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ON THE COMPLEXITY OF THE PATH-FOLLOWING METHOD FOR A TRACKING PROBLEM GOVERNED BY PARABOLIC EQUATIONS

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Abstract. The complexity of optimization methods applied to an infinite-dimensional problem in a great manner depends on the quality of finite-dimensional approximations. In this work we consider a tracking problem for a linear parabolic equation. The boundary control is assumed to have the form of a linear combination of shape-like functions. We do not consider any discretization of the differential equation. It is supposed that the solution admits a spectral representation via Fourier-like rapidly converging series involving eigenvalues and eigenfunctions of the elliptic operator, and, as a consequence, it can be rapidly calculated with machine accuracy. We show that, in this setting, the tracking problem admits an effective approximation by finite-dimensional optimization problems. The proof of the approximation theorem uses the maximum principle for parabolic equations. Based on our approximation theorem we obtain a complexity bound for the path-following method applied to the tracking problem governed by a linear parabolic equation. The result is illustrated by a series of examples showing the efficiency of the obtained complexity bound.

1. Introduction

Controlled heating and cooling are two important manufacturing processes. The emergence of new technologies requires more advanced techniques in the temperature control [5]. In this paper, we analyse complexity of the path-following method for a tracking problem governed by parabolic equations. Various aspects of the tracking problem for heat equation are discussed in [1–4, 8, 9, 11–14, 16] (see also the literature therein). This is an infinite-dimensional optimization problem and it can be considered in the frame of information-based complexity, a branch of mathematics that studies optimal algorithms and computational complexity for the continuous problems which arise in the real world applications. This theory deals

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with the intrinsic difficulty of the approximate solution of problems for which information is partial, contaminated, and priced [15]. We focus our attention on another aspect of the problem, we study it from the approximation theory point of view.

We assume that the boundary control has the form of a linear combination of shape-like functions. We use the definition of solution to a parabolic equation introduced and studied by Ladyzenskaja and her school [7]. This allows us to guarantee the existence of solution to the tracking problem and to use the maximum principle in order to obtain an effective approximation by finite-dimensional optimization problems. Based on our approximation theorem we get a complexity bound for the path-following method applied to the tracking problem governed by a linear parabolic equation. This approach is an alternative to the regularization methods (see, e.g., [6] and the literature therein).

The paper is organized in the following way. In the next section we introduce the notations used in the sequel and formulate the problem. In the third section we recall some well known results from the theory of PDEs and numerical optimization. The main results of this work are presented in the fourth section. Section five contains some auxiliary lemmas. The main results are proved in section six. In the last section we consider some illustrative examples.

2. Statement of the problem

Throughout this paper, the set of real numbers is denoted by $\mathbb{R}$. The usual inner product in $\mathbb{R}^n$ is denoted by $\langle\cdot|\cdot\rangle$ and the Euclidean norm is denoted by $|\cdot|$. Let $X \subseteq \mathbb{R}^n$ be an open set. We denote by $L_p(X, \mathbb{R})$ the space of all measurable functions on X that satisfy $\int_X |f|^p dx < \infty$ and by $\|\cdot\|_{L_p(X,\mathbb{R})}$ the norm in $L_p(X, \mathbb{R})$, $1 \leq p \leq \infty$. Let Ω be an open connected bounded subset from $\mathbb{R}^n$. Its closure is denoted by $\overline{\Omega}$. The inner product in $L_2(\Omega, \mathbb{R})$ is denoted by $\langle\cdot,\cdot\rangle$. Let $T \in \mathbb{R}$. We use the notation Q_T for the set $\Omega \times (0, T)$. Let $z \in L_p((0,T), \mathbb{R})$. We denote by $\mathbb{1}z$ the function in $L_p(Q_T, \mathbb{R})$ defined by $(\mathbb{1}z)(x,t) = z(t)$, $(x,t) \in Q_T$, $1 \leq p \leq \infty$. Let F be a function. Its gradient and its Hessian matrix are denoted by $\nabla(F)$ and $\nabla^2(F)$, respectively. We denote by $\mathcal{H}_N$ the set of piecewise constant functions $\eta \in L_2((0,T), \mathbb{R})$ taking the values $\eta(t) = \eta(\tau k)$, $t \in (\tau k, \tau(k+1)]$, $k = \overline{0, N-1}$, $\tau = T/N$. Let $W_2^2(\Omega)$ be the Hilbert space consisting of the elements $u \in L_2(\Omega, \mathbb{R})$ having generalized derivatives u_x and u_{xx}, with the norm

$$\|u\|_{W_2^2(\Omega)} = \|u\|_{L_2(\Omega,\mathbb{R})} + \|u_x\|_{L_2(\Omega,\mathbb{R})} + \|u_{xx}\|_{L_2(\Omega,\mathbb{R})}.$$

We denote by $W_2^{1,1}(Q_T)$ the Hilbert space consisting of the elements $u \in L_2(Q_T, \mathbb{R})$ having generalized derivatives u_x and u_t, with the inner product

$$(u,v)_{W_2^{1,1}(Q_T)} = \iint_{Q_T} (uv + u_x v_x + u_t v_t) dxdt$$

and by $W_2^{1,0}(Q_T)$ the Hilbert space consisting of the elements u of $L_2(Q_T, \mathbb{R})$ having generalized derivative u_x, with the inner product

$$(u,v)_{W_2^{1,0}(Q_T)} = \iint_{Q_T} (uv + u_x v_x) dxdt.$$

The Banach space consisting of the elements $u \in W_2^{1,0}(Q_T)$ having finite norm

$$|u|_{Q_T} = \text{ess} \sup_{0\le t\le T} \|u(x,t)\|_{L_2(\Omega,\mathbb{R})} + \|u_x\|_{L_2(Q_T,\mathbb{R})}$$

is denoted by $V_2(Q_T)$, and the Banach space consisting of the elements $u \in V_2(Q_T)$ such that $\|u(x,t+\Delta t) - u(x,t)\|_{L_2(\Omega,\mathbb{R})} \to 0$ when $\Delta t \to 0$ is denoted by $V_2^{1,0}(Q_T)$. The set of absolutely continuous functions is denoted by $AC([0,T],\mathbb{R})$. Fix $0 < C \in L_\infty(Q_T,\mathbb{R})$ and $0 < c \in L_\infty(Q_T,\mathbb{R})$ such that, there exists a positive number $\omega \le C$. We use the notation

$$M(\theta) = \text{div}_x(C\text{grad}_x\theta) - c\theta.$$

Let $\check{\theta} \in V_2^{1,0}(Q_T) \cap L_\infty(Q_T,\mathbb{R})$. Set $\varphi(\theta,t) = |\check{\theta}(x,t) - \theta(x,t)|^2$. Let $Q_j \in W_2^2(\overline{\Omega})$, $j = \overline{1,m}$, $K_j > 0$, $j = \overline{1,m}$, and $\theta_0 \in L_2(\Omega,\mathbb{R})$. Consider the optimal control problem

$$J(\theta, q_1, \ldots, q_m) = \iint_{Q_T} \varphi(\theta(x,t),t)dxdt \quad \to \quad \min, \tag{2.1a}$$

$$\theta_t = M(\theta), \tag{2.1b}$$

$$\theta(x,t) = \sum_{j=1}^m Q_j(x)q_j(t), \quad (x,t) \in \partial\Omega \times [0,T] \tag{2.1c}$$

$$\theta(x,0) = \sum_{j=1}^m Q_j(x)q_j(0) = \theta_0(x), \tag{2.1d}$$

$$\|\dot{q}_j\|_{L_\infty((0,T),\mathbb{R})} \le K_j, \quad j = \overline{1,m}, \tag{2.1e}$$

where $(\theta, q_1, \ldots, q_m)$ belongs to the space $V_2^{1,0}(Q_T) \times (AC([0,T],\mathbb{R}))^m$.

