



A NOTE ON SELF IMPROVEMENT OF POINCARÉ-SOBOLEV INEQUALITIES VIA GARSIA-RODEMICH SPACES

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ABSTRACT. We use the characterization of weak type inequalities via Garsia-Rodemich conditions to show self improving properties of Poincaré-Sobolev inequalities in a very general context.

1. INTRODUCTION AND SUMMARY

In this note we develop a new method to prove self-improving inequalities involving oscillations through the use of Garsia-Rodemich spaces. Although we shall apply the method to known self improving results concerning classical Sobolev-Poincaré inequalities¹, we believe that our method can be useful in other contexts as well.

One of basic problems we face here can be simply described as follows: How can we extract information about the size of a function in terms of its oscillations? The main ideas are classical and the fundamentals go back, at least, to the seminal papers of Calderón-Zygmund.

The origin of the self improving results considered in this note goes back to the classical paper by John-Nirenberg [10], where they introduced the space BMO . Their methods were later refined by many authors (cf. [5] and the references therein). Somewhat less known are some ideas that were developed by Garsia-Rodemich [8]. One possible reason that the methods of [8] are less known to the community of self improvers is the fact that the main objective of [8] lies elsewhere, moreover, the relevant results for us are only sketched at the end of [8], and then only in the one dimensional case.

We addressed some of these issues in [18] where, in particular, we extended the Garsia-Rodemich embedding to the n -dimensional case (cf. Theorem 2.1).

In this note we use Garsia-Rodemich spaces to study Poincaré inequalities in a very general context.

We shall start by recalling a construction of John-Nirenberg [10]. Let $Q_0 \subset \mathbb{R}^n$, be a fixed cube², and let

$$P(Q_0) = \{\{Q_i\}_{i \in N} : \text{countable families of subcubes } Q_i \subset Q_0,$$

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¹We refer to [18] for the corresponding study of self-improvement of BMO inequalities.

²A “cube” in this paper will always mean a cube with sides parallel to the coordinate axes.

with pairwise disjoint interiors}

Let $1 < p < \infty$. The John-Nirenberg spaces $JN_p(Q_0)$ consist of all functions $f \in L^1(Q_0)$ such that³ (cf. [10], [22])

$$(1.1) \quad \|f\|_{JN_p(Q_0)} = \sup_{\{Q_i\}_{i \in N} \in P(Q_0)} \left\{ \sum_i |Q_i| \left(\frac{1}{|Q_i|} \int_{Q_i} |f - f_{Q_i}| dx \right)^p \right\}^{1/p} < \infty.$$

John-Nirenberg [10] go on to show that

$$(1.2) \quad JN_p(Q_0) \subset L(p, \infty)(Q_0).$$

Thus, the $JN_p(Q_0)$ condition implies the following “self-improvement”: For $f \in L^1$, control of its L^1 oscillations, $\frac{1}{|Q|} \int_Q |f - f_Q| dx$, as prescribed by (1.1), allows us to conclude that f belongs to the better space $L(p, \infty)$ (cf. (1.2)). When $p \rightarrow \infty$, then, informally, we have $JN_p(Q_0) \rightarrow BMO(Q_0)$, and the corresponding limiting self-improvement is expressed by the well known John-Nirenberg Lemma [10]: Functions in BMO are exponentially integrable. Again informally, this later result corresponds to let $p \rightarrow \infty$ in (1.2), and can be formulated⁴ as

$$(1.3) \quad BMO(Q_0) \subset L(\infty, \infty)(Q_0).$$

Roughly speaking, the embeddings (1.2), (1.3), are the mechanism used by John-Nirenberg to prove self improvement when we have control of the oscillations (cf. [10], [22] and the more recent expansive survey given in [5], which contains references to many important contributions to the topic treated in this note).

In the seventies, Garsia-Rodemich [8] introduced the closely related spaces $GaRo_p(Q_0)$, $1 < p \leq \infty$, whose definition we now recall. We shall say that $f \in GaRo_p(Q_0)$, if and only if $f \in L^1(Q_0)$, and $\exists C > 0$ such that for all $\{Q_i\}_{i \in N} \in P(Q_0)$ we have

$$(1.4) \quad \sum_i \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| dx dy \leq C \left(\sum_i |Q_i| \right)^{1/p'}, \text{ where } 1/p' = 1 - 1/p,$$

and we let

$$\|f\|_{GaRo_p(Q_0)} = \inf\{C : \text{such that (1.4) holds}\}.$$

It is readily seen that (cf. [18]),

$$(1.5) \quad JN_p(Q_0) \subset GaRo_p(Q_0).$$

³Here $f_Q = \frac{1}{|Q|} \int_Q f dx$.

