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NECESSARY OPTIMALITY CONDITIONS AND A SEMI-SMOOTH NEWTON APPROACH FOR AN OPTIMAL CONTROL PROBLEM OF A COUPLED SYSTEM OF SAINT-VENANT EQUATIONS AND ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this article necessary optimality conditions for Saint-Venant equations coupled to ordinary differential equations (ODE) are derived rigorously. The Saint-Venant equations are first-order hyperbolic partial differential equations (PDE) and model here the fluid in a container that is moved by a truck that is subject to Newton's law of motion. The acceleration of the truck may be controlled.
We describe the mathematical model and the corresponding tracking-type optimal control problem. First we prove existence and uniqueness for the coupled ODE-PDE problem locally in time. For sufficiently small times, we derive the first-order necessary optimality conditions in the corresponding function spaces. Furthermore we prove existence of optimal controls.
The optimality system is formulated in the setting of a semi-smooth operator equation in Hilbert spaces which we solve numerically by a semi-smooth Newton method. We close with a numerical example for a typical driving maneuver.

1. Introduction

We consider a container with a fluid that is subject to the Saint-Venant equations, that model fluid flow in shallow water. The container is mounted on a vehicle that is subject to Newton's law of motion. Our optimal control problem (OCP), see Subsection 2.1, is to minimize tracking type functionals for the fluid height and the horizontal fluid velocity and to minimize the control costs, subject to the coupled ODE-PDE system and box constraints for the control. As a prototype example we consider a truck with a fluid container as load, which are coupled by a spring-damper element. We derive the corresponding model in Section 2 in detail. Our problem exhibits partial differential equations, i.e. the Saint-Venant equations, that are fully coupled to ordinary differential equations, given by Newton's law of motion. We may control the acceleration of the truck, that by means of the spring-damper force, acts as an indirect Neumann boundary control on the fluid height and as a distributed control on the fluid velocity. Conversely, the momentum of the moving fluid in the

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container affects the motion of the truck. The scaled problem is summarized in Subsection 2.2.

Optimal control problems for Saint-Venant equations without coupling to ODE have been considered in [3, 28], for instance. However, they consider distributed controls of the fluid level equation and only control costs. A similar control problem with a tracking-type objective for a given fluid level profile h_d, where a container is accelerated directly (representing, e.g., a container fixed on a moving horizontal conveyor) is considered by Coron *et al.* [6]. Global boundary controllability of the Saint-Venant equations between steady states is demonstrated by Gugat and Leugering [12]. Moreover, they have considered the controllability of the Saint-Venant equations in the situation of sloped canals with friction [13]. In their result, it turns out to be crucial that the considered terminal time is not too small. The latter is due to the finite propagation velocity. We note that we do not consider such a Dirichlet boundary control for h in our model.

Coupled systems involving ODEs as well as PDEs and their control have been considered only for particular examples. Hömberg *et al.* [8, 17, 18] and Gupta *et al.* [14] consider a model for laser hardening of steel, involving the heat equation and a differential equation describing phase transitions. In another example, in a gallium-arsenide crystal the phase transition of arsenic-rich droplets is modelled by an ODE for the free boundary and by the quasi-linear diffusion equation. This is further coupled to the PDE of linear elasticity, modelling mechanical stresses within the crystal. For this model and its well-posedness see [21], for the optimal control of a resulting macroscopic model see [22]. Pesch *et al.* [5, 30, 34] consider a hypersonic rocket car subject to driving dynamics and to the heat equation with state constraints on the temperature. This represents a simplified model for the re-entry of a spacecraft into atmosphere. In [23] the optimal control of a quarter car model by an electrorheological damper is considered. Here the behavior of the elastic tyre is modelled by the PDE of linear elasticity and the spring-damper element is subject to an ODE. In addition, the latter example involves a complementarity condition modelling the free road contact. In [24] an elastic crane beam subject to linear elasticity coupled to the dynamics of a pendulum, modelling the crane trolley and the applied load, is studied. To the knowledge of the authors, no analytic results have been derived for our particular kind of coupled ODE-PDE problem so far.

Our problem has been stated and solved numerically by a first-discretize-then-optimize (FDTO) approach in [11]. In contrast, in this paper we follow a first-optimize-then-discretize (FOTD) approach. Here we consider a so-called all-at-once approach, i.e. we solve for the states and the control simultaneously. We do not replace the states by the control-to-state operator as for a reduced objective that depends only on the control. First, in Section 3 we prove existence and uniqueness for the coupled state equation, that is not standard, and then apply an abstract result for the existence of optimal controls (Theorem 4.1). In Section 4 we derive analytically the necessary optimality conditions (NOC), including the adjoint differential equations, by a Lagrangian based approach. Furthermore, from the NOC we deduce the existence of the optimal control. Our problem has in common with the general situation in [16, Ch. 1], that a Tikhonov regularization is considered and that we have a tracking-type part of the objective and control box constraints.

In contrast, our problem exhibits a coupled system involving also ODEs and, in addition, terminal conditions in the objective.

We solve the NOC numerically by a semi-smooth Newton method, see Subsection 4.4. For a safety breaking maneuver of the truck-load system we present numerical results in Subsection 5.2. We close with a short discussion in Section 6.

2. Mathematical model

A typical example for Saint-Venant equations coupled to ODEs, corresponding to Newton's law of motion, is a moving truck with a fluid container as load that is not fixed permanently (see Fig. 1). We recall the model derived in [11, Sect. 1]. We consider a finite time interval $[0, T]$ with a terminal time T, that is not considered as a free parameter in this study. For ease of presentation the truck may move in one dimension only. The truck and the container are considered in a fixed coordinate system $(X_1, X_2) \in \mathbb{R}^2$. The horizontal position of the truck is represented by d_{tr} and the container is located at the x-coordinate d_w, the corresponding velocities are the time derivatives $v_{tr} := \dot{d}_{tr}$ and $v_w := \dot{d}_w$. The container has length L, height H (here defined different as in [11]), and a given (continuously differentiable) bottom profile $B(x)$, $0 \leq x \leq L$. The container could move in the horizontal direction and its mounting to the truck frame is modelled by a linear spring-damper element with damping coefficient k and spring rate c. We consider a moving coordinate system $(x_1, x_2) \in [0, L] \times [0, H]$ for the container. For keeping notation short, we write $x = x_1$. The height of the fluid in the container is represented by $h(t, x)$ and the fluid velocity in x-direction by $v(t, x)$. As domain for the fluid we introduce $Q := (0, T) \times (0, L)$ with the spatial boundary $\Gamma := (0, T) \times \{0, L\}$. From the geometry of the container, we see directly the natural state constraints

$$B(x) \leq h(t, x) + B(x) \leq H \quad \forall (t, x) \in \overline{Q}.$$

In addition, we require for our model

$$0 < \underline{h} \leq h(t, x) \leq \overline{h} < H - B(x) \quad \forall (t, x) \in \overline{Q}, \tag{2.1}$$

since we do not want to deal with issues modelling contacts (when the bottom runs dry or the fluid spills over) in this study. The fluid is assumed to be an incompressible Newtonian fluid (like water). The mass of the truck is denoted by m_{tr} and the mass of the container by m_w. The whole system may be controlled by the acceleration $u(t)$ of the truck.

We have for the force that acts between the truck and the container

$$F(d_{tr}, d_w, \dot{d}_{tr}, \dot{d}_w) := c(d_{tr} - d_w + \bar{d}) + k(\dot{d}_{tr} - \dot{d}_w). \tag{2.2}$$

For the offset $\bar{d} := d_w(0) - d_{tr}(0)$ the fluid container is at rest initially. If we assume $d_w^{(T)} = d_{tr}^{(T)} + \bar{d} = 0$, the container rests for the terminal time, too. The fluid is subject to the one-dimensional Saint-Venant equations

$$h_t + (hv)_x = 0, \qquad (t, x) \in Q, \tag{2.3}$$

$$v_t + \left(\frac{1}{2}v^2 + gh\right)_x = -gB_x - \frac{1}{m_w}F(d_{tr}, d_w, \dot{d}_{tr}, \dot{d}_w), \qquad (t, x) \in Q. \tag{2.4}$$

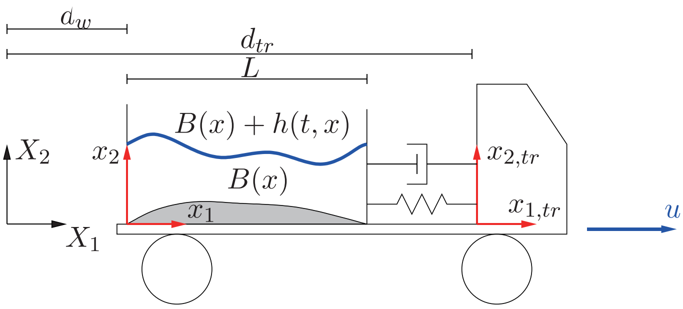

FIGURE 1. Truck with a fluid container as load, illustration of geometric quantities.

where we have exploited $h \neq 0$ in order to derive (2.4) from the conservation of linear momentum (see [11, Sect. 1]) for details). These PDEs are complemented by the initial and boundary conditions

$$h(0,x) = h^{(0)}(x), \qquad x \in [0,L], \tag{2.5}$$

$$v(0,x) = v^{(0)}(x), \qquad x \in [0,L], \tag{2.6}$$

$$h_x = -B_x - \frac{1}{gm_w} F(d_{tr}, d_w, \dot{d}_{tr}, \dot{d}_w), \qquad (t,x) \in \Gamma, \tag{2.7}$$

$$v = 0, \qquad (t,x) \in \Gamma, \tag{2.8}$$

where $h^{(0)}$ and $v^{(0)}$ are given functions. The Neumann boundary condition (2.7) follows from (2.4) and (2.8), see [11, Sect. 1] about the details.

The truck and the container observe Newton's law of motion yielding

$$m_{tr}\ddot{d}_{tr} = u - F(d_{tr}, d_w, \dot{d}_{tr}, \dot{d}_w), \qquad t \in [0,T], \tag{2.9}$$

$$m_w\ddot{d}_w = -\frac{m_w g}{L} \left[h(t,\cdot) + B\right]_0^L \qquad t \in [0,T]. \tag{2.10}$$

We set as initial conditions

$$d_{tr}(0) = d_{tr}^{(0)}, \qquad \dot{d}_{tr}(0) = v_{tr}^{(0)}, \qquad d_w(0) = d_w^{(0)}, \qquad \dot{d}_w(0) = v_w^{(0)}, \tag{2.11}$$

where $d_{tr}^{(0)}$, $v_{tr}^{(0)}$, $d_w^{(0)}$, and $v_w^{(0)}$ are given numbers.