Boundary condition (2.1c) embrace the major part of situations that one can face in practical problems. The functions Q_j, $j = \overline{1,m}$, can be considered as a partition of unity subordinate to a cover of Ω or as shape functions for a set of elements. The functions q_j, $j = \overline{1,m}$, determine the dynamics of θ in $\Omega \cap \text{supp}(Q_j)$, $j = \overline{1,m}$. Absolute continuity of q_j is a weakest natural assumption that can be imposed. Indeed, in applications q_j, $j = \overline{1,m}$, are temperatures or concentrations that have some inertia and should be rather regular functions. The derivatives $\dot{q}_j$, $j = \overline{1,m}$, are considered as control parameters. Note that the Dirichlet-like boundary condition (2.1c) is essentially used in our approach. Complexity analysis of problems with other boundary conditions (like the Neumann condition) needs another tools.

We begin the study of this problem considering an auxiliary problem with the function $\varphi : \mathbb{R} \times [0,T] \to \mathbb{R}$ being convex locally Lipschitzian with respect to θ and measurable with respect to t. Assume that $|\varphi(\theta_1,t) - \varphi(\theta_2,t)| \le K_\varphi|\theta_1 - \theta_2|$, $K_\varphi > 0$, whenever $\theta_1, \theta_2 \in [-\Theta, \Theta]$.

Put $u(x,t) = \theta(x,t) - \sum_{j=1}^{m} Q_j(x)q_j(t)$. Let $q_j^0 \in \mathbb{R}$, $j = \overline{1,m}$, be such that $\sum_{j=1}^{m} Q_j(x)q_j^0 = \theta_0(x)$. Problem (2.1a)-(2.1e) is equivalent to the following problem:

$$J(u, q_1, \dots, q_m) = \iint_{Q_T} \varphi\Big(u + \sum_{j=1}^{m} Q_j q_j, t\Big) dxdt \quad \to \quad \min, \tag{2.2a}$$

$$u_t = M(u) + \sum_{j=1}^{m} M(Q_j q_j) - \sum_{j=1}^{m} Q_j \eta_j, \tag{2.2b}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{2.2c}$$

$$u(x,0) = 0, \tag{2.2d}$$

$$\dot{q}_j = \eta_j, \quad q_j(0) = q_j^0, \quad j = \overline{1,m}, \tag{2.2e}$$

$$\|\eta_j\|_{L_\infty((0,T),\mathbb{R})} \le K_j, \quad j = \overline{1,m}, \tag{2.2f}$$

where $(u, q_1, \dots, q_m, \eta_1, \dots, \eta_m)$ belongs to the space $V_2^{1,0}(Q_T) \times (AC([0,T],\mathbb{R}))^m \times (L_\infty((0,T),\mathbb{R}))^m$. Along with this problem, we also consider the problem:

$$J(\overline{u}, \overline{q}_1, \dots, \overline{q}_m) = \iint_{Q_T} \varphi\Big(\overline{u} + \sum_{j=1}^{m} Q_j \overline{q}_j, t\Big) dxdt \quad \to \quad \min, \tag{2.3a}$$

$$\overline{u}_t = M(\overline{u}) + \sum_{j=1}^{m} M(Q_j \overline{q}_j) - \sum_{j=1}^{m} Q_j \overline{\eta}_j, \tag{2.3b}$$

$$\overline{u}(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{2.3c}$$

$$\overline{u}(x,0) = 0, \tag{2.3d}$$

$$\dot{\overline{q}}_j = \overline{\eta}_j, \quad j = \overline{1,m}, \quad \overline{q}_j(0) = q_j^0, \tag{2.3e}$$

$$\|\overline{\eta}_j\|_{L_\infty((0,T),\mathbb{R})} \le K_j, \quad j = \overline{1,m}, \tag{2.3f}$$

where $(\overline{u}, \overline{q}_1, \dots, \overline{q}_m, \overline{\eta}_1, \dots, \overline{\eta}_m)$ belongs to the space $V_2^{1,0}(Q_T) \times (AC([0,T],\mathbb{R}))^m \times (\mathcal{H}_N)^m$. Since the functions η_j, $j = \overline{1,m}$, and $\overline{\eta}_j$, $j = \overline{1,m}$, completely determine the functions $(u, q_1, \dots, q_m)$ and $(\overline{u}, \overline{q}_1, \dots, \overline{q}_m)$, respectively, we shall use the notations $J(\eta) = J(\theta, q_1, \dots, q_m) = J(u, q_1, \dots, q_m)$ and $J(\overline{\eta}) = J(\overline{\theta}, \overline{q}_1, \dots, \overline{q}_m) = J(\overline{u}, \overline{q}_1, \dots, \overline{q}_m)$.

We do not consider any discretization of differential equation (2.3b). It is suppose that the solution to (2.3b) admits a spectral representation via Fourier-like rapidly converging series involving eigenvalues and eigenfunctions of the operator M, and, as a consequence, it can be rapidly calculated with machine accuracy.

3. Background notes

Recall some classical results from the theory of PDEs and numerical optimization.

3.1. **Parabolic Equations.** Consider the following problem:

$$\left\{\begin{array}{c} u_t - \mathcal{M}u = \displaystyle\sum_{i=1}^m \frac{\partial f_i}{\partial x_i} - f, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = \psi(x). \end{array}\right. \tag{3.1}$$

where

$$\mathcal{M}u = \sum_{i=1}^m \frac{\partial}{\partial x_i}\left(\sum_{j=1}^m a_{ij}(x,t)u_{x_j} + \sum_{i=1}^m a_i(x,t)u\right) - \sum_{i=1}^m b_i(x,t)u_{x_i} - a(x,t)u,$$

and the coefficients satisfy the condition of uniform ellipticity, i.e.,

$$\nu_1 \sum_{i=1}^m \xi_i^2 \le \sum_{i,j=1}^m a_{ij}(x,t)\xi_i\xi_j \le \nu_2 \sum_{i=1}^m \xi_i^2, \quad \nu_1, \nu_2 = (\text{const}) > 0, \tag{3.2}$$

where $(\xi_1, \xi_2, \dots, \xi_m)$ is an arbitrary real vector. Set

$$\|a\|_{q,r} = \left(\int_0^T \left(\int_\Omega a^q \, dx\right)^{r/q} dt\right)^{1/r}.$$

Let q and r be real numbers satisfying the conditions

$$\left\{\begin{array}{c} \dfrac{1}{r} + \dfrac{m}{2q} = 1, \\ q \in \left(\dfrac{m}{2}, \infty\right], r \in [1, \infty), \text{ for } m \ge 2, \\ q \in [1, \infty], r \in [1, 2], \text{ for } m = 1. \end{array}\right. \tag{3.3}$$

