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FIXED POINTS FOR NONEXPANSIVE MAPPINGS AND GENERALIZED NONEXPANSIVE MAPPINGS ON BANACH LATTICES

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Abstract. Metric Fixed Point Theory on Banach lattices has attracted the interest of many researchers on Nonlinear Functional Analysis. In this paper we revise the most relevant results on this topic for nonexpansive mappings and generalized nonexpansive mappings defined on Banach lattices and we prove some new results for a class of mappings, called Suzuki type mappings, which have proved to be one of the most relevant extension of nonexpansive mappings. Nominally, we prove the existence of fixed points for Suzuki type mappings which are defined from a convex weakly compact subset of a Banach space X into itself whenever either X is isomorphic to a weakly orthogonal $r-N-$OUNC Banach lattice Y and the Banach-Mazur distance $d(X,Y)$ is less than $1/r$ or X is an N-weakly orthogonal Banach lattice and its Riesz angle α satisfies $\alpha^\ell < M/(N-1)$ where $N=\sum_{j=0}^k \epsilon_j 2^j$ with $\epsilon_j \in \{0,1\}$, $M=\sum_{j=0}^k \epsilon_j 2^j\alpha^{-j}$ and $2^{\ell-1} < \sum_{j=0}^k \epsilon_j \le 2^\ell$. This latter result is new also for nonexpansive mappings.

1. Introduction

The existence of fixed point for nonexpansive mappings and generalized nonexpansive mappings has been widely studied in the last 40 years (see, for instance, [22, 27] and references therein or [23, 28] for more recent results). A Banach space X is said to satisfy the weak fixed point property if every nonexpansive mapping defined from a convex weakly compact subset K of X into K has a fixed point. It is well known that there are some Banach spaces (and Banach lattices) failing this property [2]. Thus, a recurrent goal in Metric Fixed Point Theory is to obtain geometrical conditions on the space X which assure the weak fixed point property. In the case of Banach lattices the main geometrical tools for this purpose have been the Riesz angle, weak orthogonality and some extension of these notions as N-dimensional Riesz angle or N-weak orthogonality. In parallel, some interesting generalizations of the nonexpansivity have appeared, and so, many authors have tried to extend fixed point results which were known in the setting of nonexpansive mappings for these wider classes of mappings. In particular, some papers have appeared proving that the usual geometrical conditions to prove the weak fixed point

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property, still hold to obtain fixed point results for generalized nonexpansive mappings. In Section 3 we include some comments about the development of this topic in the last years.

Our goal in this paper is, on one hand, to extend some fixed point results on Banach lattices which are known for nonexpansive mappings and some of its generalizations as mappings satisfying condition (C) and condition (C_λ) to the setting of the, so called, mappings of Suzuki type. On the other hand, we prove the existence of fixed points for Suzuki type mappings under geometrical conditions which are weaker than the previous used for nonexpansive mappings. In order to do that we need two main lemmas which are proved in Section 4. In Section 5 we extend the results in [4,5,7,10] about existence of fixed point for mappings satisfying condition (C) or (C_λ) on OUNC Banach lattices to the case of Suzuki type mappings. In Section 6 we prove the validity of the results in [12,13] about existence of fixed points for nonexpansive mappings on N-weakly orthogonal abstract M-spaces to Suzuki type mappings defined on an arbitrary N-weakly orthogonal Banach lattices assuming that the Riesz angle is appropriately bounded from above. This result is new also for nonexpansive mappings, extends all previous results about existence of fixed point and Riesz angle and, in particular, recovers the results in [8] for the case $N = 2$. We finish this paper showing an easy example of a Banach lattice which is not an abstract M-lattice but the Riesz angle is small enough to apply our result.

2. Preliminaries and notation

Along this paper we will assume that X is a Banach lattice and by B_X we denote the unit ball of X. For notation and terminology concerning Banach lattices we refer the reader to [24, 26]. In the next lemma we collect some lattice inequalities which will be used in the sequel.

Lemma 2.1. *Let X be a Banach lattice. Then,*

(i) *for every vectors $x_1, \dots, x_N \in X$ and $w \in X$*

$$|w| \le |w - x_1| \vee \cdots \vee |w - x_N| + |x_1| \wedge \cdots \wedge |x_N|,$$

(ii) *for every $z, x_1, \dots, x_N \in X$ and $N \ge 2$, $N \in \mathbb{N}$ we have*

$$|z| \le \bigwedge_{\substack{i,j=1,\dots,N \\ i \ne j}} (|z - x_i| \vee |z - x_j|) + \sum_{\substack{i,j=1 \\ i \ne j}}^{N} |x_i| \wedge |x_j|,$$

(iii) *for all $x, y \in X$*

$$|x| - |x| \wedge |y| \le |x - y| - |y| + |x| \wedge |y|.$$

Proof. (i) By using the usual relations of the Banach lattices (see, for instance, [26, Theorem 1.1.1]) we have that $|w| \le |w - x_i| + |x_i|$ for every $i = 1, \dots, N$. Thus

$$|w| \le |w - x_1| \vee \cdots \vee |w - x_N| + |x_i|$$

for every $i = 1, \dots, N$. Taking infimum we obtain

$$|w| \le \bigwedge_{i=1}^{N} (|w - x_1| \vee \cdots \vee |w - x_N| + |x_i|)$$

$$= |w - x_1| \vee \cdots \vee |w - x_N| + |x_1| \wedge \cdots \wedge |x_N|.$$

(ii) For $i, j = 1, \ldots, N, i \neq j$ we have

$$|z| \leq |z - x_i| + |x_i| \leq |z - x_i| \vee |z - x_j| + |x_i|,$$
$$|z| \leq |z - x_j| + |x_j| \leq |z - x_i| \vee |z - x_j| + |x_j|,$$

hence $|z| \leq |z - x_i| \vee |z - x_j| + |x_i| \wedge |x_j|$ for all $i, j = 1, \ldots, N, i \neq j$. Thus, taking the infimum we have

$$\begin{aligned} |z| &\leq \bigwedge_{\substack{i,j=1,\ldots,N \\ i \neq j}} (|z - x_i| \vee |z - x_j| + |x_i| \wedge |x_j|) \\ &\leq \bigwedge_{\substack{i,j=1,\ldots,N \\ i \neq j}} (|z - x_i| \vee |z - x_j|) + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} |x_i| \wedge |x_j|. \end{aligned}$$

(iii) It is clear just splitting into the cases $|x| \wedge |y| = |x|$ and $|x| \wedge |y| = |y|$.

