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LINEAR CONTROL SYSTEMS WITH NONCONVEX INTEGRANDS ON LARGE INTERVALS

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Abstract. In this paper we study the structure of approximate solutions of optimal control problems, governed by linear equations, with autonomous nonconvex integrands, on large intervals. For these optimal control problems we prove that the turnpike phenomenon holds. We study the structure of approximate optimal trajectories in regions close to the endpoints of the time intervals. It is established that in these regions optimal trajectories converge to solutions of the corresponding infinite horizon optimal control problem which depend only on the integrand.

1. Introduction

The growing significance of the study of (approximate) solutions of variational and optimal control problems defined on infinite intervals and on sufficiently large intervals has been further realized in the recent years [2, 4–9, 12–18, 21, 26, 28, 29, 32, 33, 35, 36, 38, 39, 44]. This is due not only to theoretical achievements in this area, but also because of numerous applications to engineering [1, 11, 24, 39], models of economic dynamics [10, 11, 19, 23, 27, 31, 34, 37, 39, 41, 44], the game theory [20, 22, 39, 40, 44], models of solid-state physics [3] and the theory of thermodynamical equilibrium for materials [25, 30]. In this paper we study the structure of approximate solutions of optimal control problems, governed by linear equations, with autonomous nonconvex integrands, on large intervals. For these optimal control problems we prove that the turnpike phenomenon holds. We study the structure of approximate optimal trajectories in regions close to the endpoints of the time intervals. It is established that in these regions optimal trajectories converge to solutions of the corresponding infinite horizon optimal control problem which depend only on the integrand.

In the first part of the paper we recall the results obtained in our recent research for Lagrange problems [45]. In the second part we generalize these results for Bolza problems.

We study the structure of approximate optimal trajectories of linear control systems described by a differential equation

$$x'(t) = Ax(t) + Bu(t) \text{ for almost every (a. e.) } t \in \mathcal{I}, \tag{1.1}$$

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where $\mathcal{I}$ is either R^1 or $[T_1, \infty)$ or $[T_1, T_2]$ (here $-\infty < T_1 < T_2 < \infty$), n, m are natural numbers, $x : \mathcal{I} \to R^n$ is an absolutely continuous (a. c.) function and the control function $u : \mathcal{I} \to R^m$ is Lebesgue measurable, and A and B are given matrices of dimensions $n \times n$ and $n \times m$ with integrands $f : R^n \times R^m \to R^1$.

Note that if $\mathcal{I}$ is an unbounded interval, then $x : \mathcal{I} \to R^n$ is an absolutely continuous function if and only if it is an absolutely continuous function on any bounded subinterval of $\mathcal{I}$.

We assume that the linear system (1.1) is controllable and that the integrand f is a continuous function.

We denote by $|\cdot|$ the Euclidean norm and by $\langle \cdot, \cdot \rangle$ the inner product in the k-dimensional Euclidean space R^k. For every $s \in R^1$ set $s_+ = \max\{s, 0\}$. For every nonempty set X and every function $h : X \to R^1 \cup \{\infty\}$ set

$$\inf(h) = \inf\{h(x) : \ x \in X\}.$$

Let a_0 be a positive number and $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that

$$\lim_{t \to \infty} \psi(t) = \infty. \tag{1.2}$$

Suppose that $f : R^n \times R^m \to R^1$ is a continuous function such that the following assumption holds:

(A1)

(i) for every point $(x, u) \in R^n \times R^m$,

$$f(x, u) \geq \max\{\psi(|x|), \ \psi(|u|),$$

$$\psi([|Ax + Bu| - a_0|x|]_+)[|Ax + Bu| - a_0|x|]_+\} - a_0; \tag{1.3}$$

(ii) for every point $x \in R^n$ the function $f(x, \cdot) : \ R^m \to R^1$ is convex;

(iii) for every pair of positive numbers M, ϵ there exist positive numbers Γ, δ such that

$$|f(x_1, u_1) - f(x_2, u_2)| \leq \epsilon \max\{f(x_1, u_1), f(x_2, u_2)\}$$

for each $u_1, u_2 \in R^m$ and each $x_1, x_2 \in R^n$ which satisfy

$$|x_i| \leq M, \ |u_i| \geq \Gamma, \ i = 1, 2,$$

$$\max\{|x_1 - x_2|, |u_1 - u_2|\} \leq \delta;$$

(iv) for every positive number K there exists a positive constant a_K and an increasing function

$$\psi_K : [0, \infty) \to [0, \infty)$$

such that

$$\psi_K(t) \to \infty \text{ as } t \to \infty$$

and

$$f(x, u) \geq \psi_K(|u|)|u| - a_K$$

for every point $u \in R^m$ and every point $x \in R^n$ satisfying $|x| \leq K$.

Let $T_1 \in R^1$ and $T_2 > T_1$. A pair of an absolutely continuous function $x : [T_1, T_2] \to R^n$ and a Lebesgue measurable function $u : [T_1, T_2] \to R^m$ is called an (A, B)-trajectory-control pair if (1.1) holds with $\mathcal{I} = [T_1, T_2]$. Denote by $X(A, B, T_1, T_2)$ the set of all (A, B)-trajectory-control pairs $x : [T_1, T_2] \to R^n$, $u : [T_1, T_2] \to R^m$.

Let $T \in R^1$ and $\mathcal{I} = [T, \infty)$ be an infinite closed subinterval of R^1. Denote by $X(A, B, T, \infty)$ the set of all pairs of a.c. functions $x : [T, \infty) \to R^n$ and Lebesgue measurable functions $u : [T, \infty) \to R^m$ satisfying (1.1).

Note that a function h satisfies (A1) if $h \in C^1(R^n \times R^m)$, (A1)(i), (A1)(ii), (A1)(iv) hold, and for each $K > 0$ there exists an increasing function $\tilde{\psi} : [0, \infty) \to [0, \infty)$ such that for each $x \in R^n$ satisfying $|x| \le K$ and each $u \in R^m$,

$$\max\{|\partial h/\partial x(x, u)|,\ |\partial h/\partial u(x, u)|\} \le \tilde{\psi}(|x|)(1 + \psi_K(|u|)|u|).$$

The performance of the above control system is measured on any finite interval $[T_1, T_2] \subset [0, \infty)$ and for any $(x, u) \in X(A, B, T_1, T_2)$ by the integral functional

$$I^f(T_1, T_2, x, u) = \int_{T_1}^{T_2} f(x(t), u(t))dt. \tag{1.4}$$

We consider the following optimal control problems

$$(P_1) \qquad I^f(0, T, x, u) \to \min,$$
$$(x, u) \in X(A, B, 0, T) \text{ such that } x(0) = y,\ x(T) = z,$$

$$(P_2) \qquad I^f(0, T, x, u) \to \min,$$
$$(x, u) \in X(A, B, 0, T) \text{ such that } x(0) = y,$$

$$(P_3) \qquad I^f(0, T, x, u) \to \min,$$
$$(x, u) \in X(A, B, 0, T),$$

$$(P_4) \qquad I^f(0, T, x, u) + g(x(T)) \to \min,$$
$$(x, u) \in X(A, B, 0, T) \text{ such that } x(0) = y,$$

$$(P_5) \qquad I^f(0, T, x, u) + g(x(T)) + h(x(0)) \to \min,$$
$$(x, u) \in X(A, B, 0, T),$$

where $y, z \in R^n$, $T > 0$ and $g : R^n \to R^1$ and $h : R^n \to R^1$ are lower semicontinuous functions which are bounded on bounded sets. The study of these problems is based on the properties of solutions of the corresponding infinite horizon optimal control problem associated with the control system (1.1) and the integrand f. Problems $(P_1) - (P_3)$ where analyzed in [45] while in this paper we study problems (P_4) and (P_5).

We establish the turnpike property for the approximate solutions of problems (P_4) and (P_5). For problems (P_4) and (P_5) we show that in regions close to the right endpoint T of the time interval their approximate solutions are determined only by the pair (f, g) and are essentially independent of the choice of interval and the endpoint value y. For problems (P_5), approximate solutions are determined only by the pair (f, h) also in regions close to the left endpoint 0 of the time interval.

A number

$$\mu(f) := \inf\{\liminf_{T\to\infty} T^{-1} I^f(0,T,x,u) :\ (x,u) \in X(A,B,0,\infty)\} \tag{1.5}$$

is called the minimal long-run average cost growth rate of f. In view of (A1)(i), we have $-\infty < \mu(f)$.

We say that a pair $(\tilde{x},\tilde{u}) \in X(A,B,0,\infty)$ is (f,A,B)-overtaking optimal [39,44] if for every pair $(x,u) \in X(A,B,0,\infty)$ such that $x(0) = \tilde{x}(0)$ the inequality

$$\limsup_{T\to\infty}[I^f(0,T,\tilde{x},\tilde{u}) - I^f(0,T,x,u)] \le 0$$

holds.

We say that a pair $(x,u) \in X(A,B,0,\infty)$ is (f,A,B)-minimal [39,44] if for every positive number T,

$$I^f(0,T,x,u) \le I^f(0,T,y,v)$$

for every pair $(y,v) \in X(A,B,0,T)$ such that $y(0) = x(0)$, $y(T) = x(T)$.

Let $(x_f,u_f) \in R^n \times R^m$ satisfy

$$Ax_f + Bu_f = 0. \tag{1.6}$$

It is clear that $\mu(f) \le f(x_f,u_f)$. It is not difficult to see that the following result holds.

