 12] motivate the present work where the sequential optimality conditions are stated, via sequential subdifferential calculus, in terms of limits of sequences by using the approximate subdifferential or exact subdifferential at neabry points. So, the purpose of this paper is to establish sequential optimality conditions for a fractional optimization problem without any constraint qualifications characterizing completely a Henig properly efficient solution. Firstly, we establish a sequential Henig subdifferential calculus rule of the sums of m ($m \geq 2$) proper convex vector valued mappings with a composition of two convex vector valued mappings. This is achieved by employing a scalarization process and the epigraph of the conjugate of the sums of m ($m \geq 2$) proper convex lower semicontinuous functions. Secondly, since a multiobjective fractional problem is transformed equivalently into a nonfractional convex multiobjective problem, we apply the sequential Henig subdifferential calculus rule to obtain three different kinds of sequential optimality conditions. The first is expressed in terms of the epigraphs of the conjugate of data functions, the second is obtained by means of a sequence of ϵ-subdifferentials at a minimizer and the last kind is described, by means of the Brøndsted-Rockafellar theorem, in terms of the exact subdifferentials of the functions involved at nearby points to the minimizer.

This paper is organized as follows. In Section 2, we recall some notions and we give some preliminary results. In Section 3, we establish a sequential Henig subdifferential calculus rule of the sums of m ($m \geq 2$) proper convex vector valued mappings with a composition of two convex vector valued mappings. In Section 4, we provide some sequential efficient optimality conditions characterizing a Henig proper solution for a vector fractional optimization problem. In order to present an example illustrating our main result, we will need to establish the standard optimality conditions of a multiobjective fractional problem under a constraint qualification.

2 Preliminaries and Basic Definitions

In this section, we give some definitions and preliminary results which will be used throughout this paper. Let X, Y and Z be three real topological vector spaces and their respective continuous dual spaces X^*, Y^* and Z^* with duality paring denoted by $\langle ., . \rangle$. We will use the symbol w^* for the weak-star topology on the dual space and $\tau_{\mathbb{R}}$ for the Euclidian topology on the real line $\mathbb{R}$. Let $Z_+ \subset Z$ be a nontrivial convex cone and we assume in addition that Z_+ is pointed (i.e. $-Z_+ \cap Z_+ = \{0\}$). The nonnegative orthant of $\mathbb{R}^m$ is denoted by $\mathbb{R}^m_+$. The polar cone and strict polar cone are defined respectively as

$$Z^*_+ := \{z^* \in Z^* : \langle z^*, z \rangle \geq 0, \ \forall z \in Z_+\} \tag{2.1}$$

and

$$(Z^*_+)^\circ := \{z^* \in Z^* : \langle z^*, z \rangle > 0, \ \forall z \in Z_+ \backslash \{0\}\}. \tag{2.2}$$

On Z we define for any $z_1, z_2 \in Z$ the following relations

$$\begin{aligned} z_1 \quad &\leq_{Z_+} \quad z_2 \iff z_2 - z_1 \in Z_+, \\ z_1 \quad &\lneqq_{Z_+} \quad z_2 \iff z_2 - z_1 \in Z_+ \backslash \{0\}. \end{aligned}$$

We attach to Z an abstract maximal element, denoted by $+\infty_Z$, with respect to " $\leq_{Z_+}$ " and we denote $\overline{Z} = Z \cup \{+\infty_Z\}$. Then for every $z \in Z$ one has $z \leq_{Z_+} +\infty_Z$. The algebraic operations of Z are extended as follows

$$z + (+\infty_Z) = (+\infty_Z) + z = +\infty_Z, \quad \forall z \in Z \cup \{+\infty_Z\}, \tag{2.3}$$

$$\alpha \cdot (+\infty_Z) = +\infty_Z, \quad \forall \alpha \geq 0. \tag{2.4}$$

If $Z = \mathbb{R}$ and $K = \mathbb{R}_+$ then $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, where $+\infty = +\infty_{\mathbb{R}}$.

For a given mapping $f : X \longrightarrow \overline{Z}$, we denote respectively the effective domain and the epigraph of f by dom$f$ and epi$f$, i.e.,

$$\text{dom} f := \{x \in X : f(x) \in Z\} \tag{2.5}$$

$$\text{epi} f := \{(x, z) \in X \times Z : f(x) \leq_{Z_+} z\}. \tag{2.6}$$

We say that f is proper if dom$f \neq \emptyset$ and Z_+-epi-closed if its epigraph is closed. Let us note for each $z^* \in Z_+^*$, we put by convention $\langle z^*, f(x)\rangle = +\infty$, for any $x \notin \text{dom} f$. Thus, $z^* \circ f \ : \ X \longrightarrow \overline{\mathbb{R}}$ and $\text{dom}(z^* \circ f) = \text{dom} f$ and we say that f is strict star Z_+-lower semicontinuous if $z^* \circ f$ is lower semicontinuous for all $z^* \in (Z_+^*)^\circ$. Moreover, we recall that the mapping f is Z_+-convex if for every $\lambda \in [0, 1]$ and $x_1, x_2 \in X$ we have

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq_{Z_+} \lambda f(x_1) + (1 - \lambda) f(x_2). \tag{2.7}$$

Let "$\leq_{Y_+}$" be the partial order on Y induced by a nonempty convex cone $Y_+ \subset Y$. We say that the mapping $g \ : \ Y \longrightarrow \overline{Z}$ is (Y_+, Z_+)-nondecreasing if for each $y_1, y_2 \in Y$ we have

$$y_1 \leq_{Y_+} y_2 \Longrightarrow g(y_1) \leq_{Z_+} g(y_2). \tag{2.8}$$

Let $h \ : \ X \longrightarrow \overline{Y}$ be a mapping, then the composed mapping $g \circ h \ : \ X \longrightarrow \overline{Z}$ is defined by

$$(g \circ h)(x) := \begin{cases} g(h(x)), & \text{if } \ x \in \text{dom } h, \\ \\ +\infty_Z, & \text{otherwise.} \end{cases} \tag{2.9}$$

It is easy to see that if $g \ : \ Y \longrightarrow \overline{Z}$ is Z_+-convex, (Y_+, Z_+)-nondecreasing and $h \ : \ X \longrightarrow \overline{Y}$ is Y_+-convex, then the composed mapping $g \circ h$ is Z_+-convex.

Now, we consider the following vector minimization problem

$$(\text{VMP}) \quad \min_{x \in C} f(x) \tag{2.10}$$

where $f \ : \ X \longrightarrow \overline{Z}$ is a mapping and C is a nonempty subset of X.

Definition 2.1. Let $\overline{x}$ be a feasible point of (VMP) i.e. $\overline{x} \in C$. The point $\overline{x}$ is called properly efficient solution of the problem (VMP) in the sense of Henig if there exists a convex cone $\hat{Z}_+ \subset Z$ such that $Z_+ \setminus \{0\} \subset \text{int}\hat{Z}_+$ and

$$\nexists x \in C, \quad f(x) \lneqq_{\hat{Z}_+} f(\overline{x}). \tag{2.11}$$

The set of Henig properly efficient solutions will be denoted by $E^p(f, C)$. The above notion of Henig properly efficient solution leads us to define the notion of Henig proper subdifferential of a vector valued mapping $f : X \longrightarrow \overline{Z}$ at $\overline{x} \in \mathrm{dom} f$ (see [4]). For $A \in L(X, Z)$

$$A \in \partial^p f(\bar{x}) \text{ if there exists } \hat{Z}_+ \subsetneq Z \text{ convex cone such that } Z_+ \setminus \{0\} \subseteq \mathrm{int}\hat{Z}_+,$$
$$\nexists x \in C,\ f(x) - f(\bar{x}) \lneqq_{\hat{Z}_+} A(x - \overline{x}),\ \forall x \in X$$

where $L(X, Z)$ denotes the space of linear continuous operators from X to Z. This definition is justified by the importance of the following immediate property

$$\bar{x} \in E^p(f, X) \Longleftrightarrow 0 \in \partial^p f(\bar{x}). \tag{2.12}$$

For a given function $f : X \longrightarrow \overline{\mathbb{R}}$, the conjugate function $f^* : X^* \longrightarrow \overline{\mathbb{R}} \cup \{-\infty\}$ is defined by

$$f^*(x^*) := \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\}. \tag{2.13}$$

Recall that, for $\epsilon \geq 0$, the ϵ-subdifferential of f at $\overline{x} \in \mathrm{dom} f$ is defined by

$$\partial_\epsilon f(\overline{x}) := \{x^* \in X^* \ : \ \langle x^*, x - \overline{x}\rangle + f(\overline{x}) - \epsilon \leq f(x), \quad \forall x \in X\}. \tag{2.14}$$

If $\epsilon = 0$, the set $\partial f(\overline{x}) := \partial_0 f(\overline{x})$ is the classical subdifferential of convex analysis, that is

$$\partial f(\overline{x}) := \{x^* \in X^* \ : \ \langle x^*, x - \overline{x}\rangle + f(\overline{x}) \leq f(x), \quad \forall x \in X\}. \tag{2.15}$$

For any subset $C \subset X$, the vector indicator mapping $\delta_C^v : X \longrightarrow \overline{Z}$ of C is defined by

$$\delta_C^v(x) := \begin{cases} 0 & \text{if } x \in C \\ +\infty_Z & \text{otherwise.} \end{cases} \tag{2.16}$$

When $Z = \mathbb{R}$, the scalar indicator function is denoted by δ_C. The vector indicator mapping appears to possess properties like the scalar one. For $\epsilon \geq 0$, the ϵ-normal at $\overline{x} \in C$ is defined by

$$N_\epsilon(\overline{x}, C) := \{x^* \in X^* : \quad \langle x^*, x - \overline{x}\rangle \leq \epsilon, \quad \forall x \in C\}. \tag{2.17}$$

If $\epsilon = 0$, $N(\overline{x}, C) := N_0(\overline{x}, C)$ is the usual normal cone at $\overline{x}$. Moreover, it is easy to see that the ϵ-normal of C at $\overline{x}$ is defined as the ϵ-subdifferential of δ_C at $\overline{x}$.

Theorem 2.2 ([1])**.** *Let $f_i : X \longrightarrow \overline{\mathbb{R}}$ be m $(m \geq 2)$ proper, convex and lower semicontinuous functions such that $\bigcap_{i=1}^m \mathrm{dom} f_i \neq \emptyset$. Then*

$$\mathrm{epi}\left(\sum_{i=1}^m f_i\right)^* = \mathrm{cl}_{w^* \times \tau_{\mathbb{R}}}\left(\sum_{i=1}^m \mathrm{epi} f_i^*\right) \tag{2.18}$$

where $\mathrm{cl}_{w^* \times \tau_{\mathbb{R}}}$ denotes the weak closure on the product space $X^* \times \mathbb{R}$.

