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HIGHER-ORDER VARIATIONAL PROBLEMS OF HERGLOTZ TYPE WITH TIME DELAY

SIMÃO P. S. SANTOS, NATÁLIA MARTINS, AND DELFIM F. M. TORRES

Abstract. We study, using an optimal control point of view, higher-order variational problems of Herglotz type with time delay. Main results are higher-order Euler–Lagrange and DuBois–Reymond necessary optimality conditions as well as a higher-order Noether type theorem for delayed variational problems of Herglotz type.

1. Introduction

The generalized variational problem proposed by Herglotz in 1930 [11] can be formulated as follows: determine the trajectories $x \in C^1([a,b];\mathbb{R}^m)$ and the function $z \in C^1([a,b];\mathbb{R})$ such that

$$(H^1) \qquad \begin{gathered} z(b) \longrightarrow \text{extr}, \\ \text{where the pair } (x(\cdot), z(\cdot)) \text{ satisfies} \\ \dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)), \quad t \in [a,b], \\ \text{subject to } x(a) = \alpha, \quad z(a) = \gamma, \end{gathered}$$

for some $\alpha \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}$, and where the Lagrangian L is assumed to satisfy the following assumptions:

i. $L \in C^1([a,b] \times \mathbb{R}^{2m+1};\mathbb{R})$;
ii. the functions $t \mapsto \dfrac{\partial L}{\partial x}(t, x(t), \dot{x}(t), z(t))$, $t \mapsto \dfrac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t), z(t))$ and $t \mapsto \dfrac{\partial L}{\partial z}(t, x(t), \dot{x}(t), z(t))$ are differentiable for any admissible pair $(x(\cdot), z(\cdot))$.

Observe that if the Lagrangian L does not depend on the variable z, then we get the classical problem of the calculus of variations. An advantage of formulation (H^1) is the ability to provide a variational description of non-conservative processes, even when the Lagrangian L is autonomous, something that is not possible with

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the classical variational problem. For important applications in thermodynamics see [5].

The variational problem of Herglotz attracted the interest of the mathematical community only in 1996, with the publications [8, 9]. Since then, several authors investigated such variational problems. The following generalizations of well-known classical results are available: extension of both Noether symmetry theorems for first-order problems [5–7]; Euler–Lagrange and transversality optimality conditions for higher-order variational problems of Herglotz type [14]; the first Noether theorem for first-order problems of Herglotz with time delay [15]; and the first Noether theorem for higher-order problems of Herglotz type [17]. For an optimal control approach to first-order Herglotz type problems see [16].

Dynamic systems with time delay are very important in modelling real-life phenomena in several fields, such as mathematics, biology, chemistry, economics and mechanics. Indeed, several process outcomes are determined not only by variables at present time, but also by its behaviour in the past. Motivated by the importance of problems with time delay, many authors generalized the classical results of the calculus of variations to the delayed case. The first work in this direction seems to have been published by Èl'sgol'c [3]. For some recent works on optimal control see [1,2] and references therein. The importance of variational problems of Herglotz, as well as the wide applicability of problems with time delay, allied to the impossibility of applying the classical Noether theorem to these problems, constitute the main motivation to the present work.

The main goal of this paper is to generalize the results of [14–17] by considering higher-order variational problems of Herglotz type with a time delay, proving the corresponding Euler–Lagrange equations, transversality conditions, the DuBois–Reymond necessary optimality condition and Noether's first theorem. In particular, in relation to our previous work with time delay [15], we improved its results by considering a wider class of admissible functions. Moreover, we extend the results of [15] to the higher-order case. Precisely, we generalize Herglotz's problem (H^1) by considering the following variational problem with time delay.

Problem ($\mathbf{H^n_\tau}$). *Let τ be a real number such that $0 \leq \tau < b - a$. Determine the piecewise trajectories $x \in PC^n([a-\tau, b]; \mathbb{R}^m)$ and the function $z \in PC^1([a, b]; \mathbb{R})$ such that:*

$$z(b) \longrightarrow \text{extr},$$

where the pair $(x(\cdot), z(\cdot))$ satisfies the differential equation

$$\dot{z}(t) = L\left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), x(t-\tau), \dot{x}(t-\tau), \ldots, x^{(n)}(t-\tau), z(t)\right),$$

for $t \in [a, b]$, and is subject to initial conditions

$$z(a) = \gamma \in \mathbb{R} \text{ and } x^{(k)}(t) = \mu^{(k)}(t), \quad k = 0, \ldots, n-1,$$

where $\mu \in PC^n([a-\tau, a]; \mathbb{R}^m)$ is a given initial function. The Lagrangian L is assumed to satisfy the following hypotheses:

i. $L \in C^1([a, b] \times \mathbb{R}^{2mn+1}; \mathbb{R})$;
ii. *functions $t \mapsto \frac{\partial L}{\partial z}[x; z]^n_\tau(t)$, $t \mapsto \frac{\partial L}{\partial x^{(k)}}[x; z]^n_\tau(t)$ and $t \mapsto \frac{\partial L}{\partial x^{(k)}_\tau}[x; z]^n_\tau(t)$ are differentiable for any admissible pair $(x(\cdot), z(\cdot))$, $k = 0, \ldots, n$,*

where, to simplify expressions, we use the notation $x_\tau^{(k)}(t)$, $k = 0, \ldots, n$, to denote the kth derivative of x evaluated at $t - \tau$ (often we use $x_\tau(t)$ for $x_\tau^{(0)}(t) = x(t-\tau)$ and $\dot{x}_\tau(t)$ for $x_\tau^{(1)}(t) = \dot{x}(t-\tau)$) and

$$[x;z]_\tau^n(t) := \left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), x_\tau(t), \dot{x}_\tau(t), \ldots, x_\tau^{(n)}(t), z(t)\right).$$

The structure of the paper is as follows. In Section 2 we recall the necessary background: the well-known Pontryagin's maximum principle, the DuBois–Reymond necessary optimality condition, and an extension of the classical Noether's theorem for optimal control problems. In Section 3 we formulate and prove our main results: the higher-order Euler–Lagrange equations and transversality conditions for generalized variational problems with time delay (Theorem 3.4); the DuBois–Reymond optimality condition (Theorem 3.8); and Noether's theorem for higher-order variational problems of Herglotz type with time delay (Theorem 3.11). We end with Section 4 of conclusions and possible future work.