□

The Riesz angle for X is defined by

$$\alpha(X) = \sup\{\||x| \vee |y|\| : \|x\| \leq 1, \|y\| \leq 1\}.$$

When $\alpha(X) = 1$, the space X is said to be an abstract M-space. The most relevant examples of abstract M-spaces are the spaces of continuous functions on a compact set K with the maximum norm. In fact, every abstract M-space can be represented isometrically and lattice isomorphically by a subspace of the space $C(K)$ where K is a certain compact Hausdorff space [20]. At the opposite extreme we can find abstract L-lattices, i.e. lattices which satisfy $\|x + y\| = \|x\| + \|y\|$ for every nonnegative vectors $x, y \in X$. Typical representatives of abstract L-spaces are the Lebesgue spaces $L^1(\Omega)$ where Ω is any measure space.

The following definition is introduced in [5].

Definition 2.2. Let $r \in (0, 1]$. A Banach lattice X is said to be r-order uniformly noncreasy ($r - OUNC$) if for every $u, v \in (1/2)B_X$ we have either $\||u| \vee |v|\| \leq r$ or for every $y \in X$ the conditions $|y| \leq |u - v|$, $\|y\| \geq r$ imply that $\||u - v| - |y|\| \leq r$. A Banach lattice X is order uniformly noncreasy ($OUNC$) if it is $r - OUNC$ for some $r \in (0, 1)$.

In [7] a generalization of the Riesz angle was introduced in the following way:

Definition 2.3. Let X be a Banach lattice and $N \in \mathbb{N}$, $N \geq 2$. The N-dimensional Riesz angle of X is defined as

$$\alpha_N(X) = \sup\left\{\left\|\bigwedge_{\substack{i,j=1,\ldots,N \\ i \neq j}} (x_i \vee x_j)\right\| : \ x_1, \ldots, x_N \in B_X, \ x_1, \ldots, x_N \geq 0\right\}.$$

Of course $\alpha_N(X) \geq 1$. Moreover, $\alpha_2(X) = \alpha(X)$ and $\alpha_N(X) \leq \alpha_{N-1}(X)$ for every $N \geq 3$. Hence $\alpha_N(X) \leq \alpha(X)$. It was shown in [7] that

$$\alpha_N(X) \leq \frac{N}{N-1}$$

for every Banach lattice X and every natural $N \geq 2$.

The following class of Banach lattices was also introduced in [7].

Definition 2.4. Let $r \in (0,1]$. A Banach lattice X is said to be r-N-order uniformly noncreasy (r-N-OUNC) if for all $u_1, \ldots, u_N \in \frac{N-1}{N} B_X$ such that $\|u_i - u_j\| \leq 1$ we have either

$$\left\| \bigwedge_{\substack{i,j=1,\ldots,N \\ i \neq j}} (|u_i| \vee |u_j|) \right\| \leq r$$

or there exist $i \neq j$ such that for every $y \in X$ the conditions $|y| \leq |u_i - u_j|$, $\|y\| \geq r$ imply $\| |u_i - u_j| - |y| \| \leq r$.

A Banach lattice X is N-order uniformly noncreasy (N-OUNC) if it is r-N-OUNC for some $r \in (0,1)$.

Weak orthogonality will also play a fundamental role in this paper. The following definition was introduced by Borwein and Sims in [8].

Definition 2.5. A Banach lattice X is said to be weakly orthogonal if

$$\liminf_{n\to\infty} \liminf_{m\to\infty} \| |x_n| \wedge |x_m| \| = 0$$

whenever (x_n) is a sequence in X which converges weakly to 0.

This notion is generalized in [12]:

Definition 2.6. Let X be a subspace of a Banach lattice. We say that X is N-weakly orthogonal if for every weakly null sequence $(x_n) \subset X$ we have

$$\liminf_{n_N\to\infty} \ldots \liminf_{n_1\to\infty} \| |x_{n_N}| \wedge \cdots \wedge |x_{n_1}| \| = 0.$$

A very important notion in this paper will be the Banach-Mazur distance of two isomorphic Banach spaces X and Y, which is defined by the formula

$$d(X,Y) = \inf \|S\| \|S^{-1}\|$$

where the infimum is taken over all linear isomorphisms S of X onto Y.

Remark 1. Assume that X is a Banach lattice with Riesz angle $\alpha(X)$ and Y is a Banach space isomorphic to X such that $d(X,Y) < d$. Let $U : Y \to X$ be an isomorphism such that $\|U\| = 1$, $\|U^{-1}\| \leq d$. Then, Y can be also considered a Banach lattice defining $y_1 < y_2$ if $U(y_1) < U(y_2)$. It is easy to check that $U(y_1 \wedge y_2) = U(y_1) \wedge U(y_2)$, $U(y_1 \vee y_2) = U(y_1) \vee U(y_2)$ and $U(|y|) = |U(y)|$. Thus, Y can be understood as a Banach lattice with Riesz angle $\alpha(Y) \leq d\alpha(X)$. Furthermore, we have $\|Uv\| \leq \|v\| \leq d\|Uv\|$ for every $v \in Y$. Let $r \in (0,1]$ and assume that X is an r-N-OUNC Banach lattice. Let $u_1, \ldots, u_N \in \frac{N-1}{N} B_Y$ such that $\|u_i - u_j\| \leq 1$. Then, $Uu_1, \ldots, Uu_N \in \frac{N-1}{N} B_X$ and $\|Uu_i - Uu_j\| \leq 1$. If

$$\left\| \bigwedge_{\substack{i,j=1,\ldots,N \\ i \neq j}} (|u_i| \vee |u_j|) \right\| > rd$$

then,

$$\left\| \bigwedge_{\substack{i,j=1,\dots,N\\ i\neq j}} (|Uu_i| \vee |Uu_j|) \right\| > r.$$

