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REGULARITY AND SINGULARITY PHENOMENA FOR ONE-DIMENSIONAL VARIATIONAL PROBLEMS WITH SINGULAR ELLIPTICITY

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Dedicated to memory of Victor Mizel: entuziastic mathematician and human person

Abstract. In this paper we develop regularity theory for one-dimensional variational problems with singular ellipticity. As is known the classical indirect methods can not be applied in this case since the Euler equation is not defined. However it is still possible to suggest certain direct methods. Such a theory is the content of this paper.

1. Introduction

In this paper we consider classical one-dimensional variational problems

$$J(u) = \int_a^b L(x, u(x), \dot{u}(x))dx \to \min \tag{1.1}$$

$$u(a) = A, u(b) = B. \tag{1.2}$$

We assume that $L : [a, b] \times R \times R \to R$ is continuous and $L(x, u, v)$ is convex in v. These assumptions on the integrand L will be regarded as *basic* throughout. Under these assumptions, given a function $u \in W^{1,1}[a, b]$ we have that the function $L(\cdot, u(\cdot), \dot{u}(\cdot))$ is measurable and its negative part is integrable. Therefore, the integral $J(u)$ is defined and is either a finite value or $+\infty$.

In the case when the solution $u : [a, b] \to R$ is Lipschitz and $L \in C^1$ the Euler-Lagrange equation holds:

$$\frac{d}{dx} L_v(x, u(x), \dot{u}(x)) = L_u(x, u(x), \dot{u}(x)). \tag{1.3}$$

In case additionally $L \in C^2$ and $L_{vv} > 0$ we have $u \in C^2[a, b]$ and the equation (1.3) can be resolved with respect to the second derivative of the function u:

$$u'' = \frac{L_u - L_{xv} - L_{uv}\dot{u}}{L_{vv}}. \tag{1.4}$$

The classical theory of solvability of problems (1.1-2) and of regularity of the solutions is based on indirect methods, see e.g. [3], [20, §1,2]. In the case $L_{vv} \geq 0$ only,

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the equation (1.4) is not defined, but the theory of solvability of the problems (1.1-2) and of regularity of the corresponding solutions can be based on direct methods.

Presentation of this theory is just the purpose of this paper. Earlier, in the case of classical ellipticity such methods were developed in the papers of Sychev and Mizel [14, 16].

In multi-dimensional case the authors of [22] proved the existence of Lipschitz solutions of anisotropic elliptic equations in convex domains of special form. However C^1-regularity of solutions is still an open problem, see [9].

We will consider the following assumptions on the integrand L:

(H) for each compact set $K \subset R \times R \times R$ there exist $c > 0$, $\alpha > 0$ such that if $(x, u, v), (y, w, v) \in K$ then

$$|L(x, u, v) - L(y, w, v)| \leq c(|x - y| + |u - w|)^{\alpha},$$

and there also exist $\mu > 0$, $p > 1$ such that if $(x, u, v_1), \in K$ then for some $l \in \partial L_v(x, u, v_1)$ we have for all $(x, u, v_2) \in K$ that

$$L(x, u, v_2) - L(x, u, v_1) - (l, v_2 - v_1) \geq \mu|v_2 - v_1|^p.$$

The ellipticity condition stated in (H) will be called *singular ellipticity* everywhere further in the paper.

The first theorem is the following theorem of existence and regularity "in small".

Theorem 1.1. *Let L satisfy the condition (H), $G \subset R \times R$ be a compact set. For each $M > 0$ there exist $\epsilon_0 > 0$, $\delta_0 > 0$ such that for every $\epsilon \leq \epsilon_0$, $\delta \leq \delta_0$ the problem (1.1-2) in the class of functions $u \in W^{1,1}[a, b]$ such that $(a, A) \in K$, $|B - A|/|b - a| \leq M$, $|b - a| \leq \delta$, $|u(x) - A| \leq \epsilon$ for all $x \in [a, b]$, is solvable. Moreover, the solutions are bounded in the space $C^{1,\gamma}$, where $\gamma = \gamma(M)$ does not depend on $\delta \leq \delta_0$, $\epsilon \leq \epsilon_0$, $(a, A) \in G$.*

Remark 1.2. The exponent γ could be taken equal to α/p, where $\alpha > 0$, $p > 1$ correspond to the compact set $K' = G' \times [-M - 1, M + 1]$, $G' = \{(y, w) : |y - a| \leq \delta_0, |w - A| \leq \epsilon_0, (a, A) \in G\}$.

Note that Clarke and Vinter proved C^1-regularity of solutions for integrands L strictly convex with respect to v, but they needed Lipschitz regularity with respect to u, see [5]. As we see, singular ellipticity implies better regularity even if L is only Holder continuous in (x, u). In fact Holder continuity is the optimal assumption for C^1-regularity since continuity of L is no longer sufficient, as follows from a recentpaper by Gratwick and Preiss [12].

Note that the effect of regularity/nonregularity was known in the context of parametric problems already in the 70th years of the past century, see Chapter 6 of the book of Reshetnyak [15].

The next fundamental theorem is a corollary of Theorem 1.1.

Definition 1.3. Consider the class of functions $\Xi = \{\xi : [a_\xi, b_\xi] \to \bar{R} = R \cup \{-\infty, +\infty\}\}$ such that each function $\xi : [a_\xi, b_\xi] \to \bar{R}$ is continuous. We say that the family Ξ is conditionally equa-continuous if for every $M > 0$, $\epsilon > 0$ there exists $\delta = \delta(M, \epsilon) > 0$ such that if $|\xi(x_0)| \leq M$ then $|\xi(x) - \xi(x_0)| \leq \epsilon$ for $|x - x_0| \leq \delta$.

Theorem 1.4. *Let L satisfy the condition (H) and let $G \subset R \times R$ be a compact set. Consider the set U of those solutions of problems (1.1-2) with graphs lying inside G. Then the derivatives of functions in U form a conditionally equa-continuous family.*

Therefore Theorem 1.4 guarantees an a priori regularity of solutions of problems (1.1-2). As for the existence, this follows from the following famous results of Tonelli proved at the beginning of the 20th century.

Lemma 1.5. *Let L satisfy the basic assumptions and $u_k \rightharpoonup u$ in $W^{1,1}[a,b]$ ("$\rightharpoonup$" means weak convergence). Then*

$$\liminf_{k\to\infty} J(u_k) \geq J(u).$$

For a proof see e.g. [21].

The next theorem is a corollary of this lemma and the criteria of the weak convergence in $W^{1,1}$

Theorem 1.6 (Tonelli, [23])**.** *Let L satisfy the basic assumptions and $L(x,u,v)$ have superlinear growth in v, i.e. $L(x,u,v) \geq \theta(v)$ where θ is a convex function such that $\theta(v)/|v| \to \infty$, $|v| \to \infty$. Then each problem (1.1-2) admits a solution.*

The examples of Davie from [8] allow us to assert that Theorem 1.4 is precise. But in these examples the Lavrentiev phenomenon is present:

$$\begin{aligned} &\inf\{J(u) : u(a) = A, u(b) = B, u \in W^{1,1}[a,b]\} \\ &< \inf\{J(u) : u(a) = A, u(b) = B, u \in C^1[a,b]\}. \end{aligned}$$

It is well-known that the Lavrentiev phenomenon is connected with failure of the standard growth of

$L(x,u,v)$ with respect to v. Recall that L has the standard growth if there exists a convex function $\theta : R \to R$ such that $\theta(v)/|v| \to \infty$ as $|v| \to \infty$, and

$$\theta(v) \leq L(x,u,v) \leq c(1+\theta(v)),\ c > 0. \tag{1.5}$$

In the case of the standard growth of L the Lavrentiev phenomenon is not present, see e.g. [18]. At the same time for every two convex functions $\theta_1, \theta_2 : R \to R$ such that $\theta_1(v)/|v| \to \infty$, $\theta_2(v)/|v| \to \infty$, and $\theta_2(v)/\theta_1(v) \to \infty$ as $|v| \to \infty$, there exists an integrand $L = \theta_1(v) + f(x,u)\theta_2(v)$ with $f \geq 0$ for which the Lavrentiev phenomenon is present in a certain problem (1.1-2), see again [18].

Gratwick proved that Theorem 1.4 is precise also in the case of validity of the conditions of standard growth, which improves the results of the paper [10].

Theorem 1.7. *Let $\theta : R \to R$ be a strictly convex function of the class C^∞ such that $\theta(v)/|v| \to \infty$, as $|v| \to \infty$. Let also $E \subset [a,b]$ be a compact set of zero measure. Then there exists a function $L(x,u,v) \in C^\infty$, strictly convex with respect to v, such that*

$$\theta(v) \leq L(x,u,v) \leq c(1+\theta(v)),\ c > 0,$$

and a solution of a problem (1.1-2) with infinite derivative in E. The function L can be taken of the form $\theta(v) + F(x,u,v)$ where $F \geq 0$ is convex in v.