2. Preliminaries

We begin by recalling the problem of optimal control in Bolza form:

$$(P)\qquad \mathcal{J}(x(\cdot), u(\cdot)) = \int_a^b f(t, x(t), u(t))dt + \phi(x(b)) \longrightarrow \text{extr}$$
$$\text{subject to } \dot{x}(t) = g(t, x(t), u(t)),$$

with some initial condition on x, where $f \in C^1([a,b] \times \mathbb{R}^m \times \Omega; \mathbb{R})$, $\phi \in C^1(\mathbb{R}^m; \mathbb{R})$, $g \in C^1([a,b] \times \mathbb{R}^m \times \Omega; \mathbb{R}^m)$, $x \in PC^1([a,b]; \mathbb{R}^m)$ and $u \in PC([a,b]; \Omega)$, with $\Omega \subseteq \mathbb{R}^r$ an open set. Usually x and u are called the state and control variables, respectively, while ϕ is known as the payoff or salvage term. It is clear that the classical problem of the calculus of variations is a particular case of problem (P) with $\phi(x) \equiv 0$, $g(t,x,u) = u$ and $\Omega = \mathbb{R}^m$. Next we present Pontryagin's maximum principle, one of the main tools for this paper.

Theorem 2.1 (Pontryagin's maximum principle for problem (P) [13])**.** *If a pair $(x(\cdot), u(\cdot))$ with $x \in PC^1([a,b]; \mathbb{R}^m)$ and $u \in PC([a,b]; \Omega)$ is a solution to problem (P) with the initial condition $x(a) = \alpha$, $\alpha \in \mathbb{R}^m$, then there exists $\psi \in PC^1([a,b]; \mathbb{R}^m)$ such that the following conditions hold:*

- *the optimality condition*

$$(2.1)\qquad \frac{\partial H}{\partial u}(t, x(t), u(t), \psi(t)) = 0;$$

- *the adjoint system*

$$(2.2)\qquad \begin{cases} \dot{x}(t) = \frac{\partial H}{\partial \psi}(t, x(t), u(t), \psi(t)) \\ \dot{\psi}(t) = -\frac{\partial H}{\partial x}(t, x(t), u(t), \psi(t)); \end{cases}$$

- *the transversality condition*

$$(2.3)\qquad \psi(b) = grad(\phi(x))(b);$$

where the Hamiltonian H is defined by

$$(2.4)\qquad H(t, x, u, \psi) = f(t, x, u) + \psi \cdot g(t, x, u).$$

Definition 2.2 (Pontryagin extremal to (P))**.** A triplet $(x(\cdot), u(\cdot), \psi(\cdot))$ with $x \in PC^1([a,b];\, \mathbb{R}^m)$, $u \in PC([a,b];\Omega)$ and $\psi \in PC^1([a,b];\mathbb{R}^m)$ is called a Pontryagin extremal to problem (P) if it satisfies the necessary optimality conditions (2.1)–(2.3).

Now we present the following necessary optimality condition that is used in the proof of our Theorem 3.8.

Theorem 2.3 (DuBois–Reymond condition of optimal control [13])**.** *If* $(x(\cdot), u(\cdot), \psi(\cdot))$ *is a Pontryagin extremal to problem* (P)*, then the Hamiltonian* (2.4) *satisfies the equality*

$$\frac{dH}{dt}(t, x(t), u(t), \psi(t)) = \frac{\partial H}{\partial t}(t, x(t), u(t), \psi(t)), \quad t \in [a,b].$$

Many years before the publication of the celebrated result of Pontryagin *et al.* in [13], Emmy Noether proved two remarkable theorems that relate the invariance of a variational integral with the corresponding Euler–Lagrange equations. Several extensions of the two Noether theorems were proved in different contexts. In this paper we are concerned with the first Noether theorem, also known simply by Noether's theorem. The Noether theorem [12] is a fundamental tool of the calculus of variations [22], optimal control [18,19,21] and modern theoretical physics [4]. This theorem guarantees that when an optimal control problem is invariant under a one parameter family of transformations, then there exists a corresponding conservation law: an expression that is conserved along all the Pontryagin extremals of the problem (see [18, 19, 21] and references therein). Here we use Noether's theorem as stated in [18], which is formulated for problems of optimal control in Lagrange form, that is, for problem (P) with $\phi \equiv 0$. In order to apply the results of [18] to the Bolza problem (P), we rewrite it in the following equivalent Lagrange form:

$$\begin{gathered}
\mathcal{I}(x(\cdot), y(\cdot), u(\cdot)) = \int_a^b [f(t, x(t), u(t)) + y(t)]\, dt \longrightarrow \text{extr}, \\
\begin{cases} \dot{x}(t) = g\left(t, x(t), u(t)\right), \\ \dot{y}(t) = 0, \end{cases} \\
x(a) = \alpha, \;\; y(a) = \frac{\phi(x(b))}{b-a}.
\end{gathered} \tag{2.5}$$

To present Noether's theorem for the optimal control problem (P), we need to define the concept of invariance. In this paper we follow the definition of invariance found in [18] applied to the equivalent optimal control problem (2.5). In Definition 2.4 we use the little-o notation.

Definition 2.4 (Invariance of problem (P))**.** Let h^s be a one-parameter family of invertible C^1 maps

$$\begin{gathered}
h^s : [a,b] \times \mathbb{R}^m \times \Omega \longrightarrow \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^r, \\
h^s(t,x,u) = \left(\mathcal{T}^s(t,x,u), \mathcal{X}^s(t,x,u), \mathcal{U}^s(t,x,u)\right), \\
h^0(t,x,u) = (t,x,u) \text{ for all } (t,x,u) \in [a,b] \times \mathbb{R}^m \times \Omega.
\end{gathered}$$

Problem (P) is said to be invariant under transformations h^s if for all $(x(\cdot), u(\cdot))$ the following two conditions hold:

$$\left[f \circ h^s(t, x(t), u(t)) + \frac{\phi(x(b))}{b-a} + \xi s + o(s)\right] \frac{d\mathcal{T}^s}{dt}(t, x(t), u(t)) = f(t, x(t), u(t)) + \frac{\phi(x(b))}{b-a}$$

for some constant ξ; and

$$\frac{d\mathcal{X}^s}{dt}(t, x(t), u(t)) = g \circ h^s(t, x(t), u(t)) \frac{d\mathcal{T}^s}{dt}(t, x(t), u(t)).$$

As a direct consequence of Noether's theorem proved in [18] and Pontryagin's maximum principle (Theorem 2.1), we get the following result that is central to prove our Theorem 3.11.