Assume that the conditions

$$\left\|\sum_{i=1}^m a_i^2\right\|_{q,r} \le \mu_1, \quad \left\|\sum_{i=1}^m b_i^2\right\|_{q,r} \le \mu_1, \quad \|a\|_{q,r} \le \mu_1, \tag{3.4}$$

are satisfied. Assume that

$$\left(\iint_{Q_T} \sum_{i=1}^m f_i^2 \, dxdt\right)^{1/2} \le \mu_2, \quad \|f\|_{q,r} \le \mu_2, \tag{3.5}$$

where

$$\left\{\begin{array}{c} \dfrac{1}{r} + \dfrac{m}{2q} = 1 + \dfrac{m}{4}, \\ q \in \left[\dfrac{2m}{m+2}, 2\right], r \in [1, 2], \text{ for } m \ge 3, \\ q \in (1, 2], r \in [1, 2), \text{ for } m = 2, \\ q \in [1, 2], r \in \left[1, \dfrac{4}{3}\right], \text{ for } m = 1. \end{array}\right. \tag{3.6}$$

Set

$$I(t_1; u, \phi) = \int_\Omega u(x,t_1)\phi(x,t_1)dx - \int_0^{t_1}\int_\Omega u\phi_t dxdt + \int_0^{t_1} (\mathcal{L}_1(u,\phi) + \mathcal{L}_2(f,\phi))\, dt,$$

where

$$\mathcal{L}_1(u,\phi)=\int_\Omega\left(\sum_{i=1}^m\left(\sum_{j=1}^m a_{ij}u_{x_j}+a_iu\right)\phi_{x_i}+\left(\sum_{i=1}^m b_iu_{x_i}+au\right)\phi\right)dx$$

and

$$\mathcal{L}_2(f,\phi)=\int_\Omega\left(\sum_{i=1}^m f_i\phi_{x_i}+f\phi\right)dx.$$

We say that a function $u\in V_2^{1,0}(Q_T)$ is a solution of problem (3.1) if the equality

$$I(t_1;u,\phi)=\int_\Omega\psi(x)\phi(x,0)dx$$

holds for all $t_1\in[0,T]$, $\phi\in W_2^{1,1}(Q_T)$ with $\phi(x,t)=0$, $(x,t)\in(\partial\Omega\times[0,T])$.

Recall the following results (see, [7, ch. 3]).

Theorem 3.1 ([7])**.** *If $\psi\in L_2(\Omega,\mathbb{R})$, then problem (3.1) has a unique solution $u\in V_2^{1,0}(Q_T)$.*

Theorem 3.2 ([7])**.** *Assume that for all problems*

$$\begin{cases} u_t-\mathcal{M}^mu=\displaystyle\sum_{i=1}^m\frac{\partial f_i^m}{\partial x_i}-f^m,\\ u(x,t)=0,\quad (x,t)\in\partial\Omega\times[0,T],\\ u(x,0)=\psi^m(x). \end{cases}\tag{3.7}$$

where

$$\mathcal{M}^mu=\sum_{i=1}^m\frac{\partial}{\partial x_i}\left(\sum_{j=1}^m a_{ij}^m(x,t)u_{x_j}+a_i^m(x,t)u\right)-\sum_{i=1}^m b_i^m(x,t)u_{x_i}-a^m(x,t)u,$$

conditions (3.3)-(3.6) are satisfied with the same constants. Assume that the sequence of functions a_{ij}^m converges almost everywhere to a_{ij} and the functions a_i^m, b_i^m, a^m, f_i^m, f^m, and ψ^m converge to a_i, b_i, a, f_i, f, and ψ, respectively, in the norms of the spaces to which they belong according to conditions (3.3)-(3.6). Then the solutions u^m of problems (3.7) converge strongly in $V_2^{1,0}(Q_T)$ to the solution u of problem (3.1).

Set $U=(\partial\Omega\times[0,T])\cup(\Omega\times\{0\})$.

Theorem 3.3 ([7])**.** *Let $u\in V_2^{1,0}(Q_T)$ be a solution of problem (3.1). Assume that the following conditions are satisfied*

(1) *the coefficients a_{ij}, b_i, and a satisfy (3.3)-(3.4),*
(2) *$a(x,t)\geq 0$,*
(3) *$a_i=f_i=f=0$.*

Then

$$\min\{0,\operatorname{ess\,inf}_U u(x,t)\}\leq u(x,t)\leq\max\{0,\operatorname{ess\,sup}_U u(x,t)\}$$

for almost all (x,t) from Q_T.

3.2. **Path-following Minimization Method.** Let $P : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Consider the problem:

$$\text{(3.8)} \qquad \begin{cases} P(x) \to \min, \\ x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n, \\ x_i^2 \le K_i, \quad i = \overline{1,n}. \end{cases}$$

To apply path-following method to this problem, consider the following mathematical programming problem:

$$\text{(3.9)} \qquad \begin{cases} \sigma \to \min, \\ x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n, \\ P(x) \le \sigma, \\ x_i^2 \le K_i, \quad i = \overline{1,n}. \end{cases}$$

Let F be a function defined by

$$F(x,\sigma) = -\ln(\sigma - P(x)) - \ln(\overline{\sigma} - \sigma) - \sum_{i=1}^{n} \ln(K_i - x_i^2),$$

where $\overline{\sigma} = \max_{\{x|x_i^2 \le K_i, i=\overline{1,n}\}} P(x)$, and let $b = (0,1) \in \mathbb{R}^n \times \mathbb{R}$. Let $v \in \mathbb{R}^{n+1}$ be a vector. We use the notation $\|v\|_y^F = \langle [\nabla^2(F)(y)]^{-1} v | v \rangle^{1/2}$ whith $y = (x,\sigma)$.

Let $\beta \in (0, (3-\sqrt{5})/2)$, and let $\gamma > 0$ be such that $\gamma \le \dfrac{\sqrt{\beta}}{1+\sqrt{\beta}} - \beta$. The path-following method is presented in Table I (see, [10]).

Initialization: Set $\alpha_0 = 0$. Choose an accuracy $\varepsilon > 0$, $x_0 \in \mathbb{R}^n$, and $\sigma_0 \in \mathbb{R}$ such that $$\|\nabla(F)(x_0,\sigma_0)\|^F_{(x_0,\sigma_0)} \le \beta.$$
Step k: Set $$\alpha_{k+1} = \alpha_k + \frac{\gamma}{\|b\|^F_{(x_k,\sigma_k)}},$$ $$(x_{k+1},\sigma_{k+1}) = (x_k,\sigma_k) - [\nabla^2(F)(x_k,\sigma_k)]^{-1}(\alpha_{k+1} b + \nabla(F)(x_k,\sigma_k)).$$
Stop the process if $$n + 1 + \frac{(\beta + \sqrt{n+1})\beta}{1-\beta} \le \varepsilon \alpha_k.$$

Table 1. Path-following method

An auxiliary method to find an initial point satisfying $\|\nabla(F)(x_0,\sigma_0)\|^F_{(x_0,\sigma_0)} \le \beta$ is discussed in [10].