⁴We use the somewhat unconventional notation $L(\infty, \infty)$ (also often denoted by W) to define the weak- L^∞ space (cf. [4])

$$L(\infty, \infty) = \{f : \sup_t \{f^{**}(t) - f^*(t)\} < \infty\}.$$

Here f^* denotes the non-increasing rearrangement of f and $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds$, (cf. [4]). As was shown in [3], $L(\infty, \infty)$ is the “rearrangement invariant hull” of BMO . For further generalizations cf. [20].

A remarkable result of Garsia-Rodemich shows that (cf. [8], and [18] for the n -dimensional version of the result that we use here) as sets,

$$(1.6) \quad GaRo_p(Q_0) = L(p, \infty)(Q_0).$$

Therefore, the gist of the matter is that the weak type spaces $L(p, \infty)$ themselves can be characterized by oscillation conditions! In other words, the underlying method to prove (1.6) provides an effective method to compute the weak type norm of a function if we have control of its oscillations, and avoids the (somewhat harder!) intermediate step of showing the $JN_p(Q_0)$ condition. As a bonus, we will also show that, when applied to the self improvement of Poincaré-Sobolev inequalities, this method leads to sharp results⁵.

In the last section of this note we included a brief discussion of related methods that can be used to study self-improving inequalities; e.g. methods based on rearrangement inequalities (cf. [11]), methods based on K -functional inequalities as they relate to reverse Hölder inequalities (cf. [19], [16], [13]), and K -functional inequalities applied to Poincaré-Sobolev (cf. [14] and [17]).

Finally, we refer to [4] for background information on rearrangements and covering lemmas.

2. $GaRo_p = L(p, \infty)$

We consider a qualitative version of the Garsia-Rodemich [8] equality

$$GaRo_p = L(p, \infty).$$

We start recalling the n dimensional version as given in [18].

Theorem 2.1. *Let $1 < p < \infty$, and let $Q_0 \subset R^n$ be a fixed cube. Then*

(i) $JN_p(Q_0) \subset GaRo_p(Q_0)$. In fact,

$$\|f\|_{GaRo_p(Q_0)} \leq 2JN_p(f, Q_0).$$

(ii) $GaRo_p(Q_0) = L(p, \infty)(Q_0)$. In fact, if we let

$$\|f\|_{L(p, \infty)}^* = \sup_t f^*(t)t^{1/p},$$

then we have,

$$(2.1) \quad \|f\|_{GaRo_p(Q_0)} \leq \frac{2p}{p-1} \|f\|_{L(p, \infty)}^*,$$

$$(2.2) \quad \sup_t t^{1/p} (f^{**}(t) - f^*(t)) \leq 2^{n/p'+1} \|f\|_{GaRo_p(Q_0)} + \left(\frac{4}{|Q_0|}\right)^{1/p'} \|f\|_{L^1}.$$

The following form of Theorem 2.1 will be useful for the applications we develop in this note.

Corollary 2.2. *Let $1 < p < \infty$. Then,*

$$\|f - f_{Q_0}\|_{GaRo_p(Q_0)} \leq \frac{2p}{p-1} \|f - f_{Q_0}\|_{L(p, \infty)}^*,$$

⁵Our results in this direction ought to be compared with those presented in the recent survey [5] and the references therein (cf. e.g. Corollary 3.5 in [5]).

$$(2.3) \quad \|f - f_{Q_0}\|_{L^{(p,\infty)}}^* \leq c(n, p) \|f - f_{Q_0}\|_{GaRo_p(Q_0)}.$$