Theorem 2.3 ([5])**.** *Let $f : X \longrightarrow \overline{\mathbb{R}}$ be a proper convex lower semicontinuous function and let $\overline{x} \in \mathrm{dom} f$. Then*

$$\mathrm{epi} f^* = \bigcup_{\epsilon \geq 0} \{(x^*, \langle x^*, \overline{x}\rangle + \epsilon - f(\overline{x})) \ : \ x^* \in \partial_\epsilon f(\overline{x})\}. \tag{2.19}$$

Theorem 2.4 ([16]). *Let X be a Banach space and $f : X \longrightarrow \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function. Then for any real $\epsilon > 0$ and $\overline{x}^* \in \partial_\epsilon f(\overline{x})$, there exist $x \in \text{dom} f$ and $x^* \in \partial f(x)$ such that*

1. $\|x - \overline{x}\| \leq \sqrt{\epsilon}$,
2. $\|x^* - \overline{x}^*\| \leq \sqrt{\epsilon}$,
3. $|f(x) - f(\overline{x}) - \langle x^*, x - \overline{x}\rangle| \leq 2\epsilon$.

Let us recall the following important scalarization theorem can be found in [4].

Theorem 2.5. *Let X, Z be two Hausdorff topological vector spaces and $f : C \subseteq X \longrightarrow \overline{Z}$ be Z_+-convex vector valued mapping with Z_+ is a closed convex pointed cone of Z and C is a nonempty convex subset of X. Then*

1. $$E^p(f, C) = \bigsqcup_{z^* \in (Z_+^*)^\circ} \underset{x \in C}{\mathit{argmin}} \langle z^*, f(x)\rangle. \tag{2.20}$$

2. *For $\bar{x} \in X$, we have*
$$\partial^p f(\bar{x}) = \bigcup_{z^* \in (Z_+^*)^\circ} \{A \in L(X, Z) : z^* \circ A \in \partial(z^* \circ f)(\bar{x})\}. \tag{2.21}$$

Proof. 1. Let us show the direct inclusion in (2.20). Let $\overline{x} \in E^p(f, C)$, then there exists a convex cone $\hat{Z}_+ \subsetneq Z$ satisfies $Z_+ \backslash \{0\} \subseteq \text{int}\hat{Z}_+$ and $\nexists x \in C$ such that $f(x) \lneq_{\hat{Z}_+} f(\overline{x})$. Since, f being Z_+-convex and $Z_+ \backslash \{0\} \subseteq \text{int}\hat{Z}_+$, obviously f remains $\hat{Z}_+$-convex. So, the direct inclusion is obtained.

For the reverse inclusion, let $z^* \in (Z_+^*)^\circ$ and $\overline{x} \in \underset{x \in C}{\text{argmin}} \langle z^*, f(x)\rangle$. The pointed convex cone
$$\hat{Z}_+ := \{z \in Z : \langle z^*, z\rangle > 0\} \cup \{0\}$$
satifies $\hat{Z}_+ \neq Z$ and $Z_+ \backslash l(Z_+) \subseteq \text{int}\hat{Z}_+$. If $\overline{x} \notin E^p(f, C)$, there exist $x \in C$ such that $f(x) \lneq_{\hat{Z}_+} f(\overline{x})$, i.e, $f(\overline{x}) - f(x) \in \hat{Z}_+ \backslash \{0\}$, and hence, $\langle z^*, f(\overline{x}) - f(x)\rangle > 0$ contradicting the choice of $\overline{x}$.

2. If $\overline{x} \notin \text{dom} f$, then all the previous sets are empty. Suppose that $\overline{x} \in \text{dom} f$. By applying (2.20) and by using scalarization formula (2.12), we obtain
$$\begin{aligned} A \in \partial^p f(\overline{x}) &\Leftrightarrow A \in L(X, Z) : \ \overline{x} \in E_p(f - A, X) \\ &\Leftrightarrow A \in L(X, Z), \exists z^* \in (Z_+^*)^\circ : \ z^* \circ A \in \partial(z^* \circ f)(\overline{x}) \\ &\Leftrightarrow A \in \bigcup_{z^* \in (Z_+^*)^\circ} \{A \in L(X, Z) : z^* \circ A \in \partial(z^* \circ f)(\bar{x})\}. \end{aligned}$$

□

3 Sequential Pareto Proper Subdifferential of the Sums of Convex Vector Valued Mappings with a Composition of Two Convex Vector Valued Mappings

In what follows $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ stand for two real reflexive Banach spaces, $(Z, \|.\|_Z)$ be a real normed vector space and $(X^*, \|.\|_{X^*})$, $(Y^*, \|.\|_{Y^*})$, $(Z^*, \|.\|_{Z^*})$ their respective topological dual spaces. On $X \times Y$ we use the norm $\|(x, y)\| = \sqrt{\|x\|^2 + \|y\|^2}$ for any $(x, y) \in X \times Y$. Similarly, we define the norm on $X^* \times Y^*$. Further, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X (resp. $(x_n^*)_{n\in\mathbb{N}}$ be a sequence in X^*) and $x \in X$ (resp. $x^* \in X^*$), we write $x_n \xrightarrow{\|\cdot\|_X} x$ (resp. $x_n^* \xrightarrow{\|\cdot\|_{X^*}} x^*$) if $\|x_n - x\|_X \longrightarrow 0$ (resp. $\|x_n^* - x^*\|_{X^*} \longrightarrow 0$) as $n \longmapsto +\infty$. Our aim in this section is to formulate in the absence of constraint qualifications, a formula for the Henig proper subdifferential of the convex mapping $\sum_{i=1}^m (f_i + g \circ h)$, where $f_i : X \longrightarrow \overline{Z}$ $(i = 1, \ldots, m)$ be a proper and Z_+-convex mappings, $h : X \longrightarrow \overline{Y}$ be a proper and Y_+-convex mapping, and $g : Y \longrightarrow \overline{Z}$ be a proper, Z_+-convex and (Z_+, Y_+)-nondecreasing mapping. Let us consider the following auxiliary mappings

$$\begin{array}{rcccl} F_i & : & X \times Y & \longrightarrow & \overline{Z} \\ & & (x, y) & \longrightarrow & F_i(x, y) := f_i(x), \end{array} \quad (i = 1, \ldots, m) \tag{3.1}$$

$$\begin{array}{rcccl} G & : & X \times Y & \longrightarrow & \overline{Z} \\ & & (x, y) & \longrightarrow & G(x, y) := g(y), \end{array} \tag{3.2}$$

$$\begin{array}{rcccl} H & : & X \times Y & \longrightarrow & \overline{Z} \\ & & (x, y) & \longrightarrow & H(x, y) := \delta^v_{\mathrm{epi}h}(x, y). \end{array} \tag{3.3}$$

Lemma 3.1. *Let $z^* \in (Z_+^*)^\circ$ and $(x^*, y^*, s) \in X^* \times Y^* \times \mathbb{R}$. Then, we have*

1. $(x^*, y^*, s) \in \mathrm{epi}(z^* \circ F_i)^* \Longleftrightarrow (x^*, s) \in \mathrm{epi}(z^* \circ f_i)^*$ *and* $y^* = 0, \quad (i = 1, \ldots, m)$.
2. $(x^*, y^*, s) \in \mathrm{epi}(z^* \circ G)^* \Longleftrightarrow x^* = 0$ *and* $(y^*, s) \in \mathrm{epi}(z^* \circ g)^*$.
3. $(x^*, y^*, s) \in \mathrm{epi}(z^* \circ H)^* \Longleftrightarrow (x^*, s) \in \mathrm{epi}(-y^* \circ h)^*$ *and* $-y^* \in Y_+^*$.

Proof. It is easy to see that the conjugate functions associated to the functions $z^* \circ F_i$ $(i = 1, \ldots, m)$, $z^* \circ G$ and $z^* \circ H$ are given for any $(x^*, y^*) \in X^* \times Y^*$, by

$$(z^* \circ F_i)^*(x^*, y^*) = (z^* \circ f_i)^*(x^*) + \delta_{\{0\}}(y^*), \quad (i = 1, \ldots, m) \tag{3.4}$$

$$(z^* \circ G)^*(x^*, y^*) = (z^* \circ g)^*(y^*) + \delta_{\{0\}}(x^*) \tag{3.5}$$

$$(z^* \circ H)^*(x^*, y^*) = (-y^* \circ h)^*(x^*) + \delta_{Y_+^*}(-y^*). \tag{3.6}$$

1) Let $(x^*, y^*, s) \in \mathrm{epi}(z^* \circ F_i)^*$, $(i = 1, \ldots, m)$, then we have

$$(z^* \circ F_i)^*(x^*, y^*) \leq s \tag{3.7}$$

and by (3.4) we get

$$(z^* \circ f_i)^*(x^*) + \delta_{\{0\}}(y^*) \leq s, \quad (i = 1, \ldots, m), \tag{3.8}$$

i.e.

$$(x^*, s) \in \mathrm{epi}(z^* \circ f_i)^* \text{ and } y^* = 0, \quad (i = 1, \ldots, m). \tag{3.9}$$

By applying the same arguments as above we obtain easily 2) and 3). □

Now, we state the sequential Henig proper subdifferential of the convex mapping $\sum_{i=1}^{m} f_i + g\circ h$ by means of the epigraphs of the conjugate of data vector valued mappings.