In [19] Sychev constructed examples of integrands $L(x,u,v)$ with quadratic growth in v which are uniformly elliptic in v ($0 < \mu_1 \leq L_{vv} \leq \mu_2 < \infty$) and admit singular solutions. In these examples the singular set is a singleton and it is not known how large can the singular set in general. The conjecture is that an arbitrary compact set of zero measure, like in Theorem 1.7. Previously even some experts thought that in uniformly elliptic problems solutions are C^1-regular, see e.g. [7].

Certainly, a fundamental question is when solutions of the problems (1.1-2) are always regular. The next theorem gives an answer to this question.

Theorem 1.8. *Let L satisfy the condition (H) and let $L(x,u,v)$ have superlinear growth in v.*

For $(a,A,b,B) \in R^4$ we let

$$S(a,A,b,B) = \inf\{J(u) : u(a) = A, u(b) = B, u \in W^{1,1}[a,b]\}.$$

Then in the problem (1.1-2) with boundary conditions (a_0,A_0,b_0,B_0) all solutions are bounded in certain neighbourhoods of the points a_0 and b_0 in C^1-norm if and only if S is Lipschitz at the point (a_0,A_0,b_0,B_0).

Due to the last theorem we may assert that the solutions of all the problems (1.1-2) are Lipschitz (and even of the class $C^{1,\gamma}$) if and only if the function S is Lipschitz at each point. However, due to the results of Ball and Mizel [2] (see also Davie [8] and theorem 1.7 of Gratwick) S could be non-Lipschitz. Ball and Nadirashvili suggested clarifying how large could be the set V which consists of all the points (x_0,u_0) for which there exist boundary conditions (1.2) such that the corresponding solution has the properties $u(x_0) = u_0$, $|\dot{u}(x_0)| = \infty$ (this set was called by them the universal singular set), see [2]. It turns out that this set can not be large.

Theorem 1.9. *Let L satisfy (H) and let $L(x,u,v)$ have superlinear growth in v. Then the corresponding universal singular set V is of first Baire category and has zero 2-d Lebesgue measure.*

Theorem 1.9 was proved in [17] for C^1-regular integrands and in [6] for locally Lipschitz ones. Recall that local Holder continuity is the optimal assumption for partial regularity of minimizers as follows from results of the paper [12].

There is a possibility to characterize universal singular sets in terms of pure unrectifiability. The set $S \subset R^2$ is called purely unrectifiable if for each Lipschitz curve $\gamma : R \to R^2$ the intersection of γ and S takes place in a set of zero linear measure. Note that a purely unrectifiable compact set could be of Hausdorff dimension 2. In Theorem 6 of [6] the authors showed that the universal singular sets are always "almost" purely unrectifiable, that also covers our situation considered in Theorem 1.9. Gratwick showed that theorem 1.10 is also valid.

Theorem 1.10. *For each strictly convex function $\theta : R \to R$ of C^∞ class and with superlinear growth and for each purely unrectifiable compact set $S \subset R^2$ there exists an integrand $L(x,u,v) = \theta(v) + F(x,u,v)$ with $F \geq 0$ of C^∞ class and convex in v such that the universal singular set corresponding to L contains S. Moreover, L has the standard growth*

$$\theta(v) \leq L(x,u,v) \leq c(1 + \theta(v)),\ c > 0.$$

The paper will be organized as follows. Theorems 1.1 and 1.4 will be proved in §2. Theorem 1.8 in §3 and Theorem 1.9 in §4. As for proofs of Theorems 1.7 and 1.10 they will be presented in a forthcoming paper of Gratwick [11]. Results of this paper were previously announced in the note [13].

2. Proofs of theorems 1.1, 1.4

Given a function $u : [a,b] \to \mathbf{R}$ and given $x_1, x_2 \in [a,b]$ with $x_2 > x_1$ we define $l_{x_1,x_2} : [x_1,x_2] \to \mathbf{R}$ as the affine function which fits the same boundary data at x_1, x_2 as the function u, i.e. $l_{x_1,x_2}(x_1) = u(x_1)$, $l_{x_1,x_2}(x_2) = u(x_2)$. Then we have

$$\dot{l}_{x_1,x_2} = \frac{u(x_2)-u(x_1)}{x_2-x_1}, \quad x \in [x_1,x_2].$$

In this section we use the notation

$$J(u;[x_1,x_2]) := \int_{x_1}^{x_2} L(x,u(x),\dot{u}(x))dx.$$

Lemma 2.1. *Let $K \subset \mathbf{R}^2$ be a compact set and let $M > 0$. Assume u to be a M-Lipschitz function with the graph staying in K. Assume also $x_1 \le x_3 \le x_4 \le x_2$ be such points of $[a,b]$ that $|x_4 - x_3|e \ge |x_2 - x_1|$, $|x_2 - x_1| \le 1$ (e is Napier's number) and the graphs of the functions l_{x_1,x_2}, l_{x_3,x_4} stay in K.*

Let L be Hölder continuous in (x,u) in the set $K \times [-M,M]$, i.e.

$$|L(x,u,v) - L(\bar{x},\bar{u},v)| \le c(|x-\bar{x}| + |u-\bar{u}|)^\alpha, \; c,\alpha > 0, \tag{2.1}$$

for $(x,u),(\bar{x},\bar{u}) \in K$, $|v| \le M$, and let L be singularly elliptic in v when restricting to $K \times [-M,M]$, i.e. there exists $\mu > 0$, $p > 1$ such that

$$L(x,u,\bar{v}) - L(x,u,v) - l(\bar{v}-v) \ge \mu|\bar{v}-v|^p, \tag{2.2}$$

where $(x,u) \in K$, $|v|,|\bar{v}| \le M$ and l is an element of the subgradient of the function $L(x,u,\cdot)$ at v.

Assume also that

$$J(u;[x_1,x_2]) \le J(l_{x_1,x_2};[x_1,x_2]), \tag{2.3}$$

$$J(u;[x_3,x_4]) \le J(l_{x_3,x_4};[x_3,x_4]). \tag{2.4}$$

Then

$$\Big|\frac{u(x_2)-u(x_1)}{x_2-x_1} - \frac{u(x_4)-u(x_3)}{x_4-x_3}\Big| \le 2e\Big[\frac{2c(1+M)^\alpha}{\mu}\Big]^{1/p}|x_2-x_1|^{\alpha/p}. \tag{2.5}$$

Proof. We first obtain an estimate of the excess of the derivative in $[x_1,x_2]$ and in $[x_3,x_4]$, i.e. the estimates of the integrals

$$\int_{x_1}^{x_2} |\dot{u}(x) - \dot{l}_{x_1,x_2}|^p dx, \; \int_{x_3}^{x_4} |\dot{u}(x) - \dot{l}_{x_3,x_4}|^p dx.$$

Because of (2.1) we have

$$|L(x_1,u(x_1),\dot{u}(x)) - L(x,u(x),\dot{u}(x))| \le c(|x_1-x| + M|x_1-x|)^\alpha,$$

$$|L(x_1,u(x_1),\dot{l}_{x_1,x_2}(x)) - L(x,l_{x_1,x_2}(x),\dot{l}_{x_1,x_2}(x)| \le c(|x_1-x| + M|x_1-x|)^\alpha.$$

Therefore if we define $\tilde{L}(\cdot) := L(x_1, u(x_1), \cdot)$ then because of (2.1), (2.3) for the integral functional $\tilde{J}$ with the integrand $\tilde{L}$ we have

$$
\begin{aligned}
\tilde{J}(u;[x_1,x_2]) &\le J(u;[x_1,x_2]) + c(1+M)^\alpha |x_2-x_1|^{1+\alpha} \\
&\le J(l_{x_1,x_2};[x_1,x_2]) + c(1+M)^\alpha |x_2-x_1|^{1+\alpha} \\
&\le \tilde{J}(l_{x_1,x_2};[x_1,x_2]) + 2c(1+M)^\alpha |x_2-x_1|^{1+\alpha}.
\end{aligned} \tag{2.6}
$$

Note that if $l \in \partial\tilde{L}(\dot{l}_{x_1,x_2})$ then in view of (2.2)

$$
\begin{aligned}
\tilde{J}(u;[x_1,x_2]) - \tilde{J}(l_{x_1,x_2};[x_1,x_2]) &= \int_{x_1}^{x_2} \{\tilde{L}(\dot{u}(x)) - \tilde{L}(\dot{l}_{x_1,x_2}) - l(\dot{u} - \dot{l}_{x_1,x_2})\} dx \\
&\ge \mu \int_{x_1}^{x_2} |\dot{u} - \dot{l}_{x_1,x_2}|^p dx.
\end{aligned}
$$