Proof. The first inequality follows applying (2.1) to $f - f_{Q_0}$. To prove (2.3), let $g = f - f_{Q_0}$. Then, since $g \in L^1(Q_0)$, we see that $g^{**}(t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore, by the fundamental theorem of calculus, we can write⁶

$$g^{**}(t) = \int_t^\infty (g^{**}(s) - g^*(s)) \frac{ds}{s}.$$

Combining with (2.2) we find

$$\begin{aligned} g^{**}(t) &\leq c \|g\|_{GaRo_p(Q_0)} \int_t^\infty s^{-1/p} \frac{ds}{s} + \left(\frac{4}{|Q_0|}\right)^{1/p'} \|g\|_{L^1} \int_t^\infty s^{-1/p} \frac{ds}{s} \\ &= p \left(c \|g\|_{GaRo_p(Q_0)} + \left(\frac{4}{|Q_0|}\right)^{1/p'} \|g\|_{L^1} \right) t^{-1/p}. \end{aligned}$$

Thus,

$$(2.4) \quad \|g\|_{L^{(p,\infty)}}^* \leq \sup_t g^{**}(t) t^{1/p} \leq p \left(c \|g\|_{GaRo_p(Q_0)} + \left(\frac{4}{|Q_0|}\right)^{1/p'} \|g\|_{L^1} \right).$$

Now, since $\int_{Q_0} g = 0$, and $\{Q_0\} \in P(Q_0)$,

$$\begin{aligned} \int_{Q_0} |g(x)| dx &= \int_{Q_0} \left| g(x) - \frac{1}{|Q_0|} \int_{Q_0} g(y) dy \right| dx \\ &= \int_{Q_0} \frac{1}{|Q_0|} \left| \int_{Q_0} (g(x) - g(y)) dy \right| dx \\ &\leq \frac{1}{|Q_0|} \int_{Q_0} \int_{Q_0} |g(x) - g(y)| dy dx \\ &\leq |Q_0|^{1/p'} \|g\|_{GaRo_p(Q_0)}. \end{aligned}$$

Inserting this information in (2.4) we find,

$$\|g\|_{L^{(p,\infty)}}^* \leq p \left(c + \left(\frac{4}{|Q_0|}\right)^{1/p'} |Q_0|^{1/p'} \right) \|g\|_{GaRo_p(Q_0)},$$

concluding the proof. □

For further use below let us also note the corresponding end point result for $p = \infty$.

Lemma 2.3.

$$(2.5) \quad GaRo_\infty(Q_0) = BMO(Q_0).$$

Proof. First let us note that, as is well known, and readily verified (cf. [8]),

$$(2.6) \quad f \in BMO(Q_0) \Leftrightarrow \|f\|_* = \sup_{Q \subset Q_0} \frac{1}{|Q|^2} \int_Q \int_Q |f(x) - f(y)| dx dy < \infty,$$

⁶Recall that $\frac{d}{dt} (g^{**}(t)) = \frac{(g^*(t) - g^{**}(t))}{t}$.

where in the expression defining $\|f\|_*$ above, the sup is taken over all subcubes $Q \subset Q_0$. This given, let us suppose first that $f \in GaRo_\infty(Q_0)$. Then, since for any subcube $Q \subset Q_0$ we have $\{Q\} \in P(Q_0)$, it follows that

$$\frac{1}{|Q|} \int_Q \int_Q |f(x) - f(y)| \, dx dy \leq |Q| \|f\|_{GaRo_\infty(Q_0)}.$$

Thus, by (2.6),

$$\|f\|_* \leq \|f\|_{GaRo_\infty(Q_0)}.$$

Conversely, for any $\{Q_i\}_{i \in N} \in P(Q_0)$, we can estimate

$$\begin{aligned} \sum_i \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx dy &= \sum_i |Q_i| \frac{1}{|Q_i|^2} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx dy \\ &\leq \left(\sum_i |Q_i| \right) \|f\|_* . \end{aligned}$$

Whence,

$$\|f\|_{GaRo_\infty(Q_0)} \leq \|f\|_* .$$

□

Summarizing, just like the JN_p conditions, the $GaRo_p$ conditions form a scale joining the weak type Marcinkiewicz $L(p, \infty)$ spaces and BMO . Moreover, we have

$$JN_p \subset GaRo_p = L(p, \infty), \text{ for } p \in (1, \infty),$$

and

$$JN_\infty = GaRo_\infty = BMO.$$

3. POINCARÉ INEQUALITIES

Let $p \in (1, \infty)$. We consider $S_p(Q_0)$, the class of functions of functions $f \in L^1(Q_0)$, such that there exists a constant $c(f) > 0$, and $g \in L^p(Q_0)$, such that for all subcubes $Q \subset Q_0$,

$$\frac{1}{|Q|} \int_Q |f - f_Q| \, dx \leq c(f) l(Q) \left\{ \frac{1}{|Q|} \int_Q |g|^p \, dx \right\}^{1/p},$$

where $l(Q)$ = length of the sides of Q .