2.1. **Optimal control problem.** Motivated by the coupling force (2.2), we simplify the equation system by replacing d_{tr} by the horizontal distance between truck and container position

$$d_\Delta := d_{tr} - d_w + \bar{d}.$$

We consider the velocities $v_\Delta = \dot{d}_\Delta$ and $v_w = \dot{d}_w$ as independent variables, such that only first order time-derivatives remain in our problem. The PDE states are (h,v) and the ODE states $(d_\Delta, d_w, v_\Delta, v_w)$. We refer to all state variables by $y = (h, v, d_\Delta, d_w, v_\Delta, v_w)$. Consistently, we abbreviate for the initial conditions $y^{(0)} := (h^{(0)}, v^{(0)}, d_\Delta^{(0)}, d_w^{(0)}, v_\Delta^{(0)}, v_w^{(0)})$. Moreover, we write $[\xi]_T := \xi(T) - \xi^{(T)}$ for the difference of a function ξ at terminal time T to a given terminal value $\xi^{(T)}$.

For suitable weights α_1, $\alpha_3 \geq 0$, α_2, α_4, $\alpha_5 > 0$ and σ_4, $\sigma_5 \in \mathbb{R}$, we would like to minimize the objective function

$$J(y,u) := \frac{\alpha_1}{2}\int_Q |h(t,x) - h_d(x)|^2\,dx\,dt + \frac{\alpha_2}{2}\int_0^T u(t)^2\,dt \\ + \frac{\alpha_3}{2}\int_Q v(t,x)^2\,dx\,dt + \sum_{I\in\{\Delta;w\}}\left(\frac{\alpha_4}{2}[d_I]_T^2 + \sigma_4[d_I]_T\right) \\ + \sum_{I\in\{\Delta;w\}}\left(\frac{\alpha_5}{2}[v_I]_T^2 + \sigma_5[v_I]_T\right), \tag{2.12}$$

modelling a tracking type term for a given fluid level h_d, the control effort, the kinetic energy of the fluid, and penalties for achieving given terminal positions $d_\Delta^{(T)}$, $d_w^{(T)}$ and velocities $v_\Delta^{(T)}$, $v_w^{(T)}$ of the truck-container distance and the container, respectively. The control u, being the acceleration of the truck, is subject to the control constraints

$$u_{min} \leq u \leq u_{max}, \tag{2.13}$$

representing that an infinite acceleration is technically not realizable. Note that a negative acceleration corresponds to braking. For the height of the fluid level, we have to require the state constraints (2.1) in principle. However, it turns out in our numerics that they never get active, so we may ignore the state constraints here. For a numerical study incorporating these state constraints into a FDTO approach and the resulting minor effects, see [11, Sect. 4].

The optimal control problem (OCP) reads

$$\min_{Y\times U} J(y,u), \tag{2.14}$$

subject to the system (2.3) - (2.11) and the control constraints (2.13). We work with the state spaces

$$Y_1 := L^2(0,T;H^1(0,L)) \times L^2(0,T;H_0^1(0,L)) \tag{2.15}$$

for the PDE states and

$$Y = Y_1 \times [H^2(0,T)]^2 \times [H^1(0,T)]^2. \tag{2.16}$$

for all states. Since we can prove further regularity of the states (see Section 3), we introduce a space $\tilde{Y}$ for all states in (3.1). The control space is $U = L^2(0,T)$.

Our OCP exhibits an indirect Neumann boundary control for the PDE (2.3) and an indirect distributed control for the PDE (2.4), where the control is acting by means of the force F. This coupling force F is determined by an ODE system, that is controlled directly in (2.9). A back coupling takes place via the Dirichlet boundary values of h entering into the ODE (2.10).

2.2. **Artificial viscosity and rescaled problem.** We regularize the hyperbolic equations (2.3) and (2.4) by introducing an artificial viscosity ε, $0 < \varepsilon \ll 1$. The motivation for this is that hyperbolic conservation laws exhibit non-unique solutions. The physically correct solution (satisfying the entropy principle) is selected [25, Example 2.2.6/Th. 3.3.28] by considering the regularized system, that is semi-linear parabolic and has a unique solution, in the limit of vanishing viscosity ε.

Furthermore, by this regularization we avoid to deal with shocks and rare-faction waves, that are typically encountered for these hyperbolic conservation laws [7, Subsection 11.3.2]. It turns out that for a given ε, T has to be chosen sufficiently small for our analytic existence and uniqueness results (Th. 3.2), locally in time, to hold. Thus there is a trade-off between a good approximation of the original Saint-Venant equations, i.e. $\varepsilon \to 0$, and the validity of the well-posedness of our model in time. The regularized solutions for fixed ε are again denoted by h and v.

In order to formulate our coupled system in a simpler form, in particular more accessible for numerics, we perform some scalings. We introduce a new time $\tilde{t} \in (0,1)$ by $t = T\tilde{t}$. In this study, the dedimensionalization of time simplifies the existence and uniqueness proof for the states and, moreover, turns out to have numerical advantages. Coherently, we write $\tilde{Q} := (0,1) \times (0,L)$ and $\tilde{\Gamma} := (0,1) \times \{0,L\}$. We introduce the mass ratio $\eta = m_{tr}/m_w$ and, in addition, scale $\tilde{u} := u/m_{tr}$, $\tilde{c} := c/m_{tr}$, $\tilde{k} := k/m_{tr}$. Finally, we write for the scaled counter-force on the spring-damper system

$$F_s(d_\Delta, v_\Delta) := -\frac{1}{g}(\tilde{c}d_\Delta + \tilde{k}v_\Delta) \tag{2.17}$$

and for the force on the container due to gravity of the fluid

$$F_f(h) := -\frac{g}{L}\left[h(t,\cdot) + B(\cdot)\right]_0^L . \tag{2.18}$$

For ease of presentation we abbreviate

$$F_c(d_\Delta, v_\Delta) = -B_x + \eta F_s(d_\Delta, v_\Delta). \tag{2.19}$$

This leads to the following initial-boundary value problem, following from (2.3) – (2.11),

$$h_t + T(hv)_x - \varepsilon T h_{xx} = 0, \qquad (t,x) \in \tilde{Q}, \tag{2.20}$$

$$h_x = F_c(d_\Delta, v_\Delta), \qquad (t,x) \in \tilde{\Gamma}, \tag{2.21}$$

$$h(0,x) = h^{(0)}(x), \qquad x \in [0,L], \tag{2.22}$$

$$v_t + T\left(\frac{1}{2}v^2 + gh\right)_x - \varepsilon T v_{xx} = TgF_c(d_\Delta, v_\Delta), \qquad (t,x) \in \tilde{Q}, \tag{2.23}$$

$$v = 0, \qquad (t,x) \in \tilde{\Gamma}, \tag{2.24}$$

$$v(0,x) = v^{(0)}(x), \qquad x \in [0,L], \tag{2.25}$$

$$\dot{d}_\Delta = T v_\Delta, \qquad t \in [0,1], \tag{2.26}$$

$$d_\Delta(0) = d_\Delta^{(0)} := 0, \tag{2.27}$$

$$\dot{d}_w = T v_w, \qquad t \in [0,1], \tag{2.28}$$

$$d_w(0) = d_w^{(0)}, \tag{2.29}$$

$$\dot{v}_\Delta = T\tilde{u} + TgF_s(d_\Delta, v_\Delta) + TF_f(h), \qquad t \in [0,1], \tag{2.30}$$

$$v_\Delta(0) = v_\Delta^{(0)} := v_{tr}^{(0)} - v_w^{(0)}, \tag{2.31}$$

$$\dot{v}_w = TF_f(h) \qquad t \in [0,1], \tag{2.32}$$

$$v_w(0) = v_w^{(0)}. \tag{2.33}$$

Note that on the right-hand-side of (2.30), the (time-scaled) effective acceleration, i.e. control plus acceleration due to fluid motion in the container, appears (see [11, Sect. 1] for details). For ease of notation, we drop the tildes on $\tilde{t}$, $\tilde{Q}$, $\tilde{\Gamma}$, $\tilde{u}$, $\tilde{c}$, and $\tilde{k}$ in the following.

3. Existence and uniqueness of states

(2.20) and (2.23) are semi-linear parabolic equations that are fully coupled to the ODEs (2.26), (2.28), (2.30), and (2.32). We start by considering existence and uniqueness for this coupled problem that is non-standard assuming a given control $u \in U = L^2(0,1)$ and a given terminal time $T > 0$.

An ODE can be identified as a special case of a PDE (elliptic 1st order), see for instance [19, Subsect. 2.6]. We consider the ODE-PDE system as PDE system. As state spaces we work here with

$$\begin{aligned}\tilde{Y}_1 :=&[H^1(0,1;L^2(0,L))]^2 \cap [L^2(0,1;H^2(0,L))]^2 \\ &\cap [L^\infty(0,1;H^1(0,L)) \times L^\infty(0,1;H_0^1(0,L))] \subset Y_1\end{aligned}$$

for the PDE states and

$$\tilde{Y} = \tilde{Y}_1 \times [H^2(0,1)]^2 \times [H^1(0,1)]^2 \subset Y \tag{3.1}$$

for all states. Both the state spaces Y_1 and Y (here to be understood with $T \equiv 1$) and the control space U are separable Hilbert spaces, while $\tilde{Y}_1$ and $\tilde{Y}$ are not. Note that we have the structure of a Gelfand triple $Y \subset U = U^* \subset Y^*$ that will be exploited for the derivation of optimality conditions below. For Bochner spaces we use established abbreviations like $L^2H^1 := L^2(0,1;H^1(0,L))$ or $L^2L^2 := L^2(Q)$ in the following.

For $\varepsilon > 0$ the regularized system for h and v is semi-linear parabolic and mathematically well-posed in $\tilde{Y}_1$ for right-hand sides in L^2 and initial data in H_0^1, see [7, Subsect. 7.1.3] for a proof in case of homogeneous Dirichlet boundary conditions. As a preliminary we need

Lemma 3.1 (Estimate for the trace with explicit constant for 1d time-space intervals)**.** *For a function $h \in L^2(0,1;H^1(0,L))$ there holds*

$$\|h\|^2_{L^2(\Gamma)} \le \max\{4/L; L/2\}\, \|h\|^2_{L^2H^1}.$$

The proof is straightforward and relies on Hölder's inequality, the fundamental theorem of calculus and the Cauchy-Schwarz inequality. For details see, e.g., the proof of the similar result [20, Th. II.1 b)].