Theorem 2.5 (Noether's theorem for the optimal control problem (P))**.** *If problem (P) is invariant in the sense of Definition 2.4, then the quantity*

$$(b-t)\xi + \psi(t) \cdot X(t, x(t), u(t)) - \left[H(t, x(t), u(t), \psi(t)) + \frac{\phi(x(b))}{b-a}\right] \cdot T(t, x(t), u(t))$$

is constant in t along every Pontryagin extremal $(x(\cdot), u(\cdot), \psi(\cdot))$ of problem (P), where H is defined by (2.4) *and*

$$T(t, x(t), u(t)) = \frac{\partial \mathcal{T}^s}{\partial s}(t, x(t), u(t)) \bigg|_{s=0},$$

$$X(t, x(t), u(t)) = \frac{\partial \mathcal{X}^s}{\partial s}(t, x(t), u(t)) \bigg|_{s=0}.$$

3. Main results

We begin by introducing some definitions for the variational problem of Herglotz with time delay $(\mathbf{H}_\tau^\mathbf{n})$.

Definition 3.1 (Admissible pair to problem $(\mathbf{H}_\tau^\mathbf{n})$)**.** We say that $(x(\cdot), z(\cdot))$ with $x(\cdot) \in PC^n([a-\tau, b]; \mathbb{R}^m)$ and $z(\cdot) \in PC^1([a, b]; \mathbb{R})$ is an admissible pair to problem $(\mathbf{H}_\tau^\mathbf{n})$ if it satisfies the equation

$$\dot{z}(t) = L[x; z]_\tau^n(t), \quad t \in [a, b],$$

subject to

$$z(a) = \gamma, x^{(k)}(t) = \mu^{(k)}(t)$$

for all $k = 0, 1, \ldots, n-1$, $t \in [a-\tau, a]$ and $\gamma \in \mathbb{R}$.

Definition 3.2 (Extremizer to problem $(\mathbf{H}_\tau^\mathbf{n})$)**.** We say that an admissible pair $(x^*(\cdot), z^*(\cdot))$ is an extremizer to problem $(\mathbf{H}_\tau^\mathbf{n})$ if $z(b) - z^*(b)$ has the same signal for all admissible pairs $(x(\cdot), z(\cdot))$ that satisfy $\|z - z^*\|_0 < \epsilon$ and $\|x - x^*\|_0 < \epsilon$ for some positive real ϵ, where $\|y\|_0 = \max_{a \le t \le b} |y(t)|$.

3.1. **Reduction to a non-delayed problem.** We generalize the technique of reduction of a delayed first-order optimal control problem to a non-delayed problem proposed by Guinn in [10] to our higher-order delayed problem. In order to reduce the higher-order problem of Herglotz with time delay to a non-delayed first-order problem, we assume, without loss of generality, the initial time to be zero ($a = 0$) and the final time to be an integer multiple of τ, that is, $b = N\tau$ for $N \in \mathbb{N}$ (see Remark 3.3). We divide the interval $[a, b]$ into N equal parts and fix $t \in [0, \tau]$. We also introduce the variables $x^{k;i}$ and z_j with $k = 0, \ldots, n$, $i = 0, \ldots, N$, and $j = 1, \ldots, N + 1$. The variable k is related to the order of the derivative of x, i is related to the ith subinterval of $[-\tau, N\tau]$, and j is related to the jth subinterval of $[0, (N + 1)\tau]$ as follows:

$$\begin{aligned} x^{k;i}(t) &= x^{(k)}(t + (i-1)\tau), \quad z_j(t) = z(t + (j-1)\tau), \\ \dot{z}_j(t) &= L_j(t), \quad x^{k;N+1}(t) = 0, \quad \dot{z}_{N+1}(t) = L_{N+1} = 0 \end{aligned} \tag{3.1}$$

with

$$L_j(t) := L\left(t + (j-1)\tau, x^{0;j}(t), \ldots, x^{n;j}(t), x^{0;j-1}(t), \ldots, x^{n;j-1}(t), z_j(t)\right).$$

Finally, the higher-order problem of Herglotz with time delay ($\mathbf{H}_\tau^\mathbf{n}$) can be written as an optimal control problem without time delay as follows:

$$z_N(\tau) \longrightarrow \text{extr}, \quad \text{subject to}$$

$$\begin{cases} \dot{x}^{k;i}(t) = x^{k+1;i}(t), \\ x^{k;N+1}(t) = 0, \\ \dot{z}_j(t) = L_j(t), \\ \dot{z}_{N+1}(t) = L_{N+1}(t) = 0 \end{cases} \tag{3.2}$$

for all $t \in [0, \tau]$ and with the initial conditions

$$\begin{gathered} x^{k;0}(0) = \mu^{(k)}(-\tau), \quad x^{k;i}(0) = x^{k;i-1}(\tau), \\ z_1(0) = \gamma, \quad \gamma \in \mathbb{R}, \quad z_j(0) = z_{j-1}(\tau) \end{gathered}$$

for $k = 0, \ldots, n-1$, $i = 0, \ldots, N$ and $j = 1, \ldots, N$. In this form we look to $x^{k;i}$ and z_j as state variables and to $u_i := x^{n;i}$ as the control variables.

Remark 3.3. We considered the case of b being an integer multiple of τ. If b is not an integer multiple of τ, then there is an integer N such that $(N-1)\tau < b < N\tau$. In that case, the only modification required in the change of variables given in (3.1) is to consider the variables $x^{k;N}$, $k = 0, \ldots, n$, and $\dot{z}_N$ as defined in (3.1) for $t \in [0, b - (N-1)\tau]$ and zero for $t \in]b - (N-1)\tau, \tau]$. With this slight change, the function to be extremized remains the same and we can consider, without loss of generality, b to be an integer multiple of τ.

3.2. **Higher-order Euler–Lagrange and DuBois–Reymond optimality conditions with time delay.** We begin by proving a necessary condition for a pair $(x(\cdot), z(\cdot))$ to be an extremizer to problem ($\mathbf{H}_\tau^\mathbf{n}$). Along the proofs we sometimes suppress arguments for expressions whose arguments have been clearly stated before.