Therefore (2.6) implies

$$
\int_{x_1}^{x_2} |\dot{u}(x) - \dot{l}_{x_1,x_2}|^p dx \le \frac{2c(1+M)^\alpha}{\mu} |x_2-x_1|^{1+\alpha}. \tag{2.7}
$$

Analogously

$$
\int_{x_3}^{x_4} |\dot{u}(x) - \dot{l}_{x_3,x_4}|^p dx \le \frac{2c(1+M)^\alpha}{\mu} |x_4-x_3|^{1+\alpha}. \tag{2.8}
$$

Then by Hölder inequality and by (2.7), (2.8) we have

$$
\begin{aligned}
\int_{x_3}^{x_4} |\dot{l}_{x_1,x_2} - \dot{l}_{x_3,x_4}| dx &\le \int_{x_3}^{x_4} |\dot{l}_{x_3,x_4} - \dot{u}(x)| dx + \int_{x_1}^{x_2} |\dot{l}_{x_1,x_2} - \dot{u}(x)| dx \\
&\le \Big(\int_{x_3}^{x_4} |\dot{l}_{x_3,x_4} - \dot{u}(x)|^p dx\Big)^{1/p} |x_4-x_3|^{(p-1)/p} \\
&\quad + \Big(\int_{x_1}^{x_2} |\dot{l}_{x_1,x_2} - \dot{u}(x)|^p dx\Big)^{1/p} |x_2-x_1|^{(p-1)/p} \\
&\le \Big[\frac{2c(1+M)^\alpha}{\mu}\Big]^{1/p} \{|x_4-x_3|^{1+\alpha/p} + |x_2-x_1|^{1+\alpha/p}\} \\
&\le \Big[\frac{2c(1+M)^\alpha}{\mu}\Big]^{1/p} 2|x_2-x_1|^{1+\alpha/p}
\end{aligned}
$$

and, therefore,

$$
|\dot{l}_{x_1,x_2} - \dot{l}_{x_3,x_4}| \le \Big[\frac{2c(1+M)^\alpha}{\mu}\Big]^{1/p} 2e|x_2-x_1|^{\alpha/p},
$$

i.e. (2.5) is established. □

In the proof of Theorem 1.1 we will use Proposition 2.2 which relies on four lemmas 2.3-6. The statement, proof, and application of the following result is adapted from [5].

Proposition 2.2. *Let $S \subseteq R^2$ be compact, $L\colon S \times R \to R$ be continuous, $L = L(x,u,v)$ be convex in v, and satisfy condition (H), and suppose that there exist $\alpha > 0$ and $\beta \in R$ such that*

$$
L(x,u,v) \ge \alpha|v| + \beta
$$

for all $(x,u,v) \in S \times R$. *Let* $r_0 > 0$.

Then there exist real numbers $r_1, \delta, \alpha_1, \alpha_2$, *and* γ *and satisfying* $r_1 > r_0$, $\alpha_2 > \alpha_1 > 0$, *and* $\delta > 0$, *such that for each* $k > r_1$ *there exists* $H_k \colon S \times R \to R$ *such that the following conditions hold:*

(2.2a) H_k *satisfies condition (H);*
(2.2b) $H_k(x,u,v) = L(x,u,v)$ *whenever* $|v| \le r_0$*;*
(2.2c) $H_k(x,u,v) \le L(x,u,v) - \delta$ *whenever* $r_1 \le |v| \le k$*;*
(2.2d) $H_k(x,u,v) \le \alpha_2|v| + \gamma$*;*
(2.2e) $H_k(x,u,v) \ge \alpha_1|v| + \beta$*; and*
(2.2f) $H_k(x,u,v) \le L(x,u,v) + k^{-1}$ *whenever* $r_0 \le |v| \le r_1$.

The proof of this result relies on a number of lemmas. Throughout we assume that the conditions of the proposition hold. We begin by defining $r_1 := r_0 + 1$.

Lemma 2.3. *There exists* $\eta > 0$ *such that for all* $(x,u) \in S$, *all* $v_1 \in R$ *satisfying* $|v_1| \le r_0$, *there exists* $l \in \partial_v L(x,u,v_1)$ *such that for all* $|v_2| \ge r_1$ *we have that*

$$L(x,u,v_2) - L(x,u,v_1) \ge (l, v_2 - v_1) + \eta.$$

Proof. By the singular ellipticity condition there exist $\mu > 0$ and $p > 1$ such that for all $(x,u) \in S$ and $|v_1| \le r_0$ there exists $l \in \partial_v L(x,u,v_1)$ such that for all $|v_2| = r_1$, we have that

$$L(x,u,v_2) - L(x,u,v_1) - (l, v_2 - v_1) \ge \mu |v_2 - v_1|^p \ge \mu.$$

So setting $\eta := \mu$ we get the condition required for $|v_2| = r_1$. For $|v_2| > r_1$, we choose $s_1 \ge s_2 > 0$ such that, writing $d = (v_2 - v_1)/|v_2 - v_1|$,

$$\begin{aligned} v_1 + s_1 d &= v_2, \text{ and} \\ |v_1 + s_2 d| &= r_1. \end{aligned}$$

Then for some $l \in \partial_v L(x,u,v)$, we have by convexity of L in v that

$$\begin{aligned} L(x,u,v_2) - L(x,u,v_1) &= L(x,u,v_1 + s_1 d) - L(x,u,v_1) \\ &\ge \frac{s_1}{s_2}(L(x,u,v_1 + s_2 d) - L(x,u,v_1)) \\ &= \frac{s_1}{s_2}((l, v_1 + s_2 d - v_1) + \eta) \\ &= (l, v_2 - v_1) + s_1\eta/s_2 \\ &\ge (l, v_2 - v_1) + \eta, \end{aligned}$$

as required. □

Lemma 2.4. *There exist* $\sigma > 0$, $\tilde{\gamma} \in R$, *and* $L_0 \colon S \times R \to R$ *such that*

(2.4a) L_0 *is Hölder in* (x,u) *locally uniformly in* v, *and* L_0 *is convex in* v*;*
(2.4b) $L_0(x,u,v) = \sigma|v| + \tilde{\gamma}$ *for* $|v| \ge r_1$*;*
(2.4c) $L_0(x,u,v) = L(x,u,v)$ *whenever* $|v| \le r_0$*; and*
(2.4d) $L_0(x,u,v) \ge \alpha|v| + \beta$.

Proof. Define

$$\begin{aligned} t_1 &= \min\{L(x,u,v) : (x,u) \in S,\ |v| \le r_0\} \\ t_2 &= \max\{L(x,u,v) : (x,u) \in S,\ |v| = r_1\}. \end{aligned}$$

Choosing $\sigma > \max\{\alpha, (t_2 - t_1)/(r_1 - r_0)\}$, we see that

$$\sigma r_0 - t_1 < \sigma|v| - L(x, u, v)$$

for all $(x, u) \in S$ and $|v| = r_1$, so by continuity of L we may choose $\tilde{\gamma} \in R$ and $r_2 > r_1$ such that for all $(x, u) \in S$ and $r_1 \leq |v| \leq r_2$,

$$\sigma r_0 - t_1 < -\tilde{\gamma} < \sigma|v| - L(x, u, v).$$

Define $L_0 \colon S \times R \to R$ by

$$L_0(x, u, v) = \begin{cases} \max\{\sigma|v| + \tilde{\gamma}, L(x, u, v)\} & |v| \leq r_2, \\ \sigma|v| + \tilde{\gamma} & |v| > r_2. \end{cases}$$

For $|v| \leq r_0$,

$$L(x, u, v) \geq t_1 > \sigma r_0 + \tilde{\gamma} \geq \sigma|v| + \tilde{\gamma},$$

so $L_0(x, u, v) = L(x, u, v)$, and for $r_1 \leq |v| \leq r_2$,

$$L(x, u, v) < \sigma|v| + \tilde{\gamma},$$

so $L_0(x, u, v) = \sigma|v| + \tilde{\gamma}$. Thus L_0 satisfies conditions (2.4a-c). For (2.4d), since $L_0(x, u, v) \geq L(x, u, v) \geq \alpha|v| + \beta$ for $|v| \leq r_1$, we are just required to prove that $\sigma|v| + \tilde{\gamma} \geq \alpha|v| + \beta$ for $|v| \geq r_1$, and in fact it suffices to prove this for $|v| = r_1$. But in this case, we have for any $(x, u) \in S$ that

$$\sigma|v| + \tilde{\gamma} = L_0(x, u, v) \geq L(x, u, v) \geq \alpha|v| + \beta,$$

as required. □

We now define $\alpha_1 := \alpha/2$, and $\tilde{\eta} := \min\{\eta/2, \alpha_1 r_1/2\}$.