Thus, there exists $i \neq j$ such that for every $y \in X$ the conditions $|y| \leq |Uu_i - Uu_j|$, $\|y\| \geq r$ imply $\||Uu_i - Uu_j| - |y|\| \leq r$. Thus, the conditions $|U^{-1}y| \leq |u_i - u_j|$, $\|U^{-1}y\| \geq rd$ imply $\||u_i - u_j| - |U^{-1}y|\| \leq rd$, which shows that Y is an rd-N-OUNC Banach lattice.

In this paper we will consider some classes of generalized nonexpansive mappings, which in general, are not continuous. However, we will always assume that these mappings satisfy this weaker condition:

Definition 2.7. Let (M, d) be a metric space and let $T : M \to M$ be a mapping. We will say that the graph of T is demiclosed at the diagonal if for every sequence (x_n) in M convergent to $x \in M$ such that $\lim_n d(x_n, Tx_n) = 0$ one has that $x = Tx$.

This notion is usually referred in the literature as strong demiclosedness of $I - T$ at 0. We prefer the above notation to avoid the use of the mapping $I - T$ in a metric space which is not necessarily a linear space.

Finally, we remind the following notions which are very useful in Fixed Point Theory:

Definition 2.8. Let (M, d) be a metric space and $T : M \to M$ be a mapping. A sequence (x_n) in M is said to be an *approximate fixed point sequence* (afps) for T if it satisfies

$$\lim_n d(x_n, Tx_n) = 0.$$

Definition 2.9. A Banach space X is said to satisfy the weak fixed point property (w-FPP) for a class of mappings $\mathfrak{F}$ if for every convex weakly compact subset C of X, every mapping $T : C \to C$ belonging to $\mathfrak{F}$ has a fixed point.

3. Historical background

Due to Alspach's example [2] it is known that Banach lattices do not, in general, satisfy the w-FPP for nonexpansive mappings. In fact $L^1([0, 1])$ is an abstract L-lattice failing the w-FPP and so does the abstract M-lattice $C([0, 1])$ which isometrically contains the previous space. The first results on Metric Fixed Point Theory on Banach lattices were obtained by P. Soardi [29] precisely for the case of abstract M-lattices and abstract L-lattices. However, we could say that Metric Fixed Point Theory on Banach lattices was initiated by J. Borwein and B. Sims in [8]. Their main result is the following:

Theorem 3.1. *A Banach space X satisfies the w-FPP for nonexpansive mappings if there exists a weakly orthogonal Banach lattice Y such that*

$$d(X, Y)\alpha(Y) < 2.$$

In the case of abstract M-spaces, the existence of fixed points was studied in the paper [13]. The most relevant result in this paper is the following:

Theorem 3.2. *Let Y be an abstract M-space which is p-weakly orthogonal. Assume that X is Banach space which is isomorphic to Y and such that*

$$d(X,Y) < \frac{p}{p-1}.$$

Then X has the w-FPP.

The results in [8] were extended in [5] to weakly orthogonal $OUNC$ Banach lattices:

Theorem 3.3. *A Banach space X has the w-FPP for nonexpansive mappings if there exists a weakly orthogonal $r-OUNC$ Banach lattice Y such that $d(X,Y)r < 1$. In particular, every weakly orthogonal $OUNC$ Banach lattice X has the w-FPP for nonexpansive mappings.*

One of the most relevant generalization of nonexpansiveness, related to a converse of the Banach Contraction Principle, has been given by T. Suzuki [30]:

Definition 3.4. Let (M,d) be a metric space. A mapping $T : M \to M$ is said to satisfy condition (C) if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq d(x,y).$$

In the same paper, some results about existence of fixed point for this class of mappings are proved (for instance, for Banach spaces satisfying the Opial condition). S. Dhompongsa and A. Kaewcharoen in [10] extended Theorem 3.3 for continuous mappings satisfying condition (C). Nominally, they proved:

Theorem 3.5. *A Banach space X has the w-FPP for continuous mappings satisfying condition (C) if there exists a weakly orthogonal $r - OUNC$ Banach lattice Y such that $d(X,Y)r < 1$. In particular, every weakly orthogonal $OUNC$ Banach lattice X has the w-FPP for mappings satisfying condition (C).*

Their result was generalized for the class of N-OUNC Banach lattices in [7]:

Theorem 3.6. *A Banach space X has the w-FPP for continuous mappings satisfying condition (C) if there exists $N \geq 2$ and a weakly orthogonal Banach lattice Y such that $d(X,Y)\alpha_N(Y) < \frac{N}{N-1}$. In particular, every weakly orthogonal $N-OUNC$ Banach lattice X has the w-FPP for mappings satisfying condition (C).*

A more general notion than condition (C) was defined in [17]:

Definition 3.7. Let (M,d) be a metric space and $\lambda \in (0,1)$. A mapping $T : M \to M$ is said to satisfy condition (C_λ) if

$$\lambda d(x,Tx) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq d(x,y).$$

Some fixed point results for nonexpansive mappings were extended in [17] and some other further papers ([1, 6, 9, 15, 16, 19, 25]) to the setting of mappings satisfying condition (C_λ). In particular, Theorem 3.6 was extended to the class of all continuous mappings satisfying condition (C_λ) in [4].