Now we may prove

Theorem 3.2 (Local existence and uniqueness of the coupled ODE-PDE system for a given control)**.** *For $u \in L^2(0,1)$, $h^{(0)} \in H^1(0,L)$, $v^{(0)} \in H_0^1(0,L)$, $B \in C^1(0,L)$, $\varepsilon \in \mathbb{R}^+$, and sufficiently small times $T > 0$, there exists a unique solution $(h, v, d_\Delta, d_w, v_\Delta, v_w)$ of the coupled system (2.20) – (2.33) s.t. $(h,v)^\top \in \tilde{Y}_1$, $d_\Delta, d_w \in H^2(0,1)$, and $v_\Delta, v_w \in H^1(0,1)$.*

The idea of proof relies on the Banach fixed point theorem and an estimate yielding a factor proportional to $\sqrt{T}$ in the contraction constant. This method has been used e.g. by Niethammer [29] in case of an ODE, that results from a free boundary, coupled to the Laplace PDE. A similar proof is given in [21] for a coupled ODE-PDE problem involving a free boundary, a quasi-linear diffusion PDE and linear elasticity.

Proof. We are going to apply the Banach fixed point theorem in the space

$$\mathcal{M} = \{(h,v)^\top \in Y_1, d_\Delta, d_w \in H^2(0,1), v_\Delta, v_w \in H^1(0,1) \,|\, \|v\|_{L^\infty L^\infty} \leq \kappa\},$$

where κ is a fixed arbitrary positive number. At first we determine *a priori* estimates. In the following we use frequently the Young and Hölder inequalities in order to compensate certain terms from the right-hand sides. We test equation (2.20) by h, using the boundary conditions and Lemma 3.1, and find

$$\begin{aligned}
&\sup_{t\in(0,1)} \frac{1}{2}\|h(t)\|^2_{L^2(0,L)} + \frac{\varepsilon}{4}T\|h_x\|^2_{L^2L^2} \\
&\leq \frac{1}{2}\|h^{(0)}\|^2_{L^2(0,L)} + T\left(\frac{\kappa^2}{2\varepsilon} + 2\varepsilon C_\Gamma\right)\|h\|^2_{L^2L^2} + TL\|F_c\|^2_{L^2(0,1)}
\end{aligned}$$

where $C_\Gamma := \max\{4/L, L/2\}$ is the constant appearing in Lemma 3.1. We multiply by 2 and by Gronwall's inequality this yields

$$\begin{aligned}
&\sup_{t\in(0,1)} \|h(t)\|^2_{L^2(0,L)} + \frac{\varepsilon}{2}T\|h_x\|^2_{L^2L^2} \\
&\leq \left(\|h^{(0)}\|^2_{L^2(0,L)} + 2TL\|F_c\|^2_{L^2(0,1)}\right)\exp\left(\left(\frac{\kappa^2}{\varepsilon} + 4\varepsilon C_\Gamma\right)T\right).
\end{aligned}$$

For fixed ε and $T \leq C_h$ with a sufficiently small constant C_h depending on ε, this gives an H^1 estimate on h:

$$\|h\|^2_{L^2H^1} \leq \frac{2}{\varepsilon}\|h^{(0)}\|^2_{L^2(0,L)} + 4TL\|F_c\|^2_{L^2(0,1)}.$$

We turn to estimates for v,

$$\begin{aligned}
&\sup_{t\in(0,1)} \|v(t)\|^2_{L^2(0,L)} + \varepsilon T\|v_x\|^2_{L^2L^2} \\
&\leq \|v^{(0)}\|^2_{L^2(0,L)} + \frac{T}{\varepsilon}g^2\|h\|^2_{L^2L^2} + \|v\|^2_{L^2L^2} + T^2L^2g^2\|F_c\|^2_{L^2(0,1)}.
\end{aligned}$$

Again by compensating terms and Gronwall, we have

$$\begin{aligned}
&\sup_{t\in(0,1)} \|v(t)\|^2_{L^2(0,L)} + \varepsilon T\|v_x\|^2_{L^2L^2} \\
&\leq \left(\|v^{(0)}\|^2_{L^2(0,L)} + \frac{Tg^2}{\varepsilon}\|h\|^2_{L^2L^2} + T^2L^2g^2\|F_c\|^2_{L^2(0,1)}\right)(1+\exp(1)).
\end{aligned}$$

Now we test the equation (2.23) for v with v_t and get

$$\begin{aligned}
&\|v_t\|^2_{L^2L^2} + 2\varepsilon T \sup_{t\in(0,1)} \|v_x(t)\|^2_{L^2(0,L)} \\
&\leq 2\varepsilon T\|v_x^{(0)}\|^2_{L^2(0,L)} + 4T^2\|v_x\|_{L^2L^2} + 4T^2g^2\|h_x\|^2_{L^2L^2} + 4T^2L^2g^2\|F_c\|^2_{L^2(0,1)}.
\end{aligned}$$

By Gronwall

$$\|v_t\|^2_{L^2L^2} + 2\varepsilon T \sup_{t\in(0,1)} \|v_x(t)\|^2_{L^2(0,L)}$$
$$\le \left(2\varepsilon T\|v_x^{(0)}\|^2_{L^2(0,L)} + 4T^2g^2\|h_x\|^2_{L^2L^2} + 4T^2L^2g^2\|F_c\|^2_{L^2(0,1)}\right)\exp\left(\frac{2}{\varepsilon}T\right),$$

Due to the embedding $H^1 \hookrightarrow L^\infty$ in 1d we get with a nonnegative constant C_L from the last estimate

$$\|v\|^2_{L^\infty L^\infty} \le C_L\left(\|v_x^{(0)}\|^2_{L^2(0,L)} + \frac{2}{\varepsilon}Tg^2\left(L^2\|F_c\|^2_{L^2(0,1)} + \|h_x\|^2_{L^2L^2}\right)\right)\exp\left(\frac{2}{\varepsilon}T\right).$$

We will see below that the F_c term is dominated by a factor T. This shows with a $\kappa > \max\{\sqrt{2C_L}\|v_x^{(0)}\|_{L^2(0,L)}, \delta\}$, $0 < \delta < 1$ that we have for sufficiently small $T < \varepsilon\delta^2$ that there is a map from $\mathcal{M}$ into itself. The contraction property of the fixed point map follows analogously. The estimate for the coupling force term reads

$$\|F_c\|^2_{L^2(0,1)} \le C_B + 3\frac{\eta}{g}\left(c\|d_\Delta\|^2_{L^2(0,1)} + k\|v_\Delta\|^2_{L^2(0,1)}\right), \tag{3.2}$$

where C_B is a (nonnegative) constant depending on B_x. For the ODEs we have the estimates

$$\|d_I\|^2_{L^2(0,1)} \le 2T|d_I^{(0)}|^2 + 2T^2\|v_I\|^2_{L^2(0,L)}, \quad I \in \{\Delta, w\},$$

$$\|v_\Delta\|^2_{L^2(0,1)} \le 2T|v_\Delta^{(0)}|^2 + 6T^2\left(\|u\|^2_{L^2(0,1)} + \frac{g}{\eta}\|F_c\|^2_{L^2(0,1)} + \frac{gC_\Gamma}{L}\|h\|^2_{L^2H^1}\right) \tag{3.3}$$

and

$$\|v_w\|^2_{L^2(0,1)} \le 2T|v_w^{(0)}|^2 + 2T^2\frac{gC_\Gamma}{L}\|h\|^2_{L^2H^1}. \tag{3.4}$$

In particular, a factor T^2 enters into (3.3). Thus for sufficiently small T the term with $\|h\|_{L^2H^1}$ entering $\|F_c\|_{L^2(0,1)}$ via (3.2) yields a map into $Y_{1,1}$, being the first component of $\mathcal{M}$. This yields directly $h \in L^\infty L^2 \cap L^2H^1$, thus $v_\Delta, v_w \in H^1(0,1)$ (by the ODEs), $d_\Delta, d_w \in H^2(0,1)$ (again by the ODEs) and finally $v \in L^\infty H^1 \cap H^1L^2$. Thus we have a map into $\mathcal{M}$. From the v-PDE we see that further $v_{xx} \in L^2L^2$, thus $v \in L^2H^2$, too.

So far $h \in L^\infty L^2 \cap L^2H^1$. In order to decide whether $h \in L^\infty H^1 \cap H^1L^2$, we consider another fixed point argument. Assume $\|v_x\|_{L^\infty L^\infty} \le K$. We test (2.20) with $h_t\phi$, where we choose a sufficiently smooth ϕ with compact support in $(0, L)$ s.t. no boundary terms contribute,

$$\|h_t\|^2_{L^2L^2} + \varepsilon T \sup_{t\in(0,1)} \|h_x(t)\|^2_{L^2(0,L)}$$
$$\le \varepsilon T\|h_x^{(0)}\|^2_{L^2(0,L)} + K^2T^2\|h\|_{L^2L^2} + \kappa^2T^2\|h_x\|_{L^2L^2}.$$

Again by a Gronwall argument

$$\|h_t\|^2_{L^2L^2} + \varepsilon T \sup_{t\in(0,1)} \|h_x(t)\|^2_{L^2(0,L)}$$
$$\le \left(\varepsilon T\|h_x^{(0)}\|^2_{L^2(0,L)} + K^2T^2\|h\|_{L^2L^2}\right)\exp\left(\frac{\kappa^2}{\varepsilon}T\right)$$

we get $h \in H^1L^2 \cap L^\infty H^1$ in the interior. Thus from the PDE for h, we see that further $h_{xx} \in L^2L^2$, thus $h \in L^2H^2$, too. This justifies $K < \infty$ by the v-PDE and the Banach fixed point theorem. For the second h-estimate including the boundary, assume at first $F_c = 0$, proceed as above and then add the H^1C^0-function F_s.

By Banach's fixed point theorem the solution is unique. □

We observe that the estimates for h_x, v_x, h_t, and v_t depend critically on ε, even for arbitrary small times T, while the estimates on h and v, do not. Since $T > 0$ has to be sufficiently small we have existence and uniqueness only locally in time. Due to our objective function we expect that we may assume that the control u is determined s.t. there is no blow up for finite times and we can extend the local solution to a solution for any finite time T. Higher regularity results similar as in [7, Subsect. 7.1.3, Th. 6] are not needed in the following and therefore omitted here. Note that in one (time) dimension, we have the embeddings $H^1 \hookrightarrow C^{0,1/2}$ and $H^2 \hookrightarrow C^{1,1/2}$.