Lemma 2.5. *For each $k > r_1$, there exists $F_k \colon S \times R \to R \cup \{+\infty\}$ satisfying the following conditions:*

(2.5a) *$F_k(x, u, v) = L(x, u, v)$ for $|v| \leq r_0$;*
(2.5b) *$F_k(x, u, v) = +\infty$ for $|v| > k$;*
(2.5c) *F_k is Hölder in (x, u) uniformly in v for $|v| \leq k$, and F_k is convex in v;*
(2.5d) *$F_k(x, u, v) \leq L(x, u, v)$ for $|v| \leq k$, and $F_k(x, u, v) \leq L(x, u, v) - \tilde{\eta}$ for $r_1 \leq |v| \leq k$; and*
(2.5e) *$F_k(x, u, v) \geq \alpha_1|v| + \beta$.*

Proof. Define $f_k \colon S \times R \to R \cup \{+\infty\}$ by

$$f_k(x, u, v) = \begin{cases} L(x, u, v) & |v| < r_1, \\ L(x, u, v) - \tilde{\eta} & r_1 \leq |v| \leq 2k, \\ +\infty & |v| > 2k, \end{cases}$$

and define $\tilde{f}_k \colon S \times R \to R \cup \{+\infty\}$ by

$$\tilde{f}_k(x, u, v) = \inf\left\{\sum_{i=1}^{n} \lambda_i f_k(x, u, v_i) : \sum_{i=1}^{n} \lambda_i v_i = v,\ \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geq 0,\ n \in N\right\}.$$

By definition of f_k, we can restrict to $|v_i| \leq 2k$ in the definition of $\tilde{f}_k$. Since L is Hölder in (x, u) uniformly in v on $S \times [-2k, 2k]$, we see that $\tilde{f}_k$ is Hölder in (x, u) uniformly in v on $S \times [-k, k]$.

We now define $F_k\colon S \times R \to R \cup \{+\infty\}$ by

$$F_k(x,u,v) = \begin{cases} \tilde{f}_k(x,u,v) & |v| \leq k, \\ +\infty & |v| > k. \end{cases}$$

So condition (2.5b) holds by definition, and conditions (2.5c) and (2.5d) hold by the corresponding properties of f_k. To check condition (2.5a), we let $(x,u) \in S$ and $|v| \leq r_0$. It suffices to prove that for some $l \in \partial_v L(x,u,v)$, any convex combination $\sum_{i=1}^n \lambda_i v_i = v$ satisfies

$$f_k(x,u,v_i) \geq L(x,u,v) + (l, v_i - v).$$

Given this, it follows that

$$F_k(x,u,v) = \tilde{f}_k(x,u,v) \geq L(x,u,v),$$

which combined with (2.5d) gives the required equality.

So, to prove this claim, let $l \in \partial_v L(x,u,v)$ witness the conclusion of lemma 2.3, and consider some convex combination $\sum_{i=1}^n \lambda_i v_i = v$. For $|v_i| \geq r_1$, we apply lemma 2.3 to see that, by the choice of $\tilde{\eta}$,

$$\begin{aligned} f_k(x,u,v_i) &= L(x,u,v_i) - \tilde{\eta} \\ &\geq L(x,u,v) + (l, v_i - v) + \eta - \tilde{\eta} \\ &> L(x,u,v) + (l, v_i - v). \end{aligned}$$

For $|v_i| < r_1$, we see on the other hand by the usual subgradient inequality that

$$f_k(x,u,v_i) = L(x,u,v_i) \geq L(x,u,v) + (l, v_i - v),$$

as required.

To show condition (2.5e) we are required to prove that $f_k(x,u,v) \geq \alpha_1|v| + \beta$, since the condition then follows by the definition of F_k. We see that his follows trivially for $|v| > 2k$, and by the assumption on L if $|v| \leq r_1$. So it in fact suffices to prove that $L(x,u,v) - \tilde{\eta} \geq \alpha_1|v| + \beta$ for $|v| \geq r_1$. By the assumption on L this in turn amounts to proving that

$$\alpha|v| + \beta - \tilde{\eta} \geq \alpha_1|v| + \beta$$

for $|v| \geq r_1$. But this was guaranteed by the choice of $\tilde{\eta}$ such that $\alpha_1 r_1/2 \geq \tilde{\eta}$. □

The following lemma is quoted precisely from [5], where the (elementary) proof may be found.

Lemma 2.6. *Let $f, g\colon R \to R \cup \{+\infty\}$ be convex, and suppose they are equal and finite on an open interval C, and let $h\colon R \to R \cup \{+\infty\}$ be the convex hull of f and g.*

Then $h = f = g$ on C.

We are now in a position to prove proposition 2.2.

Proof of Proposition 2.2. Fix $k > r_1$. Define $G_k\colon S \times R \to R \cup \{+\infty\}$ by

$$G_k(x,u,v) = \inf\{\lambda L_0(x,u,v_1) + (1-\lambda)F_k(x,u,v_2) : 0 \leq \lambda \leq 1,\ \lambda v_1 + (1-\lambda)v_2 = v\},$$

where L_0 is as given by lemma 2.4 and F_k is as given by lemma 2.5. By (2.4c), (2.5a) and lemma 2.6 we have that $G_k(x,u,v) = L(x,u,v)$ for $|v| \leq r_0$. Furthermore, G_k is convex in v by definition, and is easily seen to be Hölder in (x,u) locally uniformly in v, by the corresponding properties of L_0 and F_k, where the latter is finite.

Conditions (2.4d) and (2.5e) imply that $G_k(x,u,v) \geq \alpha_1|v|+\beta$. Conditions (2.5d) and (2.4b) imply that

$$G_k(x,u,v) \leq \begin{cases} L(x,u,v) & |v| \leq k, \\ \sigma|v| + \tilde{\gamma} & |v| > k, \end{cases}$$

and moreover that $G_k(x,u,v) \leq L(x,u,v) - \tilde{\eta}$ for $r_1 \leq |v| \leq k$.

It just remains to construct an elliptic function to add to G_k and thereby define H_k. For this purpose, we define $m_0 := 0$ and $m_n := \sum_{i=1}^n 2^{-k}$, and set $\tau_n := (m_{n+1} - m_n)/2$ for $n \geq 0$. Define $\omega_n \colon [n, n+1] \to R$ by $\omega_0(v) = \tau_0 v^2$ and for $n \geq 1$,

$$\omega_n(v) = \omega_{n-1}(n) + m_n(v-n) + \tau_n(v-n)^2,$$

and finally define $\omega \colon [0,\infty) \to [0,\infty)$ by

$$\omega(v) = \omega_n(v) \text{ for } v \in [n, n+1).$$

Then we see easily that ω is a continuous, differentiable, and convex function with $\omega'(n) = m_n$ for all $n \geq 0$. Moreover an easy induction shows that

$$\omega(n) = \omega_{n-1}(n) = \sum_{i=0}^{n-1}(m_i + \tau_i) = \sum_{i=0}^{n-1}(m_{i+1} + m_i)/2 \leq \sum_{i=0}^{n-1} 1 = n,$$

and hence that $\omega(v) \leq v$ for all $v \in [0,\infty)$. Moreover, ω satisfies the local ellipticity condition in (H) for $p = 2$.

We now define $\Psi_k \colon R \to [0,\infty)$ by

$$\Psi_k(v) = \begin{cases} 0 & v \in [-r_0, r_0], \\ \min\{\tilde{\eta}/2k, 1/k^2, \sigma\}\omega(v - r_0) & v \in (r_0, \infty), \\ \min\{\tilde{\eta}/2k, 1/k^2, \sigma\}\omega(-v - r_0) & v \in (-\infty, -r_0). \end{cases}$$

Then Ψ_k satisfies the local ellipticity condition on $(-\infty, -r_0]$ and $[r_0, \infty)$, and is identically 0 on $[-r_0, r_0]$. For $r_0 \leq v \leq k$, we see that

$$\Psi_k(v) \leq \min\{\tilde{\eta}/2k, 1/k^2\}(v - r_0) \leq \min\{\tilde{\eta}/2, 1/k\}v/k \leq \min\{\tilde{\eta}/2, 1/k\},$$

and similarly for $-k \leq v \leq -r_0$. For all $v \in [r_0, \infty)$, we see that $\Psi_k(v) \leq \sigma\omega(v - r_0) \leq \sigma v$, and similarly $\Psi_k(v) \leq \sigma|v|$ for $v \in (-\infty, -r_0]$.

We may now define $H_k \colon S \times R \to R$ by $H_k(x,u,v) = G_k(x,u,v) + \Psi_k(v)$, and $\delta := \tilde{\eta}/2$, and $\alpha_2 := 2\sigma$.

Then H_k is Hölder in (x,u) locally uniformly in v, since G_k is, and satisfies the local singular ellipticity condition since $H_k = G_k = L$ for $|v| \leq r_0$, and otherwise $H_k = G_k + \Psi_k$ where G_k is convex in v and Ψ_k is locally elliptic on $(-\infty, -r_0]$ and $[r_0, \infty)$. Hence condition (2.2a) holds.