A further generalization of (C_λ)-condition was introduced in [11]:

Definition 3.8. Let (X,d) be a metric space and $K \subset X$. We say that mapping $T : K \to X$ is of Suzuki type if there exists a convex nondecreasing function $\psi : (0,\infty) \to (0,\infty)$ such that

$$d(x,Tx) - \psi(d(x,Tx)) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq d(x,y),$$

for all $x, y \in K$.

In [11] the authors extend some fixed point results for mappings satisfying condition (C) and condition (C_λ) to the setting of Suzuki type mappings. In particular, they prove that spaces with normal structure satisfy the w-FPP for Suzuki type mappings.

4. Some technical results

We will list in this section some previous fixed point results which will be necessary tools in this paper.

Lemma 4.1 ([11, Lemma 2.4])**.** *Let K be a closed, bounded and convex subset of a linear normed space X and $T : K \to K$ a mapping of Suzuki type. Then,*

$$\inf\{\|x - Tx\| : x \in K\} = 0,$$

i.e. there exists an apfs for T in K.

Theorem 4.2 ([11, Theorem 3.1])**.** *Let K be a closed convex bounded subset of a Banach space X. Assume that $T : K \to K$ is a mapping of Suzuki type such that the graph is demiclosed at the diagonal. Then, at least one of the following statement is true:*

1. *T has a fixed point,*
2. *For any afps $\{x_n\}$ for T in K and each $x \in K$ we have*

$$\limsup_{n\to\infty} \|x_n - Tx\| \leq \limsup_{n\to\infty} \|x_n - x\|.$$

The following condition, introduced in [25] lets obtain some consequences of the previous theorem.

Definition 4.3. A mapping $T : K \to K$ satisfies condition (L) if the following two conditions are fulfilled:

1. If $C \subset K$ is nonempty closed convex and T-invariant, then there exists an afps for T in C.
2. For any afps $\{x_n\}$ of T in K and each $x \in K$ one has

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \leq \limsup_{n\to\infty} \|x_n - x\|.$$

Thus, Lemma 4.1 and Theorem 4.2 give us the following:

Corollary 4.4. *Let K be a closed convex bounded subset of a Banach space X. Assume that $T : K \to K$ is a mapping of Suzuki type such that the graph is demiclosed at the diagonal. Then, at least one of the following statement is true:*

1. *T has a fixed point,*
2. *T satisfies condition (L).*

The following theorem can be understood as the counterpart of Goebel-Karlovitz Lemma [18, 21] for generalized nonexpansive mappings:

Theorem 4.5 ([25]). *Let C be a nonempty weakly compact convex subset of a Banach space X. Let $T : C \to C$ be a mapping satisfying condition (L). Let K be a minimal invariant subset of C for T. Then there exists $k \in \mathbb{R}$ such that for any afps $\{x_n\}$ for T in K and any $x \in K$,*

$$\limsup_{n\to\infty} \|x_n - x\| = k.$$

Hence, Corollary 4.4 and Theorem 4.5 let us conclude the following:

Corollary 4.6. *Let C be a nonempty weakly compact convex subset of a Banach space X. Let $T : C \to C$ be a mapping of Suzuki type such that the graph is demiclosed at the diagonal and T is fixed point free. Let K be a minimal invariant subset of C for T. Then there exists $k > 0$ such that for any afps $\{x_n\}$ for T in K and any $x \in K$,*

$$\limsup_{n\to\infty} \|x_n - x\| = k.$$

5. Main lemmas

The following lemma can be seen as a counterpart of the Banach Contraction Principle for mappings of Suzuki type.

Lemma 5.1. *Let (M, d) be a complete metric space and let $T : M \to M$ be a mapping satisfying*

$$(I - \psi)(d(x, Tx)) \le d(x, y) \quad \Rightarrow \quad d(Tx, Ty) \le r d(x, y)$$

for all $x, y \in M$, where $r \in (0, 1)$ and $\psi : (0, +\infty) \to (0, +\infty)$ is a nondecreasing function. Assume that the graph of T is demiclosed at the diagonal. Then there exists a (unique) fixed point of T.

Proof. Since for every $x \in M$

$$(I - \psi)(d(Tx, x)) \le d(x, Tx),$$

then

$$d(T^2x, Tx) \le r d(Tx, x).$$

Fix $u \in X$ and define a sequence (u_n) in M by $u_n = T^n u$. Then for all $n \in \mathbb{N}$

$$d(u_{n+1}, u_n) \le r d(u_n, u_{n-1})$$

and

$$d(u_{n+1}, u_n) \le r^n d(Tu, u).$$

Hence $\lim_{n\to\infty} d(u_{n+1}, u_n) = 0$. Moreover, for $m > n$

$$d(u_m, u_n) \le \sum_{i=n}^{m-1} d(u_i, u_{i+1}) \le \sum_{i=n}^{m-1} r^i d(Tu, u),$$

which implies that (u_n) is a Cauchy sequence.

Since (M, d) is a complete metric space there exists $z \in M$ such that $\lim_{n\to\infty} u_n = z$. Due to the demiclosedness of the graph we have $z = Tz$.

□

The following lemma, inspired on Theorem 3.6 in [7] is the main tool in our paper.

Lemma 5.2. *Let X be a Banach space and let K be a convex weakly compact subset of X which is minimal invariant for a mapping $T : K \to K$ which is of Suzuki type and such that the graph is demiclosed at the diagonal. Let $\{x_n(1)\}, \dots, \{x_n(N)\}$ be approximate fixed point sequences for T in K such that*

$$\lim_{n\to\infty} \|x_n(i) - x_n(j)\| = 1$$

for every $i, j \in \{1, \dots, N\}$, $i \neq j$ and

$$\lim_{n\to\infty} \|x_n(i) - x\| = 1$$

for every $i \in \{1, \dots, N\}$ and every $x \in K$. Let $\lambda_1, \dots, \lambda_N$ be non-negative numbers such that $\sum_{k=1}^{N} \lambda_k = 1$. Then, there exist an increasing sequence $\{n_k\}$ of positive integers and an approximate fixed point sequence $\{z_k\}$ for T in K such that

$$\lim_{n\to\infty} \|x_{n_k}(i) - z_k\| = 1 - \lambda_i$$

for every $i \in \{1, \dots, N\}$.