Proposition 1.1 (Proposition 3.1 of [45])**.** *Assume that $\mu(f) = f(x_f,u_f)$ and let $x(t) = x_f$, $u(t) = u_f$ for all $t \in [0,\infty)$. Then the pair $(x,u) \in X(A,B,0,\infty)$ is (f,A,B)-minimal.*

We suppose that the following assumption holds.

(A2) $\mu(f) = f(x_f,u_f)$ and if $(x,u) \in R^n \times R^m$ satisfies

$$Ax + Bu = 0,\ \mu(f) = f(x,u),$$

then $x = x_f$.

In [45] we proved the following result.

Proposition 1.2 (Proposition 3.4 of [45])**.** *For every trajectory-control pair $(x,u) \in X(A,B,0,\infty)$ either*

$$I^f(0,T,x,u) - T\mu(f) \to \infty \text{ as } T \to \infty$$

or $\sup\{|I^f(0,T,x,u) - T\mu(f)| :\ T > 0\} < \infty$.

A trajectory-control pair $(x,u) \in X(A,B,0,\infty)$ is called (f,A,B)-good [39,44] if

$$\sup\{|I^f(0,T,x,u) - T\mu(f)| :\ T > 0\} < \infty.$$

We have the following result.

Proposition 1.3 (Proposition 3.5 of [45])**.** *For any (f,A,B)-good pair*

$$(x,u) \in X(A,B,0,\infty)$$

the inequality

$$\sup\{|x(t)| :\ t \in [0,\infty)\} < \infty$$

holds.

We suppose that the following assumption holds.
(A3) For every (f, A, B)-good trajectory-control pair

$$(x, u) \in X(A, B, 0, \infty)$$

the equality $\lim_{t\to\infty} x(t) = x_f$ holds.

Assumptions (A1)-(A3) were introduced in Section 3.1 of [45] which also contains several examples of integrands satisfying these assumptions. These assumptions are common in the turnpike theory [39, 41, 43, 44]. In particular, we need the growth condition (A1)(i) and the convexity assumption (A1)(ii) in order to guarantee the existence of solutions of optimal control problems on finite intervals. Assumptions (A2) means that there exists a unique good stationary trajectory-control pair. This assumption is necessary if one intends to obtain a turnpike property for which the turnpike is a singleton. Assumption (A3) is called in the literature as the asymptotic turnpike [39,41,43–45]. It is also necessary for turnpike properties on finite intervals. It was shown in Chapter 7 of [45] that in the space of integrands satisfying (A1) and (A2) most integrands satisfy also (A3).

2. Turnpike results for problems (P_1) and (P_2)

We use the notation, definitions and assumptions introduced in Section 1.
Let $T > 0$ and $y, z \in R^n$. Set

$$\sigma(f, y, z, T) = \inf\{I^f(0, T, x, u) :$$

$$(x, u) \in X(A, B, 0, T) \text{ and } x(0) = y,\ x(T) = z\}, \tag{2.1}$$

$$\sigma(f, y, T) = \inf\{I^f(0, T, x, u) :\ (x, u) \in X(A, B, 0, T) \text{ and } x(0) = y\}, \tag{2.2}$$

$$\widehat{\sigma}(f, z, T) = \inf\{I^f(0, T, x, u) :\ (x, u) \in X(A, B, 0, T) \text{ and } x(T) = z\}, \tag{2.3}$$

$$\sigma(f, T) = \inf\{I^f(0, T, x, u) :\ (x, u) \in X(A, B, 0, T)\}. \tag{2.4}$$

The results of this section were obtained in [45]. The following theorem establishes the turnpike property of approximate solutions of problems (P_1) and (P_2).

Theorem 2.1 (Theorem 3.7 of [45]). *Let ϵ, $M_0, M_1 > 0$. Then there exist $L > 0$, $\delta \in (0, \epsilon)$ such that for each $T > 2L$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies for each $S \in [0, T - L]$,*

$$I^f(S, S + L, x, u) \le \sigma(f, x(S), x(S + L), L) + \delta$$

and satisfies at least one of the following conditions below

(a) $|x(0)|,\ |x(T)| \le M_0,\ I^f(0, T, x, u) \le \sigma(f, x(0), x(T), T) + M_1$;
(b) $|x(0)| \le M_0,\ I^f(0, T, x, u) \le \sigma(f, x(0), T) + M_1$;
(c) $I^f(0, T, x, u) \le \sigma(f, T) + M_1$

there exist $p_1 \in [0, L]$, $p_2 \in [T - L, T]$ such that

$$|x(t) - x_f| \le \epsilon \text{ for all } t \in [p_1, p_2].$$

Moreover if $|x(0) - x_f| \le \delta$, then $p_1 = 0$ and if $|x(T) - x_f| \le \delta$, then $p_2 = T$.

Theorem 2.2 (Theorem 3.8 of [45]). *Let $x_0 \in R^n$. Then there exists an (f, A, B)-overtaking optimal trajectory-control pair $(x, u) \in X(A, B, 0, \infty)$ satisfying $x(0) = x_0$.*

The next result describes the limit behavior of overtaking optimal trajectories.

Theorem 2.3 (Theorem 3.9 of [45]). *Let $M, \epsilon > 0$. Then there exists $L > 0$ such that for any (f, A, B)-overtaking optimal trajectory-control pair $(x, u) \in X(A, B, 0, \infty)$ which satisfies $|x(0)| \le M$ the inequality*

$$|x(t) - x_f| \le \epsilon$$

holds for all numbers $t \ge L$. Moreover, there exists $\delta > 0$ such that for any (f, A, B)-overtaking optimal trajectory-control pair $(x, u) \in X(A, B, 0, \infty)$ satisfying $|x(0) - x_f| \le \delta$, the inequality

$$|x(t) - x_f| \le \epsilon$$

holds for all numbers $t \ge 0$.

The next result shows the equivalence of the optimality criterions introduced above.

Theorem 2.4 (Theorem 3.10 of [45]). *Assume that $(x, u) \in X(A, B, 0, \infty)$. Then the following conditions are equivalent:*

(i) *(x, u) is (f, A, B)-overtaking optimal;*
(ii) *(x, u) is (f, A, B)-minimal and (f, A, B)-good;*
(iii) *(x, u) is (f, A, B)-minimal and*

$$\lim_{t \to \infty} x(t) = x_f;$$

(iv) *(x, u) is (f, A, B)-minimal and $\liminf_{t\to\infty} |x(t)| < \infty$.*

3. Auxiliary results

We use the notation, definitions and assumptions introduced in Sections 1 and 2.

For every point $z \in R^n$ denote by $\Lambda(z)$ the collection of all (f, A, B)-overtaking optimal pairs $(x, u) \in X(A, B, 0, \infty)$ such that $x(0) = z$, which is nonempty in view of Theorem 2.2.

Let $z \in R^n$. Define

$$\pi^f(z) = \liminf_{T \to \infty} [I^f(0, T, x, u) - T\mu(f)], \tag{3.1}$$

where $(x, u) \in \Lambda(z)$. By Proposition 1.2 and Theorem 2.4, $\pi^f(z)$ is finite, well defined and does not depend on the choice of $(x, u) \in \Lambda(z)$. The following results were obtained in Section 3.3 of [45].

Proposition 3.1 (Proposition 3.11 of [45]). *1. Let $(x, u) \in X(A, B, 0, \infty)$ be (f, A, B)-good. Then*

$$\pi^f(x(0)) \le \liminf_{T \to \infty} [I^f(0, T, x, u) - T\mu(f)]$$

and for each pair of numbers $S > T \ge 0$,

$$\pi^f(x(T)) \le I^f(T, S, x, u) - (S - T)\mu(f) + \pi^f(x(S)). \tag{3.2}$$

2. Let $S > T \ge 0$ and $(x, u) \in X(A, B, T, S)$. Then (3.2) holds.

Proposition 3.2 (Proposition 3.12 of [45]). *Let $(x,u) \in X(A,B,0,\infty)$ be an (f,A,B)-overtaking optimal pair. Then for each pair of numbers $S > T \geq 0$,*

$$\pi^f(x(T)) = I^f(T,S,x,u) - (S-T)\mu(f) + \pi^f(x(S)).$$

Proposition 3.3 (Propositions 3.13, 3.14, 3.16 and 3.17 of [45]). *$\pi^f(x_f) = 0$, the function π^f is continuous at x_f, the function π^f is lower semicontinuous and for each $M > 0$ the set $\{x \in R^n : \pi^f(x) \leq M\}$ is bounded.*

Proposition 3.4 (Proposition 3.15 of [45]). *Let $(x,u) \in X(A,B,0,\infty)$ be (f,A,B)-overtaking optimal. Then*

$$\pi^f(x(0)) = \lim_{T\to\infty}[I^f(0,T,x,u) - T\mu(f)].$$

Proposition 3.5 (Proposition 3.18 of [45]). *Let $(x,u) \in X(A,B,0,\infty)$ be (f,A,B)-good pair such that for all $T > 0$,*

$$I^f(0,T,x,u) - T\mu(f) = \pi^f(x(0)) - \pi^f(x(T)).$$

Then $(x,u) \in X(A,B,0,\infty)$ is (f,A,B)-overtaking optimal.

Consider a linear control system

$$x'(t) = -Ax(t) - Bu(t), \quad x(0) = x_0 \tag{3.3}$$

which is also controllable. For the triplet $(f,-A,-B)$ we use all the notation and definitions introduced for the triplet (f,A,B). It is not difficult to see that assumption (A1) holds for the triplet $(f,-A,-B)$.