Condition (2.2b) follows by the definition of Ψ_k and the corresponding property of G_k observed above. For condition (2.2c), we see for $r_1 \leq |v| \leq k$ that

$$H_k(x,u,v) \leq L(x,u,v) - \tilde{\eta} + \Psi_k(v) \leq L(x,u,v) - \tilde{\eta}/2 = L(x,u,v) - \delta.$$

Condition (2.2d) follows since for $|v| > k$ we have that

$$H_k(x,u,v) \leq \sigma|v| + \tilde{\gamma} + \Psi_k(v) \leq 2\sigma|v| + \tilde{\gamma} = \alpha_2|v| + \tilde{\gamma};$$

setting $\gamma = \max\{\sup_{S\times[-k,k]} H_k(x,u,v), \tilde{\gamma}\}$ gives the required statement.

For condition (2.2e) we note that

$$H_k(x,u,v) \geq G_k(x,u,v) \geq \alpha_1|v| + \beta,$$

and finally for condition (2.2f), we see that for $r_0 \leq |v| \leq r_1$,

$$H_k(x,u,v) \leq L(x,u,v) + \Psi_k(v) \leq L(x,u,v) + k^{-1},$$

as required. □

Proof of Theorem 1.1. We select $\epsilon_0 > 0$ in such a way that if $(a,A) \in G$, $l \in \partial_v L(a,A,0)$ then the integrand $\tilde{L}(x,u,v) := L(x,u,v) - lv$ achieves its minima in v for each $(x,u) \in [a-\epsilon_0, a+\epsilon_0] \times [A-\epsilon_0, A+\epsilon_0]$. In this case we also have $\tilde{L}(x,u,v) \geq \alpha|v| + \beta$, $\alpha > 0$ for (x,u) under consideration. Note that minimizers in the problems (1.1-2) with L and $\tilde{L}$ integrands are the same since the integral of $l\dot{u}$ is the same for all functions having the same boundary conditions (1.2). As for $\delta_0 > 0$ it will be selected later and will satisfy the inequality $\delta_0 \leq \epsilon_0$.

Take $N \geq M$ and consider a minimization problem (1.1-2) with $(a,A) \in G$, $|a-b| \leq \delta \leq \delta_0$, $|B-A|/|b-a| \leq M$ in the class of N-Lipschitz u with $|u(\cdot) - A| \leq \epsilon \leq \epsilon_0$ in $[a,b]$. By Lemma 1.5 solutions u of such a problem exist. Now we will study regularity of such solutions u. For this we will use Lemma 2.1 since u satisfy (2.3) in any interval $[x_1,x_2] \subset [a,b]$.

Let $[x_3,x_4] \subset [x_1,x_2]$. Then there exists $k \in N$ such that

$$\frac{|x_2 - x_1|}{e^k} \leq |x_4 - x_3| \leq \frac{|x_2 - x_1|}{e^{k-1}}$$

and we can select a collection of imbedded intervals $[x_1^i, x_2^i]$, $i \in \{0,\ldots,k-1\}$, such that $x_1^0 = x_1$, $x_2^0 = x_2$, $|x_2^i - x_1^i| = |x_2^{i-1} - x_1^{i-1}|/e$ for all i under consideration and $[x_3,x_4] \subset [x_1^{k-1}, x_2^{k-1}]$, where $|x_4 - x_3| \geq |x_1^{k-1} - x_2^{k-1}|/e$. By (2.5) we then have

$$\begin{aligned}
&\Big|\frac{u(x_2)-u(x_1)}{x_2-x_1} - \frac{u(x_4)-u(x_3)}{x_4-x_3}\Big| \\
&\leq \sum_{i=1}^{k-1}\Big|\frac{u(x_2^i)-u(x_1^i)}{x_2^i-x_1^i} - \frac{u(x_2^{i-1})-u(x_1^{i-1})}{x_2^{i-1}-x_1^{i-1}}\Big| \\
&\quad + \Big|\frac{u(x_2^{k-1})-u(x_1^{k-1})}{x_2^{k-1}-x_1^{k-1}} - \frac{u(x_4)-u(x_3)}{x_4-x_3}\Big| \qquad (2.10)\\
&\leq \sum_{i=1}^{k} 2e\Big[\frac{2c(N)(1+N)^{\alpha(N)}}{\mu(N)}\Big]^{1/p(N)}\Big|\frac{x_2-x_1}{e^i}\Big|^{\alpha(N)/p(N)} \\
&\leq \sum_{i=1}^{k} 2e\Big[\frac{2c(N)(1+N)^{\alpha(N)}}{\mu(N)}\Big]^{1/p(N)}\frac{1}{e^{i\alpha(N)/p(N)}}|x_2-x_1|^{\alpha(N)/p(N)},
\end{aligned}$$

where $c(N)$, $\alpha(N)$, $p(N)$, $\mu(N)$ correspond in (H) to the compact set $\tilde{K} := \tilde{G} \times [-N,N]$ with $\tilde{G} := \{(x,u) : |x-a| \leq \epsilon_0, |u-A| \leq \epsilon_0, (a,A) \in G\}$.

Finally we obtain that for $[x_3, x_4] \subset [x_1, x_2]$ we have

$$\Big|\frac{u(x_2)-u(x_1)}{x_2-x_1} - \frac{u(x_4)-u(x_3)}{x_4-x_3}\Big| \le \tilde{c}(N)|x_2-x_1|^{\gamma(N)}, \tag{2.11}$$

where

$$\tilde{c}(N) = \sum_{i=1}^{\infty} 2e\Big[\frac{2c(N)(1+N)^{\alpha(N)}}{\mu(N)}\Big]^{1/p(N)} \frac{1}{e^{i\alpha(N)/p(N)}}, \gamma(N) = \alpha(N)/p(N). \tag{2.12}$$

This allows us to state that u is differentiable everywhere. In fact if k is a derivative number at x_0 then by (2.11) we obtain $\dot{u}(x_0) = k$. Again by (2.11) we infer that

$$|\dot{u}(y) - \dot{u}(x)| \le 2\tilde{c}(N)|y-x|^{\gamma(N)}, \tag{2.13}$$

i.e. that $u \in C^{1,\gamma(N)}$.

Note now that for $\delta_0 > 0$ sufficiently small we can show that $||\dot{u}||_{C[a,b]} \le M+1$. Indeed, due to the inequality $|B-A|/|b-a| \le M$ there exists $x_0 \in [a,b]$ such that $|\dot{u}(x_0)| \le M$. Consider the maximal interval $[x_1,x_2] \subset [a,b]$ for which $x_0 \in [x_1,x_2]$ and $|\dot{u}(x)| \le M+1$ for $x \in [x_1,x_2]$. For y in this interval we have due to (2.13)

$$|\dot{u}(y) - \dot{u}(x_0)| \le 2\tilde{c}(M+1)|x_2-x_1|^{\gamma(M+1)},$$

i.e.

$$|\dot{u}(y)| \le M + 2\tilde{c}(M+1)|x_2-x_1|^{\gamma(M+1)}.$$

Therefore $|\dot{u}(y)| \le M+1$ if $2\tilde{c}(M+1)|x-y|^{\gamma(M+1)} \le 1$. This determines the choice of δ_0:

$$\delta_0 \le (1/2\tilde{c}(M+1))^{1/\gamma(M+1)}. \tag{2.14}$$

We proved that for $\delta \le \delta_0$ with $\delta_0 \le \epsilon_0$ satisfying (2.14) all solutions of the problems under consideration have derivatives bounded in C-norm by $M+1$ and all solutions are bounded in $C^{1,\gamma(M+1)}$-norm. This is independent of $N \ge M$. Therefore solutions in the class of Lipschitz functions exist and for each such solution u we have $||\dot{u}||_C \le M+1$. It remains to prove that solutions in the class of Lipschitz functions are the only solutions in the wider class of $W^{1,1}$ functions.

We have proved that for a fixed compact $G \subseteq R^2$ and $M > 0$, there exist $\epsilon_0 > 0$ and $\delta_0 \in (0, \epsilon_0]$ such that for all $0 < \delta \le \delta_0$ and $0 < \epsilon \le \epsilon_0$, the following holds. For any $(a,A) \in G$ and (b,B) satisfying $|a-b| \le \delta$, $|B-A| \le \epsilon$, and $|B-A|/|b-a| \le M$, the minimizer u in the class of Lipschitz functions $u\colon [a,b] \to R$ satisfying $u(a) = A$, $u(b) = B$, and $|u(x) - A| \le \epsilon$ for all $x \in [a,b]$ of the functional J exists and satisfies $\|\dot{u}\|_C \le M+1$. It just remains to prove that the Lipschitz minimizers are the only minimizers over $W^{1,1}[a,b]$ for such a problem. Let u be such a minimizer over Lipschitz functions.