Theorem 3.2. *Let $f_1,\dots,f_m : X \longrightarrow \overline{Z}$ be m ($m \geq 2$) proper, Z_+-convex and strict star Z_+-lower semicontinuous mappings, $g : Y \longrightarrow \overline{Z}$ be proper, Z_+-convex, strict star Z_+-lower semicontinuous and (Y_+, Z_+)-nondecreasing mapping and $h : X \longrightarrow \overline{Y}$ be proper, Y_+-convex and Y_+-epi-closed mapping. Let $\overline{x} \in (\bigcap_{i=1}^{m} \mathrm{dom} f_i) \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g)$. Then, $A \in \partial^p \left(\sum_{i=1}^{m} f_i + g\circ h\right)(\overline{x})$ if and only if, there exist $z^* \in (Z_+^*)^\circ$, $(x_{i,n}^*, r_{i,n}) \in \mathrm{epi}(z^*\circ f_i)^*$, $(i = 1,\dots,m)$, $(y_n^*, s_n) \in \mathrm{epi}(z^*\circ g)^*$, $v_n^* \in -Y_+^*$ and $(u_n^*, t_n) \in \mathrm{epi}(-v_n^*\circ h)^*$ such that*

$$\begin{cases} \displaystyle\sum_{i=1}^{m} x_{i,n}^* + u_n^* \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_{X^*}} z^*\circ A \\ y_n^* + v_n^* \xrightarrow[n\longmapsto+\infty]{\|\cdot\|_{Y^*}} 0 \\ \displaystyle\sum_{i=1}^{m} r_{i,n} + s_n + t_n \xrightarrow[n\longmapsto+\infty]{} (z^*\circ A)(\overline{x}) - \sum_{i=1}^{m}(z^*\circ f_i)(\overline{x}) - (z^*\circ g)(h(\overline{x})). \end{cases}$$

Proof. Let $A \in \partial^p \left(\sum_{i=1}^{m} f_i + g\circ h\right)(\overline{x})$. According to scalarization Theorem 2.5, there exists $z^* \in (Z_+^*)^\circ$ such that

$$z^*\circ A \in \partial\left(\sum_{i=1}^{m} z^*\circ f_i + z^*\circ g\circ h\right)(\overline{x}). \tag{3.10}$$

By introducing the scalar indicator function $\delta_{\mathrm{epi}h}$ and by adopting the convention $z^*(+\infty_Z) = +\infty$, it easy to check that $z^*\circ\delta^v_{\mathrm{epi}h} = \delta_{\mathrm{epi}h}$ and by using the monotonicity of the mapping g it follows that (3.10) becomes equivalent to

$$(z^*\circ A, 0) \in \partial\left(\sum_{i=1}^{m} z^*\circ F_i + z^*\circ G + z^*\circ H\right)(\overline{x}, h(\overline{x})) \tag{3.11}$$

i.e.

$$\left(\sum_{i=1}^{m} z^*\circ F_i + z^*\circ G + z^*\circ H\right)^*(z^*\circ A, 0) + \left(\sum_{i=1}^{m} z^*\circ F_i + z^*\circ G + z^*\circ H\right)(\overline{x}, h(\overline{x})) = \langle (z^*\circ A, 0), (\overline{x}, h(\overline{x}))\rangle = \langle z^*\circ A, \overline{x}\rangle$$

and hence we get

$$\left((z^*\circ A, 0), \langle z^*\circ A, \overline{x}\rangle - \left(\sum_{i=1}^{m} z^*\circ F_i + z^*\circ G + z^*\circ H\right)(\overline{x}, h(\overline{x}))\right) \in \mathrm{epi}\left(\sum_{i=1}^{m} z^*\circ F_i + z^*\circ G + z^*\circ H\right)^*. \tag{3.12}$$

It is easy to see that the mappings F_i, $(i = 1,\dots,m)$, G and H are proper, Z_+-convex and strict star Z_+-lower semicontinuous on $X\times Y$ and as z^* is Z_+-nondecreasing, it follows that the scalar functions $z^*\circ F_i$, $(i = 1,\dots,m)$, $z^*\circ G$ and $z^*\circ H$ are proper, convex and lower semicontinuous. Let us note that $\mathrm{dom}(z^* \circ F_i) = \mathrm{dom} f_i \times Y$, $(i = 1, \ldots, m)$, $\mathrm{dom}(z^* \circ G) = X \times \mathrm{dom} g$ and $\mathrm{dom}(z^* \circ H) = \mathrm{epi} h$ and the condition $\overline{x} \in (\bigcap_{i=1}^m \mathrm{dom} f_i) \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g)$ can be written equivalently as $(\overline{x}, h(\overline{x})) \in (\bigcap_{i=1}^m \mathrm{dom} F_i) \cap \mathrm{dom} G \cap \mathrm{dom} H$. Thus, the functions $z^* \circ F_i$ $(i = 1, \ldots, m)$, $z^* \circ G$ and $z^* \circ H$, satisfy together all the assumptions of Theorem 2.2 and hence it follows from (3.12) that

$$\begin{aligned}
&\left((z^* \circ A, 0), \langle z^* \circ A, \overline{x}\rangle - \left(\sum_{i=1}^m z^* \circ F_i + z^* \circ G + z^* \circ H\right)(\overline{x}, h(\overline{x}))\right) \\
&\qquad \in \mathrm{cl}_{w^* \times \tau_{\mathbb{R}}} \left(\sum_{i=1}^m \mathrm{epi}(z^* \circ F_i)^* + \mathrm{epi}(z^* \circ G)^* + \mathrm{epi}(z^* \circ H)^*\right) \\
&\qquad = \mathrm{cl}_{\|\cdot\|_{X^* \times Y^* \times \tau_{\mathbb{R}}}} \left(\sum_{i=1}^m \mathrm{epi}(z^* \circ F_i)^* + \mathrm{epi}(z^* \circ G)^* + \mathrm{epi}(z^* \circ H)^*\right)
\end{aligned}$$

and therefore there exist $\left((x^*_{i,n}, y^*_{i,n}), r_{i,n}\right)$, $(i = 1, \ldots, m)$, $((x^*_n, y^*_n), s_n)$ and $((u^*_n, v^*_n), t_n) \in X^* \times Y^* \times \mathbb{R}$, satisfying

$$\left.\begin{aligned}
&\left((x^*_{i,n}, y^*_{i,n}), r_{i,n}\right) \in \mathrm{epi}(z^* \circ F_i)^*, \ (i = 1, \ldots, m) \\
&((x^*_n, y^*_n), s_n) \in \mathrm{epi}(z^* \circ G)^* \\
&((u^*_n, v^*_n), t_n) \in \mathrm{epi}(z^* \circ H)^*
\end{aligned}\right\} \tag{3.13}$$

such that

$$\begin{aligned}
&\sum_{i=1}^m \left((x^*_{i,n}, y^*_{i,n}), r_{i,n}\right) + ((x^*_n, y^*_n), s_n) + ((u^*_n, v^*_n), t_n) \\
&\xrightarrow{\|\cdot\|_{X^* \times Y^*}} \left((z^* \circ A, 0), \langle z^* \circ A, \overline{x}\rangle - \left(\sum_{i=1}^m z^* \circ F_i + z^* \circ G + z^* \circ H\right)(\overline{x}, h(\overline{x}))\right).
\end{aligned} \tag{3.14}$$

By applying Lemma 3.1, (3.13) may be rewritten as

$$\begin{cases}
\left((x^*_{i,n}, y^*_{i,n}), r_{i,n}\right) \in \mathrm{epi}(z^* \circ F_i)^* \Longleftrightarrow (x^*_{i,n}, r_{i,n}) \in \mathrm{epi}(z^* \circ f_i)^* \text{ and } y^*_{i,n} = 0, \\
((x^*_n, y^*_n), s_n) \in \mathrm{epi}(z^* \circ G)^* \Longleftrightarrow x^*_n = 0 \text{ and } (y^*_n, s_n) \in \mathrm{epi}(z^* \circ g)^* \\
((u^*_n, v^*_n), t_n) \in \mathrm{epi}(z^* \circ H)^* \Longleftrightarrow (u^*_n, t_n) \in \mathrm{epi}(-v^*_n \circ h)^* \text{ and } -v^*_n \in Y^*_+.
\end{cases}$$

Since $x^*_n = 0$, $y^*_{i,n} = 0$, $(i = 1, \ldots, m)$ and $(\overline{x}, h(\overline{x})) \in \mathrm{epi} h$, then the expression (3.14) becomes equivalent to

$$\begin{cases}
\displaystyle\sum_{i=1}^m x^*_{i,n} + u^*_n \xrightarrow{\|\cdot\|_{X^*}} z^* \circ A \\
y^*_n + v^*_n \xrightarrow{\|\cdot\|_{Y^*}} 0 \\
\displaystyle\sum_{i=1}^m r_{i,n} + s_n + t_n \longrightarrow \langle z^* \circ A, \overline{x}\rangle - \sum_{i=1}^m z^* \circ f_i(\overline{x}) - z^* \circ g(h(\overline{x})).
\end{cases}$$

The proof is complete. □

4 Sequential Proper Efficiency Optimality Conditions

In this section, we are concerned with the general multiobjective fractional programming problem

$$\text{(P)} \quad \inf_{\substack{x\in C\\ h(x)\in -Y_+}} \left\{ \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_m(x)}{g_m(x)} \right) \right\},$$

with $C \subset X$ be a nonempty, closed and convex subset and $Y_+ \subset Y$ be a nonempty closed convex cone. The functions f_i, -$g_i : X \longrightarrow \mathbb{R}$, $(i = 1, \ldots, m)$ are convex, lower semicontinuous and $h : X \longrightarrow \overline{Y}$ is a proper, Y_+-convex and Y_+-epi-closed mapping. We assume that $f_i(x) \geq 0,\ g_i(x) > 0\ \ (i = 1, \ldots, m)$. Let e_i denote the ith unit coordinate vector and e the vector of ones in $\mathbb{R}^m$. For $\epsilon \geq 0$, the positive hull of the subset $S := \{e_i + \epsilon e : i = 1, \cdots, m\}$ is defined by

$$K_\epsilon := \left\{ \sum_{i=1}^{m} \alpha_i(e_i + \epsilon e) :\ \alpha_i \geq 0 \right\}. \tag{4.1}$$

In fact K_ϵ is a convex cone and the origin belongs to K_ϵ. The positive polar cone of K_ϵ is denoted by

$$K^*_\epsilon := \{v \in \mathbb{R}^m :\ \langle v, y\rangle \geqq 0, \quad \forall y \in K_\epsilon\}. \tag{4.2}$$

It is easy to see that

$$K_\epsilon\backslash\{0\} \subset \text{int}(\mathbb{R}^m_+) \subset \mathbb{R}^m_+\backslash\{0\} \subset \text{int}(K^*_\epsilon). \tag{4.3}$$

We endow the finite-dimensional space $Z := \mathbb{R}^m$ with its natural order induced by the nonnegative orthant $Z_+ := \mathbb{R}^m_+$, and we shall use the following characterization of proper efficiency (see Luc-Soleimani-damaneh [9]).