4. Necessary optimality conditions and existence of an optimal control

The control effort serves as well as a Tikhonov regularization, therefore we require $\alpha_2 > 0$. We emphasize that for our proof of existence of optimal controls, see Th. 4.1, it is crucial to treat the terminal conditions for the positions d_Δ, d_w and the velocities v_Δ, v_w as penalties.

We minimize (2.14), subject to the regularized ODE-PDE system (2.20) – (2.33) with all boundary and initial conditions and the point-wise control constraints, following from (2.13),

$$u \in U_{ad} := \{u \in L^2(0,1) \mid u_{min} \le u \le u_{max}\}.$$

The adjoints that are introduced in the Subsection 4.1 live in the space W, that turns out in our example that it can be identified with Y, furthermore let

$$\begin{aligned}\Xi &= Y^* \times H^1(0,L) \times H^1_0(0,L) \times \mathbb{R}^4 \\ &= L^2(H^1)^* \times L^2H^{-1} \times [(H^2)^*]^2 \times [(H^1)^*]^2 \times H^1(0,L) \times H^1_0(0,L) \times \mathbb{R}^4,\end{aligned}$$

the latter six factor spaces representing initial conditions. Then the weak formulation of the differential equations yields a bounded operator

$$e : (y,u) \in Y \times U \mapsto \begin{pmatrix} E(y,u) \\ y(0) - y^{(0)} \end{pmatrix} \in \Xi \tag{4.1}$$

where the operator representing the PDE-ODE system is of the following form

$$E(y,u) := \begin{pmatrix} h_t + A_1(y) \\ v_t + A_2(y) \\ \dot{d}_\Delta - v_\Delta \\ \dot{d}_w - v_w \\ \dot{v}_\Delta + A_5 y + B_5 u \\ \dot{v}_w + A_6 y \end{pmatrix} \in W^*$$

with nonlinear operators A_l, $l = 1,2$ and linear operators B_5, A_l, $l = 5,6$.

Theorem 4.1 (Existence of optimal controls)**.** *For sufficiently small $T > 0$, there exists an optimal solution $(\hat{y}, \hat{u})$ of our optimal control problem.*

Proof. U_{ad} is a convex, bounded, and closed subset of U. Obviously, the embedding $H^1(0, L) \to L^p(0, L)$ is compact for any $1 < p < \infty$. Thus, weak convergence in Y_1 implies strong convergence in $[L^2L^4]^2 \times [H^2(0, 1)]^2 \times [H^1(0, 1)]^2$, thus h^2, v^2 have a spatial L^2 regularity and the nonlinear terms $h_x v$, hv_x and vv_x multiplied with a test function may be bounded in L^2L^2. Therefore, with $[L^2(Q)]^* = L^2(Q)$ and $[L^2(Q)]^2 \subset Y_1^*$, the map $E : Y \times U \to W^*$ is continuous under weak convergence. Moreover, the state equation $E(y, u) = 0$ has a bounded control-to-state operator $S : u \in U_{ad} \mapsto y = S(u) \in Y$, see our local existence and uniqueness result Th. 3.2 for the states. The considered objective J is sequentially weakly lower semi-continuous. Now we may apply [16, Th. 1.45]. □

However, in order to compute optimal controls it is favorable to solve numerically necessary optimality conditions, see Subsection 4.3.

4.1. **Lagrangian based approach.** We start with formal Lagrange techniques as in [32, Kap. 3.1]. It would also be possible to follow a Hamiltonian approach in order to derive NOCs by Pontryagin's minimum principle, see e.g. [31] for the optimal control of a nonlinear parabolic equations.

The Lagrange function $\mathcal{L} : Y \times U \times W \to \mathbb{R}$ is defined as the objective J coupled to the weak formulation of the PDE-ODE constraints by Lagrange multipliers λ,

$$\mathcal{L}(y, u, \lambda) := J(y, u) + \langle \lambda, E(y, u)\rangle_{W,W^*}\,. \tag{4.2}$$

Here the multipliers are functions $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)^\top$ and are the so-called adjoints. We insert the modified version (reflecting the scaling in Subsection 2.2) of the objective (2.12) and the weak formulation of the differential equations into (4.2)

$$\begin{aligned}
\mathcal{L}(y, u, \lambda) = {} & \frac{\alpha_1 T}{2} \int_Q |h - h_d|^2 \, dx\, dt + \frac{\alpha_2 T}{2} \int_0^1 u^2 \, dt + \frac{\alpha_3 T}{2} \int_Q v^2 \, dt \\
& + \sum_{I \in \{\Delta, w\}} \Big(\frac{\alpha_4}{2} [d_I]_1^2 + \sigma_4 [d_I]_1 + \frac{\alpha_5}{2} [v_I]_1^2 + \sigma_5 [v_I]_1 \Big) + \varepsilon T \int_0^1 F_c [\lambda_1]_0^L \, dt \\
& - \int_Q h_t \lambda_1 - T(hv - \varepsilon h_x)\lambda_{1,x} \, dx\, dt - T \int_0^1 [(gh - \varepsilon v_x)\lambda_2]_0^L \, dt \\
& - \int_Q (v_t - TgF_c)\lambda_2 - T\Big(\frac{v^2}{2} + gh - \varepsilon v_x\Big)\lambda_{2,x} \, dx\, dt \\
& - \int_0^1 (\dot{d}_\Delta - Tv_\Delta)\lambda_3 + (\dot{d}_w - Tv_w)\lambda_4 + (\dot{v}_\Delta - T(u + gF_s + F_f))\lambda_5 \, dt \\
& - \int_0^1 (\dot{v}_w - TF_f)\lambda_6 \, dt.
\end{aligned} \tag{4.3}$$

Let $\hat{y}$, $\hat{u}$, and λ be a solution of the optimal control problem. We expect the necessary optimality conditions

$$\langle \mathcal{L}_y(\hat{y}, \hat{u}, \lambda), y\rangle_{Y^*,Y} = 0 \qquad \forall y \text{ with } y(0) = 0, \tag{4.4}$$

$$\langle \mathcal{L}_u(\hat{y}, \hat{u}, \lambda), u - \hat{u} \rangle_{U^*,U} \geq 0 \qquad \forall u \in U_{ad}. \tag{4.5}$$

They are derived rigorously below, see in Th. 4.3. The derivative of the Lagrange function w.r.t. the states yields after integrations by parts for all time and space derivatives of states (using in particular (2.17) – (2.19))

$$\begin{aligned}
\langle \mathcal{L}_y(\hat{y}, \hat{u}, \lambda), y \rangle_{Y^*,Y} &= \int_Q h \left(\alpha_1 T(\hat{h} - h_d) + \lambda_{1,t} + T\hat{v}\lambda_{1,x} + \varepsilon T \lambda_{1,xx} \right) dx\, dt \\
&- \int_0^L (h\lambda_1)(1, \cdot)\, dx - T \int_0^1 \varepsilon [h\lambda_{1,x} + gh\lambda_2]_0^L\, dt + Tg \int_Q h\lambda_{2,x}\, dx\, dt \\
&- T\frac{g}{L} \int_0^1 [h]_0^L (\lambda_5 + \lambda_6)\, dt + \int_Q v\, (\alpha_3 T\hat{v} + \lambda_{2,t} + T\hat{v}\lambda_{2,x} + \varepsilon T \lambda_{2,xx})\, dx\, dt \\
&- \int_0^L (v\lambda_2)(1, \cdot)\, dx + \varepsilon T \int_0^1 [v_x \lambda_2]_0^L\, dt + T \int_Q \hat{h} v \lambda_{1,x}\, dx\, dt + \int_0^1 d_\Delta \dot{\lambda}_3\, dt \\
&+ (\alpha_4 [\hat{d}_\Delta]_1 + \sigma_4 - \lambda_3(1)) d_\Delta(1) - Tc \int_0^1 d_\Delta \left(\eta \left(\frac{\varepsilon}{g} [\lambda_1]_0^L + \int_0^L \lambda_2\, dx \right) + \lambda_5 \right) dt \\
&+ (\alpha_4 [\hat{d}_w]_1 + \sigma_4 - \lambda_4(1)) d_w(1) + \int_0^1 d_w \dot{\lambda}_4\, dt + (\alpha_5 [\hat{v}_\Delta]_1 + \sigma_5 - \lambda_5(1))\, v_\Delta(1) \\
&+ \int_0^1 v_\Delta \dot{\lambda}_5\, dt - T \int_0^1 v_\Delta \left(k \left(\eta \left(\frac{\varepsilon}{g} [\lambda_1]_0^L + \int_0^L \lambda_2 \right) + \lambda_5 \right) - \lambda_3 \right) dt \\
&+ (\alpha_5 [\hat{v}_w]_1 + \sigma_5 - \lambda_6(1))\, v_w(1) + \int_0^1 v_w \dot{\lambda}_6\, dt + T \int_0^1 v_w \lambda_4\, dt.
\end{aligned}$$

From (4.5) we obtain

$$\langle \mathcal{L}_u(\hat{y}, \hat{u}, \lambda), u - \hat{u} \rangle_{U^*,U} = T \int_0^1 (\alpha_2 \hat{u} + \lambda_5)(u - \hat{u})\, dt \geq 0 \quad \forall u \in U_{ad}.$$

Here we have the structure $\mathcal{L}_u(y, u, \lambda) = \tilde{\mu} u + G(y, u, \lambda)$ with $\tilde{\mu} = \alpha_2 > 0$ and $G(y, u, \lambda) = \lambda_5$ continuously Fréchet-differentiable from $Y \times L^2(0,1) \times W \to L^2(0,1)$ and locally Lipschitz-continuous from $Y \times L^2(0,1) \times W \to L^p(0,1)$, $p > 2$, as required in [10, Assumpt. 4.2 (b)]. $U = L^2(0,1)$ is a Hilbert space and $U_{ad} \subset U$ is nonempty, closed, and convex. By means of a superposition operator $\Pi : Y \times U \times W \to U$,

$$\Pi(y, u, \lambda)(t, x) := u(t) - P_{U_{ad}}(u(t) - \alpha_2^{-1} \mathcal{L}_u(y(t,x), u(t), \lambda(t,x))), \tag{4.6}$$

where $P_{U_{ad}}$ is the Euclidean projection onto U_{ad}, we may rewrite [16, Corollary 1.2]) the variational inequality (4.5) as

$$\hat{u} = P_{U_{ad}} \left(-\frac{1}{\alpha_2} \lambda_5 \right). \tag{4.7}$$

As a check, by

$$\langle \mathcal{L}_\lambda(\hat{y}, \hat{u}, \lambda), \lambda \rangle_{W^*,W} = 0 \tag{4.8}$$

we recover the differential equations and its Neumann boundary conditions. Here we apply the fundamental lemma of calculus of variations (also known as Du Bois-Reymond lemma) in order to deduce that the integrands themselves are zero.