Let $\mathcal{N}$ be the largest integer satisfying

$$\mathcal{N} \leq \frac{\ln\left(\frac{(1+\beta)(n+1)+(\beta+\sqrt{n+1})\beta}{\gamma(1-2\beta)\varepsilon}\|b\|^F_{(\tilde{x},\tilde{\sigma})}\right)}{\ln\left(1+\frac{\gamma}{\beta+\sqrt{n+1}}\right)}+1,$$

where $(\tilde{x},\tilde{\sigma}) = \text{argmin}\,(F)$.

Theorem 3.4 ([10])**.** *The path-following method terminates no more than after $\mathcal{N}$ steps. At the moment of termination we have $|P(x_{\mathcal{N}}) - P(\hat{x})| < \varepsilon$, where $\hat{x}$ solution of problem (3.8).*

4. Main results

Let us formulate the main results of this work.

Theorem 4.1. *Problem (2.1a)-(2.1e) has a solution.*

Theorem 4.2. *Let $\hat{\theta}$ be an optimal solution of (2.1a)-(2.1e) and $\hat{\overline{\theta}} = \hat{\overline{u}} + \sum_{j=1}^m Q_j \hat{\overline{q}}_j$ be an optimal solution of (2.3a)-(2.3f). Then*

$$|J(\hat{\overline{\theta}}, \hat{\overline{q}}_1, \ldots, \hat{\overline{q}}_m) - J(\hat{\theta}, \hat{q}_1, \ldots, \hat{q}_m)| \leq \frac{T^2}{N}\left(V_\Omega K_\varphi \sum_{j=1}^m \|Q_j\|_{L_\infty(\Omega,\mathbb{R})} K_j\right),$$

where $V_\Omega = \int_\Omega dx$.

Set

$$\mathcal{F}(\overline{\eta},\sigma) = -\ln(\sigma - J(\overline{\eta})) - \ln(\overline{\sigma}-\sigma) - \sum_{i=1}^m \sum_{k=0}^{N-1} \ln(K_i^2 - (\overline{\eta}_i(\tau k))^2), \tag{4.1}$$

where

$$\overline{\sigma} = \max_{\{\eta\in\mathcal{H}_N \mid \|\eta_i\|_{L_\infty((0,T),\mathbb{R})}\leq K_i, i=\overline{1,m}\}} J(\eta).$$

Let $\varepsilon > 0$,

$$N > \frac{2\left(V_\Omega K_\varphi \sum_{j=1}^m \left(\|Q_j\|_{L_\infty(\Omega,\mathbb{R})} K_j\right)\right) T^2}{\varepsilon}, \tag{4.2}$$

$$\mathcal{C} = \frac{1}{\sqrt{2}} T^2 V_\Omega K_\varphi \sum_{j=1}^m \|Q_j\|_{L_\infty(Q_T,\mathbb{R})} K_i, \tag{4.3}$$

and

$$\mathcal{N} \geq \frac{\ln\left(2\frac{(1+\beta)(Nm+1)+(\beta+\sqrt{Nm+1})\beta}{\gamma(1-2\beta)\varepsilon}\mathcal{C}\right)}{\ln\left(1+\frac{\gamma}{\beta+\sqrt{Nm+1}}\right)}+1. \tag{4.4}$$

Let $\hat{\eta}$ be an optimal control of (2.2a)-(2.2f) and consider $\mathcal{H}_N$ be the set of piecewise constant functions $\eta \in L_2((0,T),\mathbb{R})$ taking values $\eta(t) = \eta(\tau k)$, $t \in (\tau k, \tau(k+1)]$, $k = \overline{0, N-1}$, $\tau = T/N$.

Theorem 4.3. *The path-following method finds $\overline{\eta} \in \mathcal{H}_N$ satisfying $|J(\overline{\eta})-J(\hat{\eta})| < \varepsilon$, no more than after $\mathcal{N}$ iterations.*

5. Auxiliary lemmas

We will need the following auxiliary results.

Lemma 5.1. *Let $\eta = (\eta_1, \eta_2, \ldots, \eta_m) \in L_\infty((0,T), \mathbb{R}^m)$ be a function satisfying $\|\eta_j\|_{L_\infty((0,T),\mathbb{R})} \leq K_j$, and let $(u, q_1, \ldots, q_m)$ be a solution of problem (2.2b)-(2.2f). Then the inequalities*

$$\left\| \sum_{j=1}^m Q_j q_j \right\|_{L_\infty(Q_T,\mathbb{R})} \leq \|\theta_0\|_{L_\infty(\Omega,\mathbb{R})} + T \left\| \sum_{j=1}^m Q_j \eta_j \right\|_{L_\infty(Q_T,\mathbb{R})},$$

and

$$\|u\|_{L_\infty(Q_T,\mathbb{R})} \leq 2 \left(\|\theta_0\|_{L_\infty(\Omega,\mathbb{R})} + T \left\| \sum_{j=1}^m Q_j \eta_j \right\|_{L_\infty(Q_T,\mathbb{R})} \right),$$

hold.

Proof. From equation (2.2e), we obtain

$$\sum_{j=1}^m Q_j(\cdot) q_j(t) = \sum_{j=1}^m Q_j(\cdot) \left(q_j(0) + \int_0^t \eta_j(s) ds \right).$$

Therefore we have

$$\begin{aligned} &\left\| \sum_{j=1}^m Q_j q_j \right\|_{L_\infty(Q_T,\mathbb{R})} \\ &\leq \left\| \sum_{j=1}^m Q_j q_j(0) \right\|_{L_\infty(\Omega,\mathbb{R})} + \int_0^T \left\| \sum_{j=1}^m Q_j \eta_j \right\|_{L_\infty(Q_T,\mathbb{R})} ds \\ &= \|\theta_0\|_{L_\infty(\Omega,\mathbb{R})} + T \left\| \sum_{j=1}^m Q_j \eta_j \right\|_{L_\infty(Q_T,\mathbb{R})}. \end{aligned}$$

Since $u(x,t) = \theta(x,t) - \sum_{j=1}^m Q_j(x) q_j(t)$, we obtain

$$\|u\|_{L_\infty(Q_T,\mathbb{R})} \leq \|\theta\|_{L_\infty(Q_T,\mathbb{R})} + \left\| \sum_{j=1}^m Q_j q_j \right\|_{L_\infty(Q_T,\mathbb{R})}.$$

By Theorem 3.3, we get

$$\|\theta\|_{L_\infty(Q_T,\mathbb{R})} \leq \left\| \sum_{j=1}^m Q_j q_j \right\|_{L_\infty(Q_T,\mathbb{R})}$$

and the end of the proof. □

Lemma 5.2. *Let* $\eta^{(1)} = (\eta_1^{(1)}, \eta_2^{(1)}, \ldots, \eta_m^{(1)}) \in L_\infty((0,T),\mathbb{R}^m)$ *and* $\eta^{(2)} = (\eta_1^{(2)}, \eta_2^{(2)}, \ldots, \eta_m^{(2)}) \in L_\infty((0,T),\mathbb{R}^m)$ *be functions satisfying* $\|\eta_j^{(1)}\|_{L_\infty((0,T),\mathbb{R})} \leq K_j$ *and* $\|\eta_j^{(2)}\|_{L_\infty((0,T),\mathbb{R})} \leq K_j$*. Then the inequality*

$$|J(\eta^{(1)}) - J(\eta^{(2)})| \leq T^2 V_\Omega K_\varphi \left\| \sum_{j=1}^m Q_j \left(\eta_j^{(1)} - \eta_j^{(2)} \right) \right\|_{L_\infty(Q_T,\mathbb{R})}$$

holds.