Proposition 4.1. *A point $\overline{x} \in C \cap h^{-1}(-Y_+)$ is properly efficient solution of the problem (P) if and only if there exists some $\epsilon > 0$ and there is no $x \in C \cap h^{-1}(-Y_+)$ such that*

$$\left(\frac{f_1(x)}{g_1(x)} - \frac{f_1(\overline{x})}{g_1(\overline{x})}, \ldots, \frac{f_m(x)}{g_m(x)} - \frac{f_m(\overline{x})}{g_m(\overline{x})} \right) \in -K^*_\epsilon. \tag{4.4}$$

We associate to problem (P) the multiobjective convex minimization problem

$$(P_{\overline{x}}) \qquad \inf_{x\in C\cap h^{-1}(-Y_+)} \left\{ (f_1(x) - v_1 g_1(x), \ldots, f_m(x) - v_m g_m(x)) \right\}, \tag{4.5}$$

where $\overline{x} \in C \cap h^{-1}(-Y_+)$ and $\nu_i := \frac{f_i(\overline{x})}{g_i(\overline{x})}$ $(i = 1, \ldots, m)$. The problem $(P_{\overline{x}})$ is intimately related to (P). The crucial relationship between (P) and $(P_{\overline{x}})$, which will serve our purposes, is stated in the following lemma

Lemma 4.2. *A point $\overline{x} \in C \cap h^{-1}(-Y_+)$ is Henig properly efficient solution for problem (P) if and only if, $\overline{x}$ is Henig properly efficient solution for problem $(P_{\overline{x}})$.*

Proof. $(\Longrightarrow)$. Suppose in the contrary that $\overline{x}$ is not Henig proper efficient solution for $(P_{\overline{x}})$, then it follows from Proposition 4.1 that for any $\epsilon > 0$, there exists some $x_0 \in C \cap h^{-1}(-Y_+)$ such that

$$(f_1(x_0) - \nu_1 g_1(x_0), \ldots, f_m(x_0) - \nu_m g_m(x_0)) \in -K^*_\epsilon, \tag{4.6}$$

i.e.

$$\sum_{i=1}^{m}(f_i(x_0)-\nu_i g_i(x_0))\left(\alpha_i+\epsilon\sum_{j=1}^{m}\alpha_j\right)\leq 0,\quad \forall(\alpha_1,\ldots,\alpha_m)\in\mathbb{R}_+^m. \tag{4.7}$$

By using the fact that $g_i(x_0)>0$ $(i=1,\ldots,m)$ and by substituting respectively in (4.7) α_i and α_j by $\frac{\alpha_i}{g_i(x_0)}$ and $\frac{\alpha_j}{g_i(x_0)}$ $(j=1,\ldots,m)$ we obtain that for any $\epsilon>0$, we have

$$\sum_{i=1}^{m}\left(\frac{f_i(x_0)}{g_i(x_0)}-\nu_i\right)\left(\alpha_i+\epsilon\sum_{j=1}^{m}\alpha_j\right)\leq 0,\ \forall(\alpha_1,\ldots,\alpha_m)\in\mathbb{R}_+^m, \tag{4.8}$$

which means for any $\epsilon>0$

$$\left(\frac{f_1(x_0)}{g_1(x_0)}-\nu_1,\ldots,\frac{f_m(x_0)}{g_m(x_0)}-\nu_m\right)\in -K_\epsilon^*.$$

According to Proposition 4.1 this contradicts the fact that $\overline{x}$ is Henig proper efficient solution for the problem (P).

($\Longleftarrow$). We proceed by contradiction. Assume that $\overline{x}$ is not Henig proper efficient solution of the problem (P), then according to Proposition 4.1, we have for any $\epsilon>0$, there exists some $x_0\in C\cap h^{-1}(-Y_+)$ such that

$$\left(\frac{f_1(x_0)}{g_1(x_0)}-\nu_1,\ldots,\frac{f_m(x_0)}{g_m(x_0)}-\nu_m\right)\in -K_\epsilon^*.$$

So, for any $(\alpha_1,\ldots,\alpha_m)\in\mathbb{R}_+^m$, we have

$$\sum_{i=1}^{m}\left(\frac{f_i(x_0)-\nu_i g_i(x_0)}{g_i(x_0)}\right)\left(\alpha_i+\epsilon\sum_{j=1}^{m}\alpha_j\right)\leq 0. \tag{4.9}$$

Since $g_i(x_0)>0$ $(i=1,\ldots,m)$, then by substituting respectively in (4.9) α_i and α_j by $\alpha_i g_i(x_0)$ and $\alpha_j g_i(x_0)$ $(j=1,\ldots,m)$ we get that for any $\epsilon>0$,

$$\sum_{i=1}^{m}(f_i(x_0)-\nu_i g_i(x_0))\left(\alpha_i+\epsilon\sum_{j=1}^{m}\alpha_j\right)\leq 0,\quad \forall(\alpha_1,\ldots,\alpha_m)\in\mathbb{R}_+^m, \tag{4.10}$$

which yields that $\epsilon>0$,

$$(f_1(x_0)-\nu_1 g_1(x_0),\ldots,f_i(x_0)-\nu_i g_i(x_0))\in -K_\epsilon^*, \tag{4.11}$$

this contradicts, by virtue of Proposition 4.1 the fact that $\overline{x}$ is Henig proper efficient solution of the problem $(P_{\overline{x}})$.

□

Now, by using this equivalence result in conjunction with Theorem 3.2, we can establish the first characterization of sequential optimality conditions for problem (P) by means of the epigraphs of the conjugate of data functions.

Theorem 4.3. *Let $\overline{x} \in C \cap h^{-1}(-Y_+)$ and $\nu_i := \frac{f_i(\overline{x})}{g_i(\overline{x})}$ $(i = 1, \dots, m)$. Then, $\overline{x}$ is a properly efficient solution of the problem (P) in the sense of Henig, if and only if, there exist $(\lambda_1, \dots, \lambda_m) \in (\mathbb{R}_+ \setminus \{0\})^m$, $(x^*_{i,n}, a_{i,n}) \in \operatorname{epi}(\lambda_i f_i)^*$, $(w^*_{i,n}, b_{i,n}) \in \operatorname{epi}(\lambda_i \nu_i(-g_i))^*$, $(c^*_n, d_n) \in \operatorname{epi}\delta^*_C$, $y^*_n \in Y^*_+$, $s_n \in \mathbb{R}_+$, $v^*_n \in -Y^*_+$ and $(u^*_n, t_n) \in \operatorname{epi}(-v^*_n \circ h)^*$ such that*

$$\begin{cases} \displaystyle\sum_{i=1}^m x^*_{i,n} + \sum_{i=1}^m w^*_{i,n} + c^*_n + u^*_n \xrightarrow{\|\cdot\|_{X^*}} 0 \\ y^*_n + v^*_n \xrightarrow{\|\cdot\|_{Y^*}} 0 \\ \displaystyle\sum_{i=1}^m a_{i,n} + \sum_{i=1}^m b_{i,n} + d_n + t_n + s_n \longrightarrow 0. \end{cases}$$

Proof. According to Lemma 4.2, $\overline{x}$ is Henig properly efficient solution for problem (P) if and only if, $\overline{x}$ is Henig properly efficient solution for problem $(P_{\overline{x}})$. By introducing the vector indicator mappings δ^v_C and $\delta^v_{-Y_+}$, the problem $(P_{\overline{x}})$ may be written equivalently as

$$\inf_{x \in X} \left\{ F_{\overline{x}}(x) + \delta^v_C(x) + (\delta^v_{-Y_+} \circ h)(x) \right\} \tag{4.12}$$

where $F_{\overline{x}} : X \longrightarrow \mathbb{R}^m$ is defined for any $x \in X$, by

$$F_{\overline{x}}(x) := (f_1(x) - \nu_1 g_1(x), \dots, f_m(x) - \nu_m g_m(x)). \tag{4.13}$$

Hence $\overline{x}$ is Henig properly efficient solution for problem (P) if and only if,

$$0 \in \partial^p \left(F_{\overline{x}} + \delta^v_C + \delta^v_{-Y_+} \circ h \right)(\overline{x}). \tag{4.14}$$

Let us consider the following vector mappings $L_i : X \longrightarrow \overline{\mathbb{R}^m}$, $(i = 1, \dots, 2m+1)$ defined by

$$L_i(x) := \begin{cases} (0, \dots, f_i(x), \dots, 0) & \text{if } (i = 1, \dots, m) \\ (0, \dots, \nu_{m-i}(-g_{m-i}(x)), \dots, 0) & \text{if } (i = m, \dots, 2m) \\ \delta^v_C(x) & \text{if } i = 2m+1, \end{cases} \tag{4.15}$$

where the effective domain of the mappings L_i, $(i = 1, \dots, 2m+1)$ are given by

$$\operatorname{dom} L_i := \begin{cases} \operatorname{dom} f_i = X & \text{if } (i = 1, \dots, m) \\ \operatorname{dom}[\nu_{i-m}(-g_{i-m})] = X & \text{if } (i = m+1, \dots, 2m) \\ \operatorname{dom}\delta^v_C = C & \text{if } i = 2m+1. \end{cases} \tag{4.16}$$

It is easy to see that the mappings $L_i : X \longrightarrow \overline{\mathbb{R}^m}$, $(i = 1, \dots, 2m+1)$ are proper, $\mathbb{R}^m_+$-convex and strict star $\mathbb{R}^m_+$-lower semicontinuous. By means of these notations the expression (4.14) may be written equivalently as

$$0 \in \partial^p \left(\sum_{i=1}^{2m+1} L_i + \delta^v_{-Y_+} \circ h \right)(\overline{x}). \tag{4.17}$$