Proof. For each $n \in \mathbb{N}$, let

$$\begin{aligned} \delta_n &= \max\{\|x_n(i) - Tx_n(i)\| : \ i = 1, \dots, N\} \\ d_n &= \max\{\|x_n(i) - x_n(j)\| : \ i, j = 1, \dots, N, i \neq j\}. \end{aligned}$$

Choose (ε_n) such that $\varepsilon_n \in (0,1)$ and $\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \frac{\delta_n}{\varepsilon_n} = 0$. Let $\eta_n = \frac{1-\varepsilon_n}{\varepsilon_n}\delta_n$. For $n \in \mathbb{N}$, we put

$$K_n = \{z \in K : \ \|x_n(i) - z\| \leq (1 - \lambda_i)d_n + \eta_n \ \text{ for all } i = 1, \dots, N\}.$$

All sets K_n are closed convex and nonempty because $\sum_{k=1}^{N} \lambda_k x_n(k)$ belongs to K_n. Define a mapping $T_n : K_n \to K$ by

$$T_n z = (1 - \varepsilon_n)Tz + \varepsilon_n \sum_{j=1}^{N} \lambda_j x_n(j).$$

We will show that T_n maps K_n into K_n for large n. Let $z \in K_n$. Fix $\varepsilon \in (0, \frac{1}{2})$. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$(I - \psi)(\|Tx_n(i) - x_n(i)\|) \leq \|Tx_n(i) - x_n(i)\| \leq \varepsilon$$

and

$$\|x_n(i) - z\| \geq 1 - \varepsilon.$$

Hence

$$(I - \psi)(\|Tx_n(i) - x_n(i)\|) \leq \varepsilon < 1 - \varepsilon \leq \|x_n(i) - z\|,$$

so for all $n \geq n_0$

$$\|Tx_n(i) - Tz\| \leq \|x_n(i) - z\|.$$

Using the equality $(1 - \varepsilon_n)\delta_n = \varepsilon_n\eta_n$, we obtain

$$\|x_n(i) - T_n z\| \ = \ \left\|(1 - \varepsilon_n)(x_n(i) - Tz) + \varepsilon_n\Big(x_n(i) - \sum_{j=1}^{N} \lambda_j x_n(j)\Big)\right\|$$

$$
\begin{aligned}
&\leq (1-\varepsilon_n)(\|x_n(i)-Tx_n(i)\|+\|Tx_n(i)-Tz\|)\\
&+ \varepsilon_n \sum_{j\neq i} \lambda_j \|x_n(i)-x_n(j)\|\\
&\leq (1-\varepsilon_n)\left(\delta_n+(1-\lambda_i)d_n+\eta_n\right)+\varepsilon_n(1-\lambda_i)d_n\\
&= (1-\lambda_i)d_n+\eta_n.
\end{aligned}
$$

This means that $T_n(K_n) \subset K_n$ for every $n \geq n_0$.

Let

$$\alpha_n = \inf_{x\in K_n} \|x-T_nx\|.$$

We will show that $\liminf_{n\to\infty} \alpha_n = 0$. Assuming the contrary, we can find $\alpha_0 > 0$ such that $\alpha_n > \alpha_0$ for every n larger than some $n_1 \geq n_0$. Since $\lim_{n\to\infty} \varepsilon_n = 0$, there exists $n_2 \geq n_1$ such that for every $n \geq n_2$ we have $\varepsilon_n \operatorname{diam}(K) < \min\{\frac{\alpha_0}{2}, \frac{1}{8}\psi(\frac{\alpha_0}{2})\}$ and $\varepsilon_n\psi(\operatorname{diam}(K)) < \frac{1}{2}\psi(\frac{\alpha_0}{2})$.

Let $n \geq n_2$ and let $x, y \in K_n$ be such that $(I - \frac{1}{2}\psi)(\|x - T_nx\|) \leq \|x - y\|$. We will show that $\|T_nx - T_ny\| \leq (1 - \varepsilon_n)\|x - y\|$. First, note that

$$
\begin{aligned}
\alpha_0 \leq \|T_nx-x\| &\leq (1-\varepsilon_n)\|Tx-x\|+\varepsilon_n\left\|\sum_{i=1}^{N}\lambda_i x_n(i)-x\right\|\\
&\leq \|Tx-x\|+\varepsilon_n \operatorname{diam}(K)\\
&\leq \|Tx-x\|+\frac{\alpha_0}{2}
\end{aligned}
$$

and hence

$$\|Tx-x\| \geq \frac{\alpha_0}{2}.$$

We have

$$
\begin{aligned}
\|x-y\| &\geq (I-\frac{1}{2}\psi)(\|T_nx-x\|)=\|T_nx-x\|-\frac{1}{2}\psi(\|T_nx-x\|)\\
&\geq \|Tx-x\|-2\varepsilon_n \operatorname{diam}(K)-\frac{1}{2}\psi((1-\varepsilon_n)\|Tx-x\|+\varepsilon_n \operatorname{diam}(K))\\
&\geq \|Tx-x\|-2\varepsilon_n \operatorname{diam}(K)-(1-\varepsilon_n)\frac{1}{2}\psi(\|Tx-x\|)-\varepsilon_n\frac{1}{2}\psi(\operatorname{diam}(K))\\
&\geq (I-\psi)(\|Tx-x\|)+\frac{1}{2}\psi(\|Tx-x\|)-2\varepsilon_n \operatorname{diam}(K)-\varepsilon_n\frac{1}{2}\psi(\operatorname{diam}(K))\\
&\geq (I-\psi)(\|Tx-x\|)+\frac{1}{2}\psi\left(\frac{\alpha_0}{2}\right)-\frac{1}{4}\psi\left(\frac{\alpha_0}{2}\right)-\frac{1}{4}\psi\left(\frac{\alpha_0}{2}\right)\\
&= (I-\psi)(\|Tx-x\|)
\end{aligned}
$$

and thus

$$\|Tx-Ty\| \leq \|x-y\|$$

so

$$\|T_nx-T_ny\| = (1-\varepsilon_n)\|Tx-Ty\| \leq (1-\varepsilon_n)\|x-y\|.$$

From Lemma 5.1 T_n has a fixed point x_n in K_n, which contradicts our assumption that $\alpha_n > \alpha_0 > 0$.