4.2. **Adjoint differential equations.** The so far only formally derived NOC (4.4) yield the adjoint system for λ. We would have the adjoint PDE

$$\lambda_{1,t} + T\hat{v}\lambda_{1,x} + \varepsilon T\lambda_{1,xx} + Tg\lambda_{2,x} = -\alpha_1 T(\hat{h} - h_d), \qquad (t,x) \in Q, \tag{4.9}$$

$$\lambda_{1,x} = -\frac{g}{\varepsilon L}(\lambda_5 + \lambda_6), \qquad (t,x) \in \Gamma, \tag{4.10}$$

$$\lambda_1(1,x) = 0, \qquad x \in [0,L], \tag{4.11}$$

$$\lambda_{2,t} + T\hat{v}\lambda_{2,x} + \varepsilon T\lambda_{2,xx} + T\hat{h}\lambda_{1,x} = -T\alpha_3\hat{v}, \qquad (t,x) \in Q, \tag{4.12}$$

$$\lambda_2 = 0, \qquad (t,x) \in \Gamma, \tag{4.13}$$

$$\lambda_2(1,x) = 0, \qquad x \in [0,L], \tag{4.14}$$

and the adjoint ODE

$$\dot{\lambda}_3 = Tc\left(\eta\left(\frac{\varepsilon}{g}[\lambda_1]_0^L + \int_0^L \lambda_2\, dx\right) + \lambda_5\right), \qquad t \in (0,1), \tag{4.15}$$

$$\lambda_3(1) = \alpha_4\left(\hat{d}_\Delta(1) - d_\Delta^{(T)}\right) + \sigma_4, \tag{4.16}$$

$$\lambda_4 = \alpha_4\left(\hat{d}_w(1) - d_w^{(T)}\right) + \sigma_4, \qquad t \in (0,1], \tag{4.17}$$

$$\dot{\lambda}_5 = -T\lambda_3 + Tk\left(\eta\left(\frac{\varepsilon}{g}[\lambda_1]_0^L + \int_0^L \lambda_2\, dx\right) + \lambda_5\right), \qquad t \in (0,1), \tag{4.18}$$

$$\lambda_5(1) = \alpha_5\left(\hat{v}_\Delta(1) - v_\Delta^{(T)}\right) + \sigma_5, \tag{4.19}$$

$$\lambda_6 = T\lambda_4(1-t) + \alpha_5\left(\hat{v}_w(1) - v_w^{(T)}\right) + \sigma_5, \qquad t \in (0,1], \tag{4.20}$$

where we have exploited that $\dot{\lambda}_4 = 0$ and that we may integrate $\dot{\lambda}_6$ in time. We abbreviate the adjoint terminal conditions by

$$\lambda^{(1)} := -(0, 0, \alpha_4[\hat{d}_\Delta]_1 + \sigma_4, \alpha_4[\hat{d}_w]_1 + \sigma_4, \alpha_5[\hat{v}_\Delta]_1 + \sigma_5, \alpha_5[\hat{v}_w]_1 + \sigma_5)^\top.$$

Remark 4.2 (Coupling structure)**.** In the ODE-PDE system we have a control that acts on the ODE system (state v_Δ). All ODE states enter the PDE or the boundary conditions for the states h and v, that are fully coupled. Finally, h enters into the ODE states v_Δ and v_w.

In the adjoint system the adjoints λ_5 and λ_6 (corresponding to v_Δ and v_w) enter via the Neumann boundary condition for λ_1 (corresponding to h), fully coupled with λ_2 (corresponding to v) and all PDE adjoints (λ_1 and λ_2) enter into the ODE for the adjoints λ_3 and λ_5. The adjoint λ_5 (corresponding to v_Δ) determines the control. We notice a reversed coupling in the adjoint system.

4.3. **Derivation of necessary optimality conditions.** In order to prove the necessary optimality conditions we combine the concept and notation of [32, Kap. 5.5] and [16, Sect. 1.7], who have demonstrated this for the Neumann boundary control as well as for the distributed control in case of a single parabolic PDE.

Theorem 4.3 (First-order necessary optimality conditions)**.** *Let $T > 0$ be a sufficiently small time. Then for our optimal control problem (2.14) subject to (2.20) – (2.33) and (2.13), the necessary optimality conditions (4.4), (4.7) and (4.8) hold,* *i.e. an optimal solution $(\hat{y}, \hat{u})$ fulfills: i) the adjoint system (4.9) – (4.20), the ii) optimality condition (4.7), and iii) the PDE-ODE system in the weak form (4.8).*

Proof. We consider box constraints and U_{ad} is a closed, convex, nonempty subset of an open Banach space U. Furthermore, according to Th. 3.2 there exists a control-to-state operator $S : U \to Y, u \mapsto S(u)$ such that the reduced objective $\mathcal{J}(u) = J(S(u), u)$ is well-defined on an open neighborhood V of U_{ad} and Gâteaux differentiable around $\hat{u}$. Thus we may apply [16, Th. 1.46] yielding (4.7).

We consider the operator for the state equation $E(y, u) = 0$, $E : Y \times U \to W^*$ in (4.1). Since we consider control constraints, [16, Th. 1.48, Corollary 1.3] shows that it suffices to require only

1) Continuous F-differentiability of $J : Y \times U \to \mathbb{R}$ and $E : Y \times U \to W^*$,
2) Unique solvability of the state equation in $V \subset U$, and
3) $E_y(y(u), u) \in \mathcal{L}(Y, W^*)$ has a bounded inverse for all $u \in V \supset U_{ad}$.

We check:

1) The statement follows from the imbedding $Y \hookrightarrow [C^0([0,1]; L^2(0,L))]^2 \times [C^1([0,1])]^2 \times [C^0([0,1])]^2$. The terminal conditions on d_Δ, d_w, v_Δ, and v_w are well-defined, too.
2) Let $S : V \to Y$ denote the control-to-state operator (solution operator) of the differential equation system. This control-to-state operator S is well-defined, since for every u we have a solution, see Th. 3.2.
3) The linearized problem in strong form is obtained by linearizing (2.20) – (2.33)

$$\begin{aligned}
\tilde{h}_t + T(\hat{h}\tilde{v} + \tilde{h}\hat{v})_x - \varepsilon T \tilde{h}_{xx} &= 0, && (t,x) \in Q,\\
\varepsilon T \tilde{h}_x &= \varepsilon T \eta F_s(\tilde{d}_\Delta, \tilde{v}_\Delta), && (t,x) \in \Gamma,\\
\tilde{h}(0,x) &= 0, && x \in [0,L],\\
\tilde{v}_t + (T\hat{v}\tilde{v} + g\tilde{h})_x - \varepsilon T \tilde{v}_{xx} &= T g \eta F_s(\tilde{d}_\Delta, \tilde{v}_\Delta), && (t,x) \in Q,\\
\tilde{v} &= 0 && (t,x) \in \Gamma,\\
\tilde{v}(0,x) &= 0, && x \in [0,L]
\end{aligned}$$

and

$$\begin{aligned}
\dot{\tilde{d}}_\Delta &= T\tilde{v}_\Delta, && t \in (0,1), && \tilde{d}_\Delta(0) = 0,\\
\dot{\tilde{d}}_w &= T\tilde{v}_w, && t \in (0,1), && \tilde{d}_w(0) = 0,\\
\dot{\tilde{v}}_\Delta - T g F_s(\tilde{d}_\Delta, \tilde{v}_\Delta) &= Tu - T\frac{g}{L}[\tilde{h}(t,\cdot)]_0^L, && t \in (0,1), && \tilde{v}_\Delta(0) = 0,\\
\dot{\tilde{v}}_w &= -T\frac{g}{L}[\tilde{h}(t,\cdot)]_0^L, && t \in (0,1), && \tilde{v}_w(0) = 0.
\end{aligned}$$

Here we have scaled the Neumann boundary condition for h by a factor of εT corresponding to the conormal derivative. Note that for well-balanced equations in our numerics, we scale the Neumann boundary condition for λ_1 analogously by εT. $\langle E_y, \tilde{y} \rangle_{Y^*,Y}$ is the linearized problem in weak form, that follows from integrating by parts.

For the linearized problem we have the same structure of estimates as for the full problem. The linearized problem has a unique solution for every u and every initial data $h^{(0)} \in H^1(0,L)$, $v^{(0)} \in H_0^1(0,L)$, and $(d_\Delta^{(0)}, d_w^{(0)}, v_\Delta^{(0)}, v_w^{(0)})^\top \in \mathbb{R}^4$. Thus the linearized control-to-state operator $\tilde{S}(\hat{u})$ is a well-defined unique linear continuous (bounded) affine operator and, in particular, is surjective (not a proper dense subset of the image). It has a bounded inverse, according to the theorem of the inverse mapping [1, Satz 5.8] (for a suitable neighborhood $V \supset U_{ad}$).

□

The adjoint system has the same structure as the linearized original ODE-PDE problem and possesses hence a unique solution. Note that the adjoint equations are solved backwards in time. Consistently, we have terminal conditions in the parabolic PDEs for λ_1 and λ_2 and the Laplacian operator has the opposite sign in the adjoint PDE (compared to the state PDE).

Using the same ingredients (Hölder/Young inequalities, trace theorem, Gronwall inequality, compensation for sufficiently small T) as for state equations, we get the following regularity from estimates for the adjoint PDEs:

$$(\lambda_1, \lambda_2)^\top \in \tilde{Y}_1 \subset Y_1, \quad \lambda \in \tilde{Y} \subset Y \simeq W, \tag{4.21}$$

indeed, showing the higher regularity for λ.

Remark 4.4 (Convexity and direct approach)**.** For a convex problem the necessary conditions would be also sufficient. Note that due to the Saint-Venant equations the problem is in general not convex.

For a semi-linear parabolic equation with an objective convex w.r.t. u, the necessary optimality conditions are proven in [32, Satz 5.8/Satz 5.15] under reasonable assumptions. The proofs use, among other things, the convexity and closedness of U_{ad}, the lower semi-continuity of the objective, a Hilbert space structure, and the solvability of the adjoint equation. The necessary optimality conditions are derived directly.