Proof. Let $(u^{(i)}, q_1^{(i)}, \ldots, q_m^{(i)}) \in V_2^{1,0}(Q_T) \times AC([0,T],\mathbb{R})^m$, be functions satisfying

$$u_t^{(i)} = M(u^{(i)}) + \sum_{j=1}^m M(Q_j q_j^{(i)}) - \sum_{j=1}^m Q_j \eta_j^{(i)},$$

$$u^{(i)}(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T],$$

$$u^{(i)}(x,0) = 0,$$

$$\dot{q}_j^{(i)} = \eta_j^{(i)}, \quad q_j^{(i)}(0) = q_j^0, \quad i \in \{1,2\}.$$

Put $\theta^{(1)}(x,t) = u^{(1)}(x,t) + \sum_{j=1}^m Q_j(x) q_j^{(1)}(t)$ and $\theta^{(2)}(x,t) = u^{(2)}(x,t) + \sum_{j=1}^m Q_j(x) q_j^{(2)}(t)$. Then we have

$$\begin{aligned} |J(\eta^{(1)}) - J(\eta^{(2)})| &= \left| \iint_{Q_T} \left(\varphi(\theta^{(1)}(x,t),t) - \varphi(\theta^{(2)}(x,t),t) \right) dxdt \right| \\ &\leq \iint_{Q_T} K_\varphi \left| \theta^{(1)}(x,t) - \theta^{(2)}(x,t) \right| dxdt \\ &\leq T V_\Omega K_\varphi \|\theta^{(1)} - \theta^{(2)}\|_{L_\infty(Q_T,\mathbb{R})} \end{aligned}$$

By Lemma 5.1 we obtain

$$\left\| \sum_{j=1}^m Q_j (q_j^{(1)} - q_j^{(2)}) \right\|_{L_\infty(Q_T,\mathbb{R})} \leq T \left\| \sum_{j=1}^m Q_j \left(\eta_j^{(1)} - \eta_j^{(2)} \right) \right\|_{L_\infty(Q_T,\mathbb{R})}.$$

Therefore, by Theorem 3.3 we get

$$\|\theta^{(1)} - \theta^{(2)}\|_{L_\infty(Q_T,\mathbb{R})} \leq T \left\| \sum_{j=1}^m Q_j \left(\eta_j^{(1)} - \eta_j^{(2)} \right) \right\|_{L_\infty(Q_T,\mathbb{R})}.$$

Thus we have

$$|J(\eta^{(1)}) - J(\eta^{(2)})| \leq T^2 V_\Omega K_\varphi \left\| \sum_{j=1}^m Q_j \left(\eta_j^{(1)} - \eta_j^{(2)} \right) \right\|_{L_\infty(Q_T,\mathbb{R})}.$$

∎

6. Proof of the main results

Proof of Theorem 4.1. Let ℓ be the infimum of problem (2.2a)-(2.2f) and $\{u^l\}$ and $\{q^l\}$ be solutions of problems

$$(6.1)\qquad \begin{cases} u^l_t = M(u^l) + \sum_{j=1}^m M(Q_j q^l_j) - \sum_{j=1}^m Q_j \eta^l_j, \\ u^l(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \\ u^l(x,0) = 0 \\ \dot{q}^l_j = \eta^l_j, \quad q^l_j(0) = q^0_j, \\ \|\eta^l_j\|_{L_\infty((0,T),\mathbb{R})} \le K_j, \end{cases}$$

where $(\theta, q_1, \dots, q_m, \eta_1, \dots, \eta_m)$ belongs to the space $V_2^{1,0}(Q_T) \times (AC([0,T],\mathbb{R}))^m \times (L_\infty((0,T),\mathbb{R}))^m$, such that $J(u^l(\cdot,\cdot), q^l(\cdot)) \to \ell$. Since $\{\eta^l\}$ is bounded in $L_2((0,T),\mathbb{R}^m)$, there exists a subsequence $\{\eta^{l_k}\}$ weakly convergent in $L_2((0,T),\mathbb{R}^m)$. Let η^0 be the weak limit of $\{\eta^{l_k}\}$. By Mazur's theorem, there exists a sequence of convex combinations of $\{\eta^{l_k}\}$, $\xi^p = \sum_{i=p}^{I(p)} \lambda_i \eta^{l_i}$, such that $\xi^p \to \eta^0$ in $L_2((0,T),\mathbb{R}^m)$. Since equation (2.2b) is linear, by Theorem 3.2 we get $\sum_{i=p}^{I(p)} \lambda_i u^{l_i} \to u^0$ and $\sum_{i=p}^{I(p)} \lambda_i q^{l_i} \to q^0$. Hence, since J is convex, from Lemma 5.2 we obtain

$$J(u^0, q^0) = \lim_{p\to\infty} J\left(\sum_{i=p}^{I(p)} \lambda_i u^{l_i}, \sum_{i=p}^{I(p)} \lambda_i q^{l_i}\right) \le \lim_{p\to\infty} \sum_{i=p}^{I(p)} \lambda_i J(u^{l_i}, q^{l_i}) = \ell.$$

Therefore (u^0, q^0) is the solution of (2.2a)-(2.2f). □

In what follows, we denote by $(\hat\theta, \hat q_j)$ an optimal process from (2.1a)-(2.1e). Problem (2.3a)-(2.3f) also has a solution. This follows from the compactness of the set of controls and Theorem 3.2. Let us show that the difference between $J(\hat{\hat\theta})$ and $J(\hat\theta)$, where $\hat{\hat\theta} = \hat{\hat u} + \sum_{j=1}^m Q_j \hat{\hat q}_j$ is an optimal solution to (2.3a)-(2.3f), can be done arbitrary small whenever N is sufficiently large. Consider the functions $\tilde q_j(t)$, $j = \overline{1,m}$ defined by

$$\tilde q_j(0) = \hat q_j(0),$$

and

$$\tilde q_j(t) = \tilde q_j(\tau k) + \frac{t - \tau k}{\tau} \int_{\tau k}^{\tau(k+1)} \dot{\hat q}_j(s)ds,$$
$$j = \overline{1,m}, \quad t \in (\tau k, \tau(k+1)], \quad \tau = \frac{T}{N}.$$

Obviously these functions are piecewise linear, continuous and satisfy $|\dot{\tilde q}_j(t)| \le K_j$, $t \in [0,T]$, $j = \overline{1,m}$. The function $\sum_{j=1}^m Q_j \tilde q_j$ is close to the function $\sum_{j=1}^m Q_j \hat q_j$ in the following sense.