We therefore see that $\liminf_{n\to\infty} \alpha_n = 0$ and hence there exists subsequence $\{n_k\}$ for which $\lim_{k\to\infty} \alpha_{n_k} = 0$. Choose $z_k \in K_{n_k}$ so that $\|z_k - T_{n_k} z_k\| \le \alpha_{n_k} + \frac{1}{k}$. Thus

$$\begin{aligned} \|z_k - Tz_k\| &\le \|z_k - T_{n_k} z_k\| + \|T_{n_k} z_k - Tz_k\| \\ &\le \alpha_{n_k} + \frac{1}{k} + \varepsilon_{n_k} \left\| \sum_{i=1}^{N} \lambda_i x_{n_k}(i) - Tz_k \right\| \\ &\le \alpha_{n_k} + \frac{1}{k} + \varepsilon_{n_k} \operatorname{diam}(K). \end{aligned}$$

This implies that $\lim_{k\to\infty} \|z_k - Tz_k\| = 0$. Taking again a subsequence, if necessary, we have that there exists

$$\lim_{k\to\infty} \|x_{n_k}(i) - z_k\| \le 1 - \lambda_i$$

for all $i = 1, \dots, N$. □

6. Fixed points for Suzuki type mappings on r-OUNC Banach lattices

Now we extend the result from [4] for mappings of Suzuki type.

Theorem 6.1. *Let C be a convex weakly compact subset of a Banach space X and let $T : C \to C$ be a mapping of Suzuki type such that the graph is demiclosed at the diagonal. If there exists a weakly orthogonal r'-N-OUNC Banach lattice Y such that $r'd(X,Y) < 1$, then T has a fixed point.*

Proof. By Remark 1 we can assume that X is an r-N-OUNC Banach lattice where $r = r'd(X,Y) < 1$. Assume that the theorem is false. Then there exist a nonempty weakly compact convex subset K of X and a mapping of Suzuki type $T : K \to K$ which has no fixed point and such that the graph is demiclosed at the diagonal. We can assume that K is minimal and T-invariant. By Lemma 4.1 there exists an afps $\{x_n\}$ for T in K. By Corollary 4.6 we have $\rho = \lim_{n\to\infty} \|x_n - x\|$ for every $x \in K$. There is no loss of generality in assuming that $\rho = 1$ and $\{x_n\}$ converges weakly to 0. In particular $0 \in K$.

Choose $\varepsilon > 0$ such that $r < \frac{1-\varepsilon}{1+\varepsilon}$. Similarly as in [7] we find subsequences $\{x_n(1)\}, \dots, \{x_n(N)\}$ of $\{x_n\}$ satisfying for all $i, j = 1, \dots, N$, $i \ne j$

$$\lim_{n\to\infty} \| |x_n(i)| \wedge |x_n(j)| \| = 0$$

and

$$\lim_{n\to\infty} \|x_n(i) - x_n(j)\| = \lim_{n\to\infty} \|x_n(i) - x\| = 1$$

for $i, j = 1, \dots, N$, $i \ne j$ and $x \in K$. From Lemma 5.2 there exist subsequences of $\{x_n(1)\}, \dots, \{x_n(N)\}$, denoted again $\{x_n(1)\}, \dots, \{x_n(N)\}$, and a sequence $\{z_n\}$ such that for $i = 1, \dots, N$ we have

$$\lim_{n\to\infty} \|z_n - x_n(i)\| \le \frac{N-1}{N} \qquad \text{and} \qquad \lim_{n\to\infty} \|z_n\| = 1.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that for all natural $n \ge n_0$ and $i, j = 1, \dots, N, i \ne j$ we have

$$\|x_n(i) - x_n(j)\| \le 1 + \varepsilon,\ \|x_n(i)\| > 1 - \varepsilon/2,$$
$$\|x_n(i) - z_n\| \le \frac{N-1}{N}(1 + \varepsilon),$$

$$\|z_n\| > 1 - \varepsilon/2,$$
$$\||x_n(i)| \wedge |x_n(j)|\| \le \varepsilon/(2N(N-1)) < \varepsilon/2.$$

Fix $n \ge n_0$ and put $u_n(i) = (z_n - x_n(i))/(1+\varepsilon)$ and $y_n(i,j) = (|x_n(i)| - |x_n(i)| \wedge |x_n(j)|)/(1+\varepsilon)$ for $i,j = 1,\dots,N, i<j$. Then for all $i,j = 1,\dots,N, i<j$ we have

$$\|u_n(i)\| \le \frac{N-1}{N} \quad \text{and} \quad \|u_n(i) - u_n(j)\| \le 1.$$

Furthermore, since $1 - \varepsilon/2 \le \|x_n(i)\|$ we have

$$\begin{aligned}\|y_n(i,j)\| \ge&(\|x_n(i)\| - \||x_n(i)| \wedge |x_n(j)|\|)/(1+\varepsilon)\\ \ge&(1-\varepsilon/2-\varepsilon/2)/(1+\varepsilon)\\ =&(1-\varepsilon)/(1+\varepsilon) > r.\end{aligned}$$