Let $T_1 \in R^1$, $T_2 > T_1$. A pair of an absolutely continuous function $x : [T_1,T_2] \to R^n$ and a Lebesgue measurable function $u : [T_1,T_2] \to R^m$ is called an $(-A,-B)$-trajectory-control pair if (3.3) holds for a. e. $t \in [T_1,T_2]$. Denote by $X(-A,-B,T_1,T_2)$ the set of all $(-A-B)$-trajectory-control pairs $x : [T_1,T_2] \to R^n$, $u : [T_1,T_2] \to R^m$.

Let $T \in R^1$. Denote by $X(-A,-B,T,\infty)$ the set of all pairs of a. c. functions $x : [T,\infty) \to R^n$ and Lebesgue measurable functions $u : [T,\infty) \to R^m$ satisfying (3.3) for a. e. $t \geq T$, which are called $(-A,-B)$-trajectory-control pairs.

Assume that $S_1 \in R^1$, $S_2 > S_1$ and that $(x,u) \in X(A,B,S_1,S_2)$. For all $t \in [S_1,S_2]$ set

$$\bar{x}(t) = x(S_2 - t + S_1),\ \bar{u}(t) = u(S_2 - t + S_1). \tag{3.4}$$

By (1.1) and (3.4) for a. e. $t \in [S_1,S_2]$,

$$\begin{aligned}\bar{x}'(t) &= -x'(S_2 - t + S_1) = -Ax(S_2 - t + S_1) - Bu(S_2 - t + S_1)\\ &= -A\bar{x}(t) - B\bar{u}(t), \quad (\bar{x},\bar{u}) \in X(-A,-B,S_1,S_2).\end{aligned} \tag{3.5}$$

In view of (3.4),

$$\begin{aligned}\int_{S_1}^{S_2} f(\bar{x}(t),\bar{u}(t))dt &= \int_{S_1}^{S_2} f(x(S_2 - t + S_1), u(S_2 - t + S_1))dt\\ &= \int_{S_1}^{S_2} f(x(t),u(t))dt.\end{aligned} \tag{3.6}$$

For every pair of numbers $T_2 > T_1$ and every trajectory-control pair $(x,u) \in X(-A,-B,T_1,T_2)$ define

$$I^f(T_1,T_2,x,u) = \int_{T_1}^{T_2} f(x(t),u(t))dt. \tag{3.7}$$

For every pair of points $y,z \in R^n$ and every positive number T define

$$\sigma_-(f,y,z,T) = \inf\{I^f(0,T,x,u) : \\ (x,u) \in X(-A,-B,0,T) \text{ and } x(0)=y,\ x(T)=z\}, \tag{3.8}$$

$$\sigma_-(f,y,T) = \inf\{I^f(0,T,x,u) : \\ (x,u) \in X(-A,-B,0,T) \text{ and } x(0)=y\}, \tag{3.9}$$

$$\widehat{\sigma}_-(f,z,T) = \inf\{I^f(0,T,x,u) : \\ (x,u) \in X(-A,-B,0,T) \text{ and } x(T)=z\}, \tag{3.10}$$

$$\sigma_-(f,T) = \inf\{I^f(0,T,x,u) :\ (x,u) \in X(-A,-B,0,T)\}. \tag{3.11}$$

The following auxiliary results were proved in Section 3.3 of [45].

Proposition 3.6. *Let $S_2 > S_1$ be real numbers, $M \geq 0$ and that $(x_i,u_i) \in X(A,B,S_1,S_2)$, $i=1,2$. Then*

$$I^f(S_1,S_2,x_1,u_1) \geq I^f(S_1,S_2,x_2,u_2) - M$$

if and only if $I^f(S_1,S_2,\bar{x}_1,\bar{u}_1) \geq I^f(S_1,S_2,\bar{x}_2,\bar{u}_2) - M$.

Proposition 3.7. *Let $S_2 > S_1$ be real numbers and*

$$(x,u) \in X(A,B,S_1,S_2).$$

Then the following assertions hold:

$$I^f(S_1,S_2,x,u) \leq \sigma(f,S_2-S_1)+M$$
$$\text{if and only if } I^f(S_1,S_2,\bar{x},\bar{u}) \leq \sigma_-(f,S_2-S_1)+M;$$
$$I^f(S_1,S_2,x,u) \leq \sigma(f,x(S_1),x(S_2),S_2-S_1)+M$$
$$\text{if and only if } I^f(S_1,S_2,\bar{x},\bar{u}) \leq \sigma_-(f,\bar{x}(S_1),\bar{x}(S_2),S_2-S_1)+M;$$
$$I^f(S_1,S_2,x,u) \leq \sigma(f,x(S_1),S_2-S_1)+M$$
$$\text{if and only if } I^f(S_1,S_2,\bar{x},\bar{u}) \leq \widehat{\sigma}_-(f,\bar{x}(S_2),S_2-S_1)+M;$$
$$I^f(S_1,S_2,x,u) \leq \widehat{\sigma}(f,x(S_2),S_2-S_1)+M$$
$$\text{if and only if } I^f(S_1,S_2,\bar{x},\bar{u}) \leq \sigma_-(f,\bar{x}(S_1),S_2-S_1)+M.$$

Define

$$\mu_-(f) = \inf\{\liminf_{T\to\infty} T^{-1} I^f(0,T,x,u) \in X(-A,-B,0,\infty)\}. \tag{3.12}$$

Proposition 3.8. $\mu_-(f) = \mu(f) = f(x_f,u_f)$.

Proposition 3.9. *For any $(f,-A,-B)$-good trajectory-control pair $(x,u) \in X(-A,-B,0,\infty)$,*

$$\lim_{t\to\infty} x(t) = x_f.$$

Therefore $(f, -A, -B)$ satisfies all the assumptions posed for the triplet (f, A, B) and all the results stated above for the triplet (f, A, B) are also true for $(f, -A, -B)$.

For every point $z \in R^n$ define

$$\pi_-^f(z) = \liminf_{T\to\infty}[I^f(0, T, x, u) - T\mu(f)],$$

where $(x, u) \in X(-A, -B, 0, \infty)$ is an $(f, -A, -B)$-overtaking optimal pair such that $x(0) = z$.

4. Spaces of integrands

We use the notation, definitions and assumptions introduced in Sections 1-3. Recall that $a_0 > 0$ and $\psi : [0, \infty) \to [0, \infty)$ is an increasing function such that

$$\lim_{t\to\infty} \psi(t) = \infty.$$

We continue to study the structure of optimal trajectories of the controllable linear control system

$$x' = Ax + Bu,$$

where A and B are given matrices of dimensions $n\times n$ and $n\times m$, with the continuous integrand $f : R^n \times R^m \to R^1$ which satisfy assumptions (A1)-(A3) and (1.6).

Denote by $\mathfrak{M}$ the set of all borelian functions $g : R^{n+m+1} \to R^1$ which satisfy

$$g(t, x, u) \geq \max\{\psi(|x|),\ \psi(|u|),$$

$$\psi([|Ax + Bu| - a_0|x|]_+)[|Ax + Bu| - a_0|x|]_+\} - a_0 \tag{4.1}$$

for each $(t, x, u) \in R^{n+m+1}$.

We equip the set $\mathfrak{M}$ with the uniformity which is determined by the following base:

$$E(N, \epsilon, \lambda) = \{(f, g) \in \mathfrak{M} \times \mathfrak{M} :\ |f(t, x, u) - g(t, x, u)| \leq \epsilon$$
$$\text{for each } (t, x, u) \in R^{n+m+1} \text{ satisfying } |x|, |u| \leq N\}$$
$$\cap\{(f, g) \in \mathfrak{M} \times \mathfrak{M} :\ (|f(t, x, u)| + 1)(|g(t, x, u)| + 1)^{-1} \in [\lambda^{-1}, \lambda]$$
$$\text{for each } (t, x, u) \in R^{n+m+1} \text{ satisfying } |x| \leq N\}, \tag{4.2}$$

where $N > 0$, $\epsilon > 0$ and $\lambda > 1$.

It is clear that the uniform space $\mathfrak{M}$ is Hausdorff and has a countable base. Therefore $\mathfrak{M}$ is metrizable. It is not difficult to show that the uniform space $\mathfrak{M}$ is complete.

Denote by $\mathfrak{M}_b$ the set of all functions $g \in \mathfrak{M}$ which are bounded on bounded subsets of R^{n+m+1}. Clearly, $\mathfrak{M}_b$ is a closed subset of $\mathfrak{M}$. We consider the topological subspace $\mathfrak{M}_b \subset \mathfrak{M}$ equipped with the relative topology.

For each a pair of numbers $T_1 \in R^1$, $T_2 > T_1$, each $(x, u) \in X(A, B, T_1, T_2)$ and each borelian bounded from below function $g : [T_1, T_2] \times R^n \times R^m \to R^1$ set

$$I^g(T_1, T_2, x, u) = \int_{T_1}^{T_2} g(t, x(t), u(t))dt.$$

We consider the following optimal control problems

$$I^g(T_1, T_2, x, u) \to \min,$$

$$(x,u) \in X(A,B,T_1,T_2) \text{ such that } x(T_1) = y,\ x(T_2) = z,$$
$$I^g(T_1,T_2,x,u) \to \min,$$
$$(x,u) \in X(A,B,T_1,T_2) \text{ such that } x(T_1) = y,$$
$$I^g(T_1,T_2,x,u) \to \min,$$
$$(x,u) \in X(A,B,T_1,T_2),$$

where $y, z \in R^n$, $\infty > T_2 > T_1 > -\infty$ and $g \in \mathfrak{M}$.