Theorem 3.4 (Higher-order delayed Euler–Lagrange and transversality conditions)**.** *If* $(x(\cdot), z(\cdot))$ *is an extremizer to problem* $(\mathbf{H}_\tau^\mathbf{n})$ *that satisfies the conditions* $x^{(k)}(t) = \mu^{(k)}(t)$*,* $k = 0, \dots, n-1$ *and* $t \in [a-\tau, a]$*, with* $\mu \in PC^n([a-\tau, a]; \mathbb{R}^m)$*, then the two Euler–Lagrange equations*

$$\sum_{l=0}^{n} (-1)^l \frac{d^l}{dt^l} \left(\psi_z(t) \frac{\partial L}{\partial x^{(l)}}[x;z]_\tau^n(t) + \psi_z(t+\tau) \frac{\partial L}{\partial x_\tau^{(l)}}[x;z]_\tau^n(t+\tau) \right) = 0, \tag{3.3}$$

for $t \in [a, b-\tau]$*, and*

$$\sum_{l=0}^{n} (-1)^l \frac{d^l}{dt^l} \left(\psi_z(t) \frac{\partial L}{\partial x^{(l)}}[x;z]_\tau^n(t) \right) = 0, \tag{3.4}$$

for $t \in [b-\tau, b]$ *and* ψ_z *defined by*

$$\psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z}[x;z]_\tau^n(\theta) d\theta}, \quad t \in [a, b],$$

hold. Furthermore, the following transversality conditions hold:

$$\sum_{l=0}^{n-k} (-1)^l \frac{d^l}{dt^l} \left(\psi_z(t) \frac{\partial L}{\partial x^{(l+k)}}[x;z]_\tau^n(t) \right) \Bigg|_{t=b} = 0, \tag{3.5}$$

$k = 1, \dots, n.$

Proof. In order to prove both Euler–Lagrange equations consider problem $(\mathbf{H}_\tau^\mathbf{n})$ in the optimal control form (3.2). Applying Pontryagin's maximum principle for problem (P) to problem $(\mathbf{H}_\tau^\mathbf{n})$ in the form (3.2), we conclude that there are multipliers $\phi_{k;i}$ and ψ_j for $k = 1, \dots, n$, $i = 0, \dots, N$ and $j = 1, \dots, N+1$, such that, with the Hamiltonian defined by

$$H = \sum_{l=1}^{n} \left(\sum_{i=0}^{N} \phi_{l;i}(t) \cdot x^{l;i}(t) \right) + \sum_{j=1}^{N+1} \psi_j(t) L_j(t), \tag{3.6}$$

the following conditions hold:

- the optimality conditions

$$\frac{\partial H}{\partial u_i} = 0,$$

- the adjoint system

$$\begin{cases} \dot{x}^{k-1;i} = \frac{\partial H}{\partial \phi_{k;i}}, \\ \dot{z}_j = \frac{\partial H}{\partial \psi_j}, \\ \dot{\phi}_{k;i} = -\frac{\partial H}{\partial x^{k-1;i}}, \\ \dot{\psi}_j = -\frac{\partial H}{\partial z_j}, \end{cases}$$

- the transversality conditions

$$\begin{cases} \phi_{k;i}(\tau) = 0, \\ \psi_j(\tau) = 1. \end{cases}$$

Observe that the forth equation in the adjoint system is equivalent to the differential equation $\dot{\psi}_j = -\psi_j \frac{\partial L_j}{\partial z_j}$. Together with the transversality condition, we obtain that the multipliers ψ_j, $j = 1, \ldots, N+1$, are given by

$$\psi_j(t) = e^{\int_t^\tau \frac{\partial L_j}{\partial z_j} d\theta}.$$

From the third equation in the adjoint system, we obtain that

$$\dot{\phi}_{k;i} = -\phi_{k-1;i} - \psi_i \frac{\partial L_i}{\partial x^{k-1;i}} - \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{k-1;i}}, \tag{3.7}$$

$k, i = 1, \ldots, n$, which for the particular case of $k = n$ reduces to

$$\dot{\phi}_{n;i} = -\phi_{n-1;i} - \psi_i \frac{\partial L_i}{\partial x^{n-1;i}} - \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{n-1;i}}.$$

This equality, together with the differentiation of the optimality condition

$$\begin{aligned}\dot{\phi}_{n;i} &= -\frac{d}{dt}\left(\psi_i \frac{\partial L_i}{\partial u_i}\right) - \frac{d}{dt}\left(\psi_{i+1} \frac{\partial L_{i+1}}{\partial u_i}\right) \\ &= -\frac{d}{dt}\left(\psi_i \frac{\partial L_i}{\partial x^{n;i}} + \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{n;i}}\right),\end{aligned}$$

leads to

$$\phi_{n-1;i} = -\psi_i \frac{\partial L_i}{\partial x^{n-1;i}} - \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{n-1;i}} + \frac{d}{dt}\left(\psi_i \frac{\partial L_i}{\partial x^{n;i}} + \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{n;i}}\right).$$

By differentiation of the previous expression and comparison with (3.7) for $k = n-1$, we find the expression for $\phi_{n-2;i}$:

$$\begin{aligned}\phi_{n-2;i} = &-\psi_i \frac{\partial L_i}{\partial x^{n-2;i}} - \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{n-2;i}} \\ &+ \frac{d}{dt}\left(\psi_i \frac{\partial L_i}{\partial x^{n-1;i}} + \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{n-1;i}}\right) - \frac{d^2}{dt^2}\left(\psi_i \frac{\partial L_i}{\partial x^{n;i}} + \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{n;i}}\right).\end{aligned}$$

Using recursively the technique of derivation of $\phi_{k;i}$ and comparison with (3.7), we find the expression for $\phi_{k;i}$ ($k = 1, \ldots, n$):

$$\phi_{k;i} = \sum_{l=0}^{n-k} (-1)^{l+1} \frac{d^l}{dt^l}\left(\psi_i \frac{\partial L_i}{\partial x^{l+k;i}} + \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{l+k;i}}\right), \quad i = 1, \ldots, N. \tag{3.8}$$