Suppose for a contradiction that there exists an admissible $\tilde{u} \in W^{1,1}[a,b]$ such that $J(\tilde{u}) \le J(u)$ and $\|\dot{u}\|_{L^\infty} = \infty$. We recall that we may assume that there exist $\alpha > 0$ and $\beta \in R$ such that

$$L(x,u,v) \ge \alpha|v| + \beta$$

for all $(x,u) \in [a-\delta_0, a+\delta_0] \times [A-\epsilon_0, A+\epsilon_0]$. We apply proposition 2.2 with these values of α, β, compact set $S = [a-\delta_0, a+\delta_0] \times [A-\epsilon_0, A+\epsilon_0]$, and $r_0 \ge M+1$, to get functions $H_k\colon S \times R \to R$ as in the lemma, for $k > r_1 > r_0$.

Mimicking the proof for L with the integrand H_k, we see that a minimizer u_k exists over the class of Lipschitz functions $u\colon [a,b] \to R$ satisfying $u(a) = A$, $u(b) = B$, and $|u(x) - A| \le \epsilon$ of the functional $\int_a^b H_k(x,u,\dot{u})$, and satisfies $\|\dot{u}_k\|_C \le M+1$. Using condition (2.2b) we see that

$$\begin{aligned}\int_a^b H_k(x,u_k,\dot{u}_k) \le \int_a^b H_k(x,u,\dot{u}) &= \int_a^b L(x,u,\dot{u}) \\ &\le \int_a^b L(x,u_k,\dot{u}_k) \\ &= \int_a^b H_k(x,u_k,\dot{u}_k),\end{aligned}$$

so $\int_a^b H_k(x,u_k,\dot{u}_k) = \int_a^b H_k(x,u,\dot{u}) = \int_a^b L(x,u,\dot{u})$, and u is a minimizer for the integrand H_k over admissible Lipschitz functions. There exist admissible Lipschitz functions $v_i \in W^{1,\infty}[a,b]$ such that $v_i \to \tilde{u}$ in $W^{1,1}[a,b]$ and, owing to the linear growth conditions

$$\alpha_1|v| + \beta \le H_k(x,u,v) \le \alpha_2|v| + \gamma,$$

we have as $i \to \infty$ that

$$\int_a^b H_k(x,v_i,\dot{v}_i) \to \int_a^b H_k(x,\tilde{u},\dot{\tilde{u}}).$$

Now, by our assumption on the derivative of $\tilde{u}$, we know that the set $\{|\dot{\tilde{u}}| > r_1\}$ has positive measure. Then by conditions (2.2f), (2.2b), (2.2c), and (2.2d), we see that

$$\begin{aligned}\int_a^b H_k(x,\tilde{u},\dot{\tilde{u}}) &\le \int_{\{|\dot{\tilde{u}}|\le r_1\}} (L(x,\tilde{u},\dot{\tilde{u}}) + k^{-1}) + \int_{\{r_1<|\dot{\tilde{u}}|\le k\}} (L(x,\tilde{u},\dot{\tilde{u}}) - \delta) \\ &\qquad + \int_{\{|\dot{\tilde{u}}|>k\}} (\alpha_2|\dot{\tilde{u}}| + \gamma) \\ &\to \int_{\{|\dot{\tilde{u}}|\le r_1\}} L(x,\tilde{u},\dot{\tilde{u}}) + \int_{\{|\dot{\tilde{u}}|>r_1\}} (L(x,\tilde{u},\dot{\tilde{u}}) - \delta) \quad \text{as } k \to \infty \\ &= \int_a^b L(x,\tilde{u},\dot{\tilde{u}}) - \delta \text{meas}(\{|\dot{\tilde{u}}| > r_1\}).\end{aligned}$$

But on the other hand,

$$\int_a^b L(x,u,\dot{u}) = \int_a^b H_k(x,u_k,\dot{u}_k) \le \int_a^b H_k(x,v_i,\dot{v}_i)$$

for each i and each k. So

$$\begin{aligned}\int_a^b L(x,u,\dot{u}) &\le \lim_{i\to\infty} \int_a^b H_k(x,v_i,\dot{v}_i) \\ &= \int_a^b H_k(x,\tilde{u},\dot{\tilde{u}}) \\ &\le \int_a^b L(x,\tilde{u},\dot{\tilde{u}}) - \delta \text{meas}(\{|\dot{\tilde{u}}| > r_1\}),\end{aligned}$$

which contradicts the choice of $\tilde{u}$ as a minimizer for J over $W^{1,1}[a,b]$.

The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.1 would be simplified if instead of Proposition 2.2 we would use Lemma 2.7.

Lemma 2.7. *Let $L(x,u,v); [a,b]\times[A,B]\times R\to R$ satisfy condition (H) and $L\geq \alpha|v|+\beta$, $\alpha>0$. Let also $N>0$.*

Then there exists $G(x,u,v):[a,b]\times[A,B]\times R\to R$ which also satisfy condition (H), $L=G$ for $|v|\leq N$, $G\leq L$ everywhere when $G<L$ for $|v|$ sufficiently large. Moreover G has linear growth at infinity:

$$c_1|v|+c_2\leq G(x,u,v)\leq c_3|v|+c_4,\ c_3\geq c_1>0.$$

Unfortunatelly we were unable to prove this lemma. Note that another proof of Theorem 1.1 could be taken from [14].

Proof of Theorem 1.4. Let u be a minimizer with graph in K. Let $x_0\in[a,b]$ be such that for $x_1^n\to x_0$, $x_2^n\to x_0$ with $x_0\in[x_1^n,x_2^n]$, $n\in N$, we have

$$\liminf_{n\to\infty}\Big|\frac{u(x_2^n)-u(x_1^n)}{x_2^n-x_1^n}\Big|\leq M<\infty. \tag{2.15}$$

Then by Theorem 1.1 u is C^1-regular in a neighbourhood of x_0. Therefore u is C^1-regular in an open set of full measure. At other points x_0, where (2.15) does not hold, we obviously have that $\dot u(x_0)=\infty$ or $\dot u(x_0)=-\infty$. We show now that $\dot u:[a,b]\to\bar R$ is continuous. For this it is enough to show that if $\dot u(x_0)=\infty$ then $\dot u(x)\to\infty$ for $x\to x_0$ (the case of $-\infty$ can be considered analogously). Assume that $x_n\to x_0$ and $\dot u(x_n)\leq M<\infty$, $n\in N$. Since $((u(x_0)-u(x_n))/(x_0-x_n)\to\infty$ we infer existence of $y_n\in[x_n,x_0]$ such that $((u(y_n)-u(x_n))/(y_n-x_n)=M+2$, $n\in N$. Since $|y_n-x_n|\to 0$ by Theorem 1.1 we have that the oscillations of the derivative in the interval $[x_n,y_n]$ do not exceed 1 for $n\in N$ sufficiently large. Then $\dot u(y)\geq M+1$ for $y\in[x_n,y_n]$ with such $n\in N$. But $\dot u(x_n)\leq M$ which gives the contradiction. This way we proved that $\dot u(x_n)\to\infty$ as $x_n\to x_0$, i.e. that $\dot u:[a,b]\to\bar R$ is a continuous function.

The last assertion we have to prove is that derivatives of the minimizers under consideration belong to a certain conditionally equa-continuous family, i.e. that for each $M>0$ and $\epsilon>0$ there exists $\delta=\delta(M,\epsilon)>0$ such that for a minimizer u we have $|\dot u(y)-\dot u(x_0)|\leq\epsilon$ provided $|\dot u(x_0)|\leq M$ and $|y-x_0|\leq\delta$. Indeed, we can consider the maximal interval I such that $x_0\in I$ and $|\dot u(y)|\leq M+1$ for $y\in I$. In this interval the modulus of continuity of $\dot u$ does not exceed $w(\epsilon)=c\epsilon^\gamma$ by Theorem 1.1. Then we have

$$|\dot u(y)-\dot u(x_0)|\leq c|y-x_0|^\gamma,$$

which holds for all y with $c|y-x_0|^\gamma\leq 1$, i.e. in the case when $|I|\leq(1/c)^{1/\gamma}$. Therefore we can take

$$\delta(M,\epsilon):=\min\{(1/c)^{1/\gamma},(\epsilon/c)^{1/\gamma}\}.$$

Proof of Theorem 1.4 is complete. □

3. Theorem 1.8

We will use the following lemma about the properties of a conditionally equa-continuous family, the proof of which can be found in [17].