Let $y, z \in R^n$, $T_1 \in R^1$, $T_2 > T_1$ and $g : [T_1,T_2] \times R^n \times R^m \to R^1$ be a borelian bounded from below function. Set

$$\sigma(g,y,z,T_1,T_2) = \inf\{I^g(T_1,T_2,x,u) :$$
$$(x,u) \in X(A,B,T_1,T_2) \text{ and } x(T_1) = y,\ x(T_2) = z\}, \tag{4.3}$$
$$\sigma(g,y,T_1,T_2) = \inf\{I^g(T_1,T_2,x,u) :$$
$$(x,u) \in X(A,B,T_1,T_2) \text{ and } x(T_1) = y\}, \tag{4.4}$$
$$\widehat{\sigma}(g,z,T_1,T_2) = \inf\{I^g(T_1,T_2,x,u) :$$
$$(x,u) \in X(A,B,T_1,T_2) \text{ and } x(T_2) = z\}, \tag{4.5}$$

$$\sigma(g,T_1,T_2) = \inf\{I^g(T_1,T_2,x,u) :\ (x,u) \in X(A,B,T_1,T_2)\}. \tag{4.6}$$

Recall that $f : R^n \times R^m \to R^1$ is a continuous function which satisfies (1.6) and assumptions (A1), (A2) and (A3). For each $(t,x,u) \in R^{n+m+1}$ set

$$F(t,x,u) = f(x,u). \tag{4.7}$$

The following stability results were obtained in Chapter 4 of [45]. They show that the turnpike phenomenon, for approximate solutions on large intervals, is stable under small perturbations of the objective function (integrand) f.

Theorem 4.1. *Let $\epsilon, M > 0$. Then there exist $L_0 \geq 1$ and $\delta_0 > 0$ such that for each $L_1 \geq L_0$ there exists a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$ such that the following assertion holds.*

Assume that $T > 2L_1$, $g \in \mathcal{U}$, $(x,u) \in X(A,B,0,T)$ and that a finite sequence of numbers $\{S_i\}_{i=0}^q$ satisfy

$$S_0 = 0,\ S_{i+1} - S_i \in [L_0, L_1],\ i = 0,\dots,q-1,\ S_q \in (T - L_1, T],$$
$$I^g(S_i, S_{i+1}, x, u) \leq (S_{i+1} - S_i)\mu(f) + M$$

for each integer $i \in [0, q-1]$,

$$I^g(S_i, S_{i+2}, x, u) \leq \sigma(g, x(S_i), x(S_{i+2}), S_i, S_{i+2}) + \delta_0$$

for each nonnegative integer $i \leq q-2$ and

$$I^g(S_{q-2}, T, x, u) \leq \sigma(g, x(S_{q-2}), x(T), S_{q-2}, T) + \delta_0.$$

Then there exist $p_1, p_2 \in [0,T]$ such that $p_1 \leq p_2$, $p_1 \leq 2L_0$, $p_2 > T - 2L_1$ and that

$$|x(t) - x_f| \leq \epsilon \text{ for all } t \in [p_1, p_2].$$

Moreover if $|x(0) - x_f| \leq \delta$, then $p_1 = 0$ and if $|x(T) - x_f| \leq \delta$, then $p_2 = T$.

Theorem 4.2. *Let $\epsilon \in (0,1)$, $M_0, M_1 > 0$. Then there exist $L > 0$, $\delta \in (0,\epsilon)$ and a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$ such that for each $T > 2L$, each $g \in \mathcal{U}$ and each $(x,u) \in X(A,B,0,T)$ which satisfies for each $S \in [0, T-L]$,*

$$I^g(S, S+L, x, u) \le \sigma(g, x(S), x(S+L), S, S+L) + \delta$$

and satisfies at least one of the following conditions below

(a) $|x(0)|,\ |x(T)| \le M_0, \quad I^g(0,T,x,u) \le \sigma(g, x(0), x(T), 0, T) + M_1;$
(b) $|x(0)| \le M_0, \quad I^g(0,T,x,u) \le \sigma(g, x(0), 0, T) + M_1;$
(c) $I^g(0,T,x,u) \le \sigma(g, 0, T) + M_1$

there exist $p_1 \in [0,L]$, $p_2 \in [T-L, T]$ such that

$$|x(t) - x_f| \le \epsilon \text{ for all } t \in [p_1, p_2].$$

Moreover if $|x(0) - x_f| \le \delta$, then $p_1 = 0$ and if $|x(T) - x_f| \le \delta$, then $p_2 = T$.

Theorem 4.3. *Let $\epsilon \in (0,1)$, $M_0, M_1 > 0$. Then there exist $l > 0$, an integer $Q \ge 1$ and a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$ such that for each $T > lQ$, each $g \in \mathcal{U}$ and each $(x,u) \in X(A,B,0,T)$ which satisfies at least one of the following conditions below*

(a) $|x(0)|,\ |x(T)| \le M_0, \quad I^g(0,T,x,u) \le \sigma(g, x(0), x(T), 0, T) + M_1;$
(b) $|x(0)| \le M_0, \quad I^g(0,T,x,u) \le \sigma(g, x(0), 0, T) + M_1;$
(c) $I^g(0,T,x,u) \le \sigma(g, 0, T) + M_1$

there exist strictly increasing sequences of numbers $\{a_i\}_{i=1}^q$, $\{b_i\}_{i=1}^q \subset [0,T]$ such that $q \le Q$, for all $i = 1, \dots, q$,

$$0 \le b_i - a_i \le l,$$

$b_i \le a_{i+1}$ for all integers i satisfying $1 \le i < q$ and that

$$|x(t) - x_f| \le \epsilon \text{ for all } t \in [0,T] \setminus \cup_{i=1}^q [a_i, b_i].$$

In Chapter 4 of [45] we also prove the following two stability results. They show that the convergence of approximate solutions on large intervals, in the regions close to the end points, is stable under small perturbations of the objective function (integrand) f.

Theorem 4.4. *Let $L_0 > 0$, $\epsilon \in (0,1)$, $M > 0$. Then there exist $\delta > 0$, a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$ and $L_1 > L_0$ such that for each $T \ge L_1$, each $g \in \mathcal{U}$ and each $(x,u) \in X(A,B,0,T)$ which satisfies*

$$|x(0)| \le M,\ I^g(0,T,x,u) \le \sigma(g, x(0), 0, T) + \delta$$

there exists an $(f, -A, -B)$-overtaking optimal pair

$$(x_*, u_*) \in X(-A, -B, 0, \infty)$$

such that

$$\pi_-^f(x_*(0)) = \inf(\pi_-^f),$$

$$|x(T-t) - x_*(t)| \le \epsilon \text{ for all } t \in [0, L_0].$$

Theorem 4.5. *Let $L_0 > 0$, $\epsilon \in (0,1)$. Then there exist $\delta > 0$, a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$ and $L_1 > L_0$ such that for each $T \geq L_1$, each $g \in \mathcal{U}$ and each $(x,u) \in X(A,B,0,T)$ which satisfies*

$$I^g(0,T,x,u) \leq \sigma(g,0,T) + \delta$$

there exist an (f,A,B)-overtaking optimal pair $(x_*,u_*) \in X(A,B,0,\infty)$ and an $(f,-A,-B)$-overtaking optimal pair $(\bar{x}_*,\bar{u}_*) \in X(-A,-B,0,\infty)$ such that

$$\pi^f(x_*(0)) = \inf(\pi^f),$$

$$\pi^f_-(\bar{x}_*(0)) = \inf(\pi^f_-)$$

and for all $t \in [0,L_0]$,

$$|x(t) - x_*(t)| \leq \epsilon, \ |x(T-t) - \bar{x}_*(t)| \leq \epsilon.$$

5. Bolza optimal control problems

We use the notation, definitions and assumptions introduced in Sections 1-4. Recall that $f : R^n \times R^m \to R^1$ is a continuous function which satisfy assumptions (A1)-(A3) and (1.6).

Let $a_1 > 0$. Denote by $\mathfrak{A}(R^n)$ the set of all lower semicontinuous functions $h : R^n \to R^1$ which are bounded on bounded subsets of R^n and satisfy

$$h(z) \geq -a_1 \text{ for all } z \in R^n. \tag{5.1}$$

For simplicity we set $\mathfrak{A} = \mathfrak{A}(R^n)$. We equip the set $\mathfrak{A}$ with the uniformity which is determined by the following base:

$$E(N,\epsilon) = \{(h_1,h_2) \in \mathfrak{A} \times \mathfrak{A} : \ |h_1(z) - h_2(z)| \leq \epsilon$$

$$\text{for each } z \in R^n \text{ satisfying } |z| \leq N\}, \tag{5.2}$$

where $N > 0$, $\epsilon > 0$. Clearly, the uniform space $\mathfrak{A}$ is metrizable and complete.

We consider the following optimal control problems

$$I^g(T_1,T_2,x,u) + h(x(T_2)) \to \min,$$

$$(x,u) \in X(A,B,T_1,T_2) \text{ such that } x(T_1) = y$$

and

$$I^g(T_1,T_2,x,u) + h(x(T_2)) + \xi(x(T_1)) \to \min,$$

$$(x,u) \in X(A,B,T_1,T_2),$$

where $y \in R^n$, $\infty > T_2 > T_1 > -\infty$, $g \in \mathfrak{M}$ and $h, \xi \in \mathfrak{A}$.