By Lemma 2.1(iii) $|y_n(i,j)| \le |u_n(i) - u_n(j)|$ for $i,j = 1,\dots,N, i<j$. Moreover, using again Lemma 2.1(iii), we have

$$\begin{aligned}\||u_n(i)-u_n(j)| - |y_n(i,j)|\| =&\||x_n(i)-x_n(j)| - |x_n(i)| + |x_n(i)| \wedge |x_n(j)|\|/(1+\varepsilon)\\ \ge&\||x_n(j)| - |x_n(i)| \wedge |x_n(j)|\|/(1+\varepsilon)\\ \ge&(\|x_n(j)\| - \||x_n(i)| \wedge |x_n(j)|\|)/(1+\varepsilon) > r\end{aligned}$$

and

$$\begin{aligned}1-\varepsilon/2 &< \|z_n\|\\ &\le \left\| \bigwedge_{\substack{i,j=1,\dots\\ i\ne j}} (|z_n - x_n(i)| \vee |z_n - x_n(j)|) \right\|\\ &+ \sum_{\substack{i,j=1\\ i\ne j}}^{N} \||x_n(i)| \wedge |x_n(j)|\|\\ &\le (1+\varepsilon) \left\| \bigwedge_{\substack{i,j=1,\dots,N\\ i\ne j}} (|u_n(i)| \vee |u_n(j)|) \right\| + \varepsilon/2.\end{aligned}$$

Thus

$$\left\| \bigwedge_{\substack{i,j=1,\dots,N\\ i\ne j}} (|u_n(i)| \vee |u_n(j)|) \right\| > (1-\varepsilon)/(1+\varepsilon) > r$$

which implies that X is not $r-N-OUNC$. □

Corollary 6.2. *A Banach space X has the w-FPP for mappings of the Suzuki type such that the graph is demiclosed at the diagonal if there exists $N \ge 2$ and weakly orthogonal Banach lattice Y such that $d(X,Y)\alpha_N(Y) < \frac{N}{N-1}$.*

Corollary 6.3. *Every weakly orthogonal N-OUNC Banach lattice has the w-FPP for mappings of Suzuki type such that the graph is demiclosed at the diagonal.*

7. Fixed point for Suzuki type mappings on N-weakly orthogonal Banach lattices

In this section we will extend the main result (Theorem 3.2) in [13] in two ways: Firstly, Theorem 3.2 in [13] only applies for abstract M-lattices. We will prove its validity for any arbitrary Banach lattice such that the Riesz angle has an appropriate upper bound. Secondly, while (Theorem 3.2) in [13] is also proved for nonexpansive mappings we will show that our more general version holds for Suzuki type mappings.

Theorem 7.1. *Let X be a Banach lattice which is N-weakly orthogonal, with Riesz angle α. Let $\epsilon_k, \ldots, \epsilon_0$ be the binary expression of N, i.e.*

$$N = \sum_{j=0}^{k} \epsilon_j 2^j$$

where $\epsilon_j \in \{0, 1\}$. Denote

$$M = M(N, \alpha) = \sum_{j=0}^{k} \epsilon_j 2^j \alpha^{-j} \le N$$

and choose $\ell = \ell(N, \alpha)$ such that

$$2^{\ell-1} < \sum_{j=0}^{k} \epsilon_j \le 2^{\ell}.$$

Assume that $\alpha^{\ell} < M/(N-1)$. Then, X satisfies the w-FPP for Suzuki type mappings such that the graph is demiclosed at the diagonal.

Proof. Denote

$$a_j = 1 - \frac{N-1}{M\alpha^j}$$

and note that

$$\sum_{j=0}^{k} 2^j \epsilon_j a_j = \sum_{j=0}^{k} 2^j \epsilon_j - \frac{N-1}{M} \sum_{j=0}^{k} \left(\frac{2}{\alpha}\right)^j \epsilon_j = N - \frac{N-1}{M} M = 1. \tag{7.1}$$

Furthermore

$$1 \le \alpha^l < \frac{M}{N-1} \le \frac{M\alpha^j}{N-1}$$

which implies $a_j > 0$.

Denote

$$J = \{j \in \{0, 1, \ldots, k\} : \epsilon_j = 1\}.$$

Since $N = \sum_{j \in J} 2^j$, we can express the set $\{1, \ldots, N\}$ as

$$\{a(i, j) : i = 1, \ldots, 2^j; j \in J\}.$$

Choose scalars $\lambda(i, j) = a_j$ for every $i = 1, \ldots, 2^j$ and $j \in J$. By (7.1) we have that

$$\sum_{j \in J} \sum_{i=1}^{2^j} \lambda(i, j) = \sum_{j \in J} 2^j a_j = 1.$$

Choose $\delta > 0$ small enough such that $\delta + \alpha^l \frac{N-1}{M} < 1$ and assume, by contradiction, that T fails to have a fixed point. By weak compactness, and using Zorn's Lemma, we can find a subset K of C which is convex weakly compact T-invariant and minimal with these properties. By Lemma 4.1 there exists an approximate fixed point sequence in K. By Lemma 4.6 there exists a number r such $\lim_{n\to\infty} \|u_n - x\| = r$ for every afps $\{u_n\}$ of T in K and every $x \in K$. By multiplication we can assume $r = 1$. Furthermore, by weak compactness and translation we can assume that $\{u_n\}$ is weakly null. Since X is N-weakly orthogonal we have

$$\liminf_{n_N\to\infty} \ldots \liminf_{n_1\to\infty} \| |u_{n_N}| \wedge \cdots \wedge |u_{n_1}| \| = 0.$$