Considering $\phi_{1;i}$ given by the previous equation and comparing it with

$$\phi_{1;i} = -\dot{\phi}_{2;i} - \psi_i \frac{\partial L_i}{\partial x^{1;i}} - \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{1;i}},$$

given by (3.7) for $k = 2$, we obtain that

$$\sum_{l=0}^{n} (-1)^l \frac{d^l}{dt^l}\left(\psi_i \frac{\partial L_i}{\partial x^{l;i}} + \psi_{i+1} \frac{\partial L_{i+1}}{\partial x^{l;i}}\right) = 0, \quad i = 1, \ldots, N. \tag{3.9}$$

Since $L_{N+1} = 0$, the previous equation for $i = N$ reduces to

$$\sum_{l=0}^{n} (-1)^l \frac{d^l}{dt^l}\left(\psi_N \frac{\partial L_N}{\partial x^{l;N}}\right) = 0. \tag{3.10}$$

The final step is to rewrite the results obtained inverting the changes of variables (3.1). For this purpose, define $\psi_z(t)$, $t \in [0, b+\tau]$, by

$$\psi_z(t) = \psi_i(t-(i-1)\tau), \quad (i-1)\tau \leq t \leq i\tau, \quad i = 1, \ldots, N+1,$$

and $\phi_k(t)$, $k = 1, \ldots, n$, $t \in [-\tau, b]$, by

$$\phi_k(t) = \phi_{k;i}(t-(i-1)\tau), \quad (i-1)\tau \leq t \leq i\tau, \quad i = 1, \ldots, N.$$

This allows to write

$$\psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z}[x;z]^n_\tau(\theta)d\theta}, \quad t \in [a,b], \tag{3.11}$$

and

$$\phi_k(t) = \sum_{l=0}^{n-k}(-1)^{l+1}\frac{d^l}{dt^l}\left(\psi_z(t+\tau)\frac{\partial L}{\partial x_\tau^{(l+k)}}[x;z]^n_\tau(t+\tau)\right), \quad t \in [a-\tau, a],$$

$$\begin{aligned}\phi_k(t) = \sum_{l=0}^{n-k}(-1)^{l+1}\frac{d^l}{dt^l}\Bigg(&\psi_z(t)\frac{\partial L}{\partial x^{(l+k)}}[x;z]^n_\tau(t) \\ &+\psi_z(t+\tau)\frac{\partial L}{\partial x_\tau^{(l+k)}}[x;z]^n_\tau(t+\tau)\Bigg), \quad t \in [a,b],\end{aligned} \tag{3.12}$$

$k = 1, \ldots, n$. Note that if $t \in [b-\tau, b]$, then $L[x;z]^n_\tau(t+\tau)$ is, by definition, null. Finally, equations (3.9)–(3.10) lead to the Euler–Lagrange equations for the higher-order problem of Herglotz with time delay $(\mathbf{H}^{\mathbf{n}}_\tau)$:

$$\sum_{l=0}^{n}(-1)^l\frac{d^l}{dt^l}\left(\psi_z(t)\frac{\partial L}{\partial x^{(l)}}[x;z]^n_\tau(t) + \psi_z(t+\tau)\frac{\partial L}{\partial x_\tau^{(l)}}[x,z]^n_\tau(t+\tau)\right) = 0$$

for $t \in [a, b-\tau]$ and

$$\sum_{l=0}^{n}(-1)^l\frac{d^l}{dt^l}\left(\psi_z(t)\frac{\partial L}{\partial x^{(l)}}[x;z]^n_\tau(t)\right) = 0$$

for $t \in [b-\tau, b]$. From (3.8) and the transversality conditions for $\phi_{k;i}$, we obtain the transversality conditions $\phi_k(b) = 0$, that is,

$$\sum_{l=0}^{n-k}(-1)^l\frac{d^l}{dt^l}\left(\psi_z(t)\frac{\partial L}{\partial x^{(l+k)}}[x;z]^n_\tau(t)\right)\Bigg|_{t=b} = 0,$$

$k = 1, \ldots, n$. □

Definition 3.5 (Extremal to problem $(\mathbf{H}^{\mathbf{n}}_\tau)$)**.** We say that an admissible pair $(x(\cdot), z(\cdot))$ is an extremal to problem $(\mathbf{H}^{\mathbf{n}}_\tau)$ if it satisfies the Euler–Lagrange equations (3.3)–(3.4) and the transversality conditions (3.5).

Theorem 3.4 gives a generalization of the Euler–Lagrange equation and transversality conditions for the higher-order problem of Herglotz presented by the authors in [14]. It is also a generalization of the results in [16, 17].

Corollary 3.6 (cf. [14,17]). *If $(x(\cdot), z(\cdot))$ is an extremizer to the higher-order problem of Herglotz*

$$z(b) \longrightarrow extr,$$
$$\dot{z}(t) = L\left(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), z(t)\right), \quad t \in [a, b], \tag{3.13}$$
$$z(a) = \gamma \in \mathbb{R}, \quad x^{(k)}(a) = \alpha_k, \quad \alpha_k \in \mathbb{R}^m, \quad k = 0, \ldots, n-1,$$

then the Euler–Lagrange equation

$$\sum_{l=0}^{n}(-1)^l \frac{d^l}{dt^l}\left(\psi_z(t)\frac{\partial L}{\partial x^{(l)}}[x; z]_0^n(t)\right) = 0$$

holds for $t \in [a, b]$, where ψ_z is defined in (3.11). *Furthermore, the following transversality conditions hold:*

$$\sum_{l=0}^{n-k}(-1)^l \frac{d^l}{dt^l}\left(\psi_z(t)\frac{\partial L}{\partial x^{(l+k)}}[x; z]_0^n(t)\right)\bigg|_{t=b} = 0,$$

$k = 1, \ldots, n$.

Proof. Consider Theorem 3.4 with no delay, that is, with $\tau = 0$. □

Theorem 3.4 is also a generalization of the Euler–Lagrange equations for the first-order problem of Herglotz with time delay obtained in [15].