Let $y, z \in R^n$, $T_1 \in R^1$, $T_2 > T_1$, $g : [T_1,T_2] \times R^n \times R^m \to R^1$ be a borelian bounded from below function and $h, \xi \in \mathfrak{A}$. Set

$$\sigma(g,h,y,T_1,T_2) = \inf\{I^g(T_1,T_2,x,u) + h(x(T_2)) :$$

$$(x,u) \in X(A,B,T_1,T_2) \text{ and } x(T_1) = y\}, \tag{5.3}$$

$$\sigma(g,h,\xi,T_1,T_2) = \inf\{I^g(T_1,T_2,x,u) + h(x(T_2)) + \xi(x(T_1)) :$$

$$(x,u) \in X(A,B,T_1,T_2)\}, \tag{5.4}$$

$$\widehat{\sigma}(g,\xi,z,T_1,T_2) = \inf\{I^g(T_1,T_2,x,u) + \xi(x(T_1)) :$$

$$(x,u) \in X(A,B,T_1,T_2) \text{ and } x(T_2)=z\}, \tag{5.5}$$

$$\sigma(g,h,\xi,y,z,T_1,T_2) = \inf\{I^g(T_1,T_2,x,u) + h(x(T_2))$$

$$+\xi(x(T_1)) : (x,u) \in X(A,B,T_1,T_2),\ x(T_2)=z,\ x(T_1)=y\}. \tag{5.6}$$

We prove the following turnpike results for our Bolza optimal control problems which show that the turnpike phenomenon, for approximate solutions on large intervals, is stable under small perturbations of the objective functions.

Theorem 5.1. *Let $\epsilon \in (0,1)$, $M_0, M_1, M_2 > 0$. Then there exist $l > 0$, an integer $Q \geq 1$ and a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$ such that for each $T > lQ$, each $g \in \mathcal{U}$, each $h, \xi \in \mathfrak{A}$ satisfying*

$$|h(x_f)|,\ |\xi(x_f)| \leq M_2$$

and each $(x,u) \in X(A,B,0,T)$ which satisfies at least one of the following conditions below

(a) $|x(0)| \leq M_0, \quad I^g(0,T,x,u) + h(x(T)) \leq \sigma(g,h,x(0),0,T) + M_1$;
(b) $I^g(0,T,x,u) + h(x(T)) + \xi(x(0)) \leq \sigma(g,h,\xi,0,T) + M_1$

there exist strictly increasing sequences of numbers $\{a_i\}_{i=1}^q$, $\{b_i\}_{i=1}^q \subset [0,T]$ such that $q \leq Q$, for all $i = 1,\dots,q$,

$$0 \leq b_i - a_i \leq l,$$

$b_i \leq a_{i+1}$ for all integers i satisfying $1 \leq i < q$ and that

$$|x(t) - x_f| \leq \epsilon \text{ for all } t \in [0,T] \setminus \cup_{i=1}^q [a_i,b_i].$$

Theorem 5.2. *Let $\epsilon \in (0,1)$, $M_0, M_1, M_2 > 0$. Then there exist $L > 0$, $\delta \in (0,\epsilon)$ and a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$ such that for each $T > 2L$, each $g \in \mathcal{U}$, each $h, \xi \in \mathfrak{A}$ satisfying*

$$|h(x_f)|,\ |\xi(x_f)| \leq M_2$$

and each $(x,u) \in X(A,B,0,T)$ which satisfies for each $S \in [0, T-L]$,

$$I^g(S,S+L,x,u) \leq \sigma(g,x(S),x(S+L),S,S+L) + \delta$$

and satisfies at least one of the following conditions below

(a) $|x(0)| \leq M_0, I^g(0,T,x,u) + h(x(T)) \leq \sigma(g,h,x(0),0,T) + M_1$;
(b) $I^g(0,T,x,u) + h(x(T)) + \xi(x(0)) \leq \sigma(g,h,\xi,0,T) + M_1$

there exist $p_1 \in [0,L]$, $p_2 \in [T-L,T]$ such that

$$|x(t) - x_f| \leq \epsilon \text{ for all } t \in [p_1,p_2].$$

Moreover if $|x(0) - x_f| \leq \delta$, then $p_1 = 0$ and if $|x(T) - x_f| \leq \delta$, then $p_2 = T$.

In this paper we also prove the following two stability results for our Bolza optimal control problems. They show that the convergence of approximate solutions on large intervals, in the regions close to the end points, is stable under small perturbations of the objective functions.

Theorem 5.3. *Let $L_0 > 0$, $\epsilon \in (0,1)$, $M > 0$ and $h \in \mathfrak{A}$. Then there exist $\delta > 0$, a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$, a neighborhood $\mathcal{V}$ of h in $\mathfrak{A}$ and $L_1 > L_0$ such that for each $T \geq L_1$, each $g \in \mathcal{U}$, each $\xi \in \mathcal{V}$ and each $(x,u) \in X(A,B,0,T)$ which satisfies*

$$|x(0)| \leq M,$$

$$I^g(0,T,x,u) + \xi(x(T)) \leq \sigma(g,\xi,x(0),0,T) + \delta$$

there exists an $(f,-A,-B)$-overtaking optimal pair

$$(x_*,u_*) \in X(-A,-B,0,\infty)$$

such that

$$(\pi_-^f + h)(x_*(0)) = \inf(\pi_-^f + h),$$

$$|x(T-t) - x_*(t)| \leq \epsilon \text{ for all } t \in [0,L_0].$$

Theorem 5.4. *Let $L_0 > 0$, $\epsilon \in (0,1)$, $h_1, h_2 \in \mathcal{A}$. Then there exist $\delta > 0$, a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$, a neighborhood $\mathcal{V}_i$ of h_i, $i = 1,2$ in $\mathfrak{A}$ and $L_1 > L_0$ such that for each $T \geq L_1$, each $g \in \mathcal{U}$, each $\xi_i \in \mathcal{V}_i$, $i = 1,2$ and each $(x,u) \in X(A,B,0,T)$ which satisfies*

$$I^g(0,T,x,u) + \xi_1(x(T)) + \xi_2(x(0)) \leq \sigma(g,\xi_1,\xi_2,0,T) + \delta$$

there exist an (f,A,B)-overtaking optimal pair $(x_*,u_*) \in X(A,B,0,\infty)$ and an $(f,-A,-B)$-overtaking optimal pair $(\bar{x}_*,\bar{u}_*) \in X(-A,-B,0,\infty)$ such that

$$(\pi^f + h_2)(x_*(0)) = \inf(\pi^f + h_2),$$

$$(\pi_-^f + h_1)(\bar{x}_*(0)) = \inf(\pi_-^f + h_1)$$

and for all $t \in [0,L_0]$,

$$|x(t) - x_*(t)| \leq \epsilon, \ |x(T-t) - \bar{x}_*(t)| \leq \epsilon.$$

6. Auxiliary results for Theorems 5.1 and 5.2

In the sequel we use the following auxiliary results.

Proposition 6.1 (Proposition 3.27 of [45]). *Let $T > 0$ and $y,z \in R^n$. Then there exists $(x,u) \in X(A,B,0,T)$ such that*

$$x(0) = y, \ x(T) = z,$$

$$I^f(0,T,x,u) = \sigma(f,y,z,T).$$

Proposition 6.2 (Proposition 4.5 of [43], Proposition 3.28 of [45]). *Let $M, \tau > 0$. Then*

$$\sup\{|\sigma(f,y,z,\tau)| : \ y,z \in R^n, \ |y|,|z| \leq M\} < \infty.$$

Proposition 6.3 (Proposition 2.9 of [43]). *Let $g \in \mathfrak{M}$, $0 < c_1 < c_2$ and $D, \epsilon > 0$. Then there exists a neighborhood $\mathcal{U}$ of g in $\mathfrak{M}$ such that for each $h \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \in [T_1 + c_1, T_1 + c_2]$ and each trajectory-control pair $(x,u) \in X(A,B,T_1,T_2)$ which satisfies*

$$\min\{I^g(T_1,T_2,x,u), \ I^h(T_1,T_2,x,u)\} \leq D$$

the inequality

$$|I^g(T_1,T_2,x,u) - I^h(T_1,T_2,x,u)| \leq \epsilon$$

holds.

Proposition 6.4 (Proposition 4.2 of [43]). *Let $T_2 > T_1$ be real numbers, $\{(x_j, u_j)\}_{j=1}^{\infty} \subset X(A, B, T_1, T_2)$ and let the sequence $\{I^f(T_1, T_2, x_j, u_j)\}_{j=1}^{\infty}$ be bounded. Then there exist a subsequence $\{(x_{j_k}, u_{j_k})\}_{k=1}^{\infty}$ and $(x, u) \in X(A, B, T_1, T_2)$ such that*

$$x_{j_k}(t) \to x(t) \text{ as } k \to \infty \text{ uniformly in } [T_1, T_2],$$
$$u_{j_k} \to u \text{ as } k \to \infty \text{ weakly in } L^1(R^m; (T_1, T_2)),$$
$$I^f(T_1, T_2, x, u) \leq \liminf_{k\to\infty} I^f(T_1, T_2, x_{j_k}, u_{j_k}).$$

Proposition 6.5 (Proposition 4.6 of [43]). *Let $M, \tau, \epsilon > 0$. Then there exists a number $\delta > 0$ such that for each $y_1, y_2, z_1, z_2 \in R^n$ satisfying*

$$|y_i|, |z_i| \leq M,\ i = 1, 2,\ |y_1 - y_2|,\ |z_1 - z_2| \leq \delta$$

the following relation holds:

$$|\sigma(f, y_1, z_1, \tau) - \sigma(f, y_2, z_2, \tau)| \leq \epsilon.$$

Proposition 6.6 (Proposition 2.7 of [43]). *Let $M_1 > 0$ and $0 < \tau_0 < \tau_1$. Then there exists a positive number M_2 such that for each $T_1 \in R^1$, each $T_2 \in [T_1 + \tau_0, T_1 + \tau_1]$ and each $(x, u) \in X(A, B, T_1, T_2)$ satisfying*

$$I^f(T_1, T_2, x, u) \leq M_1$$

the inequality $|x(t)| \leq M_2$ holds for all $t \in [T_1, T_2]$.