We could also follow a direct approach with a reduced objective as in [16].

4.4. **Semi-smooth Newton method in a Hilbert space.** We introduce $\bar{Z} = Y \times U \times W$ and aggregate all unknowns in

$$z = (y, u, \lambda)^\top = (h, v, d_\Delta, d_w, v_\Delta, v_w, u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)^\top \in \bar{Z}.$$

The necessary optimality conditions (4.4), (4.7), and (4.8) yield the following non-smooth system:

$$f(z) = \begin{pmatrix} \mathcal{L}_\lambda(z) \\ \Pi(z) \\ \mathcal{L}_y(z) \end{pmatrix} \stackrel{!}{=} 0, \tag{4.22}$$

where $\mathcal{L}_\lambda(z)$ represents the PDE-ODE system for the states and $\mathcal{L}_y(z)$ yields the adjoint PDE-ODE system.

In order to continue analogously as in [10], where elliptic problems are considered, we semi-discretize in time. First semi-discretizing in time and then discretizing in space is a standard technique and is e.g. applied by Hinze *et al.* in [15, Section 2] to optimal flow. We divide the time interval $[0, 1]$ by $N + 1$ time steps with constant increment $\Delta t = 1/N$. For ease of presentation we keep a similar notation as in the continuous case. Let $Z = H^1(0,L) \times H^1_0(0,L) \times \mathbb{R}^4 \times U \times H^1(0,L) \times H^1_0(0,L) \times \mathbb{R}^4$ and

$$z := (y^0, u^0, \lambda^0, \dots, y^N, u^N, \lambda^N)^\top \in Z^{N+1}.$$

Then the time-discretized PDE-ODE system is of the following form

$$y^i = y^{i-1} + \Delta t\, \Phi(y^i, u^i), \qquad i = 1, \dots, N, \tag{4.23}$$

with initial conditions $y^0 = \phi := (h^{(0)}, v^{(0)}, d^{(0)}_\Delta, d^{(0)}_w, v^{(0)}_\Delta, v^{(0)}_w)$. Analogously, the adjoint system semi-discretized in time reads

$$\lambda^i = \lambda^{i+1} + \Delta t\, \Psi(y^i, u^i, \lambda^{i+1}), \qquad i = 0, \dots, N-1, \tag{4.24}$$

with terminal conditions $\lambda^N = \psi := \lambda^{(1)}$. Note that the state equations are solved forward in time by an implicit Euler method, while the adjoint equations are solved backward in time, see e.g. [4, Sect. 3.3]. Note that the coupled forward/adjoint system can be interpreted as a Hamiltonian system [2, Sect. 2.2] and that it is computed here by a symplectic method [9, Th. 5.3.3]. In particular, the discrete adjoint backward update is explicit w.r.t. λ and implicit w.r.t. y.

As a possibility to speed up our numerics, we could try a more elaborated scheme of Crank-Nicolson type. For evolution equations where the spatial differential operator is self-adjoint it has been demonstrated that FDTO and FOTD approaches commute for certain variants of Crank-Nicolson schemes [2], but it is not clear how to apply this framework to our problem. Alternatively, we could start with a semi-discretization in space, too.

From (4.22) this yields the equation for the vector $f \in [Z^*]^{N+1}$

$$f = (f_0, \dots, f_i, \dots, f_N)^\top = 0 \tag{4.25}$$

where

$$f_0 = \begin{pmatrix} -y^0 + \phi \\ u^0 - P_{[u_{min}, u_{max}]}(-\lambda^0_5/\alpha_2) \\ -(\lambda^0 - \lambda^1) + \Delta t\, \Psi(y^0, u^0, \lambda^1) \end{pmatrix},$$

$$f_i = \begin{pmatrix} -(y^i - y^{i-1}) + \Delta t\, \Phi(y^i, u^i) \\ u^i - P_{[u_{min}, u_{max}]}(-\lambda^i_5/\alpha_2) \\ -(\lambda^i - \lambda^{i+1}) + \Delta t\, \Psi(y^i, u^i, \lambda^{i+1}) \end{pmatrix}, \quad i = 1, \dots, N-1,$$

and

$$f_N = \begin{pmatrix} -(y^N - y^{N-1}) + \Delta t\, \Phi(y^N, u^N) \\ u^N - P_{[u_{min}, u_{max}]}(-\lambda^N_5/\alpha_2) \\ \lambda^N - \psi(y^N) \end{pmatrix}.$$

Let $\mathcal{L}^i$ denote the semi-discretized Lagrange function. The assumptions on the structure of $\mathcal{L}^i_u$ [10, Assumpt. 4.2], yielding the semi-smoothness of f_i, and on the uniform invertibility of the Newton matrices M_i [10, Assumpt. 2.3], due to the Lax-Milgram theorem are fulfilled. Furthermore, we assume that the Tikhonov parameter α_2 is sufficiently large [10, Assumpt. 3.1 (a)], such that a descent direction w.r.t. the merit function $\Theta = \|f\|^2_{[Z^*]^{N+1}}$ is always obtained. Thus we may apply the globalized semi-smooth Newton method derived in [10] to compute a zero of f.

Let I_Y denote the identity operator in the Banach space Y. The Newton matrix M is a block-tridiagonal matrix of the form

$$M := \begin{pmatrix} M_1 & R_1 & & & \\ L_2 & M_2 & R_2 & & \\ & \ddots & \ddots & \ddots & \\ & & L_{N-1} & M_{N-1} & R_{N-1} \\ & & & L_N & M_N \end{pmatrix} \tag{4.26}$$

with

$$\begin{aligned} M_i &\in \partial_C f_i, & i &= 1, \ldots, N, \\ L_i &:= diag(I_Y, 0, 0), & i &= 2, \ldots, N, \\ R_i &:= diag(0, 0, I_W), & i &= 1, \ldots, N-1. \end{aligned}$$

The set $\partial_C f_i$ consists of all matrices $M_i \in L(Z, Z^*)$, $i = 0, \ldots, N$. M_i has the structure

$$M_i(y,u,\lambda) = \begin{pmatrix} \Delta t\, \Phi_y(y,u) - I_Y & \Delta t\, E_u^i(y,u) & 0 \\ 0 & 1 & (0,0,0,0,D,0) \\ \mathcal{L}^i_{y,y}(y,u,\lambda) & \mathcal{L}^i_{y,u}(y,u,\lambda) & \mathcal{L}^i_{y,\lambda}(y,u,\lambda) - I_W \end{pmatrix}, \tag{4.27}$$

where the generalized differential $D \in L^\infty(0,1)$ is chosen such that

$$D(t) \in \partial_C P_{[u_{min}, u_{max}]}(-\lambda_5(t)/\alpha_2), \quad t \in (0,1). \tag{4.28}$$

The subdifferential $\partial_C P_{[u_{min}, u_{max}]}$ takes its values in $\{\{0\}, [0,1], \{1\}\}$, whereupon at a non-differentiability point we may choose a fixed value in the interval $[0,1]$.

Here we have matrix blocks of the following structure, for the PDE-ODE system (incorporating the boundary conditions for the h-PDE into the first/last components of the corresponding block suitably)

$$\Phi_y(y,u) =$$

$$\left(\begin{array}{cc|c|ccc} & & & Tc\eta\frac{\varepsilon}{g} & Tk\eta\frac{\varepsilon}{g} & \\ \multicolumn{2}{c|}{-T(\hat{v}\partial_x + \hat{v}_x) + \varepsilon T \partial_{xx}} & -T(\hat{h}\partial_x + \hat{h}_x) & & & \\ & & & -Tc\eta\frac{\varepsilon}{g} & -Tk\eta\frac{\varepsilon}{g} & \\ \hline \multicolumn{2}{c|}{-Tg\partial_x} & -T\hat{v}\partial_x + \varepsilon T\partial_{xx} & -Tc\eta & -Tk\eta & \\ \hline & & & & T & \\ & & & & & T \\ \frac{Tg}{L} & \frac{-Tg}{L} & & -Tc & -Tk & \\ \frac{Tg}{L} & \frac{-Tg}{L} & & & & \end{array}\right),$$

and for the transposed adjoint system

$$\mathcal{L}^i_{y,\lambda}(y,u,\lambda) = \Delta t\, \Psi_\lambda(y,u)^\top =$$

$$\Delta t \left(\begin{array}{cc|c|ccc} & & & Tc\eta\frac{\varepsilon}{g} & Tk\eta\frac{\varepsilon}{g} & \\ \multicolumn{2}{c|}{T\hat{v}\partial_x + \varepsilon T \partial_{xx}} & T\hat{h}\partial_x & & & \\ & & & -Tc\eta\frac{\varepsilon}{g} & -Tk\eta\frac{\varepsilon}{g} & \\ \hline \multicolumn{2}{c|}{Tg\partial_x} & T\hat{v}\partial_x + \varepsilon T\partial_{xx} & -Tc\eta & -Tk\eta & \\ \hline & & & & T & \\ & & & & & T \\ \frac{Tg}{L} & \frac{-Tg}{L} & & -Tc & -Tk & \\ \frac{Tg}{L} & \frac{-Tg}{L} & & & & \end{array}\right).$$

Furthermore, $E_u^i = (0,0,0,0,T,0)^\top$, and for the blocks corresponding to the objective we have

$$\mathcal{L}^i_{y,y}(y,u,\lambda) = \Delta t \begin{pmatrix} T\alpha_1 & T\lambda_{1,x} & & & & \\ T\lambda_{1,x} & T\alpha_3 + T\lambda_{2,x} & & & & \\ & & \alpha_4\delta_1 & & & \\ & & & \alpha_4\delta_1 & & \\ & & & & \alpha_5\delta_1 & \\ & & & & & \alpha_5\delta_1 \end{pmatrix}$$

(δ_1 here denoting the Dirac distribution that is 1 for the terminal time $t = 1$ and 0 otherwise) and $\mathcal{L}_{y,u}(y,u,\lambda)$ vanishes. This yields set valued mappings $\partial_C f_i : Z \rightrightarrows L(Z, Z^*)$ where the map has values in the set of all M fulfilling (4.27). Note that the the subscript "C" is due to the close relation in finite dimensions to Qi's C-subdifferential.

The semi-smooth Newton method with a suitable globalization strategy [10, Algorithm 3.3], there applied to semi-linear elliptic equations, is here adapted to the semi-discretization in time of semi-linear parabolic equations.