Corollary 3.7 (cf. [15]). *If $(x(\cdot), z(\cdot))$ is an extremizer to the first-order problem of Herglotz with time delay*

$$z(b) \longrightarrow extr,$$
$$\dot{z}(t) = L\left(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau), z(t)\right), \quad t \in [a, b], \tag{3.14}$$
$$z(a) = \gamma \in \mathbb{R}, \quad x(t) = \mu(t), \quad t \in [a-\tau, a],$$

for a given piecewise initial function μ, then the Euler–Lagrange equations

$$\begin{aligned}\psi_z(t)\frac{\partial L}{\partial x}[x; z]_\tau^1(t) + \psi_z(t+\tau)\frac{\partial L}{\partial x_\tau}[x, z]_\tau^1(t+\tau)\\ - \frac{d}{dt}\left(\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x; z]_\tau^1(t) + \psi_z(t+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x, z]_\tau^1(t+\tau)\right) = 0,\end{aligned}$$

for $t \in [a, b-\tau]$, and

$$\psi_z(t)\frac{\partial L}{\partial x}[x; z]_\tau^1(t) - \frac{d}{dt}\left(\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x; z]_\tau^1(t)\right) = 0,$$

for $t \in [b-\tau, b]$, hold.

Proof. Consider Theorem 3.4 with $n = 1$. □

Theorem 3.8 (Higher-order delayed DuBois–Reymond condition). *If the pair $(x(\cdot), z(\cdot))$ is an extremal to problem $(\mathbf{H}_\tau^\mathbf{n})$, then*

$$\frac{d}{dt}\left(\sum_{k=1}^{n}\phi_k(t)\cdot x^{(k)}(t) + \psi_z(t)L[x; z]_\tau^n(t)\right) = \psi_z(t)\frac{\partial L}{\partial t}[x; z]_\tau^n(t), \tag{3.15}$$

where ψ_z and ϕ_k are defined by (3.11) *and* (3.12)*, respectively.*

Proof. Consider problem ($\mathbf{H}^{\mathbf{n}}_{\tau}$) in the formulation given by (3.2). Theorem 2.3 asserts that $\frac{dH}{dt} = \frac{\partial H}{\partial t}$ for H given by (3.6). We obtain (3.15) by writing H in the variables ϕ_k and ψ_z. □

Theorem 3.8 is also a generalization of the DuBois–Reymond condition presented in [15] for the first-order problem of Herglotz with time delay. In that paper, for technical reasons, we added an additional hypothesis that we are able to avoid here.

Corollary 3.9 (cf. [15])**.** *If $(x(\cdot), z(\cdot))$ is an extremizer to the first-order problem of Herglotz with time delay* (3.14)*, then*

$$\psi_z(t)\frac{\partial L}{\partial t}[x;z]^1_\tau(t) = \frac{d}{dt}\Bigg(\psi_z(t)L[x;z]^1_\tau(t) - \left(\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x;z]^1_\tau(t) + \psi_z(t+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]^1_\tau(t+\tau)\right)\dot{x}(t)\Bigg),$$

where ψ_z is defined by (3.11).

Proof. Consider Theorem 3.8 with $n = 1$. □

3.3. **Higher-order Noether's symmetry theorem with time delay.** Before presenting a Noether theorem to problem ($\mathbf{H}^{\mathbf{n}}_{\tau}$), we introduce the notion of invariance.

Definition 3.10 (Invariance of problem ($\mathbf{H}^{\mathbf{n}}_{\tau}$))**.** Let h^s be a one-parameter family of invertible C^1 maps $h^s : [a-\tau, b] \times \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}$,

$$h^s(t, x(t), z(t)) = (\mathcal{T}^s[x;z]^n_\tau(t), \mathcal{X}^s[x;z]^n_\tau(t), \mathcal{Z}^s[x;z]^n_\tau(t)),$$
$$h^0(t,x,z) = (t,x,z), \quad \forall (t,x,z) \in [a-\tau, b] \times \mathbb{R}^m \times \mathbb{R}.$$

Problem ($\mathbf{H}^{\mathbf{n}}_{\tau}$) is said to be invariant under the transformations h^s, if for all admissible pairs $(x(\cdot), z(\cdot))$ the following two conditions hold:

$$\left(\frac{z(b)}{b-a} + \xi s + o(s)\right)\frac{d\mathcal{T}^s}{dt}[x;z]^n_\tau(t) = \frac{z(b)}{b-a} \tag{3.16}$$

for some constant ξ and

$$\begin{aligned}\frac{d\mathcal{Z}^s}{dt}[x;z]^n_\tau(t) = \frac{d\mathcal{T}^s}{dt}[x;z]^n_\tau(t)\, L\Bigg(&\mathcal{T}^s[x;z]^n_\tau(t), \mathcal{X}^s[x;z]^n_\tau(t),\\ &\frac{d\mathcal{X}^s}{d\mathcal{T}^s}[x;z]^n_\tau(t), \ldots, \frac{d^n\mathcal{X}^s}{d(\mathcal{T}^s)^n}[x;z]^n_\tau(t), \mathcal{X}^s[x,z]^n_\tau(t-\tau),\\ &\frac{d\mathcal{X}^s}{d\mathcal{T}^s}[x,z]^n_\tau(t-\tau), \ldots, \frac{d^n\mathcal{X}^s}{d(\mathcal{T}^s)^n}[x,z]^n_\tau(t-\tau), \mathcal{Z}^s[x;z]^n_\tau(t)\Bigg),\end{aligned} \tag{3.17}$$

where

$$\frac{d\mathcal{X}^s}{d\mathcal{T}^s}[x;z]^n_\tau(t) = \frac{\frac{d\mathcal{X}^s}{dt}[x;z]^n_\tau(t)}{\frac{d\mathcal{T}^s}{dt}[x;z]^n_\tau(t)},$$

$$\frac{d^k\mathcal{X}^s}{d(\mathcal{T}^s)^k}[x;z]^n_\tau(t) = \frac{\frac{d}{dt}\left(\frac{d^{k-1}\mathcal{X}^s}{d(\mathcal{T}^s)^{k-1}}[x;z]^n_\tau(t)\right)}{\frac{d\mathcal{T}^s}{dt}[x;z]^n_\tau(t)},$$

$k = 2, \dots, n$.

Now we generalize the higher-order Noether's theorem of [17] to the more general case of variational problems of Herglotz type with time delay.