Lemma 3.1. *Let Ξ be a conditionally equa-continuous family and let $\delta(M,\epsilon)$ be the associated function (modulus of conditional equa-continuity). Then*

1) *if ξ_n is a sequence in the family which is bounded in L^1 then there exists a subsequence ξ_k that converges uniformly in any compact set of a certain open set of full measure to a function ξ_0. In this case $\Xi \cup \{\xi_0\}$ is also conditional equa-continuous family with the same modulus δ;*
2) *if a conditional equa-continuous family consists of functions which are weakly compact in L^1 then this family is also a strongly compact in L^1;*
3) *if $\xi \in \Xi$ is such that $\xi(x) \geq M > 0$ ($\xi(x) \leq -M$) then for y with $|x-y| \leq \delta(M/2, M/2)$ we have $\xi(y) \geq M/2$ ($\xi(y) \leq -M/2$).*

Proof of theorem 1.8. Let for boundary conditions (a_0, A_0, b_0, B_0) all minimizers have bounded derivatives at a_0 and b_0. Then there exists $M > 0$ such that for every minimizer u we have $|\dot{u}(a_0)| \leq M$, $|\dot{u}(b_0)| \leq M$. Otherwise we can find a sequence of minimizers $u_n : [a_0, b_0] \to R$ such that $|\dot{u}_n(a_0)| \to \infty$ (the case of b_0 can be considered analogously). Then the limit function u_0 is also a minimizer by lemma 1.5. By Lemma 3.1 we have $\dot{u}_0(a_0) = \infty$ which is a contradiction. Therefore we proved the existence of $M > 0$ such that $|\dot{u}(a_0)| \leq M$, $|\dot{u}(b_0)| \leq M$ for all minimizers u of the problem.

Then there exists $\eta > 0$ such that $|\dot{u}(x)| \leq M + 1$ for $x \in [a_0, a_0 + \eta]$ and $x \in [b_0 - \eta, b_0]$ and all minimizers u since their derivatives belong to a conditionally equa-continuous family by Theorem 1.4.

Note that S is continuous at (a_0, A_0, b_0, B_0). In fact it is lower semicontinuous by the lower semicontinuity lemma 1.5. If we have $(a_n, A_n, b_n, B_n) \to (a_0, A_0, b_0, B_0)$ then we can take $u_n(x) = u(x)$ for $x \in [a_0 + \eta_n, b_0 - \eta_n]$, $u_n(a_n) = A_n$, $u_n(b_n) = B_n$ and u_n be affine in the intervals $[a_n, a_0 + \eta_n]$, $[b_0 - \eta_n, b_n]$. Then obviously $J(u_n; [a_n, b_n]) \to J(u; [a_0, b_0])$ for appropriate $\eta_n \to 0$, which implies continuity of S at (a_0, A_0, b_0, B_0).

Now we can state that for all (a_n, A_n, b_n, B_n) sufficiently close to (a_0, A_0, b_0, B_0) and the minimizer u_n associated with the n-th conditions we have $|\dot{u}(a_n)| \leq M+1$, $|\dot{u}(b_n)| \leq M + 1$. Indeed, $\dot{u}_n$ converges weakly to $\dot{u}_0$ where u_0 is a minimizer in the problem with the 0-th boundary datum because of continuity of S at (a_0, A_0, b_0, B_0). By Lemma 3.1 then $\dot{u}_n(a_n) \to \dot{u}_0(a_0)$, $\dot{u}_n(b_0) \to \dot{u}(b_0)$. This implies the result.

Since all minimizers associated with the n-datum sufficiently close to 0-th datum have derivatives in a certain conditionally equa-continuous family we obtain that for appropriate $\eta > 0$ there holds $|\dot{u}_n(x)| \leq M + 2$ for $x \in [a_n, a_n + \eta]$, $x \in [b_n - \eta, b_n]$, $n \in N$.

Now it is easy to derive Lipschitz continuity of S at (a_0, A_0, b_0, B_0). Indeed if u_n is a minimizer with $n-th$ boundary conditions then $u_n \to u_0$, where u_0 is a minimizer with 0-boundary conditions. Now we can define $\tilde{u}_n = u_0$ for $x \in [a_n + \eta_n, b_n - \eta_n]$, $\tilde{u}_n(a_n) = A_n$, $\tilde{u}(b_n) = B_n$ and $\tilde{u}_n$ be affine in the intervals $[a_n, a_n + \eta_n]$, $[b_n - \eta_n, b_n]$.

We have that $|\dot{\tilde{u}}_n(x)| \leq 2M+5$ in those intervals if

$$2\{|a_0-a_n|+|b_0-b_n|+|A_0-A_n|+|B_0-B_n|\}=\eta_n.$$

Indeed, in the interval $[a_n, a_n+\eta_n]$ the modulus of the derivative of $\tilde{u}_n$ is equal to $|u_0(a_n+\eta_n)-A_n|/\eta_n$ and we have

$$\begin{aligned}\frac{|u_0(a_n+\eta_n)-A_n|}{\eta_n} &\leq \frac{|u_0(a_n+\eta_n)-A_0|}{\eta_n}+\frac{|A_0-A_n|}{\eta_n}\\ &\leq 2(M+2)+1=2M+5.\end{aligned}$$

Analogously $|\dot{\tilde{u}}_n(x)| \leq 2M+5$ for $x \in [b_n-\eta_n, b_n]$.

Therefore

$$\begin{aligned}&S(a_n,A_n,b_n,B_n)-S(a_0,A_0,b_0,B_0)\\ &\leq J(\tilde{u}_n;[a_n,b_n])-J(u_0;[a_0,b_0])\\ &\leq \Big|\int_{a_n}^{a_n+\eta_n} L(x,\tilde{u}_n(x),\dot{\tilde{u}}_n(x))dx\Big|\\ &\quad+\Big|\int_{b_n-\eta_n}^{b_n} L(x,\tilde{u}_n(x),\dot{\tilde{u}}_n(x))dx\Big|\\ &\quad+\Big|\int_{a_0}^{a_n+\eta_n} L(x,u_0(x),\dot{u}_0(x))dx\Big|\\ &\quad+\Big|\int_{b_n-\eta_n}^{b_0} L(x,u_0(x),\dot{u}_0(x))dx\Big|\\ &\leq 6\eta_n \max\{|L(x,u,v):(x,u)\in K, |v|\leq 2M+5\}\\ &\leq const\{|a_n-a_0|+|b_n-b_0|+|A_n-A_0|+|B_n-B_0|\}.\end{aligned}$$

Here $K \subset R^2$ is a compact set such that it contains graphs of all functions under consideration. Analogously

$$S(a_0,A_0,b_0,B_0)-S(a_n,A_n,b_n,B_n) \leq const\{|a_n-a_0|+|b_n-b_0|+|A_n-A_0|+|B_n-B_0|\}.$$

Therefore Lipschitz continuity of S at (a_0,A_0,b_0,B_0) holds.

The last point to prove is that if S is Lipschitz continuous at (a_0,A_0,b_0,B_0) then all associated minimizers are bounded in the neighborhoods of a_0 and b_0 in C^1-norm. Assume otherwise, i.e. we can assume without loss of generality that $\dot{u}(b_0)=\infty$. Then we can consider (a_0,A_0,b_0,B_n) boundary conditions with $B_n<B_0$ and u_n such that if $u(x_n)=B_n$ and $u(x)\geq B_n$ for $x\in[x_n,b_0]$ then $u_n(x)=B_n$ for $x\in[x_n,b_0]$, for $x\in[a_0,x_n]$ we assume $u_n(x)=u(x)$. We have

$$\begin{aligned}S(a_0,A_0,b_0,B_0)-S(a_0,A_0,b_0,B_n) &\geq \int_{x_n}^{b_0} L(x,u(x),\dot{u}(x))dx\\ &\quad-\int_{x_n}^{b_0} L(x,u_n(x),\dot{u}_n(x))dx\\ &\geq \int_{x_n}^{b_0}\theta(\frac{u(b_0)-u(x_n)}{|b_0-x_n|})dx-c(b_0-x_n),\end{aligned}$$

where $\theta: R\to R$ is a convex function with superlinear growth.

Then

$$\{S(a_0, A_0, b_0, B_0) - S(a_0, A_0, b_0, B_n)\}/|B_0 - B_n| \to \infty,\ n \to \infty.$$

This is a contradiction with the assumption that S is Lipschitz continuous at the point (a_0, A_0, b_0, B_0).

The proof is completed. □

Remark 3.2. It follows from the proof that if S is Lipschitz at certain point (a_0, A_0, b_0, B_0) then it is also uniformly Lipschitz in an appropriate neigborhood of the point.

The advantage of Theorem 1.8 is that it is possible to prove Lipschitz continuity of S for some classes of integrands: when $L(x, u, v)$ is independent on x or u, or when it is jointly convex in (u, v), see [14, §5], as well as in some other cases. Note also that Lipschitz continuity of minimizers was proved by A.Cellina in [4] in the case of autonomous $L = L(u, v)$ under more general growth conditions.