Lemma 6.7. *Let $M_0, M_1, M_2 > 0$. Then there exist $L_0 > 0$, $M_3 > M_0$ and a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$ such that for each $T \geq L_0$, each $g \in \mathcal{U}$, each $h, \xi \in \mathfrak{A}$ satisfying*

$$|h(x_f)|,\ |\xi(x_f)| \leq M_2,$$

each $(x, u) \in X(A, B, 0, T)$ and each $T_1, T_2 \in [0, T]$ which satisfy $T_1 < T_2$ and satisfy at least one of the following conditions below

(a) $|x(0)| \leq M_0, \quad I^g(0, T, x, u) + h(x(T)) \leq \sigma(g, h, x(0), 0, T) + M_1,$ $|x(t)| \geq M_3,\ t \in [T_1, T_2], \quad T_2 = T,\ |x(T_1)| = M_3;$

(b) $I^g(0, T, x, u) + h(x(T)) + \xi(x(0)) \leq \sigma(g, h, \xi, 0, T) + M_1, \quad |x(t)| \geq M_3,\ t \in [T_1, T_2]$

and either $T_1 = 0$, $|x(T_2)| = M_3$ or $T_2 = T$, $|x(T_1)| = M_3$ or $T_1 = 0$, $T_2 = T$.

Then $T_2 - T_1 \leq L_0$.

Proof. By (1.2) there exists a number M_3 such that

$$M_3 > M_0 + |x_f|,$$

$$\psi(M_3) > |f(x_f, u_f)| + 2 + a_0. \tag{6.1}$$

There exists a neighborhood $\mathcal{U}_0$ of F in $\mathfrak{M}_b$ such that

$$|g(t, x_f, u_f) - f(x_f, u_f)| \leq 4^{-1} \text{ for all } t \in R^1 \text{ and all } g \in \mathcal{U}_0. \tag{6.2}$$

By Proposition 6.2, there exists $M_4 > 0$ such that

$$|\sigma(f, y, z, 1)| \leq M_4 \text{ for all } y, z \in R^n \text{ satisfying } |y|, |z| \leq M_3. \tag{6.3}$$

By Proposition 6.3, there exists a neighborhood

$$\mathcal{U} \subset \mathcal{U}_0 \tag{6.4}$$

of F in $\mathfrak{M}_b$ such that for each $h \in \mathcal{U}$, each $\tau \in R^1$ and each trajectory-control pair $(x, u) \in X(A, B, \tau, \tau + 1)$ which satisfies

$$\min\{I^h(\tau, \tau + 1, x, u),\ I^f(\tau, \tau + 1, x, u)\} \le M_4 + 1$$

the inequality

$$|I^h(\tau, \tau + 1, x, u) - I^f(\tau, \tau + 1, x, u)| \le 1 \tag{6.5}$$

holds.

Fix

$$L_0 > 4 + 2M_2 + 2a_1 + 2M_1 + 2M_4. \tag{6.6}$$

Assume that

$$T \ge L_0,\ g \in \mathcal{U}, h, \xi \in \mathfrak{A},\ |h(x_f)|,\ |\xi(x_f)| \le M_2, \tag{6.7}$$

$(x, u) \in X(A, B, 0, T)$, $T_1, T_2 \in [0, T]$, $T_1 < T_2$,

$$|x(t)| \ge M_3,\ t \in [T_1, T_2] \tag{6.8}$$

and that at least one of the conditions (a), (b) holds. We show that $T_2 - T_1 \le L_0$.

Assume the contrary. Then

$$T_2 - T_1 > L_0. \tag{6.9}$$

We construct $(y, v) \in X(A, B, T_1, T_2)$. If (b) holds and

$$T_1 = 0,\ T_2 = T,$$

then we set

$$y(t) = x_f,\ v(t) = u_f,\ t \in [0, T]. \tag{6.10}$$

If

$$T_2 = T,\ |x(T_1)| = M_3,$$

then in view of Proposition 6.1 there exists $(y, v) \in X(A, B, T_1, T)$ such that

$$y(T_1) = x(T_1),\ y(T_1 + 1) = x_f,$$

$$I^f(T_1, T_1 + 1, y, v) = \sigma(f, x(T_1), x_f, 1), \tag{6.11}$$

$$y(t) = x_f,\ v(t) = u_f,\ t \in [T_1 + 1, T].$$

If (b) holds and $T_1 = 0$, $|x(T_2)| = M_3$, then in view of Proposition 6.1, there exists $(y, v) \in X(A, B, 0, T_2)$ such that

$$y(T_2) = x(T_2),\ y(T_2 - 1) = x_f,$$

$$I^f(T_2 - 1, T_2, y, v) = \sigma(f, x_f, x(T_2), 1),$$

$$y(t) = x_f,\ v(t) = u_f,\ t \in [T_1, T_2 - 1]. \tag{6.12}$$

Set

$$y(t) = x(t),\ v(t) = u(t) \text{ for all } t \in [0, T] \setminus [T_1, T_2]. \tag{6.13}$$

Clearly,

$$(y, v) \in X(A, B, 0, T). \tag{6.14}$$

It follows from (4.1), (6.1) and (6.8) that for all $t \in [T_1, T_2]$,

$$g(t, x(t), u(t)) \geq \psi(M_3) - a_0 > |f(x_f, u_f)| + 2. \tag{6.15}$$

Relations (6.2), (6.4) and (6.7) imply that

$$|g(t, x_f, u_f)| \leq |f(x_f, u_f)| + 4^{-1} \text{ for all } t \in R^1. \tag{6.16}$$

Assume that (b) holds and $T_1 = 0,\ T_2 = T$. Then by (6.10), (6.15) and (6.16),

$$I^g(0, T, x, u) \geq T(|f(x_f, u_f)| + 2),$$

$$I^g(0, T, y, v) \leq T(|f(x_f, u_f)| + 4^{-1}).$$

By the relations above, condition (b), (5.1), (6.7) and (6.10),

$$\begin{aligned} T(|f(x_f, u_f)| + 2) - 2a_1 &\leq I^g(0, T, x, u) + h(x(T)) + \xi(x(0)) \\ &\leq I^g(0, T, y, v) + h(y(T)) + \xi(y(0)) + M_1 \\ &\leq T(|f(x_f, u_f)| + 4^{-1}) + 2M_2 + M_1 \end{aligned}$$

and in view of (6.6),

$$T_2 - T_1 = T \leq 2M_2 + 2a_1 + M_1 < L_0.$$

This contradicts (6.9).

Assume that

$$T_2 = T,\ |x(T_1)| = M_3. \tag{6.17}$$

By (6.1), (6.3), (6.11) and (6.17),

$$|I^f(T_1, T_1 + 1, y, v)| = |\sigma(f, x(T_1), x_f, 1)| \leq M_4. \tag{6.18}$$

In view of the choice of $\mathcal{U}$, (6.5), (6.7) and (6.18),

$$I^g(T_1, T_1 + 1, y, v) \leq M_4 + 1. \tag{6.19}$$

In the case (a), it follows from (5.1), (6.7), (6.15) and (6.17) that

$$\begin{aligned} I^g(0, T, x, u) + h(x(T)) &\geq I^g(0, T_1, x, u) + I^g(T_1, T, x, u) - a_1 \\ &\geq I^g(0, T_1, x, u) - a_1 + (T - T_1)(|f(x_f, u_f)| + 2), \end{aligned} \tag{6.20}$$

by (6.6), (6,7), (6.9), (6.11), (6.13), (6.17) and (6.19),

$$\begin{aligned} I^g(0, T, y, v) + h(y(T)) &= I^g(0, T_1, x, u) + I^g(T_1, T, y, v) + h(x_f) \\ &\leq I^g(0, T_1, x, u) + I^g(T_1, T_1 + 1, y, v) \\ &\quad + I^g(T_1 + 1, T, y, v) + M_2 \\ &\leq I^g(0, T_1, x, u) + M_4 + 1 \\ &\quad + (T - T_1 - 1) \sup\{g(t, x_f, u_f) :\ t \in [T_1 + 1, T]\} \\ &\quad + M_2 \\ &\leq I^g(0, T_1, x, u) + M_4 + 1 \\ &\quad + (T - T_1 - 1)(|f(x_f, u_f)| + 4^{-1}) + M_2, \end{aligned} \tag{6.21}$$

and in view of condition (a), (6.11), (6.13), (6.20) and (6.21),

$$\begin{aligned} M_2 + I^g(0,T_1,x,u) + M_4 + 1 + (T-T_1-1)(|f(x_f,u_f)| + 4^{-1}) \\ &\geq I^g(0,T,y,v) + h(y(T)) \\ &\geq -M_1 + I^g(0,T,x,u) + h(x(T)) \\ &\geq -M_1 - a_1 + I^g(0,T_1,x,u) + (T-T_1)(|f(x_f,u_f)|+2) \end{aligned}$$

and

$$T_2 - T_1 = T - T_1 \leq M_2 + M_4 + 1 + a_1 < L_0.$$

This contradicts (6.9).