Let us note that the mapping $\delta^v_{-Y_+}$ is proper, $\mathbb{R}^m_+$-convex and strict star $\mathbb{R}^m_+$-lower semicontinuous since Y_+ is a nonempty convex and closed cone. Moreover, let us recall that $\delta^v_{-Y_+}$

is $(Y_+, \mathbb{R}^m_+)$-nondecreasing (see [4]) and the condition $\bar{x} \in C \cap h^{-1}(-Y_+)$, can be equivalently rewritten as $\bar{x} \in (\bigcap_{i=1}^{2m+1} \mathrm{dom} L_i) \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom}\delta^v_{-Y_+})$. Hence, the mapping L_i $(i = 1, \dots, 2m+1)$, $\delta^v_{-Y_+}$ and h satisfy together all the assumptions of Theorem 3.2 and then there exist $z^* = (\lambda_1, \dots, \lambda_m) \in ((\mathbb{R}^m_+)^*)^\circ = (\mathbb{R}_+ \setminus \{0\})^m$, $(\overline{x}^*_{i,n}, r_{i,n}) \in \mathrm{epi}(z^* \circ L_i)^*$ $(i = 1, \dots, 2m+1)$, $(y^*_n, s_n) \in \mathrm{epi}(z^* \circ \delta^v_{-Y_+})^*$, $v^*_n \in -Y^*_+$ and $(u^*_n, t_n) \in \mathrm{epi}(-v^*_n \circ h)^*$ such that

$$\begin{cases} \displaystyle\sum_{i=1}^{2m+1} \overline{x}^*_{i,n} + u^*_n \xrightarrow{\|\cdot\|_{X^*}} 0 & (4.18) \\ y^*_n + v^*_n \xrightarrow{\|\cdot\|_{Y^*}} 0 \\ \displaystyle\sum_{i=1}^{2m+1} r_{i,n} + t_n + s_n \longrightarrow -\sum_{i=1}^{2m+1} (z^* \circ L_i)(\overline{x}) - (z^* \circ \delta^v_{-Y_+})(h(\overline{x})). & (4.19) \end{cases}$$

It is easy to check that $z^* \circ \delta^v_C = \delta_C$ and $z^* \circ \delta^v_{-Y_+} = \delta_{-Y_+}$. Therefore, for each $i = 1, \dots, 2m+1$ the conditions $(\overline{x}^*_{i,n}, r_{i,n}) \in \mathrm{epi}(z^* \circ L_i)^*$, $(y^*_n, s_n) \in \mathrm{epi}(z^* \circ \delta^v_{-Y_+})^*$ and (4.19) can be rewritten by means of data functions f_i, g_i, δ_C and δ_{-Y_+} as follows

$$(\overline{x}^*_{i,n}, r_{i,n}) \in \mathrm{epi}(z^* \circ L_i)^* \Longleftrightarrow \begin{cases} (x^*_{i,n}, a_{i,n}) := (\overline{x}^*_{i,n}, r_{i,n}) \in \mathrm{epi}(\lambda_i f_i)^*, \\ \qquad \text{if } \ i \in \{1, \dots, m\} \\ (w^*_{i,n}, b_{i,n}) := (\overline{x}^*_{i+m,n}, r_{i+m,n}) \in \mathrm{epi}(\lambda_i \nu_i(-g_i))^*, \\ \qquad \text{if } i \in \{1, \dots, m\} \\ (c^*_n, d_n) := (\overline{x}^*_{2m+1,n}, r_{2m+1,n}) \in \mathrm{epi}(z^* \circ \delta^v_C)^* = \\ \qquad \mathrm{epi}(\delta_C)^*, \text{if } i = 2m+1, \end{cases} \tag{4.20}$$

$$(y^*_n, s_n) \in \mathrm{epi}(z^* \circ \delta^v_{-Y_+})^* = \mathrm{epi}(\delta_{-Y_+})^* = \mathrm{epi}(\delta_{Y^*_+}) = Y^*_+ \times \mathbb{R}_+ \tag{4.21}$$

and

$$\begin{aligned} (4.19) \Longleftrightarrow \sum_{i=1}^m a_{i,n} + \sum_{i=1}^m b_{i,n} + d_n + t_n + s_n \longrightarrow & -\sum_{i=1}^m \lambda_i f_i(\overline{x}) - \sum_{i=1}^m \lambda_i \nu_i(-g_i)(\overline{x}) - \delta_C(\overline{x}) \\ & - \delta_{-Y_+}(h(\overline{x})) \\ = & -\sum_{i=1}^m \lambda_i (f_i(\overline{x}) + \nu_i(-g_i)(\overline{x})) - \delta_C(\overline{x}) - \delta_{-Y_+}(h(\overline{x})). \end{aligned}$$

Since $f_i(\overline{x}) + \nu_i(-g_i)(\overline{x}) = 0$, $\overline{x} \in C$ and $h(\overline{x}) \in -Y_+$ then the above limit reduces to

$$\sum_{i=1}^m a_{i,n} + \sum_{i=1}^m b_{i,n} + d_n + t_n + s_n \longrightarrow 0. \tag{4.22}$$

Moreover, (4.18) can be written as follows

$$\sum_{i=1}^m x^*_{i,n} + \sum_{i=1}^m w^*_{i,n} + c^*_n + u^*_n \xrightarrow{\|\cdot\|_{X^*}} 0. \tag{4.23}$$

This completes the proof. □

For the second characterization, by applying Theorem 4.3 and Theorem 2.3 we express the sequential optimality conditions of the problem (P) in terms of limits for the ϵ-subdifferential ($\epsilon \geq 0$) of the functions involved at the minimizer.

Theorem 4.4. *Let $\overline{x} \in C \cap h^{-1}(-Y_+)$ and $\nu_i := \frac{f_i(\overline{x})}{g_i(\overline{x})}$ $(i = 1, \ldots, m)$. Then, $\overline{x}$ is Henig properly efficient solution for problem (P), if and only if, there exist $(\lambda_1, \ldots, \lambda_m) \in (\mathbb{R}_+ \setminus \{0\})^m$, $\gamma_n \geq 0$, $x^*_{i,n} \in \partial_{\gamma_n}(\lambda_i f_i)(\overline{x})$, $w^*_{i,n} \in \partial_{\gamma_n}(\lambda_i \nu_i(-g_i))(\overline{x})$ $(i = 1, \ldots, m)$, $c^*_n \in N_{\gamma_n}(\overline{x}, C)$, $v^*_n \in -Y^*_+$, $y^*_n \in Y^*_+ \cap N_{\gamma_n}(h(\overline{x}), -Y_+)$ and $u^*_n \in \partial_{\gamma_n}(-v^*_n \circ h)(\overline{x})$ such that*

$$\begin{cases} \gamma_n \longrightarrow 0 \\ \displaystyle\sum_{i=1}^m x^*_{i,n} + \sum_{i=1}^m w^*_{i,n} + c^*_n + u^*_n \xrightarrow{\|\cdot\|_{X^*}} 0 \\ y^*_n + v^*_n \xrightarrow{\|\cdot\|_{Y^*}} 0. \end{cases}$$

Proof. By virtue of Theorem 4.3, $\overline{x}$ is properly efficient solution of the problem (P) in the sense of Henig, if and only if, there exist $(\lambda_1, \ldots, \lambda_m) \in (\mathbb{R}_+ \setminus \{0\})^m$, $(x^*_{i,n}, a_{i,n}) \in \operatorname{epi}(\lambda_i f_i)^*$, $(w^*_{i,n}, b_{i,n}) \in \operatorname{epi}(\lambda_i \nu_i(-g_i))^*$ $(i = 1, \ldots m)$, $(c^*_n, d_n) \in \operatorname{epi}\delta^*_C$, $y^*_n \in Y^*_+$, $s_n \in \mathbb{R}_+$, $v^*_n \in -Y^*_+$ and $(u^*_n, t_n) \in \operatorname{epi}(-v^*_n \circ h)^*$ such that

$$\begin{cases} \displaystyle\sum_{i=1}^m x^*_{i,n} + \sum_{i=1}^m w^*_{i,n} + c^*_n + u^*_n \xrightarrow{\|\cdot\|_{X^*}} 0 & (4.24) \\ y^*_n + v^*_n \xrightarrow{\|\cdot\|_{Y^*}} 0 & (4.25) \\ \displaystyle\sum_{i=1}^m a_{i,n} + \sum_{i=1}^m b_{i,n} + d_n + t_n + s_n \longrightarrow 0. & (4.26) \end{cases}$$

Since $(y^*_n, s_n) \in \operatorname{epi}\delta_{Y^*_+} = \operatorname{epi}\delta^*_{-Y_+}$, it follows according to Theorem 2.3, there exist $\alpha_{i,n}$, $\beta_{i,n}$, η_n, θ_n, $\epsilon_n \in \mathbb{R}_+$ such that $x^*_{i,n} \in \partial_{\alpha_{i,n}}(\lambda_i f_i)(\overline{x})$, $w^*_{i,n} \in \partial_{\beta_{i,n}}(\lambda_i \nu_i(-g_i))(\overline{x})$, $c^*_n \in N_{\eta_n}(\overline{x}, C)$, $y^*_n \in N_{\theta_n}(h(\overline{x}), -Y_+)$, $u^*_n \in \partial_{\epsilon_n}(-v^*_n \circ h)(\overline{x})$ and

$$\begin{cases} a_{i,n} = \langle x^*_{i,n}, \overline{x} \rangle + \alpha_{i,n} - (\lambda_i f_i)(\overline{x}), \ \ i = 1, \ldots, m \\ b_{i,n} = \langle w^*_{i,n}, \overline{x} \rangle + \beta_{i,n} - (\lambda_i \nu_i(-g_i))(\overline{x}), \ \ i = 1, \ldots, m \\ d_n = \langle c^*_n, \overline{x} \rangle + \eta_n \\ s_n = \langle y^*_n, h(\overline{x}) \rangle + \theta_n \\ t_n = \langle u^*_n, \overline{x} \rangle + \epsilon_n - (-v^*_n \circ h)(\overline{x}). \end{cases}$$

By adding the terms of the above equalities and using the fact that $f_i(\overline{x}) + \nu_i(-g_i)(\overline{x}) = 0$, we obtain

$$\begin{aligned}
\sum_{i=1}^m a_{i,n}+\sum_{i=1}^m b_{i,n}+d_n+t_n+s_n &= \sum_{i=1}^m[\langle x^*_{i,n},\overline{x}\rangle+\alpha_{i,n}-(\lambda_i f_i)(\overline{x})]\\
&\quad+\sum_{i=1}^m[\langle w^*_{i,n},\overline{x}\rangle+\beta_{i,n}-(\lambda_i\nu_i(-g_i))(\overline{x})]\\
&\quad+\langle c^*_n,\overline{x}\rangle+\eta_n+\langle y^*_n,h(\overline{x})\rangle+\theta_n+\langle u^*_n,\overline{x}\rangle+\epsilon_n\\
&\quad-(-v^*_n\circ h)(\overline{x}).\\
&= \left\langle\sum_{i=1}^m x^*_{i,n}+\sum_{i=1}^m w^*_{i,n}+c^*_n+u^*_n,\overline{x}\right\rangle+\langle y^*_n+v^*_n,h(\overline{x})\rangle\\
&\quad+\sum_{i=1}^m\alpha_{i,n}+\sum_{i=1}^m\beta_{i,n}+\eta_n+\theta_n+\epsilon_n.
\end{aligned}$$

It follows from (4.24), (4.25) and (4.26) that

$$\sum_{i=1}^m\alpha_{i,n}+\sum_{i=1}^m\beta_{i,n}+\eta_n+\epsilon_n\longrightarrow 0,\quad n\longmapsto+\infty. \tag{4.27}$$

Moreover, since $\alpha_{i,n}$, $\beta_{i,n}$, η_n, θ_n, $\epsilon_n\in\mathbb{R}_+$, we get from (4.27) that $\alpha_{i,n}\longrightarrow 0$, $\beta_{i,n}\longrightarrow 0$, $\eta_n\longrightarrow 0$, $\theta_n\longrightarrow 0$, $\epsilon_n\longrightarrow 0$ as $n\longmapsto+\infty$. By setting $\gamma_n:=\max_{1\le i\le m}\{\alpha_{i,n},\beta_{i,n},\eta_n,\theta_n,\epsilon_n\}$, it follows that $x^*_{i,n}\in\partial_{\gamma_n}(\lambda_i f_i)(\overline{x})$, $w^*_{i,n}\in\partial_{\gamma_n}(\lambda_i\nu_i(-g_i))(\overline{x})$, $c^*_n\in N_{\gamma_n}(\overline{x},C)$, $N_{\gamma_n}(h(\overline{x}),-Y_+)$, $u^*_n\in\partial_{\gamma_n}(-v^*_n\circ h)(\overline{x})$ with $\gamma_n\longrightarrow 0$, as $n\longmapsto+\infty$.