Algorithm 4.5 (Global semi-smooth Newton method)**.**

(i) Set $k = 0$, define $z_0 := z^{(0)} \in Z^{N+1}$, and choose $\beta \in (0,1)$, $\sigma \in (0, 1/2)$.

(ii) If $\|f\|_{[Z^*]^{N+1}} < tol$, then stop.

(iii) For fixed $M(z_k)$, $M_i(z_k) \in \partial_C f_i(z_k)$ for all $i = 0,\dots,N$ compute the search direction s_k by solving

$$M(z_k)s_k = -f(z_k)$$

(iv) Determine the smallest $i_k \in \mathbb{N}_0$ such that

$$\Theta(z_k + \beta^{i_k} s_k) \le (1 - 2\sigma\beta^{i_k})\Theta(z_k)$$

and set $\tilde{\beta}_k := \beta^{i_k}$.

(v) Update $z_{k+1} := z_k + \tilde{\beta}_k s_k$ and $k := k+1$. Goto (ii).

In Step (iv) the step-size $\tilde{\beta}_k$ is determined by an Armijo line-search, relying in particular on the merit function $\Theta(z_k)$ and that the gradient of $f(z_k)$ applied to s_k is $-2\Theta(z_k)$.

According to [10, Th. 3.4, Th. 3.5] we have

Theorem 4.6 (Accumulation points are global solutions)**.** *For α_2 sufficiently large we have that any accumulation point $\bar{z}$ of a sequence $\{z_k\}_{k\in\mathbb{N}_0}$ (with $f(z_k) \neq 0$ for all $k \in \mathbb{N}_0$) generated by Algorithm 4.5 is a zero. The sequence converges super-linearly to $\bar{z}$ in a suitable neighborhood of $\bar{z}$.*

5. Numerical methods

The discretized optimal control problem can be solved by a gradient-based optimization procedure like SQP (as in [11]), or, the discretized NOC by a semi-smooth Newton method as we do in this study.

5.1. **Fully discretized problem.** We discretize the space interval $[0, L]$ equidistant with $\Delta x = L/M$, yielding $x_j := j\Delta x$, $j = 0, \dots, M$. We introduce

$$z^{\Delta x,i} := (h_1^i, \dots, h_{M-1}^i, v_1^i, \dots, v_{M-1}^i, d_\Delta^i, d_w^i, v_\Delta^i, v_w^i)^\top, \quad i = 0, \dots, N.$$

The boundary values $h_0^i, h_M^i, v_0^i, v_M^i$, $i = 0, \dots, N$, are determined directly and, thus, are not included in the solution vectors $z^{\Delta x,i}$. The vectors in case of the full discretization are indicated by an upper index Δx. For ease of presentation we state the discretized problem for the case $B \equiv 0$.

Note that we discretize the flux terms in the Saint-Venant equations as in the Lax-Friedrichs scheme, e.g.,

$$(h\partial_x v + \partial_x hv)(x_j) = (hv)_x(x_j) \approx \frac{h_{j+1}v_{j+1} - h_{j-1}v_{j-1}}{2\Delta x}.$$

Furthermore, in consistence with the Lax-Friedrichs scheme we have set

$$\varepsilon = \frac{1}{2}\frac{(\Delta x)^2}{T\,\Delta t}$$

for the artificial viscosity. With these two considerations, for instance (2.20) is approximated by

$$\frac{h_j^{i+1} - h_j^i}{\Delta t} \approx -\frac{T}{2\Delta x}\left(h_{j+1}^i v_{j+1}^i - h_{j-1}^i v_{j-1}^i\right) + \frac{1}{2\Delta t}\left(h_{j+1}^i - 2h_j^i + h_{j-1}^i\right).$$

The Lax-Friedrichs scheme is an explicit method that is first order in time and second order in space. For the convergence, and hence the stability, of an explicit scheme the Courant-Friedrichs-Levy (CFL) condition

$$V\frac{T\Delta t}{\Delta x} < 1 \tag{5.1}$$

is necessary and sufficient [27, Sect. 8.3]. Here $V = \max_{t,x}\{v \pm \sqrt{gh}\}$ denotes the so-called group velocity at which information is exchanged within the numerical grid. But V cannot be determined *a priori* and is part of the numerical solution. Thus the CFL number $VT\Delta t/\Delta x$ has to be checked in the numerical results *a posteriori*.

Analogously as in [11] time integrals are approximated by first order Riemann sums and space integrals by the second order trapezoidal rule. This is consistent with the Lax-Friedrichs scheme that is first order in time and second order in space. Here the PDE-ODE (4.23) are fully discretized by

$$y^i = y^{i-1} + \Delta t\,\Phi^{\Delta x,i}, \qquad i = 1, \dots, N,$$

where $\Phi^{\Delta x,i}$ is here the space discretization of $\Phi(y^i, u^i)$ $(i = 1, \dots, N)$, together with the boundary conditions that follow by the method of undetermined coefficients accurately in second order,

$$\begin{aligned} h_0^i &= \frac{4}{3}h_1^i - \frac{1}{3}h_2^i - \frac{2}{3}\Delta x F_{c,0}^i, & v_0^i &= 0,\\ h_M^i &= \frac{4}{3}h_{M-1}^i - \frac{1}{3}h_{M-2}^i + \frac{2}{3}\Delta x F_{c,M}^i, & v_M^i &= 0, \end{aligned}$$

where $F^i_{c,j}$ is the discretization of (2.19) at time step t_i at the grid point x_j. The initial conditions $y^0 \in \mathbb{R}^{2(M+1)}$ enter by

$$y^0 = \phi(y^{(0)}) := \left(\begin{array}{ccc|ccc|cccc} h_1^0 & \dots & h_{M-1}^0 & v_1^0 & \dots & v_{M-1}^0 & d_{tr}^0 & d_w^0 & v_{tr}^0 & v_w^0 \end{array} \right)^\top .$$

From (4.7) we obtain for the discretized control directly

$$u^i = P_{[u_{min},u_{max}]}(-\lambda_5^i/\alpha_2), \quad i = 0, \dots, N.$$

By discretizing the NOC in time and space, the obtained FOTD-adjoints λ are different to the FDTO adjoints. We write

$$\lambda^i = (\lambda^i_{1,1}, \dots, \lambda^i_{1,M-1}, \lambda^i_{2,1}, \dots, v^i_{2,M-1}, \lambda^i_3, \lambda^i_4, \lambda^i_5, \lambda^i_6)^\top .$$

From (4.24) we find for the fully discretized adjoint equation

$$\lambda^i = \lambda^{i+1} + \Delta t\, \Psi^{\Delta x, i}, \qquad i = 0, \dots, N-1,$$

where $\Psi^{\Delta x,i}$ is here the space discretization of $\Psi(y^i, u^i, \lambda^{i+1})$ with boundary conditions

$$\begin{aligned} \lambda^i_{1,0} &= \frac{4}{3}\lambda^i_{1,1} - \frac{1}{3}\lambda^i_{1,2} + \frac{4}{3}\frac{T\Delta t}{\Delta x}\frac{g}{L}\left(\lambda^i_5 + \lambda^i_6\right), & \lambda^i_{2,0} &= 0, \\ \lambda^i_{1,M} &= \frac{4}{3}\lambda^i_{1,M-1} - \frac{1}{3}\lambda^i_{1,M-2} - \frac{4}{3}\frac{T\Delta t}{\Delta x}\frac{g}{L}\left(\lambda^i_5 + \lambda^i_6\right), & \lambda^i_{2,M} &= 0, \end{aligned}$$

and terminal conditions $\lambda^N = \psi(y^N)$. Note that in the initial guess z_0 we set $z_{Nm+2M-1} = d_\Delta^{(T)}$ and $z_{Nm+2M} = d_w^{(T)}$. We abbreviate the number of vector components at each time step by $m = 2(M+1) + 1 + 2(M+1) = 4M + 5$. Let

$$z^{\Delta x} := (y^0, u^0, \lambda^0, \dots, y^N, u^N, \lambda^N)^\top \in \mathbb{R}^{(N+1)\times m}$$

and $f_i^{\Delta x}(z^{\Delta x}) \in \mathbb{R}^m$, $i = 0, \dots, N$, are vectors for each time step with

$$f_i^{\Delta x} = \begin{pmatrix} -(y^i - y^{i-1}) + \Delta t\, \Phi^{\Delta x,i} \\ \pi(u^i, \lambda^i_5) \\ -(\lambda^i - \lambda^{i+1}) + \Delta t\, \Psi^{\Delta x,i} \end{pmatrix}, \quad i = 1, \dots, N-1,$$

but with the two exceptions

$$\begin{aligned} (f_0^{\Delta x})_{\text{1st line}} &= -y^0 + \phi, \\ (f_N^{\Delta x})_{\text{3rd line}} &= \lambda^N - \psi(y^N). \end{aligned}$$

Then it remains to solve

$$f^{\Delta x} = (f_0^{\Delta x}, \dots, f_i^{\Delta x}, \dots, f_N^{\Delta x})^\top = 0$$

by a semi-smooth Newton method. The Newton matrix $M_i^{\Delta x}$ is the space-discretized version of (4.26), where the entries are the following matrices

$$\begin{aligned} &M_i^{\Delta x} \in \partial_C f_i^{\Delta x} \quad i = 1, \dots, N, \\ &L_i^{\Delta x} := diag(Id_{2(M+1)}, 0, 0_{2(M+1)\times 2(M+1)}), \quad i = 2, \dots, N, \\ &R_i^{\Delta x} := diag(0_{2(M+1)\times 2(M+1)}, 0, Id_{2(M+1)}), \quad i = 1, \dots, N-1, \end{aligned}$$

and the subdifferential $\partial_C f_i^{\Delta x}$ consists of all matrices in $\mathbb{R}^{m\times m}$ of the form (following from (4.27))

$$\begin{pmatrix} \Delta t\, \Phi_{y^i}^{\Delta x,i} - Id_{2(M+1)} & (0_{2M}, T, 0)^\top & 0 \\ 0 & 1 & (0_{2M}, D, 0)^\top \\ (\mathcal{L}_{y^i,y^i})^{\Delta x,i} & 0 & \Delta t\, \Psi_{\lambda^i}^{\Delta x,i-1} - Id_{2(M+1)} \end{pmatrix}.$$

with $D \in [L^\infty(0,1)]$, $\quad D(t) \in \partial_C P_{[u_{min},u_{max}]}(-\lambda_5^i/\alpha_2)$, whereupon we have modified Newton matrices in the cases $i = 0$ and $i = N$.