Theorem 3.11 (Higher-order delayed Noether's theorem)**.** *If problem* $(\mathbf{H}^{\mathbf{n}}_\tau)$ *is invariant in the sense of Definition 3.10, then the quantity*

$$\sum_{k=1}^{n} \phi_k(t) \cdot X_{k-1}[x;z]^n_\tau(t) + \psi_z(t) Z[x;z]^n_\tau(t) - \left[\sum_{k=1}^{n} \phi_k(t) \cdot x^{(k)}(t) + \psi_z(t) L[x;z]^n_\tau(t)\right] T[x;z]^n_\tau(t)$$

is constant in t *along all extremals of problem* $(\mathbf{H}^{\mathbf{n}}_\tau)$*, where the generators of the one-parameter family of maps are given by*

$$T = \left.\frac{\partial \mathcal{T}^s}{\partial s}\right|_{s=0}, \quad X_0 = \left.\frac{\partial \mathcal{X}^s}{\partial s}\right|_{s=0}, \quad Z = \left.\frac{\partial \mathcal{Z}^s}{\partial s}\right|_{s=0},$$

$$X_k = \frac{d}{dt} X_{k-1} - x^{(k)} \frac{d}{dt}\left(\left.\frac{\partial \mathcal{T}^s}{\partial s}\right|_{s=0}\right), \quad k = 1, \dots, n-1,$$

and ψ_z*,* ϕ_k *are defined by* (3.11)–(3.12)*.*

Proof. We start by considering problem $(\mathbf{H}^{\mathbf{n}}_\tau)$ in its non-delayed optimal control form (3.2). The first step is to prove that if problem $(\mathbf{H}^{\mathbf{n}}_\tau)$ is invariant in the sense of Definition 3.10, then (3.2) is invariant in the sense of Definition 2.4. In order to do that, observe that (3.16) is equivalent to

$$\left(\frac{z_N(\tau)}{N\tau} + \xi s + o(s)\right) \frac{d\mathcal{T}^s}{dt}[x;z]^n_\tau(t) = \frac{z_N(\tau)}{N\tau}$$

and defining $\xi_\tau := \xi N$ we have

$$\left(\frac{z_N(\tau)}{\tau} + \xi_\tau s + o(s)\right) \frac{d\mathcal{T}^s}{dt}[x;z]^n_\tau(t) = \frac{z_N(\tau)}{\tau}, \quad \text{for some } \xi_\tau. \tag{3.18}$$

Observe also that the control system of (3.2) defines $\mathcal{X}^s_k := \frac{d\mathcal{X}^s_{k-1}}{d\mathcal{T}^s}$, that is,

$$\frac{d\mathcal{X}^s_{k-1}}{dt}[x;z]^n_\tau(t) = \mathcal{X}^s_k[x;z]^n_\tau(t) \frac{d\mathcal{T}^s}{dt}[x;z]^n_\tau(t), \quad k = 1, \dots, n.$$

Let

$$\mathcal{X}_{k;i}[x;z]^n_\tau(t) := \mathcal{X}^s_k[x;z]^n_\tau(t + (i-1)\tau),$$

$$\mathcal{T}_i[x;z]^n_\tau(t) := \mathcal{T}^s[x;z]^n_\tau(t + (i-1)\tau),$$

$$\mathcal{Z}_j[x;z]^n_\tau(t) := \mathcal{Z}^s[x;z]^n_\tau(t + (j-1)\tau).$$

One has

$$\frac{d\mathcal{X}_{k;i}}{dt}[x;z]^n_\tau(t) = \mathcal{X}_{k+1;i}[x;z]^n_\tau(t)\frac{d\mathcal{T}_i}{dt}[x;z]^n_\tau(t) \tag{3.19}$$

and

$$\frac{d\mathcal{Z}_j}{dt}[x;z]^n_\tau(t) \;=\; L_j\left[\mathcal{X}^s[x;z]^n_\tau(t);\mathcal{Z}^s[x;z]^n_\tau(t)\right]^n_\tau\left(\mathcal{T}^s_j[x;z]^n_\tau(t)\right)\frac{d\mathcal{T}_j}{dt}[x;z]^n_\tau(t), \tag{3.20}$$

$k=0,\ldots,n-1$, $i=0,\ldots N$, $j=1,\ldots,N$. Equalities (3.18)–(3.20) prove that problem (3.2) is invariant in the sense of Definition 2.4. This put us in conditions to advance to the second step: to apply Theorem 2.5 to the non-delayed optimal control problem (3.2). This theorem guarantees that the quantity

$$\begin{aligned}(\tau-t)\xi_\tau &+ \sum_{k=1}^{n}\sum_{i=0}^{N}\phi_{k;i}(t)\cdot X_{k-1;i}[x;z]^n_\tau(t) + \sum_{j=1}^{N}\psi_j(t)Z_j[x;z]^n_\tau(t)\\ &- \left[\sum_{k=1}^{n}\sum_{i=0}^{N}\phi_{k;i}(t)\cdot x^{k;i}(t) + \sum_{j=1}^{N}\psi_j(t)L_j[x;z]^n_\tau(t) + \frac{z_N(\tau)}{\tau}\right]T[x;z]^n_\tau(t)\end{aligned}$$

is constant in t along the extremals of (3.2), where $X_{k;i} = \frac{\partial}{\partial s}\frac{d^k\mathcal{X}^s_{k;i}}{d(\mathcal{T}^s)^k}\Big|_{s=0}$ and $Z_i = \frac{\partial}{\partial s}\frac{d\mathcal{Z}^s_i}{d(\mathcal{T}^s)}\Big|_{s=0}$. Rewriting in the original variables, we obtain

$$\begin{aligned}(\tau-t)\xi_\tau &+ \sum_{k=1}^{n}\phi_k(t)\cdot X_{k-1}[x;z]^n_\tau(t) + \psi_z(t)Z[x;z]^n_\tau(t)\\ &- \left[\sum_{k=1}^{n}\phi_k(t)\cdot x^{(k)}(t) + \psi_z(t)L[x;z]^n_\tau(t) + \frac{z_N(\tau)}{\tau}\right]T[x;z]^n_\tau(t)\end{aligned}$$

constant in t along the extremals of (3.2). The third step is to prove that

$$(\tau-t)\xi_\tau - \frac{z_N(\tau)}{\tau}T[x;z]^n_\tau(t) \tag{3.21}$$

is constant in t. From the invariance condition (3.18), we know that

$$\left(\frac{z_N(\tau)}{\tau} + \xi_\tau s + o(s)\right)\frac{d\mathcal{T}^s}{dt}[x;z]^n_\tau(t) = \frac{z_N(\tau)}{\tau}.$$