The function g is usually called an upper gradient of f (cf. [7], [9], [11] and the references therein). As is well known, with a minor variant of this definition⁷ one can study Poincaré inequalities in metric spaces. In particular, the classical Euclidean $(1, p)$ Poincaré inequalities, correspond to the choice $|g| = |\nabla f|$.

Theorem 3.1. (i) Let $1 < p < n$. Suppose that $f \in S_p(Q_0)$ then, $f \in L(p^*, \infty)(Q_0)$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$

(ii) If $p = n$, then $f \in S_n(Q_0)$ implies that $f \in GaRo_\infty(Q_0) = BMO(Q_0)$.

⁷We need to replace *cubes* by *balls*. For more information on this point see Remark 4.1 below.

Proof. (i) Let $\{Q_i\}_{i \in N} \in P(Q_0)$. Let $1 < p < n$, and $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, $\frac{1}{p'} = 1 - \frac{1}{p}$, $\frac{1}{(p^*)'} = 1 - \frac{1}{p^*}$. Note that since $p < n$, then $p^* > p$, and we have $p^*/p > 1$. Then⁸,

$$\begin{aligned}
& \sum_i \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx dy \\
& \leq 2 \sum_i \int_{Q_i} |f(x) - f_{Q_i}| \, dx \\
& = 2 \sum_i |Q_i| \frac{1}{|Q_i|} \int_{Q_i} |f(x) - f_{Q_i}| \, dx \\
& \leq 2c(f) \sum_i |Q_i| l(Q_i) \left\{ \frac{1}{|Q_i|} \int_{Q_i} |g|^p \, dx \right\}^{1/p} \\
& = 2c(f)c_n \sum_i |Q_i|^{1-1/p+1/n} \left\{ \int_{Q_i} |g|^p \, dx \right\}^{1/p} \\
& = 2c(f)c_n \sum_i |Q_i|^{1-1/p^*} \left\{ \int_{Q_i} |g|^p \, dx \right\}^{1/p} \\
& \leq 2c(f)c_n \left\{ \sum_i |Q_i|^{\frac{1}{(p^*)'}(p^*)'} \right\}^{1/(p^*)'} \left[\left\{ \sum_i \left\{ \int_{Q_i} |g|^p \, dx \right\}^{p^*/p} \right\}^{p/p^*} \right]^{1/p} \\
& \leq 2c(f)c_n \left\{ \sum_i |Q_i| \right\}^{1/(p^*)'} \left\{ \sum_i \int_{Q_i} |g|^p \, dx \right\}^{1/p} \\
& \leq 2c(f)c_n \left\{ \sum_i |Q_i| \right\}^{1/(p^*)'} \left\{ \int_{\cup Q_i} |g|^p \, dx \right\}^{1/p} \\
& \leq 2c(f)c_n \left\{ \sum_i |Q_i| \right\}^{1/(p^*)'} \left\{ \int_{Q_0} |g|^p \, dx \right\}^{1/p}.
\end{aligned}$$

Thus,

$$(3.1) \quad \|f\|_{GaRo_{p^*}(Q_0)} \leq 2c(f)c_n \left\{ \int_{Q_0} |g|^p \, dx \right\}^{1/p}.$$

Now, since

$$\begin{aligned}
\frac{1}{|Q|} \int_Q |(f - f_{Q_0}) - (f - f_{Q_0})_Q| \, dx &= \frac{1}{|Q|} \int_Q |f - f_Q| \, dx \\
&\leq c(f)l(Q) \left\{ \frac{1}{|Q|} \int_Q |g|^p \, dx \right\}^{1/p},
\end{aligned}$$

⁸In the course of the proof we use the fact that

$$\|\{x_n\}\|_{l^{p^*/p}} \leq \|\{x_n\}\|_{l^1}.$$

we see that g is also an upper gradient of $f - f_{Q_0}$. Consequently, (3.1) holds for $f - f_{Q_0}$ and we find that

$$\|f - f_{Q_0}\|_{GaRo_{p^*}(Q_0)} \leq 2c(f)c_n \left\{ \int_{Q_0} |g|^p dx \right\}^{1/p}.$$