Conversely, suppose that there exist $(\lambda_1,\dots,\lambda_m)\in(\mathbb{R}_+\setminus\{0\})^m$, $\gamma_n\ge 0$, $x^*_{i,n}\in\partial_{\gamma_n}(\lambda_i f_i)(\overline{x})$, $w^*_{i,n}\in\partial_{\gamma_n}(\lambda_i\nu_i(-g_i))(\overline{x})$ $(i=1,\dots,m)$, $c^*_n\in N_{\gamma_n}(\overline{x},C)$, $v^*_n\in -Y^*_+$, $y^*_n\in Y^*_+\cap N_{\gamma_n}(h(\overline{x}),-Y_+)$, $u^*_n\in\partial_{\gamma_n}(-v^*_n\circ h)(\overline{x})$ and

$$\begin{cases}\gamma_n\longrightarrow 0\\ \displaystyle\sum_{i=1}^m x^*_{i,n}+\sum_{i=1}^m w^*_{i,n}+c^*_n+u^*_n\xrightarrow{\|\cdot\|_{X^*}}0\\ y^*_n+v^*_n\xrightarrow{\|\cdot\|_{Y^*}}0.\end{cases}$$

For any $x\in C\cap h^{-1}(-Y_+)$, $y\in -Y_+$ and any positive integer n, we have

$$\begin{aligned}
\lambda_i f_i(x)-\lambda_i f_i(\overline{x}) &\ge \langle x^*_{i,n},x-\overline{x}\rangle-\gamma_n,\ \ (i=1,\dots,m)\\
\lambda_i\nu_i(-g_i)(x)-\lambda_i\nu_i(-g_i)(\overline{x}) &\ge \langle w^*_{i,n},x-\overline{x}\rangle-\gamma_n,\ \ (i=1,\dots,m)\\
0 &\ge \langle c^*_n,x-\overline{x}\rangle-\gamma_n\\
0 &\ge \langle y^*_n,y-h(\overline{x})\rangle-\gamma_n\\
(-v^*_n\circ h)(x)-(-v^*_n\circ h)(\overline{x}) &\ge \langle u^*_n,x-\overline{x}\rangle-\gamma_n.
\end{aligned}$$

By adding them up and by taking $y:=h(x)$, we get

$$\begin{aligned}
&\sum_{i=1}^m\lambda_i(f_i+\nu_i(-g_i))(x)-\sum_{i=1}^m\lambda_i(f_i+\nu_i(-g_i))(\overline{x})+\langle v^*_n+y^*_n,h(\overline{x})-h(x)\rangle\\
&\qquad\ge\langle\sum_{i=1}^m x^*_{i,n}+\sum_{i=1}^m w^*_{i,n}+c^*_n+u^*_n,x-\overline{x}\rangle-5\gamma_n.
\end{aligned}$$

By taking the limit as $n \longmapsto +\infty$ in both terms of the above inequality, we deduce that

$$\sum_{i=1}^{m} \lambda_i(f_i + \nu_i(-g_i))(x) - \sum_{i=1}^{m} \lambda_i(f_i + \nu_i(-g_i))(\overline{x}) \geq 0,\ \forall x \in C \cap h^{-1}(-Y_+) \tag{4.28}$$

i.e.

$$\sum_{i=1}^{m} \lambda_i(f_i + \nu_i(-g_i))(x) + \delta_C(x) + (\delta_{-Y_+} \circ h)(x) - \sum_{i=1}^{m} \lambda_i(f_i + \nu_i(-g_i))(\overline{x}) - \delta_C(\bar{x}) - \delta_{-Y_+} \circ h(\bar{x}) \geq 0,\ \forall x \in X.$$

By setting $z^* := (\lambda_1, \dots, \lambda_m)$, it is clear that $z^* \in (\mathbb{R}_+ \backslash \{0\})^m = ((\mathbb{R}^m_+)^*)^\circ$. As $z^* \circ \delta^v_C = \delta_C$ and $z^* \circ \delta^v_{-Y_+} = \delta_{-Y_+}$, it follows that

$$0 \in \partial \Big(z^* \circ (f_1 + \nu_1(-g_1), \dots, f_m + \nu_m(-g_m)) + z^* \circ \delta^v_C + z^* \circ \delta^v_{-Y_+} \circ h \Big) (\overline{x}) \tag{4.29}$$

i.e.

$$0 \in \partial \Big(z^* \circ \Big((f_1 - \nu_1 g_1, \dots, f_m - \nu_m g_m) + \delta^v_C + \delta^v_{-Y_+} \circ h \Big) \Big) (\overline{x}). \tag{4.30}$$

The mapping $\Big((f_1 - \nu_1 g_1, \dots, f_m - \nu_m g_m) + \delta^v_C + \delta^v_{-Y_+} \circ h \Big)$ is obviously $\mathbb{R}^m_+$-convex and by virtue of scalarization Theorem 2.5, we get

$$0 \in \partial^p \Big((f_1 - \nu_1 g_1, \dots, f_m - \nu_m g_m) + \delta^v_C + \delta^v_{-Y_+} \circ h \Big) (\overline{x}), \tag{4.31}$$

which means that $\overline{x}$ is Henig properly efficient solution for problem $(P_{\overline{x}})$ and by Lemma 4.2, $\overline{x}$ is Henig properly efficient solution for problem (P). The proof is complete. □

Similarly, we establish the sequential optimality conditions for (P) in terms of the subdifferentials of the functions involved.

Theorem 4.5. *Let $\overline{x} \in C \cap h^{-1}(-Y_+)$ and $\nu_i := \frac{f_i(\overline{x})}{g_i(\overline{x})}$ $(i = 1, \dots, m)$. Then, $\overline{x}$ is Henig properly efficient solution for problem (P), if and only if, there exist $(\lambda_1, \dots, \lambda_m) \in (\mathbb{R}_+ \setminus \{0\})^m$, $x_{i,n} \in X$, $w_{i,n} \in X$, $c_n \in C$, $u_n \in \operatorname{dom}(-v^*_n \circ h)$, $y_n \in -Y_+$, $x^*_{i,n} \in \partial(\lambda_i f_i)(x_{i,n})$, $w^*_{i,n} \in \partial(\lambda_i \nu_i(-g_i))(w_{i,n})$, $c^*_n \in N(c_n, C)$, $y^*_n \in N(y_n, -Y_+)$, $v^*_n \in -Y^*_+$ and $u^*_n \in \partial(-v^*_n \circ h)(u_n)$ such that*

$$\begin{cases} x_{i,n} \xrightarrow{\|\cdot\|_X} \overline{x},\ w_{i,n} \xrightarrow{\|\cdot\|_X} \overline{x},\ c_n \xrightarrow{\|\cdot\|_X} \overline{x},\ u_n \xrightarrow{\|\cdot\|_X} \overline{x}, \\ \sum_{i=1}^{m} x^*_{i,n} + \sum_{i=1}^{m} w^*_{i,n} + c^*_n + u^*_n \xrightarrow{\|\cdot\|_{X^*}} 0,\ y^*_n + v^*_n \xrightarrow{\|\cdot\|_{Y^*}} 0, \\ \lambda_i f_i(x_{i,n}) - \langle x^*_{i,n}, x_{i,n} - \overline{x} \rangle \longrightarrow \lambda_i f_i(\overline{x})\ (i = 1, \dots, m) \\ \lambda_i \nu_i(-g_i)(w_{i,n}) - \langle w^*_{i,n}, w_{i,n} - \overline{x} \rangle \longrightarrow \lambda_i \nu_i(-g_i)(\overline{x}) \quad (i = 1, \dots, m) \\ \langle c^*_n, c_n - \overline{x} \rangle \longrightarrow 0,\ \langle y^*_n, y_n - h(\bar{x}) \rangle \longrightarrow 0 \\ \langle u^*_n, u_n - \overline{x} \rangle + \langle v^*_n, h(u_n) - h(\overline{x}) \rangle \longrightarrow 0. \end{cases}$$

Proof. According to Theorem 4.4, $\overline{x}$ is Henig properly efficient solution for problem (P), if and only if, there exist $(\lambda_1, \dots, \lambda_m) \in (\mathbb{R}_+ \setminus \{0\})^m$, $\gamma_n \geq 0$, $\overline{x}^*_{i,n} \in \partial_{\gamma_n}(\lambda_i f_i)(\overline{x})$, $\overline{w}^*_{i,n} \in$

$\partial_{\gamma_n}(\lambda_i\nu_i(-g_i))(\overline{x}),\ \overline{c}_n^* \in N_{\gamma_n}(\overline{x},C),\ v_n^* \in -Y_+^*,\ \overline{y}_n^* \in Y_+^* \cap N_{\gamma_n}(h(\overline{x}),-Y_+),\ \overline{u}_n^* \in \partial_{\gamma_n}(-v_n^* \circ h)(\overline{x})$ and

$$\begin{cases} \gamma_n \longrightarrow 0 \\ \overline{y}_n^* + v_n^* \xrightarrow{\|\cdot\|_{Y^*}} 0 & (4.32) \\ \sum_{i=1}^m \overline{x}_{i,n}^* + \sum_{i=1}^m \overline{w}_{i,n}^* + \overline{c}_n^* + \overline{u}_n^* \xrightarrow{\|\cdot\|_{X^*}} 0. & (4.33) \end{cases}$$