Lemma 6.1. *The following inequality holds:*

$$(6.2)\qquad \left\| \sum_{j=1}^m Q_j(\tilde q_j - \hat q_j) \right\|_{L_\infty(Q_T,\mathbb{R})} \le \tau \sum_{j=1}^m \left(\|Q_j\|_{L_\infty(\Omega,\mathbb{R})} K_j \right).$$

Proof. By induction we have

$$\begin{aligned}\sum_{j=1}^{m} Q_j(x)\tilde{q}_j(\tau(k+1)) &= \sum_{j=1}^{m} Q_j(x)\tilde{q}_j(\tau k) + \sum_{j=1}^{m} Q_j(x)\int_{\tau k}^{\tau(k+1)} \dot{\hat{q}}_j(s)ds \\ &= \sum_{j=1}^{m} Q_j(x)\hat{q}_j(\tau k) + \sum_{j=1}^{m} Q_j(x)\int_{\tau k}^{\tau(k+1)} \dot{\hat{q}}_j(s)ds \\ &= \sum_{j=1}^{m} Q_j(x)\hat{q}_j(\tau(k+1)).\end{aligned}$$

Thus we obtain

$$\sum_{j=1}^{m} Q_j(x)\tilde{q}_j(\tau k) = \sum_{j=1}^{m} Q_j(x)\hat{q}_j(\tau k), k = \overline{0, N} \tag{6.3}$$

Observe that for all $t \in (\tau k, \tau(k+1)]$, we have

$$\begin{aligned}\sum_{j=1}^{m} Q_j(x)\tilde{q}_j(t) &= \sum_{j=1}^{m} Q_j(x)\tilde{q}_j(\tau k) + \frac{t-\tau k}{\tau}\sum_{j=1}^{m} Q_j(x)\int_{\tau k}^{\tau(k+1)} \dot{\hat{q}}_j(s)ds \\ &= \sum_{j=1}^{m} Q_j(x)\tilde{q}_j(\tau(k+1)) + \frac{t-\tau(k+1)}{\tau}\sum_{j=1}^{m} Q_j(x)\int_{\tau k}^{\tau(k+1)} \dot{\hat{q}}_j(s)ds\end{aligned}$$

If $t \in (\tau k, \tau(k+\frac{1}{2})]$, then we obtain

$$\begin{aligned}&\left|\sum_{j=1}^{m} Q_j(x)(\tilde{q}_j(t) - \hat{q}_j(t))\right| \\ &= \left|\frac{t-\tau k}{\tau}\sum_{j=1}^{m} Q_j(x)\int_{\tau k}^{\tau(k+1)} \dot{\hat{q}}_j(s)ds - \sum_{j=1}^{m} Q_j(x)\int_{\tau k}^{t} \dot{\hat{q}}_j(s)ds\right| \\ &\le \frac{t-\tau k}{\tau}\left|\sum_{j=1}^{m} Q_j(x)\int_{\tau k}^{\tau(k+1)} \dot{\hat{q}}_j(s)ds\right| + \left|\sum_{j=1}^{m} Q_j(x)\int_{\tau k}^{t} \dot{\hat{q}}_j(s)ds\right| \\ &\le \frac{1}{2}\sum_{j=1}^{m} |Q_j(x)|\int_{\tau k}^{\tau(k+1)} \left|\dot{\hat{q}}_j(s)\right| ds + \sum_{j=1}^{m} |Q_j(x)|\int_{\tau k}^{t} \left|\dot{\hat{q}}_j(s)\right| ds \\ &\le \frac{\tau}{2}\sum_{j=1}^{m} |Q_j(x)|\, K_j + \frac{\tau}{2}\sum_{j=1}^{m} |Q_j(x)|\, K_j \\ &= \tau\sum_{j=1}^{m} |Q_j(x)|\, K_j\end{aligned}$$

and, if $t \in (\tau(k+\frac{1}{2}), \tau(k+1)]$, then we have

$$\begin{aligned}
&\left|\sum_{j=1}^{m} Q_j(x)(\tilde{q}_j(t)-\hat{q}_j(t))\right| \\
&= \left|\frac{t-\tau(k+1)}{\tau}\sum_{j=1}^{m} Q_j(x)\int_{\tau k}^{\tau(k+1)} \dot{\hat{q}}_j(s)ds - \sum_{j=1}^{m} Q_j(x)\int_{t}^{\tau(k+1)} \dot{\hat{q}}_j(s)ds\right| \\
&\le \frac{\tau(k+1)-t}{\tau}\left|\sum_{j=1}^{m} Q_j(x)\int_{\tau k}^{\tau(k+1)} \dot{\hat{q}}_j(s)ds\right| + \left|\sum_{j=1}^{m} Q_j(x)\int_{t}^{\tau(k+1)} \dot{\hat{q}}_j(s)ds\right| \\
&\le \frac{\tau}{2}\sum_{j=1}^{m} |Q_j(x)|\, K_j + \frac{\tau}{2}\sum_{j=1}^{m} |Q_j(x)|\, K_j \\
&= \tau\sum_{j=1}^{m} |Q_j(x)|\, K_j.
\end{aligned}$$

Thus we get (6.2). □

Now, we can use the maximum principle (Theorem 3.3) to find an estimate for the difference between the optimal solution of (2.2a)-(2.2f) and the optimal solution of (2.3a)-(2.3f).

Lemma 6.2. *Let* $\sum_{j=1}^{m} Q_j(x)\tilde{q}_j(t)$ *be the function defined above and* $\tilde{\theta}$ *be the solution of the problem*

$$(6.4)\qquad \begin{cases} \tilde{\theta}_t = M(\tilde{\theta}), \\ \tilde{\theta}(x,t) = \sum_{j=1}^{m} Q_j(x)\tilde{q}_j(t), \quad (x,t)\in\partial\Omega\times[0,T], \\ \tilde{\theta}(x,0) = \theta_0(x). \end{cases}$$

Then the inequality

$$\left\|\tilde{\theta}-\hat{\theta}\right\|_{L_\infty(Q_T,\mathbb{R})} \le T\sum_{j=1}^{m}\left(\|Q_j\|_{L_\infty(\Omega,\mathbb{R})} K_j\right)/N$$

holds.

Proof. Let $\delta\theta = \tilde{\theta}-\hat{\theta}$. We have

$$\delta\theta_t = M(\delta\theta),$$

$$\delta\theta(x,t) = \sum_{j=1}^{m} Q_j(x)(\tilde{q}_j(t)-\hat{q}_j(t)), \quad (x,t)\in\partial\Omega\times[0,T],$$

and

$$\delta\theta(x,0) = 0.$$

By Theorem 3.3 and Lemma 6.1, we get,

$$\|\delta\theta\|_{L_\infty(Q_T,\mathbb{R})} \le \left\|\sum_{j=1}^{m} Q_j(\tilde{q}_j-\hat{q}_j)\right\|_{L_\infty(Q_T,\mathbb{R})} \le \frac{T}{N}\sum_{j=1}^{m}\|Q_j\|_{L_\infty(\Omega,\mathbb{R})} K_j.$$