Applying (2.3) we finally arrive at

$$\|f - f_{Q_0}\|_{L(p,\infty)}^* \leq c(n,p)2c(f)c_n \left\{ \int_{Q_0} |g|^p dx \right\}^{1/p}.$$

(ii) Suppose that $p = n$. We proceed as in the first part of the proof noticing that when $p = n$, we have $1 - 1/p^* = 1$. Consequently,

$$\begin{aligned} \sum_i \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| dx dy &\leq 2c(f)c_n \sum_i |Q_i| \left\{ \int_{Q_i} |g|^n dx \right\}^{1/n} \\ &\leq 2c(f)c_n \left\{ \int_{Q_0} |g|^n dx \right\}^{1/n} \sum_i |Q_i|. \end{aligned}$$

Therefore,

$$\|f - f_{Q_0}\|_{GaRo_\infty(Q_0)} \leq 2c(f)c_n \left\{ \int_{Q_0} |g|^n dx \right\}^{1/n},$$

and we conclude by (2.5). □

Remark 3.2. The previous result shows that starting with a function in $S_p(Q_0)$, $1 < p < n$, we obtain the (weak type) improvement $f \in L(p^*, \infty)(Q_0)$. Moreover, combining this result with Maz'ya's self-improvement principle for weak type inequalities for the gradient⁹ (cf. [9]) we obtain (the strong type) improvement: if $f \in S_p(Q_0)$ then $f \in L^{p^*}(Q_0)$ or even $L(p^*, p)(Q_0)$ (cf. [14]). Since we have nothing new to add to the known methods used to show how to self improve from weak type to strong type, we shall not consider this issue here and refer to [21], [9], [15], [14], and the references therein.

4. FINAL COMMENTS AND PROBLEMS

We will show some connections with other approaches to the self improvement of Sobolev-Poincaré inequalities.

4.1. Poincaré inequalities, maximal inequalities and rearrangements. There is a close connection between Sobolev-Poincaré inequalities, rearrangement inequalities for gradients, weak type inequalities and maximal operators. Consequently all of the above can be expressed in terms of Garsia-Rodemich conditions. In this section we shall briefly explore some of these interconnections.

Let Q_0 be a fixed cube on R^n . Suppose that $f \in S_1(Q_0)$. Therefore, there exists a constant $c(f) \geq 0$, and $g \in L^1(Q_0)$, such that for all $Q \subset Q_0$ subcubes of Q_0 , we

⁹To the effect that ‘weak type’ implies ‘strong type’.

have

$$(4.1) \quad \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq c(f) \frac{|Q|^{1/n}}{|Q|} \int_Q |g(x)| dx.$$

We now reproduce the argument in [11]. We first note that if (4.1) holds then,

$$\begin{aligned} f_{1/n}^\#(x) &:= \sup_{Q_{xx}} \frac{1}{|Q|^{1+1/n}} \int_Q |f(x) - f_Q| dx \\ &\leq c(f) \sup_{Q_{xx}} \frac{1}{|Q|} \int_Q g(x) dx = c(f) M g(x), \end{aligned}$$

where M is the maximal operator of Hardy-Littlewood. Consequently, by a modification of an argument of [4], we find

$$\begin{aligned} f_{1/n}^{\#\#}(t) &\leq C_n (Mg)^{\#\#}(t) \\ &= C_n g^{\#\#}(t). \end{aligned}$$

As a consequence (cf. [4], [17]) we obtain a version of a well known rearrangement inequality for the gradient (cf. [12], [2], [11] and the references therein)

$$(4.2) \quad f^{\#\#}(t) - f^*(t) \leq c_n t^{1/n} g^{\#\#}(t), \text{ for } 0 < t < |Q_0|/2.$$

Note that (4.2) yields a weak type form of the Gagliardo-Nirenberg inequality. Indeed, we can rewrite (4.2) as

$$(f^{\#\#}(t) - f^*(t)) \leq c_n t^{1/n} \frac{1}{t} \int_0^t g^*(s) ds, \quad 0 < t < |Q_0|/2,$$

which readily implies (cf. [18])

$$\sup_{t>0} (f^{\#\#}(t) - f^*(t)) t^{1/n'} \leq c_n \|g\|_{L^1(Q_0)} + \left(\frac{|Q_0|}{2}\right)^{\frac{1}{n'}-1} \|f\|_{L^1(Q_0)}.$$