As $\overline{x}_{i,n}^* \in \partial_{\gamma_n}(\lambda_i f_i)(\overline{x}),\ \overline{w}_{i,n}^* \in \partial_{\gamma_n}(\lambda_i\nu_i(-g_i))(\overline{x}),\ \overline{c}_n^* \in N_{\gamma_n}(\overline{x},C),\ \overline{y}_n^* \in N_{\gamma_n}(h(\overline{x}),-Y_+),\ \overline{u}_n^* \in \partial_{\gamma_n}(-v_n^* \circ h)(\overline{x})$ then by applying Theorem 2.4, we obtain the existence of $x_{i,n} \in \operatorname{dom}(\lambda_i f_i) = X,\ w_{i,n} \in \operatorname{dom}(\lambda_i\nu_i(-g_i)) = X,\ c_n \in C,\ y_n \in -Y_+,\ u_n \in \operatorname{dom}(-v_n^* \circ h)$, $x_{i,n}^* \in \partial(\lambda_i f_i)(x_{i,n}),\ w_{i,n}^* \in \partial(\lambda_i\nu_i(-g_i))(w_{i,n}),\ c_n^* \in N(c_n,C),\ y_n^* \in N(y_n,-Y_+),\ u_n^* \in \partial(-v_n^* \circ h)(u_n)$ such that

$$\begin{cases} \|x_{i,n}-\bar{x}\|_X \le \sqrt{\gamma_n},\ \|w_{i,n}-\bar{x}\|_X \le \sqrt{\gamma_n},\ \|c_n-\bar{x}\|_X \le \sqrt{\gamma_n},\ \|u_n-\bar{x}\|_X \le \sqrt{\gamma_n} \\ \qquad \|y_n-h(\bar{x})\|_Y \le \sqrt{\gamma_n}, & (4.34) \\ \|x_{i,n}^*-\bar{x}_{i,n}^*\|_{X^*} \le \sqrt{\gamma_n},\ \|w_{i,n}^*-\bar{w}_{i,n}^*\|_{X^*} \le \sqrt{\gamma_n}, \|c_n^*-\bar{c}_n^*\|_{X^*} \le \sqrt{\gamma_n} \\ \quad \|u_n^*-\bar{u}_n^*\|_{X^*} \le \sqrt{\gamma_n},\ \|y_n^*-\bar{y}_n^*\|_{Y^*} \le \sqrt{\gamma_n}, & (4.35) \\ \left.\begin{array}{l} \mid \lambda_i f_i(x_{i,n}) - \langle x_{i,n}^*, x_{i,n}-\bar{x}\rangle - \lambda_i f_i(\bar{x}) \mid \le 2\gamma_n \\ \mid -\lambda_i\nu_i g_i(w_{i,n}) - \langle w_{i,n}^*, w_{i,n}-\bar{x}\rangle + \lambda_i\nu_i g_i(\bar{x}) \mid \le 2\gamma_n \\ \mid \delta_C(c_n) - \langle c_n^*, c_n-\bar{x}\rangle - \delta_C(\bar{x}) \mid \le 2\gamma_n \\ \mid \delta_{-Y_+}(y_n) - \langle y_n^*, y_n-h(\bar{x})\rangle - \delta_{-Y_+}(h(\bar{x})) \mid \le 2\gamma_n \\ \mid (-v_n^* \circ h)(u_n) - \langle u_n^*, u_n-\bar{x}\rangle - (-v_n^* \circ h)(\bar{x}) \mid \le 2\gamma_n \end{array}\right\}. & (4.36) \end{cases}$$

By letting $n \longmapsto +\infty$, we get from (4.34) and (4.36) that

$$\begin{cases} x_{i,n} \xrightarrow{\|\cdot\|_X} \bar{x},\ w_{i,n} \xrightarrow{\|\cdot\|_X} \bar{x},\ c_n \xrightarrow{\|\cdot\|_X} \bar{x}, u_n \xrightarrow{\|\cdot\|_X} \bar{x},\ y_n \xrightarrow{\|\cdot\|_Y} h(\bar{x}) \\ \lambda_i f_i(x_{i,n}) - \langle x_{i,n}^*, x_{i,n}-\bar{x}\rangle \longrightarrow \lambda_i f_i(\bar{x}), \\ -\lambda_i\nu_i g_i(w_{i,n}) - \langle w_{i,n}^*, w_{i,n}-\bar{x}\rangle \longrightarrow -\lambda_i\nu_i g_i(\bar{x}), \\ \left.\begin{array}{l} \delta_C(c_n) - \langle c_n^*, c_n-\bar{x}\rangle \longrightarrow \delta_C(\bar{x}). \\ \delta_{-Y_+}(y_n) - \langle y_n^*, y_n-h(\bar{x})\rangle \longrightarrow \delta_{-Y_+}(h(\bar{x})) \end{array}\right\} & (4.37) \\ (-v_n^* \circ h)(u_n) - \langle u_n^*, u_n-\bar{x}\rangle - (-v_n^* \circ h)(\bar{x}) \longrightarrow 0. \end{cases}$$

Since $c_n \in C,\ y_n \in -Y_+$ and $\overline{x} \in C \cap h^{-1}(-Y_+)$, the expression (4.37) reduces to

$$\langle c_n^*, c_n-\bar{x}\rangle \longrightarrow 0, \quad \langle y_n^*, y_n-h(\bar{x})\rangle \longrightarrow 0. \tag{4.38}$$

Moreover, we have

$$\|y_n^*+v_n^*\|_{Y^*} \quad = \quad \|y_n^*-\overline{y}_n^*+v_n^*+\overline{y}_n^*\|_{Y^*} \le \|y_n^*-\overline{y}_n^*\|_{Y^*} + \|\overline{y}_n^*+v_n^*\|_{Y^*},$$

and

$$\begin{array}{rl} & \|\sum_{i=1}^{m} x_{i,n}^* + \sum_{i=1}^{m} w_{i,n}^* + c_n^* + u_n^*\|_{X^*} \\ = & \|\sum_{i=1}^{m} x_{i,n}^* - \sum_{i=1}^{m} \overline{x}_{i,n}^* + \sum_{i=1}^{m} w_{i,n}^* - \sum_{i=1}^{m} \overline{w}_{i,n}^* + c_n^* - \overline{c}_n^* + u_n^* - \overline{u}_n^* \\ & + \sum_{i=1}^{m} \overline{x}_{i,n}^* + \sum_{i=1}^{m} \overline{w}_{i,n}^* + \overline{c}_n^* + \overline{u}_n^*\|_{X^*} \\ \leq & \sum_{i=1}^{m} \|x_{i,n}^* - \overline{x}_{i,n}^*\|_{X^*} + \sum_{i=1}^{m} \|w_{i,n}^* - \overline{w}_{i,n}^*\|_{X^*} + \|c_n^* - \overline{c}_n^*\|_{X^*} + \|u_n^* - \overline{u}_n^*\|_{X^*} \\ & + \|\sum_{i=1}^{m} \overline{x}_{i,n}^* + \sum_{i=1}^{m} \overline{w}_{i,n}^* + \overline{c}_n^* + \overline{u}_n^*\|_{X^*}. \end{array}$$

and hence by letting $n \longrightarrow +\infty$, it follows from (4.32), (4.33) and (4.35) that

$$\begin{cases} y_n^* + v_n^* \xrightarrow{\|\cdot\|_{Y^*}} 0, \\ \sum_{i=1}^{m} x_{i,n}^* + \sum_{i=1}^{m} w_{i,n}^* + c_n^* + u_n^* \xrightarrow{\|\cdot\|_{X^*}} 0. \end{cases}$$

Conversely, assume that the preceding sequential optimality conditions hold. Then, for any $x \in C \cap h^{-1}(-Y_+)$, $y \in -Y_+$ and any positive integer n, we have

$$\begin{array}{rcl} \lambda_i f_i(x) & \geq & \lambda_i f_i(x_{i,n}) + \langle x_{i,n}^*, x - x_{i,n}\rangle, \quad (i = 1, \ldots, m) \\ \lambda_i \nu_i(-g_i)(x) & \geq & \lambda_i \nu_i(-g_i)(w_{i,n}) + \langle w_{i,n}^*, x - w_{i,n}\rangle, \quad (i = 1, \ldots, m) \\ 0 & \geq & \langle c_n^*, x - c_n\rangle \\ 0 & \geq & \langle y_n^*, y - y_n\rangle \\ 0 & \geq & -(-v_n^* \circ h)(x) + (-v_n^* \circ h)(u_n) + \langle u_n^*, x - u_n\rangle. \end{array}$$

By adding these inequalities and taking for all $x \in C \cap h^{-1}(-Y_+)$, $y := h(x)$, we get

$$\begin{array}{rcl} \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{m} \lambda_i \nu_i(-g_i)(x) & \geq & \sum_{i=1}^{m} \big(\lambda_i f_i(x_{i,n}) - \langle x_{i,n}^*, x_{i,n} - \overline{x}\rangle\big) \\ & & + \sum_{i=1}^{m} \big(\lambda_i \nu_i(-g_i)(w_{i,n}) - \langle w_{i,n}^*, w_{i,n} - \overline{x}\rangle\big) \\ & & -\langle c_n^*, c_n - \overline{x}\rangle - \langle y_n^*, y_n - h(\overline{x})\rangle - \langle u_n^*, u_n - \overline{x}\rangle \\ & & -\langle v_n^*, h(u_n^*) - h(\overline{x})\rangle + \langle y_n^* + v_n^*, h(x) - h(\overline{x})\rangle \\ & & + \left\langle \sum_{i=1}^{m} x_{i,n}^* + \sum_{i=1}^{m} w_{i,n}^* + c_n^* + u_n^*, x - \overline{x} \right\rangle. \end{array}$$

Passing to the limit as $n \longmapsto +\infty$, we obtain

$$\sum_{i=1}^{m} \lambda_i (f_i + \nu_i(-g_i))(x) - \sum_{i=1}^{m} \lambda_i (f_i + \nu_i(-g_i))(\overline{x}) \geq 0, \ \forall x \in C \cap h^{-1}(-Y_+) \tag{4.39}$$

i.e.

$$\sum_{i=1}^{m}\lambda_i(f_i+\nu_i(-g_i))(x)+\delta_C(x)+\delta_{-Y_+}\circ h(x)-\sum_{i=1}^{m}\lambda_i(f_i+\nu_i(-g_i))(\overline{x})-\delta_C(\bar{x}) - \delta_{-Y_+}\circ h(\bar{x})\geq 0,\ \forall x\in X.$$

By using the similar arguments used in the proof of the converse of Theorem 4.4, we get

$$0\in\partial^p\left((f_1-\nu_1 g_1,\dots,f_m-\nu_m g_m)+\delta_C^v+\delta_{-Y_+}^v\circ h\right)(\overline{x}), \tag{4.40}$$

which means that $\overline{x}$ is Henig properly efficient solution for problem $(P_{\overline{x}})$ and by Lemma 4.2, $\overline{x}$ is Henig properly efficient solution for problem (P). The proof is complete. □

We now illustrate the above results with the help of an example of multiobjective fractional programming problem, where the standard Lagrange multiplier condition can not be derived due to the lack of constraint qualification and the sequential conditions hold. For this, we will need to establish the standard necessary and sufficient optimality conditions for a feasible point $\overline{x}$ to be an efficient solution for problem (P) under a constraint qualification.