4. Theorem 1.9

Proof of Theorem 1.9. The proof will rely severely on Theorem 1.4, i.e. that derivatives of minimizers with graphs lying in a certain compact set $K \subset R^2$ belong to a conditionally equa-continuous family.

Let U be the universal singular set associated with L. We will consider subsets of U: $L^\epsilon_{\rho,+}$, $L^\epsilon_{\rho,-}$, $R^\epsilon_{\rho,+}$, $R^\epsilon_{\rho,-}$, where $\rho > 0$, $\epsilon > 0$. $L^\epsilon_{\rho,+}$ ($L^\epsilon_{\rho,-}$) is a subset of U such that if $(x_0, u_0) \in L^\epsilon_{\rho,+}$ ($L^\epsilon_{\rho,-}$) then for each $\delta < \rho$ there exists boundary datum $a = x_0 - \delta$, $b = x_0$, $B = u_0$ such that the corresponding minimizer u has the properties $J(u; [a, b]) \le 1/\epsilon$, $\dot u(b) = \infty$ ($\dot u(b) = -\infty$). Analogously $R^\epsilon_{\rho,+}$ ($R^\epsilon_{\rho,-}$) is the set of $(x_0, u_0) \in U$ such that for each $\delta < \rho$ there exists a problem (1.1-2) with $a = x_0$, $A = u_0$, $b = x_0 + \delta$ with $J(u; [a, b]) \le 1/\epsilon$ and with $\dot u(a) = \infty$ ($\dot u(a) = -\infty$).

We will prove that given a closed ball $\bar B(r)$ centred at the origin and with radius $r > 0$ the sets $L^\epsilon_{\rho,+} \cap \bar B(r)$, $L^\epsilon_{\rho,-} \cap \bar B(r)$, $R^\epsilon_{\rho,+} \cap \bar B(r)$, $R^\epsilon_{\rho,-} \cap \bar B(r)$ are closed and of zero 2-d Lebesgue measure. Then they are also nowhere dense. This result is sufficient to claim that U is of first Baire category and of zero 2-d Lebesgue measure.

We prove only that $L^\epsilon_{\rho,+} \cap \bar B(r)$ is closed and of zero 2-d Lebesgue measure. For the other sets the proof is analogous.

Let $(x_n, u_n) \in L^\epsilon_{\rho,+} \cap \bar B(r)$ and $(x_n, u_n) \to (x_0, u_0)$, $n \to \infty$. We have to show that $(x_0, u_0) \in L^\epsilon_{\rho,+} \cap \bar B(r)$. For sufficiently small $\nu > 0$ we consider minimizers $u_n : [x_n - \rho + \nu, x_n] \to R$ associated with (x_n, u_n). Then $J(u_n; [x_n - \rho + \nu; x_n]) \le 1/\epsilon$. Switching if necessary to a subsequence we can assume that u_n converge uniformly to a function $u_0 : [x_0 - \rho + \nu, x_0] \to R$. By the properties of conditionally equa-continuous families, see Lemma 3.1, we have $\dot u_0(x_0) = \infty$ and there also exists $\bar x \in [x_0 - \rho + \nu, x_0 - \rho + 2\nu]$ such that $\dot u_n(\bar x) \to \dot u_0(\bar x)$, where $|\dot u_0(\bar x)| < \infty$. It turns out that there is a minimizer $\bar u$ in the problem (1.1-2) with the boundary conditions $(a, A, b, B) = (\bar x, \bar u_0(\bar x), x_0, u_0)$ such that $\dot{\bar u}(b) = \infty$. This result is sufficient to state that $L^\epsilon_{\rho,+} \cap \bar B(r)$ is closed.

We prove this fact by contradiction. If there are no such a minimizer then all minimizers of the (a, A, b, B)-problem has bounded derivative at b and if $\bar u$ is a minimizer then $J(\bar{u}) < J(u_0)$. In particular

$$\liminf_{n\to\infty} J(u_n;[\bar{x},x_n]) \geq J(u_0;[\bar{x},x_0]) > J(\bar{u};[\bar{x},x_0]). \tag{4.1}$$

But we will show existence of functions $w_n : [\bar{x}-\eta_n, x_n] \to R$ such that $J(w_n;[\bar{x}-\eta_n,x_n]) \to J(\bar{u};[\bar{x},x_0])$ with $\eta_n \to +0$ and w_n having the same boundary conditions as u_n at $\bar{x}-\eta_n$ and x_n. This will be a contradiction with (4.1) which will allow us to state that $(x_0,u_0) \in L^{\epsilon}_{\rho,+} \cap \bar{B}(r)$.

Let $\eta > 0$ be sufficiently small. Define $w_n : [\bar{x}-\eta; x_n] \to R$ be equal to $\bar{u}$ in $[\bar{x}, x_n-\eta]$ and affine in each interval $[\bar{x}-\eta,\bar{x}]$, $[x_n-\eta,x_n]$, where $w_n(\bar{x}-\eta) = u_n(\bar{x}-\eta)$, $w_n(x_n) = u_n(x_n)$. Then for all $n \in N$ sufficiently large and an $\eta > 0$ sufficiently small the derivative of w_n in the intervals $[\bar{x}-\eta,\bar{x}]$, $[x_n-\eta,x_n]$ are equa-bounded in modulus. Here we use the fact that the derivatives of u_n in the intervals $[\bar{x}-\eta,\bar{x}]$, $n \in N$, are equa-bounded since they belong to a conditionally equa-continuous family. Finally we can take $\eta_n \to 0$ instead of η keeping the derivatives in the intervals of affinity equa-bounded in modulus. Therefore $J(w_n;[\bar{x}-\eta_n,x_n]) \to J(\bar{u};[\bar{x},x_0])$ which gives the desired contradiction with (4.1).

We have established the fact that $L^{\epsilon}_{\rho,+} \cap \bar{B}(r)$ is a closed set. Now we have to prove that this set has zero 2-d Lebesgue measure. It is enough to show that if x is fixed then the set $G := \{u \in [-r,r] : (x,u) \in (L^{\epsilon}_{\rho,+} \cap \bar{B}(r))\}$ has zero 1-d Lebesgue measure. Let u_0 be such a point. Consider the set $[u_0-\delta,u_0+\delta]$. We can find $\rho > 0$ sufficiently small and A such that $A < u(x-\rho)$ for all minimizers associated with $[u_0-\delta,u_0+\delta] \cap G$. We can take $\rho > 0$ so small that if u is a minimizer in the problem (1.1-2) with $a = x-\rho$, A, $b = x$, $B \in [u_0-\delta,u_0+\delta]$ then $\dot{u}(\cdot) \geq M$, where

$$\min\{L(x,u,v) : (x,u) \in B(r), |v| \geq M\} > \max\{|L(x,u,0)| : (x,u) \in B(r)\}. \tag{4.2}$$

For boundary conditions (a,A,b,B) under consideration we can consider a function $I(B) = \inf\{J(u;[a,b]) : u \in W^{1,1}[a,b], u(a) = A, u(b) = B\}$, $B \in [u_0-\delta,u_0+\delta]$. This function is monotone increasing. In fact if $B_1, B_2 \in [u_0-\delta,u_0+\delta]$ and $B_2 > B_1$ and if u_2 is a minimizer associated with B_2 then there exists $x_2 \in [a,b]$ such that $u_2(x_2) = B_1$, $u_2(x) > B_1$ for $x \in [x_2,b]$. We can assume $w = u_2$ in $[a,x_2]$, $w(x) = B_1$ in $[x_2,b]$. Then because of (4.2) we have $I(B_2) > J(w) \geq I(B_1)$, i.e. I in indeed an increasing function. At the points $B_2 \in [u_0-\delta,u_0+\delta] \cap G$ we have $I'(B_2-0) = \infty$. Indeed

$$\begin{aligned} I(B_2) - I(B_1) &\geq \int_{x_2}^{b} L(x,u_2(x),\dot{u}_2(x))dx - \int_{x_2}^{b} L(x,w(x),0)dx \\ &\geq \int_{x_2}^{b} \theta(\frac{B_2-B_1}{b-x_2})dx - const(b-x_2), \end{aligned} \tag{4.3}$$

where $\theta : R \to R$ is a convex function with superlinear growth. Because of superlinear growth of θ and the equality $\dot{u}_2(b) = \infty$ we have in (4.3)

$$(I(B_2) - I(B_1))/(B_2-B_1) \to \infty, \ B_1 \to B_2 - 0.$$

However monotone functions can have infinite derivative only is a set of values of zero measure. Therefore u_0 is not a Lebesque point of G. This proves that the set G has zero 1-d Lebesgue measure. Therefore the set $L^{\epsilon}_{\rho,+} \cap \bar{B}(r)$ has zero 2-d Lebesgue measure.

Proof is completed. □

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