Integrating from 0 to t we conclude that

$$\begin{aligned}&\left(\frac{z_N(\tau)}{\tau} + \xi_\tau s + o(s)\right)\mathcal{T}^s[x;z]^n_\tau(t)\\ &\qquad = \frac{z_N(\tau)}{\tau}t + \left(\frac{z_N(\tau)}{\tau} + \xi_\tau s + o(s)\right)\mathcal{T}^s[x;z]^n_\tau(0).\end{aligned}$$

Differentiating this equality with respect to s, and then putting $s=0$, we get

$$\xi_\tau t + \frac{z_N(\tau)}{\tau}T[x;z]^n_\tau(t) = \frac{z_N(\tau)}{\tau}T[x;z]^n_\tau(0). \tag{3.22}$$

We conclude from (3.22) that expression (3.21) is the constant

$$\tau\xi_\tau - \frac{z_N(\tau)}{\tau} T[x;z]^n_\tau(0).$$

Hence,

$$\sum_{k=1}^{n} \phi_k(t) \cdot X_{k-1}[x;z]^n_\tau(t) + \psi_z(t) Z[x;z]^n_\tau(t) - \left[\sum_{k=1}^{n} \phi_k(t) \cdot x^{(k)}(t) + \psi_z(t) L[x;z]^n_\tau(t)\right] T[x;z]^n_\tau(t)$$

is constant in t along the extremals of problem (3.2). Finally, observe that $X_0 = \frac{\partial \mathcal{X}^s}{\partial s}\big|_{s=0}$ and

$$\begin{aligned} X_k = \frac{\partial}{\partial s} \frac{d^k \mathcal{X}^s}{d(\mathcal{T}^s)^k}\bigg|_{s=0} &= \frac{\partial}{\partial s}\left(\frac{\frac{d}{dt}\left(\frac{d^{k-1}\mathcal{X}^s}{d(\mathcal{T}^s)^{k-1}}\right)}{\frac{d\mathcal{T}^s}{dt}}\right)\Bigg|_{s=0} \\ &= \frac{d}{dt}\left(\frac{\partial}{\partial s} \frac{d^{k-1}\mathcal{X}^s}{d(\mathcal{T}^s)^{k-1}}\bigg|_{s=0}\right) - x^{(k)} \frac{d}{dt}\left(\frac{\partial \mathcal{T}^s}{\partial s}\bigg|_{s=0}\right) \\ &= \frac{d}{dt} X_{k-1} - x^{(k)} \frac{d}{dt}\left(\frac{\partial \mathcal{T}^s}{\partial s}\bigg|_{s=0}\right), \end{aligned}$$

$k = 1, \ldots, n-1$. This concludes the proof. □

Corollary 3.12 (cf. [17])**.** *If the higher-order problem of Herglotz* (3.13) *is invariant in the sense of Definition 3.10 (in* $[a,b]$*), then the quantity*

$$\sum_{k=1}^{n} \tilde{\phi}_k(t) \cdot X_{k-1}[x;z]^n_0(t) + \psi_z(t) Z[x;z]^n_0(t) - \left[\sum_{k=1}^{n} \tilde{\phi}_k(t) \cdot x^{(k)}(t) + \psi_z(t) L[x;z]^n_0(t)\right] T[x;z]^n_0(t)$$

is constant in t *along any extremal of the problem, where*

$$\tilde{\phi}_k(t) = \sum_{l=0}^{n-k} (-1)^{l+1} \frac{d^l}{dt^l}\left(\psi_z(t) \frac{\partial L}{\partial x^{(l+k)}}[x;z]^n_0(t)\right),$$

$k = 1, \ldots, n$*, and* ψ_z *is given by* (3.11)*.*

Proof. Consider Theorem 3.11 with $\tau = 0$. □

Theorem 3.11 is a generalization of Noether's theorem [15] for the first-order problem of Herglotz with time delay. Besides the improvement of dealing with piecewise functions instead of continuous, the theorem presents a similar conserved quantity but without the imposition of two additional hypotheses required in [15]. Moreover, the current definition of invariance is more general than the one considered in [15].

Corollary 3.13 (cf. [15]). *If the first-order problem of Herglotz with time delay* (3.14) *is invariant in the sense of Definition 3.10, then the quantity*

$$\left(\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x;z]^1_\tau(t)+\psi_z(t+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]^1_\tau(t+\tau)\right)X_0[x;z]^1_\tau(t) + \psi_z(t)Z[x;z]^1_\tau(t)+\left[-\left(\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x;z]^1_\tau(t) +\psi_z(t+\tau)\frac{\partial L}{\partial \dot{x}_\tau}[x;z]^1_\tau(t+\tau)\right)\dot{x}(t)+\psi_z(t)L[x;z]^1_\tau(t)\right]T[x;z]^1_\tau(t)$$

is constant in $t \in [a,b]$ *along any extremal of the problem.*

Proof. Consider Theorem 3.11 with $n = 1$. □

Remark 3.14. If $t \in [b-\tau, b]$, then $L[x;z]^n_\tau(t+\tau)$ is, by definition, null (see (3.1)) and the constant of Corollary 3.13 reduces to

$$\left(\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x;z]^1_\tau(t)\right)X_0[x;z]^1_\tau(t)+\psi_z(t)Z[x;z]^1_\tau(t) +\left[-\left(\psi_z(t)\frac{\partial L}{\partial \dot{x}}[x;z]^1_\tau(t)\right)\dot{x}(t)+\psi_z(t)L[x;z]^1_\tau(t)\right]T[x;z]^1_\tau(t)$$

for $t \in [b-\tau, b]$, which is the second constant quantity of [15].

4. Conclusion

Optimal control is a convenient tool to deal with delayed and non-delayed Herglotz type variational problems. In this work we have shown how some of the central results from the classical calculus of variations can be proved for higher-order Herglotz variational problems with time delay from analogous and well-known optimal control results. The techniques here developed can now be used to obtain other results. For example, our optimal control approach can be employed together with [20] to derive an extension of the second Noether theorem to the delayed or non-delayed Herglotz framework. This is under investigation and will be addressed elsewhere.

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Simão P. S. Santos
Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail address: spsantos@ua.pt

Natália Martins
Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail address: natalia@ua.pt

Delfim F. M. Torres
Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail address: delfim@ua.pt