Theorem 4.6. *Let $f_i,\ -g_i : X\longrightarrow\mathbb{R}$ be $2m$ convex functions such that $f_i(x)\geq 0$ for any $x\in C\cap h^{-1}(-Y_+)$ $(i=1,\dots,m)$ and $h : X\longrightarrow Y\cup\{+\infty_Y\}$ be a proper and Y_+-convex mapping. Let us consider the following constraint qualification*

$$(C.Q.M_0.R_0)\begin{cases}(X,\|.\|_X)\ \text{and}\ (Y,\|.\|_Y)\ \text{are two real reflexive Banach spaces,}\\ \exists a\in C\cap\operatorname{dom}h\ \text{such that}\\ h(a)\in-\operatorname{int}Y_+.\end{cases}$$

Suppose that $\operatorname{int}Y_+\neq\emptyset$ ($\operatorname{int}Y_+$ stands for the topological interior of Y_+) and the constraint qualification $(C.Q.M_0.R_0)$ is satisfied. Then $\overline{x}\in C\cap h^{-1}(-Y_+)$ is Henig properly efficient solution for problem (P), if and only if, there exists $y^*\in Y_+^*$ such that $\langle y^*,h(\overline{x})\rangle=0$ and

$$0\in\partial\left(\sum_{i=1}^{m}(\lambda_i(f_i-\nu_i g_i)+\delta_C+y^*\circ h\right)(\overline{x}). \tag{4.41}$$

Proof. Following the proof of Theorem 4.3, we have $\overline{x}$ is Henig properly efficient solution of (P) if and only if,

$$0\in\partial^p\left(F_{\overline{x}}+\delta_C^v+\delta_{-Y_+}^v\circ h\right)(\overline{x}) \tag{4.42}$$

where $F_{\overline{x}} : X\longrightarrow\mathbb{R}^m$ is defined for any $x\in X$ by

$$F_{\overline{x}}(x):=(f_1(x)-\nu_1 g_1(x),\dots,f_m(x)-\nu_m g_m(x)). \tag{4.43}$$

According to scalarization Theorem 2.5, there exists $z^*=(\lambda_1,\dots,\lambda_m)\in(\mathbb{R}_+\setminus\{0\})^m$ such that

$$0\in\partial(z^*\circ F_{\overline{x}}+z^*\circ\delta_C^v+z^*\circ\delta_{-Y_+}^v\circ h)(\overline{x}). \tag{4.44}$$

Since $z^*\circ\delta_C^v=\delta_C$ and $z^*\circ\delta_{-Y_+}^v=\delta_{-Y_+}$, therefore we obtain

$$0\in\partial\left(\sum_{i=1}^{m}(\lambda_i(f_i-\nu_i g_i)+\delta_C+\delta_{-Y_+}\circ h\right)(\overline{x}). \tag{4.45}$$

The constraint qualification $(C.Q.M_0.R_0)$ show that the indicator function δ_{-Y_+} is continuous at $h(a)$ and by applying a formula in [2] by Combari-Laghdir-Thibault, concerning the computation of the subdifferential of the composite of a nondecreasing convex function with a convex mapping taking values in a partially ordered topological vector space, there exists $y^* \in \partial\delta_{-Y_+}(h(\overline{x})) = N(h(\overline{x}), -Y_+)$ such that

$$0 \in \partial \left(\sum_{i=1}^{m} (\lambda_i (f_i - \nu_i g_i) + \delta_C + y^* \circ h \right) (\overline{x}). \tag{4.46}$$

The condition $y^* \in N(h(\overline{x}), -Y_+)$ is equivalent to $y^* \in Y_+^*$ and $\langle y^*, h(\overline{x}) \rangle = 0$. □

We now give an example of multiobjective fractional programming problem, where the standard optimality condition can not be derived due to the lack of constraint qualification and the sequential optimality conditions hold.

Example 4.7. Let us consider the following multiobjective fractional problem

$$(\mathrm{Q}) \begin{cases} \inf \left(\frac{2x}{y+3}, \frac{2x}{y^2+1} \right) \\ \left((\max\{0, x\})^2, \sqrt{x^2+y^2} - y \right) \in -\mathbb{R}_+^2 \\ (x, y) \in \mathbb{R}_+ \times [0, 1], \end{cases}$$

where $h(x, y) = \left((\max\{0, x\})^2, \sqrt{x^2+y^2} - y \right)$, $f_1(x, y) = 2x$, $f_2(x, y) = -2x$, $g_1(x, y) = y + 3$, $g_2(x, y) = -y^2 - 1$ and $C = \mathbb{R}_+ \times [0, 1]$. The euclidian space $Y = \mathbb{R}^2$ is equipped with the natural order induced by the nonnegative orthant $Y_+ = \mathbb{R}_+^2$. Obviously $Y_+^* = \mathbb{R}_+^2$. Let $(\overline{x}, \overline{y}) = \left(0, \frac{1}{2}\right)$ be a feasible point and $\nu_i = \frac{f_i(\overline{x}, \overline{y})}{g_i(\overline{x}, \overline{y})}$ $(i = 1, 2)$. Then $\nu_1 = \nu_2 = 0$. Let us observe that for any $(x, y) \in C$, we have $h(x, y) \in \mathbb{R}_+^2$, and hence the feasible set of problem (Q) is given by $S = \{0\} \times [0, 1]$ which yields that the constraint qualification $(C.Q.M_0.R_0)$ does not hold. By taking $(\lambda_1, \lambda_2) := (1, 1)$, $v_n^* := (0, 0)$, it follows that the epigraph of the conjugate functions turn out to be $\mathrm{epi}(\lambda_1 f_1)^* = \mathrm{epi} f_1^* = \{(2, 0)\} \times [0, +\infty[$, $\mathrm{epi}(\lambda_2 f_2)^* = \mathrm{epi} f_2^* = \{(-2, 0)\} \times [0, +\infty[$, $\mathrm{epi}(\lambda_i \nu_i(-g_i))^* = \{(0, 0)\} \times [0, +\infty[$, $\mathrm{epi}\delta_C^* = \bigcup_{\alpha > 0} \{((0, \alpha), \alpha)\} \cup \{0\} \times \mathbb{R} \times [0, +\infty[$, $\mathrm{epi}(-v_n^* \circ h)^* = \{(0, 0)\} \times [0, +\infty[$. For $i = 1, 2$, by taking $(x_{1,n}^*, a_{1,n}) := ((2, 0), \frac{1}{n}) \in \mathrm{epi}(\lambda_1 f_1)^*$, $(x_{2,n}^*, a_{2,n}) := ((-2, 0), \frac{1}{n}) \in \mathrm{epi}(\lambda_2 f_2)^*$, $(w_{i,n}^*, b_{i,n}) := ((0, 0), \frac{1}{n}) \in \mathrm{epi}(\lambda_i \nu_i(-g_i))^*$, $(c_n^*, d_n) := ((0, \frac{1}{n}), \frac{1}{n}) \in \mathrm{epi}\delta_C^*$, $y_n^* := (0, 0) \in (\mathbb{R}_+)^2$, $s_n := \frac{1}{n} \in \mathbb{R}_+$, and $(u_n^*, t_n) = ((0, 0), 0) \in \mathrm{epi}(-v_n^* \circ h)^*$ such that

$$\begin{cases} \displaystyle\sum_{i=1}^{2} x_{i,n}^* + \sum_{i=1}^{2} w_{i,n}^* + c_n^* + u_n^* = (0, \frac{1}{n}) \xrightarrow{\|\cdot\|_{\mathbb{R}^2}} 0 \\ y_n^* + v_n^* = (0, 0) \xrightarrow{\|\cdot\|_{\mathbb{R}^2}} 0 \\ \displaystyle\sum_{i=1}^{2} a_{i,n} + \sum_{i=1}^{2} b_{i,n} + d_n + t_n + s_n = \frac{6}{n} \longrightarrow 0. \end{cases}$$

Therefore, by Theorem 4.3 the point $(\overline{x}, \overline{y})$ is a Henig efficient solution for (Q).

Conclusion and Discussions. In this paper, we consider sequential necessary and sufficient optimality conditions for a constrained multiobjective fractional programming problem characterizing a Henig proper efficient solution in terms of the subdifferentials of the data functions. To get these optimality conditions, we formulate a corresponding equivalent scalar convex problem by using a parametric approach and sequential subdifferential calculus. Let us recall [4] that the set of Henig proper efficient solutions $E^p(f, C)$, the set of efficient solutions $E^e(f, C)$ and the set of weakly efficient solutions $E^w(f, C)$ satisfy the following inclusions

$$E^p(f, C) \subseteq E^e(f, C) \subseteq E^w(f, C),$$

which means that our paper can be considered as an extension of earlier papers [10] and [12].

Let us note that the subdifferential calculus have been established in the convex framework by using the crucial Brondsted-Rockafellar theorem.

To the best of our knowledge, we have not explored any work dealing with sequential subdifferential calculus in the non-convex framework. In a forthcoming work, we will try to study the same problem (P) where all the data functions $g_1, \ldots, g_m$ are supposed convex. In this case, by using a parametric approach we transform the problem (P) equivalently as a DC programming problem. Also, we will attempt to examine the robustness of the problem (P) from optimality conditions and duality point of view.

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