Our Algorithm 4.5 for the time-discretized situation is additionally equipped with a standard expansion strategy in the Armijo line-search. This expanded Armijo line-search is efficient [26, §5, Satz 1] and allows for a significant speed-up in the numerical computation of the optimal control.

5.2. **Numerical results.** The Algorithm 4.5 has been implemented in MATLAB R2015b. The $[Z^*]^{N+1}$ norm entering in the stopping criterion in step (ii) of the algorithm is discretized using again the trapezoidal rule for the spatial integrals. As parameters in this algorithm we work with $\beta = 0.9$, $\sigma = 0.001$, and $tol = 10^{-6}$.

We consider two examples, the first scenario corresponding to an optimal braking maneuver as considered in [11] and a second scenario with different parameters and weights. The following data is underlying both examples. We work with the values $d_w(0) = -5$, $d_\Delta(1) = 0$ (corresponding to $d_{tr}(1) = 100$), $d_w(1) = 95$, $v_{tr}(0) = 10$, $v_w(0) = 10$, and the parameters in Table 1. By definition of the offset $\bar{d}$, we have $d_\Delta(0) = 0$. d_Δ, d_w are measured in m, v_Δ, v_w in m/s.

TABLE 1. Parameters (unscaled)

Parameter	Value	Unit	Description
L	4	[m]	length of fluid container
b	1	[m]	width of fluid container
$h^{(0)}$	1	[m]	initial height of fluid level
ρ	1000	[kg/m^3]	density of fluid (water)
m_{tr}	2000	[kg]	mass of truck
m_w	$\rho\, b\, h^{(0)} L$	[kg]	mass of fluid container
c	40000	[N/m]	spring force constant
k	10000	[Ns/m]	damper force constant
g	9.81	[N/kg]	earth acceleration

Feasible values for the terminal time T (in s) are taken from [11]. The control u is considered between the bounds $u_{min} = -20000/m_{tr}$ and $u_{max} = 2000/m_{tr}$. We start with $u^{(0)} = u_{max}/2 = const.$ Furthermore we set $h_j^i \equiv 1$, $i = 0, \ldots, N$, $j = 1, \ldots, M-1$. The other values of $z^{\Delta x,i}$, $i = 0, \ldots N$, unless they are determined by initial values are set to zero at the start of the Newton method.

5.2.1. *Example 1, as in [11].* We consider here the situation $B \equiv 0$. For the control problem we work with the weights $\alpha_1 = 1$, $\alpha_2 = 0.01/m_{tr}^2$, $\alpha_3 = 0$, $\alpha_4 = 10^3$, $\sigma_4 = -10^{-4}$, $\alpha_5 = 100$, and $\sigma_5 = -10^{-5}$.

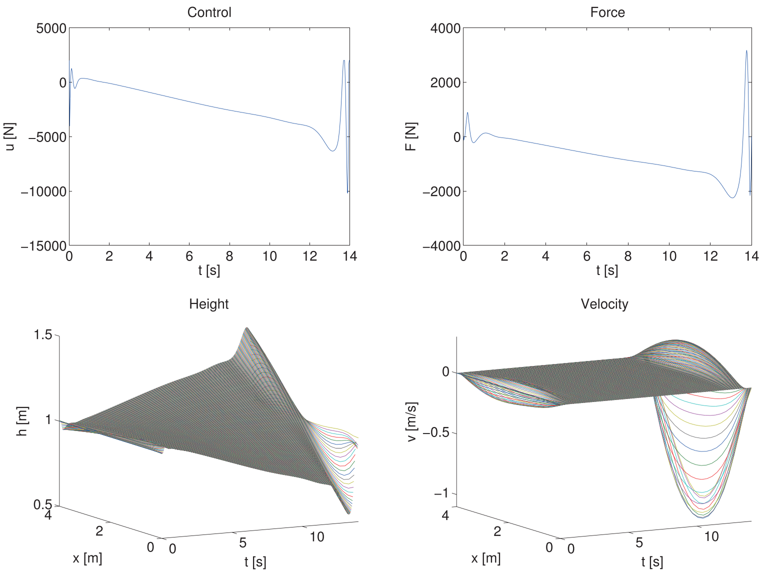

FIGURE 2. Unscaled computed optimal control $m_{tr}u$ vs. time t (top left), unscaled computed spring-damper force $m_{tr}F$ vs. time t (top right), computed fluid height h vs. time and space (t, x) (bottom left), and computed horizontal fluid velocity v vs. time and space (t, x) for a safety braking maneuver. We observe an excitation of h and v shortly before the end of the braking maneuver. This is reflected in the control, that has a general behavior turning from almost maximal acceleration to maximal deceleration, by some counteractions at the begin and at the end. The coupling force has a qualitatively similar behavior as the approximate control.

For the space discretization we consider $M = 20$ and a factor of 30 for the time discretization, yielding $N = 600$. Here the CFL number (and thus the factor 30) is suggested by the numerical results in [11, Subsection 3.3] and the CFL condition (5.1), that depends itself on the numerical solution, is verified *a posteriori*. As in [11] we find for the artificial viscosity $\varepsilon \approx 0.85714286$ for this example.

For a safety breaking maneuver, i.e. with $T = 14$, the numerical optimal control u, the spring-damper force, the vertical fluid level, and the horizontal fluid velocity are depicted in Figure 2. Our algorithm requires about 30 Newton iterations and yields a feasibility of the terminal constraints smaller than 10^{-7}.

5.2.2. *Example 2.* Now we consider the situation $B = -0.05\sin(\pi x/L)$. For the control problem we work with the weights $\alpha_1 = 5$, $\alpha_2 = 0.01/m_{tr}^2$, $\alpha_3 = 0$, $\alpha_4 = 10^3$, $\sigma_4 = 10^{-4}$, $\alpha_5 = 10$, $\sigma_5 = 10^{-6}$. For the discretization we consider again $M = 20$ and $N = 600$. Again, the CFL condition is checked numerically. Consequently, the numerical viscosity ε has the same value as in Example 1.

For a safety breaking maneuver, i.e. with $T = 14$, the numerical optimal control u, the spring-damper force, the vertical fluid level, and the horizontal fluid velocity

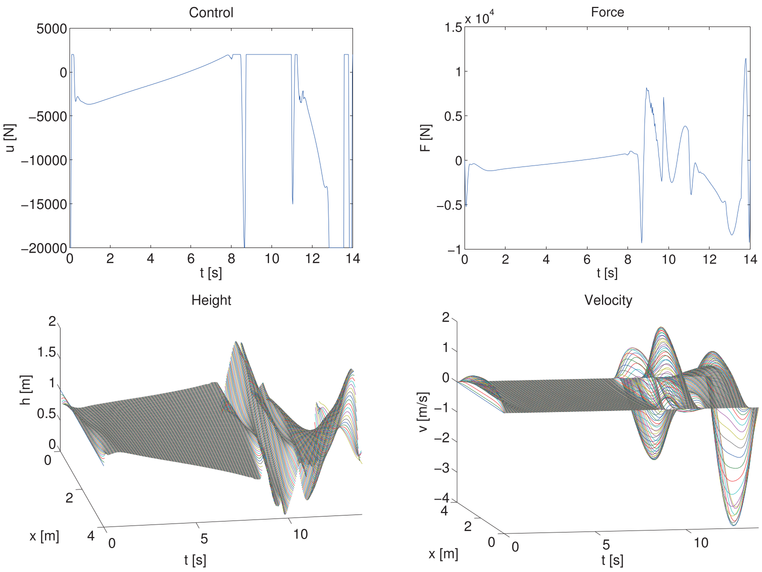

Figure 3. Unscaled computed optimal control $m_{tr}u$ vs. time t (top left), unscaled computed spring-damper force $m_{tr}F$ vs. time t (top right), computed fluid height h vs. time and space (t, x) (bottom left), and computed horizontal fluid velocity v vs. time and space (t, x) for a safety braking maneuver. We observe an excitation of h and v in the second half of the braking maneuver. This is reflected in the control, that has a general behavior reaching from maximal acceleration to maximal deceleration. In contrast to Example 1 we find oscillations that might be due to the slope of the container bottom B.

are depicted in Figure 3. Our algorithm requires about 80 Newton iterations and yields a feasibility of the terminal constraints smaller than 10^{-6}.

6. Conclusion and outlook

We compare with the results obtained in Gerdts *et al.* [11] by a FDTO approach, but for a free terminal time T with a further contribution $\alpha_0 T$ in the objective (with $\alpha_0 > 0$). The obtained numerical results in Example 1 are almost identical. Note that in [11] and in Example 1 different weights are considered as in Example 2. The mild oscillatory behavior of the spring-damper force in Example 2 might be explained by a swinging regime of the spring-damper element and the wave character of h and v, describing shallow water waves.

The initial guess $(y^0, u^0, \lambda^0)^\top$ turns out to be crucial for the performance of our algorithm. In the first Newton iterations we observe with our method the theoretically predicted super-linear convergence. However, for the last iterations this fast convergence is not always observed due to issues with the numerical precision. In particular, our algorithm terminates with less than 100 iterations, while the FDTO approach in [11] requires up to about 3900 iterations of the SQP method. Furthermore in [11] no reformulation introducing d_Δ is exploited. A FDTO optimization approach [33], taking into account the particular structure of the problem, features super-linear convergence, but for a certain range of parameters only.

The reason for considering a FOTD approach like Algorithm 4.5 is, in addition to theoretical insight, that a faster convergence, i.e. less iterations and computing times, are obtained by discretizing in the second step, not before the optimization. However, the numerical precision does not outperform our first approach. In the FDTO ansatz the Courant-Friedrichs-Levy (CFL) condition, that is required for numerical stability of the Lax-Friedrichs scheme, leads to a time discretization finer than the space discretization by a factor of 30. For the computing times of numerical optimal control this is unfavorable, but we meet again this issue in our FOTD approach.

As next step we study further the convergence properties of the global Newton method and how they could be improved by exploiting the structure of the problem. It could be interesting to consider free terminal times. This case would require to adapt our techniques to the non-linearities in T. Furthermore, more simulations for a variation of different parameter sets are of interest. In the near future, we will extend our model to the situation, where the truck moves on the surface of a three-dimensional landscape together with simulating the fluid by the 2d Saint-Venant equations. We might also think of a truck with a semitrailer, involving the drive dynamics both of the drawing vehicle and of the semitrailer with the fluid container.

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