In the case (b), it follows from (5.1), (6.7), (6.15) and (6.17) that

$$\begin{aligned} I^g(0,T,x,u) + h(x(T)) + \xi(x(0)) &\geq I^g(0,T_1,x,u) \\ &\quad + I^g(T_1,T,x,u) - a_1 + \xi(x(0)) \\ &\geq I^g(0,T_1,x,u) - a_1 \\ &\quad + (T-T_1)(|f(x_f,u_f)|+2) + \xi(x(0)), \end{aligned} \tag{6.22}$$

by (6.11), (6,13), (6.16) and (6.19),

$$\begin{aligned} I^g(0,T,y,v) + h(y(T)) + \xi(y(0)) &= I^g(0,T_1,x,u) + I^g(T_1,T,y,v) \\ &\quad + h(x_f) + \xi(x(0)) \\ &\leq I^g(0,T_1,x,u) + I^g(T_1,T_1+1,y,v) \\ &\quad + I^g(T_1+1,T,y,v) + M_2 + \xi(x(0)) \\ &\leq I^g(0,T_1,x,u) + M_4 + 1 \\ &\quad + \int_{T_1+1}^{T} g(t,x_f,u_f)dt + M_2 + \xi(x(0)) \\ &\leq I^g(0,T_1,x,u) + M_4 + 1 \\ &\quad + (T-T_1-1)(|f(x_f,u_f)| + 4^{-1}) \\ &\quad + M_2 + \xi(x(0)), \end{aligned} \tag{6.23}$$

and in view of condition (b), (6.22) and (6.23),

$$\begin{aligned} I^g(0,T_1,x,u) + \xi(x(0)) + M_4 + 1 + (T-T_1-1)(|f(x_f,u_f)| + 4^{-1}) + M_2 \\ &\geq I^g(0,T,y,v) + h(y(T)) + \xi(y(0)) \\ &\geq -M_1 + I^g(0,T,x,u) + h(x(T)) + \xi(x(0)) \\ &\geq -M_1 + I^g(0,T_1,x,u) + \xi(x(0)) - a_1 + (T-T_1)(|f(x_f,u_f)|+2) \end{aligned}$$

and by (6.17),

$$T_2 - T_1 = T - T_1 \leq M_1 + a_1 + M_4 + 1 + M_2 < L_0.$$

This contradicts (6.9).

Assume that

$$T_1 = 0, \ |x(T_2)| = M_3. \tag{6.24}$$

We need to consider only the case (b). By (6.3), (6.12), (6.14) and (6.24),

$$|I^f(T_2-1,T_2,y,v)| = |\sigma(f,x_f,x(T_2),1)| \leq M_4. \tag{6.25}$$

In view of the choice of $\mathcal{U}$, (6.5), (6.7) and (6.25),

$$I^g(T_2-1,T_2,y,v) \le M_4+1. \tag{6.26}$$

It follows from (5.1), (6.15) and (6.24) that

$$\begin{aligned} I^g(0,T,x,u)+h(x(T))+\xi(x(0)) &\ge I^g(0,T_2,x,u)+I^g(T_2,T,x,u) \\ &\quad + h(x(T))-a_1 \\ &\ge h(x(T))-a_1+I^g(T_2,T,x,u) \\ &\quad + T_2(|f(x_f,u_f)|+2), \end{aligned} \tag{6.27}$$

by (6.7), (6.12), (6.13), (6.16), (6.24) and (6.26),

$$\begin{aligned} I^g(0,T,y,v)+h(y(T))+\xi(y(0)) &= I^g(0,T_2-1,y,v)+I^g(T_2-1,T_2,y,v) \\ &\quad + I^g(T_2,T,x,u)+h(x(T)))+\xi(x_f) \\ &\le (T_2-1)(|f(x_f,u_f)|+4^{-1}) \\ &\quad + M_4+1+I^g(T_2,T,x,u) \\ &\quad + h(x(T))+M_2. \end{aligned} \tag{6.28}$$

It follows from (6.27) and (6.28) that

$$\begin{aligned} h(x(T)) - a_1 &+ I^g(T_2,T,x,u)+T_2(|f(x_f,u_f)|+2) \\ &\le I^g(0,T,x,u)+h(x(T))+\xi(x(0)) \\ &\le M_1+I^g(0,T,y,v)+h(y(T))+\xi(y(0)) \\ &\le M_1+(T_2-1)(|f(x_f,u_f)|+4^{-1}) \\ &\quad + M_4+1+I^g(T_2,T,x,u)+h(x(T))+M_2 \end{aligned}$$

and

$$T_2 \le a_1+M_1+M_4+1+M_2 < L_0.$$

This contradicts (6.9).

Thus in all the cases we have reached the contradiction which shows that $T_2-T_1 \le L_0$. Lemma 6.7 is proved. □

7. Proof of Theorem 5.1

By Lemma 6.7, there exist $L_0>0$, $M_3>M_0$ and a neighborhood $\mathcal{U}_1$ of F in $\mathfrak{M}_b$ such that the following property holds:

(P1) for each $T \ge L_0$, each $g \in \mathcal{U}_1$, each $h,\xi \in \mathfrak{A}$ satisfying

$$|h(x_f)|,\ |\xi(x_f)| \le M_2,$$

each $(x,u) \in X(A,B,0,T)$ and each $T_1,T_2 \in [0,T]$ which satisfies $T_1<T_2$ and satisfies at least one of the following conditions below

- (i) $|x(0)| \le M_0, \quad I^g(0,T,x,u)+h(x(T)) \le \sigma(g,h,x(0),0,T)+M_1,$
  $|x(t)| \ge M_3,\ t \in [T_1,T_2], \quad T_2=T,\ |x(T_1)|=M_3;$
- (ii) $I^g(0,T,x,u)+h(x(T))+\xi(x(0)) \le \sigma(g,h,\xi,0,T)+M_1$
  $|x(t)| \ge M_3,\ t \in [T_1,T_2]$

and either $T_1 = 0,\ |x(T_2)| = M_3$ or $T_2 = T,\ |x(T_1)| = M_3$ or $T_1 = 0,\ T_2 = T$, then $T_2 - T_1 \le L_0$.

By Theorem 4.3, there exist $l > L_0$, an integer $Q > 2$ and a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of F in $\mathfrak{M}_b$ such that the following property holds:

(P2) for each $T > l(Q-2)$, each $g \in \mathcal{U}$ and each $(x,u) \in X(A,B,0,T)$ which satisfies

$$|x(0)|,\ |x(T)| \le M_3,$$
$$I^g(0,T,x,u) \le \sigma(g,x(0),x(T),0,T) + M_1$$

there exist strictly increasing sequences of numbers $\{a_i\}_{i=1}^q,\ \{b_i\}_{i=1}^q \subset [0,T]$ such that $q \le Q-2$, for all $i = 1,\dots,q$,

$$0 \le b_i - a_i \le l,$$

$b_i \le a_{i+1}$ for all integers i satisfying $1 \le i < q$ and that

$$|x(t) - x_f| \le \epsilon \text{ for all } t \in [0,T] \setminus \cup_{i=1}^q [a_i,b_i].$$

Assume that

$$T > lQ,\ g \in \mathcal{U},\ h,\xi \in \mathfrak{A},\ |h(x_f)|,\ |\xi(x_f)| \le M_2 \tag{7.1}$$

and $(x,u) \in X(A,B,0,T)$ satisfies at least one of the conditions (a), (b) of Theorem 5.1.

Assume that (a) holds. Set $T_1 = 0$,

$$T_2 = \sup\{t \in [0,T] :\ |x(t)| \le M_3\}. \tag{7.2}$$

In view of (a) and the relation $M_3 > M_0$, T_2 is well-defined. Property (P1), condition (a), (7.1), (7.2) and the inequality $l > L_0$ imply that

$$T - T_2 \le L_0. \tag{7.3}$$

In view of condition (a),

$$I^g(T_1,T_2,x,u) \le \sigma(g,x(T_1),x(T_2),T_1,T_2) + M_1. \tag{7.4}$$

Assume that (b) holds. If $|x(t)| \ge M_3$ for all $t \in [0,T]$, then by (7.1) and property (P1), $T \le L_0 < l$, a contradiction. Thus there exists $t_0 \in [0,T]$ such that

$$|x(t_0)| < M_3.$$

Set

$$T_1 = \min\{t \in [0,T] :\ |x(t)| \le M_3\},$$
$$T_2 = \max\{t \in [0,T] :\ |x(t)| \le M_3\}. \tag{7.5}$$

Clearly, T_1, T_2 are well-defined and $T_1 < T_2$. Relations (7.1) and (7.5) imply that

$$T_1 \le L_0,\ T - T_2 \le L_0. \tag{7.6}$$

By (7.5), (7.6) and condition (b),

$$I^g(T_1,T_2,x,u) \le \sigma(g,x(T_1),x(T_2),T_1,T_2) + M_1. \tag{7.7}$$

Hence (7.7) is true in the both cases. In view of (7.1), (7.7) and property (P2), there exist strictly increasing sequences of numbers $\{a_i\}_{i=1}^q,\ \{b_i\}_{i=1}^q \subset [T_1,T_2]$ such that $q \le Q-2$, for all $i = 1,\dots,q$,

$$0 \le b_i - a_i \le l,$$

$b_i \le a_{i+1}$ for all integers i satisfying $1 \le i < q$ and that

$$|x(t) - x_f| \le \epsilon \text{ for all } t \in [T_1, T_2] \setminus \cup_{i=1}^{q} [a_i, b_i].$$

This completes the proof of Theorem 5.1.