Assume that $\{\delta_n\}$ is a null sequence of positive numbers. We can find $x_{n_1}(1)$, $x_{n_1}(2), \ldots, x_{n_1}(N) \in \{u_n : n \in \mathbb{N}\}$ with $\| |x_{n_1}(1)| \wedge \cdots \wedge |x_{n_1}(N)| \| \le \delta_1$ and $| \|x_{n_1}(i) - x_{n_1}(j)\| - 1| < \delta_1$ for every $i, j \in \{1, \ldots, N\}$, $i \ne j$. Thus, by an induction argument, we can construct subsequences $\{x_{n_s}(1)\}_{s\in\mathbb{N}}$, $\{x_{n_s}(2)\}_{s\in\mathbb{N}}$, $\ldots$, $\{x_{n_s}(N)\}_{s\in\mathbb{N}}$ of $\{u_n\}$, which satisfy

$$\| |x_{n_s}(1)| \wedge \cdots \wedge |x_{n_s}(N)| \| \le \delta_s$$

and $| \|x_{n_s}(i) - x_{n_s}(j)\| - 1| < \delta_s$ for every $i, j \in \{1, \ldots, N\}$, $i \ne j$. Thus, we have

$$\lim_{s\to\infty} \| |x_{n_s}(1)| \wedge \cdots \wedge |x_{n_s}(N)| \| = 0$$

and $\lim_{s\to\infty} \|x_{n_s}(i) - x_{n_s}(j)\| = 1$ for every $i, j \in \{1, \ldots, N\}$, $i \ne j$.

Applying Lemma 5.2 we can take a subsequence of $\{n_s\}$, again denoted $\{n_s\}$, for which there exists a sequence $\{z_s\}$ such that

$$\limsup_{s\to\infty} \|z_s - x_{n_s}(a(i,j))\| \le 1 - \lambda(i,j)$$

for every $j \in J$ and $i = 1, \ldots, 2^j$. Thus

$$\limsup_{s\to\infty} \|z_s - x_{n_s}(a(i,j))\| \le \frac{N-1}{M\alpha^j}$$

for every $j \in J$ and $i = 1, \ldots, 2^j$.

Choose s large enough such that

$$\left\| \bigwedge_{j\in J, i=1,\ldots,2^j} |x_{n_s}(a(i,j))| \right\| \le \delta.$$

By Lemma 2.1 we have

$$|z_s| \le \bigvee_{j\in J, i=1,\ldots,2^j} |z_s - x_{n_s}(a(i,j))| + \bigwedge_{j\in J, i=1,\ldots,2^j} |x_{n_s}(a(i,j))|$$

and so

$$\begin{aligned} \|z_s\| &\le \left\| \bigvee_{j\in J, i=1,\ldots,2^j} |z_s - x_{n_s}(a(i,j))| \right\| + \left\| \bigwedge_{j\in J, i=1,\ldots,2^j} |x_{n_s}(a(i,j))| \right\| \\ &\le \left\| \bigvee_{j\in J, i=1,\ldots,2^j} |z_s - x_{n_s}(a(i,j))| \right\| + \delta. \end{aligned}$$

Note that for every $j \in J$ we have

$$\left\| \bigvee_{i=1}^{2^j} |z_s - x_{n_s}(a(i,j))| \right\| \leq \alpha^j \left(\bigvee_{i=1}^{2^j} \|z_s - x_{n_s}(a(i,j))\| \right)$$

which implies that

$$\left\| \bigvee_{j\in J} \bigvee_{i=1}^{2^j} |z_s - x_{n_s}(a(i,j))| \right\| \leq \alpha^{j+\ell} \left(\bigvee_{j\in J} \bigvee_{i=1}^{2^j} \|z_s - x_{n_s}(a(i,j))\| \right)$$

because $\sum_{j\in J} \epsilon_j \leq 2^\ell$. Thus, taking limits, we obtain the contradiction

$$1 = \lim_{s\to\infty} \|z_s\| \leq \alpha^{j+\ell} \frac{N-1}{M\alpha^j} + \delta < 1.$$

□

Having in mind Remark 1, we easily obtain:

Corollary 7.2. *Let X be a Banach lattice with Riesz angle α which is N-weakly orthogonal and Y a Banach space isomorphic to X such that $d(X,Y) < r$. Denote M, ℓ as in Theorem 7.1 and assume that $(r\alpha)^\ell < M/(N-1)$. Then Y satisfies the w-FPP for mappings of Suzuki type such that the graph is demiclosed at the diagonal.*

Remark 2. Note that for $N = 2$, Corollary 7.2 recovers Corollary 6.2 and so, it extends the results in [8] for nonexpansive mappings to the case of Suzuki type mappings. Furthermore, for M-abstract spaces, i.e. Banach lattices such that $\alpha(X) = 1$, Theorem 7.1 and Corollary 7.2 recover and extend the results in [12, 13] for nonexpansive mappings to Suzuki type mappings.

Remark 3. Assume that X is 3-weakly orthogonal and denote α the Riesz angle of X. Since $M(3) = (2/\alpha) + 1$ and $l(3) = 1$, Theorem 7.1 implies that every mapping of Suzuki type defined on a weakly compact convex subset of X has a fixed point if $2\alpha < (2/\alpha) + 1$, or equivalently if

$$\alpha < \frac{1+\sqrt{17}}{4}.$$

Assume that $X = C(\omega^2 + 1) \oplus_\infty \ell_p$ where

$$p > \frac{\ln 2}{\ln(1+\sqrt{17}) - 2\ln 2}.$$

It is easy to check that X is 3-weakly orthogonal and $\alpha = 2^{1/p}$ which implies

$$\alpha < \frac{1+\sqrt{17}}{4}$$

due to the choice of p. It should be noted that the existence of fixed points for this space, even in the case of nonexpansive mappings, cannot be deduced from previous results in the literature on fixed point for nonexpansive mappings. Indeed, the space X is not weakly orthogonal because it contains $C(\omega^2 + 1)$ and it is known (see [12, Theorem 3.4]) that the space $C(K)$ is 2-weakly orthogonal if and only if the 2-derived set $K^{(2)}$ is empty (and note that $(\omega^2+1)^{(2)}$ is a singleton), and X is not an abstract M-lattice because it contains ℓ_p and the Riesz angle $\alpha(\ell_p)$ of any ℓ_p-space is greater than 1.

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