□

Proof of Theorem 4.2. Let $\tilde{\theta}$ be the function defined as in (6.4). As in the proof of Lemma 5.2 we obtain

$$\left|J(\tilde{\theta},\tilde{q}_1,\dots,\tilde{q}_m)-J(\hat{\theta},\hat{q}_1,\dots,\hat{q}_m)\right| = \left|\iint_{Q_T}\left(\varphi(\tilde{\theta},t)-\varphi(\hat{\theta},t)\right)dxdt\right| \le TV_\Omega K_\varphi\|\tilde{\theta}-\hat{\theta}\|_{L_\infty(Q_T,\mathbb{R})}.$$

From Lemma 6.2 we have

$$\left|J(\tilde{\theta},\tilde{q}_1,\dots,\tilde{q}_m)-J(\hat{\theta},\hat{q}_1,\dots,\hat{q}_m)\right| \le TV_\Omega K_\varphi\|\tilde{\theta}-\hat{\theta}\|_{L_\infty(Q_T,\mathbb{R})} = \frac{T^2}{N}\left(V_\Omega K_\varphi\sum_{j=1}^m\|Q_j\|_{L_\infty(\Omega,\mathbb{R})}K_j\right).$$

Since

$$J(\hat{\theta},\hat{q}_1,\dots,\hat{q}_m)\le J(\hat{\bar{\theta}},\hat{\bar{q}}_1,\dots,\hat{\bar{q}}_m)\le J(\tilde{\theta},\tilde{q}_1,\dots,\tilde{q}_m),$$

we have

$$\left|J(\hat{\bar{\theta}},\hat{\bar{q}}_1,\dots,\hat{\bar{q}}_m)-J(\tilde{\theta},\tilde{q}_1,\dots,\tilde{q}_m)\right| \le \frac{T^2}{N}\left(V_\Omega K_\varphi\sum_{j=1}^m\|Q_j\|_{L_\infty(\Omega,\mathbb{R})}K_j\right).$$

□

Proof of Theorem 4.3. Let

$$N > \frac{2\left(V_\Omega K_\varphi\sum_{j=1}^m\|Q_j\|_{L_\infty(\Omega,\mathbb{R})}K_j\right)T^2}{\varepsilon}.$$

By Theorem 4.2, we have $|J(\hat{\bar{\eta}})-J(\hat{\eta})|<\varepsilon/2$, where $\hat{\bar{\eta}}$ is an optimal process of (2.3a)-(2.3f) and $\hat{\eta}$ is an optimal process of (2.2a)-(2.2f). Using the path-following method we find $\overline{\eta}$ such that $|J(\overline{\eta})-J(\hat{\bar{\eta}})|\le\varepsilon/2$. By Theorem 3.4, the number of iterations is at most

$$\frac{\ln\left(2\dfrac{(1+\beta)(Nm+1)+(\beta+\sqrt{Nm+1})\beta}{\gamma(1-2\beta)\varepsilon}\|b\|^{\mathcal{F}}_{(\tilde{x},\tilde{\sigma})}\right)}{\ln\left(1+\dfrac{\gamma}{\beta+\sqrt{Nm+1}}\right)}+1,$$

where $b=(0,1)$. We have

$$|J(\overline{\eta})-J(\hat{\eta})|\le|J(\overline{\eta})-J(\hat{\bar{\eta}})|+|J(\hat{\bar{\eta}})-J(\hat{\eta})|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Since $\|b\|^{\mathcal{F}}_{(\tilde{x},\tilde{\sigma})}$ is the square root of the $((n+1),(n+1))^{\text{th}}$ entry of the matrix $[\nabla^2(\mathcal{F})(\tilde{x},\tilde{\sigma})]^{-1}$, using the Sherman-Morrison-Woodbury formula, we get

$$\|b\|^{\mathcal{F}}_{(\tilde{x},\tilde{\sigma})}\le\left(\frac{1}{(\tilde{\sigma}-J(\tilde{\eta}))^2}+\frac{1}{(\overline{\sigma}-\tilde{\sigma})^2}\right)^{-1/2} = \left(\frac{(\tilde{\sigma}-J(\tilde{\eta}))^2(\overline{\sigma}-\tilde{\sigma})^2}{(\tilde{\sigma}-J(\tilde{\eta}))^2+(\overline{\sigma}-\tilde{\sigma})^2}\right)^{1/2},$$

where $\overline{\sigma} = \max_{\{\eta\in\mathcal{H}_N | \|\eta_i\|_{L_\infty((0,T),\mathbb{R})}\leq K_i, i=\overline{1,m}\}} J(\eta)$ and $(\tilde{\eta},\tilde{\sigma}) = \text{argmin}\ (\mathcal{F})$ (see (4.1)). Consider the function

$$g(\lambda) = \frac{(\lambda - J(\tilde{\eta}))^2(\overline{\sigma}-\lambda)^2}{(\lambda - J(\tilde{\eta}))^2 + (\overline{\sigma}-\lambda)^2}, \quad \lambda \in [J(\tilde{\eta}), \overline{\sigma}].$$

It attains its maximum $\lambda = (\overline{\sigma} + J(\tilde{\eta}))/2$. Thus,

$$\|b\|^{\mathcal{F}}_{(\tilde{x},\tilde{\sigma})} \leq \frac{1}{2\sqrt{2}}(\overline{\sigma} - J(\tilde{\eta})).$$

Applying Lemma 5.2, we obtain the result. □

7. Tracking problem. examples

Consider the tracking problem. In this case $\varphi(\theta,t) = |\check{\theta}(x,t) - \theta(x,t)|^2$, where $\check{\theta} \in V_2^{1,0}(Q_T) \cap L_\infty(Q_T,\mathbb{R})$. The number of iterations needed to reach a given accuracy is determined by (4.2)-(4.4), where

$$K_\varphi = 2\left(\|\theta_0\|_{L_\infty(\Omega,\mathbb{R})} + T\max_{j=\overline{1,m}} K_j + \|\check{\theta}\|_{L_\infty(Q_T,\mathbb{R})}\right).$$

Note that, since the functional is strictly convex, the tracking problem has a unique solution. Moreover, the sequence of solutions to problem (2.3a)-(2.3f) converges almost everywhere to the solution of the tracking problem.

Below we present a few examples. Let us consider the problem

$$\begin{gathered} J(\theta(\cdot,\cdot),q(\cdot)) = \int_0^{10}\int_0^{\pi} |\theta(x,t) - 3\cos(\pi t)/4|^2 dxdt \quad \to \quad \min, \\ \theta_t = \theta_{xx}, \\ \theta(0,t) = \theta(\pi,t) = q(t), \\ \|\dot{q}\|_{L_\infty((0,1),\mathbb{R})} \leq 1, \\ \theta(x,0) = q(0) = 3/4. \end{gathered}$$

Introducing new function $y = \theta - \mathbb{1}q$ we get

$$\begin{gathered} y_t = y_{xx} - (\mathbb{1}q)_t, \\ y(0,t) = y(\pi,t) = 0, \\ y(x,0) = 0, \\ \dot{q} = \eta, \\ q(0) = 3/4. \end{gathered}$$

Now the problem takes the form

$$J(y(\cdot,\cdot),q(\cdot)) = \int_0^{10}\int_0^{\pi} |y(x,t) + (\mathbb{1}q) - 3\cos(\pi t)/4|^2 dxdt \quad \to \quad \min,$$
$$y_t = y_{xx} - (\mathbb{1}q)_t,$$
$$y(0,t) = y(\pi,t) = 0,$$
$$y(x,0) = 0,$$
$$\dot{q} = \eta, \quad |\eta| \le 1$$
$$q(0) = 3/4.$$