It follows that (cf. the proof of Corollary 2.2 above)

$$\|f - f_{Q_0}\|_{L^{(n',\infty)}}^* \leq c(n, |Q_0|) \left[c(f) \|g\|_{L^1(Q_0)} + \left(\frac{|Q_0|}{2}\right)^{\frac{1}{n'}-1} \|f - f_{Q_0}\|_{L^1(Q_0)} \right]$$

and since by (4.1)

$$\begin{aligned} \|f - f_{Q_0}\|_{L^1(Q_0)} &\leq c(f) |Q_0|^{1/n} \int_{Q_0} |g(x)| dx \\ &= c(f) |Q_0|^{1/n} \|g\|_{L^1(Q_0)}, \end{aligned}$$

we finally arrive at

$$\|f - f_{Q_0}\|_{L^{(n',\infty)}}^* \leq c(n) \|g\|_{L^1(Q_0)}.$$

Of course this weak type version of the Gagliardo-Nirenberg inequality can be now rewritten using Garsia-Rodemich conditions via (2.3).

The analysis for (q, p) Poincaré inequalities follows the same pattern (cf. [11]). For example, suppose that $f \in S_p(Q_0)$. Then there exists $c(f) > 0$, and g upper gradient of f , such that for all subcubes $Q \subset Q_0$,

$$\left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^q dx \right)^{1/q} \leq c(f) |Q|^{1/n} \left(\frac{1}{|Q|} \int_Q g^p(x) dx \right)^{1/p}.$$

Then

$$t^{-\frac{1}{n}} \left(\frac{1}{t} \int_0^t [f^*(s) - f^*(s)]^q ds \right)^{1/q} \leq Cc(f)[(g^p)^{**}(t)]^{1/p}, \text{ for } 0 < t < |Q_0|/2.$$

Which again can be rewritten as a Sobolev-Poincaré weak type inequality.

Remark 4.1. As was shown in [11] the analysis above holds in the general setting of metric spaces provided with a doubling measure. In particular, on a doubling metric measure space (X, μ) there exists $s =$ doubling order, or homogeneous dimension, such that for each ball $B \subset X$

$$\mu(B) \geq cr(B)^s, \text{ where } r(B) \text{ is the radius of } B.$$

The corresponding rearrangement inequality associated with the Poincaré inequality

$$\frac{1}{\mu(B)} \int_B |f(x) - f_B| d\mu(x) \leq c \frac{\mu(B)^{1/s}}{\mu(B)} \int_B |g(x)| d\mu(x),$$

takes the form (cf. [11])

$$f^{**}(t) - f^*(t) \leq c_n t^{1/s} g^{**}(t), \text{ for } 0 < t < \mu(X)/2.$$

Remark 4.2. It could be of interest to investigate the connection of the homogenous dimension of X and measures of the form $w d\mu$, with w in a Muckenhoupt class of weights. Given the connection between BMO and the A_p classes (cf. [6]) it would be interesting to study the connection with BMO and isoperimetry. Recently, an interesting connection between isoperimetry and BMO was uncovered in [1].

4.2. K-functional connection. Due to its connection to maximal operators, the K -functional of interpolation theory (cf. [4]) is a good tool to study self-improving inequalities involving averages. For example, the well known equivalence of Herz-Stein to the effect that the maximal operator of Hardy Littlewood, M , can be estimated by

$$(Mf)^*(t) \approx f^{**}(t),$$

can be effectively used to prove self improving inequalities connected with reverse Hölder inequalities (cf. [19], [16], [13], and the references therein). Moreover, this is immediately connected with the computation of the K -functional for the pair (L^1, L^∞) :

$$K(t, f; L^1, L^\infty) = t f^{**}(t).$$

In this context Gehring’s self-improving inequalities can be formulated as differential inequalities connected with the reiteration formulae of Holmstedt (cf. [19]).

Likewise, we believe that suitable reformulations of the K -functional for the pair (L^1, BMO) (cf. [4], [17]) can be used to reformulate some of the results considered in this note in terms of differential inequalities, via reiteration (cf. [19]).

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