Obviously we have $y(x,t) = \sum_{n=1}^{\infty} y_n(t)\sin(nx)$ and $\mathbb{1} = \sum_{n=1}^{\infty} b_n \sin(nx)$, where

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)dx = \frac{2}{\pi}\left(\frac{1-\cos(\pi x)}{n}\right) = \begin{cases} \dfrac{4}{n\pi}, & n = 2k-1, \\ 0, & n = 2k. \end{cases}$$

Therefore we have

$$\begin{cases} \dot{y}_k = -(2k-1)^2 y_k - \dfrac{4}{(2k-1)\pi}\eta & k = 1,2,\dots, \\ \dot{q} = \eta, \end{cases}$$
$$y_k(0) = 0, \qquad q(0) = 3/4.$$

The functional takes the form

$$J = \int_0^{10}\left(\frac{\pi}{2}\sum_{k\in\mathbb{N}} y_k^2 + 4(q - 3\cos(\pi t)/4)\sum_{k\in\mathbb{N}}\frac{y_k}{2k-1} + \pi(q - 3\cos(\pi t)/4)^2\right)dt$$

Putting

$$Y(0) = \begin{bmatrix} q(0) \\ y_1(0) \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 3/4 \\ \vdots \\ 0 \\ \vdots \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & -(2k-1)^2 & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ \vdots \\ -\dfrac{4}{(2k-1)\pi} \\ \vdots \end{bmatrix},$$

$$M = \begin{bmatrix} \pi & 0 & 0 & 0 \\ \vdots & \frac{\pi}{2} & 0 & 0 \\ \dfrac{4}{2k-1} & 0 & \frac{\pi}{2} & 0 \\ \vdots & 0 & 0 & \frac{\pi}{2} \end{bmatrix}, \quad \mu = \begin{bmatrix} -\frac{3}{2}\pi\cos(\pi t) \\ \vdots \\ -\dfrac{3\cos(\pi t)}{2k-1} \\ \vdots \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & e^{-(2k-1)^2 t} & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix},$$

we get

$$J(Y) = \int_0^{10} \left(\langle Y(t)|MY(t)\rangle + \langle Y(t)|\mu\rangle\right) dt + \frac{45\pi}{16},$$
$$Y(t) = S(t)Y(0) + \int_0^t S(t-s)B\eta(s)ds.$$

Put

$$\overline{J} = \frac{T}{N}\sum_{k=1}^{N}\left(\langle \overline{Y}_k|M\overline{Y}_k\rangle + \langle \overline{Y}_k|\mu(k/N)\rangle\right) + \frac{45\pi}{16},$$

The value of $\overline{Y}_k$ is numerically calculated as

$$\overline{Y}_k = S(k/N)Y_0 + \frac{T}{N}\sum_{i=1}^{k}\left(S((k-i)/N)B\overline{\eta}(i)\right).$$

Set $N = 10$, $\varepsilon = 10^{-12}$, $F(\overline{\eta},\sigma) = -\ln(\sigma - \overline{J}(\overline{\eta})) - \ln(\overline{\sigma} - \sigma) - \sum_{k=1}^{10}\ln(1-\overline{\eta}(k)^2)$, $\eta_{in} \equiv 0$, $\sigma_{in} = \overline{J}(\eta_{in}) + 50$, and $\overline{\sigma} = \sigma_{in} + 50$. (We need to put σ_{in} and $\overline{\sigma}$ sufficiently large to guarantee the inequality $\|\nabla(F)(\eta_{in},\sigma_{in})\|^F_{(\eta_{in},\sigma_{in})} < \beta$). Using the path-following method with $\beta = 1/9$ and $\gamma = 5/36$, we get, after 693 iterations, the solution

$$\eta = \left[-1, -1, 0.73, -1, 1, -1, 1, -1, 0.95, -1, -0.18\right]$$

and the functional takes the value 15.8919. The theoretical estimate to the number of iterations is 1048.

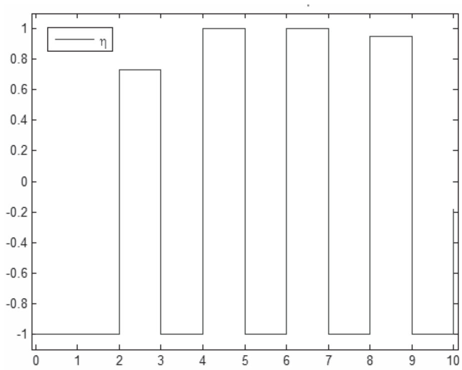

FIGURE 1. Evolution of discrete control

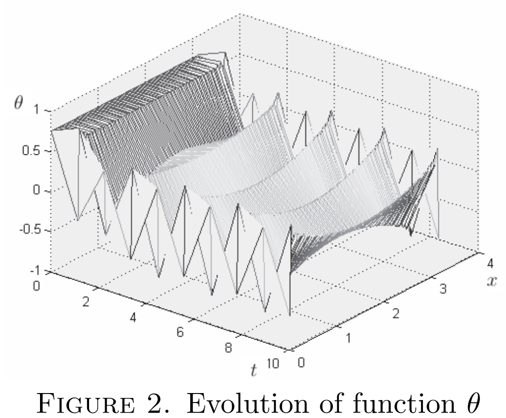

FIGURE 2. Evolution of function θ

Tables 2 and 3 shows the estimated and real number of iterations for different accuracies and other objective functions.

TABLE 2. 10 partitions; T=10;

precision	e^{-t}	$\cos(\pi t)$	$\cos(10t)$	$\|\cos(5t)\|$	estimated
10^{-1}	85	138	116	85	411
10^{-2}	110	188	158	111	469
10^{-3}	133	239	199	136	527
10^{-4}	157	289	240	159	585
10^{-5}	180	340	281	182	643
10^{-6}	203	390	323	206	701
10^{-7}	227	441	364	229	759
10^{-8}	250	491	405	252	817
10^{-9}	273	542	446	276	875
10^{-10}	297	592	488	299	933
10^{-11}	320	643	529	322	990
10^{-12}	343	693	570	345	1048
10^{-13}	367	744	611	369	1106
10^{-14}	389	794	626	390	1164

TABLE 3. 20 partitions; T=10;

precision	e^{-t}	$\cos(\pi t)$	$\cos(10t)$	$\|\cos(5t)\|$	estimated
10^{-1}	96	146	146	107	581
10^{-2}	119	195	191	136	660
10^{-3}	143	245	237	165	739
10^{-4}	167	294	284	193	818
10^{-5}	190	344	331	222	897
10^{-6}	214	393	378	251	976
10^{-7}	237	443	424	280	1055
10^{-8}	261	493	471	309	1134
10^{-9}	285	542	518	337	1213
10^{-10}	308	592	565	366	1292
10^{-11}	332	641	612	395	1371
10^{-12}	355	691	658	424	1450
10^{-13}	379	740	705	452	1529
10^{-14}	402	790	752	480	1608

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