8. Proof of Theorem 5.2

By Theorem 4.2, there exist $L_0 > 0$, $\delta \in (0, \epsilon)$ and a neighborhood $\mathcal{U}_1$ of F in $\mathfrak{M}_b$ such that the following property holds:

(P3) For each $T > 2L_0$, each $g \in \mathcal{U}_1$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies for each $S \in [0, T - L_0]$,

$$I^g(S, S + L_0, x, u) \le \sigma(g, x(S), x(S + L_0), S, S + L_0) + \delta$$

and satisfies

$$I^g(0, T, x, u) \le \sigma(g, x(0), x(T), 0, T) + M_1,$$
$$|x(0) - x_f| \le \delta, \ |x(T) - x_f| \le \delta$$

we have

$$|x(t) - x_f| \le \epsilon \text{ for all } t \in [0, T].$$

By Theorem 5.1, there exist $l_0 > 0$, an integer $Q \ge 1$ and a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of F in $\mathfrak{M}_b$ such that the following property holds:

(P4) for each $T > l_0 Q$, each $g \in \mathcal{U}$, each $h, \xi \in \mathfrak{A}$ satisfying

$$|h(x_f)|, \ |\xi(x_f)| \le M_2$$

and each $(x, u) \in X(A, B, 0, T)$ which satisfies at least one of the following conditions below

$$|x(0)| \le M_0, \ I^g(0, T, x, u) + h(x(T)) \le \sigma(g, h, x(0), 0, T) + M_1;$$
$$I^g(0, T, x, u) + h(x(T)) + \xi(x(0)) \le \sigma(g, h, \xi, 0, T) + M_1$$

there exist strictly increasing sequences of numbers $\{a_i\}_{i=1}^{q}$, $\{b_i\}_{i=1}^{q} \subset [0, T]$ such that $q \le Q$, for all $i = 1, \dots, q$,

$$0 \le b_i - a_i \le l_0,$$

$b_i \le a_{i+1}$ for all integers i satisfying $1 \le i < q$ and that

$$|x(t) - x_f| \le \delta \text{ for all } t \in [0, T] \setminus \cup_{i=1}^{q} [a_i, b_i].$$

Set

$$L = 2L_0 + 2l_0 Q. \tag{8.1}$$

Assume that

$$T > 2L, \ g \in \mathcal{U}, \ h, \xi \in \mathfrak{A} \tag{8.2}$$

satisfy

$$|h(x_f)|, \ |\xi(x_f)| \le M_2, \tag{8.3}$$

$(x, u) \in X(A, B, 0, T)$ satisfies for each $S \in [0, T - L]$,

$$I^g(S, S + L, x, u) \le \sigma(g, x(S), x(S + L), S, S + L) + \delta \tag{8.4}$$

and satisfies at least one of the conditions (a), (b) of Theorem 5.2. By (8.1), (8.3), (8.21), property (P4) and conditions (a), (b) of Theorem 5.2 there exist $S_1, S_2 \in [0, T]$ such that

$$S_1 \leq Ql_0,\ S_2 \geq T - Ql_0, \tag{8.5}$$

$$|x(S_i) - x_f| \leq \delta,\ i = 1, 2. \tag{8.6}$$

In view of (8.1), (8.2) and (8.5),

$$S_2 - S_1 \geq 2L - 2Ql_0 > L > 2L_0. \tag{8.7}$$

If $|x(0) - x_f| \leq \delta$, we may assume that $S_1 = 0$ and if $|x(T) - x_f| \leq \delta$, we may assume that $S_2 = T$. It follows from (8.1)-(8.4), (8.6), (8.7), conditions (a), (b) and property (P3) that

$$|x(t) - x_f| \leq \epsilon,\ t \in [S_1, S_2].$$

Theorem 5.2 is proved.

9. Auxiliary results for Theorem 5.3

The following result easily follows from Proposition 3.3.

Lemma 9.1. *Let $h \in \mathfrak{A}$. Then the function $\pi^f + h$ is lower semicontinuous and bounded from below, for every number M the set*

$$\{z \in R^n :\ (\pi^f + h)(z) \leq M\}$$

is bounded and there exists $z_* \in R^n$ satisfying

$$(\pi^f + h)(z_*) = \inf(\pi^f + h).$$

Lemma 9.2. *Let $h \in \mathfrak{A}$, $S_0 > 0$, $\epsilon \in (0, 1)$. Then there exists $\delta \in (0, \epsilon)$ such that for each $(x, u) \in X(A, B, 0, S_0)$ which satisfies*

$$(\pi^f + h)(x(0)) \leq \inf(\pi^f + h) + \delta,$$

$$I^f(0, S_0, x, u) - S_0\mu(f) - \pi^f(x(0)) + \pi^f(x(S_0)) \leq \delta$$

there exists an (f, A, B)-overtaking optimal pair $(x_*, u_*) \in X(A, B, 0, \infty)$ such that

$$(\pi^f + h)(x_*(0)) = \inf(\pi^f + h),$$

$$|x(t) - x_*(t)| \leq \epsilon \text{ for all } t \in [0, S_0].$$

Proof. Assume that the lemma does not hold. Then there exist a sequence $\{\delta_k\}_{k=1}^{\infty} \subset (0, 1]$ and a sequence $\{(x_k, u_k)\}_{k=1}^{\infty} \subset X(A, B, 0, S_0)$ such that

$$\lim_{k \to \infty} \delta_k = 0 \tag{9.1}$$

and that for all integer $k \geq 1$,

$$(\pi^f + h)(x_k(0)) \leq \inf(\pi^f + h) + \delta_k, \tag{9.2}$$

$$I^f(0, S_0, x_k, u_k) - S_0\mu(f) - \pi^f(x_k(0)) + \pi^f(x_k(S_0)) \leq \delta_k \tag{9.3}$$

and that the following property holds:

(i) for each (f, A, B)-overtaking optimal pair $(y, v) \in X(A, B, 0, \infty)$ satisfying

$$(\pi^f + h)(y(0)) = \inf(\pi^f + h)$$

we have

$$\sup\{|x_k(t) - y(t)| : t \in [0, S_0]\} > \epsilon.$$

In view of (9.2) and (9.3) and the boundedness from below of the functions π^f, h the sequences $\{\pi^f(x_k(0))\}_{k=1}^{\infty}$, $\{h(x_k(0))\}_{k=1}^{\infty}$, $\{I^f(0, S_0, x_k, u_k)\}_{k=1}^{\infty}$ are bounded. By Proposition 6.4, extracting a subsequence and re-indexing if necessary, we may assume without loss of generality that there exists $(x, u) \in X(A, B, 0, S_0)$ such that

$$x_k(t) \to x(t) \text{ as } k \to \infty \text{ uniformly on } [0, S_0], \tag{9.4}$$

$$I^f(0, S_0, x, u) \le \liminf_{k\to\infty} I^f(0, S_0, x_k, u_k). \tag{9.5}$$

It follows from (9.2), (9.4) and the lower semicontinuity of π^f, h that

$$\pi^f(x(0)) \le \liminf_{k\to\infty} \pi^f(x_k(0)),\ h(x(0)) \le \liminf_{k\to\infty} h(x_k(0)),$$

$$(\pi^f + h)(x(0)) \le \liminf_{k\to\infty}(\pi^f + h)(x_k(0)) = \inf(\pi^f + h). \tag{9.6}$$

In view of (9.2) and (9.6),

$$\pi^f(x(0)) = \lim_{k\to\infty} \pi^f(x_k(0)),\ h(x(0)) = \lim_{k\to\infty} h(x_k(0)). \tag{9.7}$$

By (9.4) and the lower semicontinuity of π^f,

$$\pi^f(x(S_0)) \le \liminf_{k\to\infty} \pi^f(x_k(S_0)). \tag{9.8}$$

It follows from (9.1), (9.3), (9.5) and (9.7) that

$$\begin{aligned}
&I^f(0, S_0, x, u) - S_0\mu(f) - \pi^f(x(0)) + \pi^f(x(S_0))\\
&\le \liminf_{k\to\infty}[I^f(0, S_0, x_k, u_k) - S_0\mu(f)] - \lim_{k\to\infty} \pi^f(x_k(0)) + \liminf_{k\to\infty} \pi^f(x_k(S_0))\\
&\le \liminf_{k\to\infty}[I^f(0, S_0, x_k, u_k) - S_0\mu(f) - \pi^f(x_k(0)) + \pi^f(x_k(S_0))] \le 0.
\end{aligned}$$

In view of the inequality above and Proposition 3.1,

$$I^f(0, S_0, x, u) - S_0\mu(f) - \pi^f(x(0)) + \pi^f(x(S_0)) = 0. \tag{9.9}$$

Theorem 2.2 implies that there exists an (f, A, B)-overtaking optimal pair $(\tilde{x}, \tilde{u}) \in X(A, B, 0, \infty)$ such that

$$\tilde{x}(0) = x(S_0). \tag{9.10}$$

For all $t > S_0$ set

$$x(t) = \tilde{x}(t - S_0),\ u(t) = \tilde{u}(t - S_0). \tag{9.11}$$

It is not difficult to see that the pair $(x, u) \in X(A, B, 0, \infty)$ is an (f, A, B)-good pair. By (9.11), (9.9) and Propositions 3.1 and 3.2,

$$I^f(0, S, x, u) - S\mu(f) - \pi^f(x(0)) + \pi^f(x(S)) = 0 \text{ for all } S > 0.$$

Combined with Proposition 3.5 and (9.6) this implies that

$$(x, u) \in X(A, B, 0, \infty)$$

is an (f, A, B)-overtaking optimal pair satisfying

$$(\pi^f + h)(x(0)) = \inf(\pi^f + h).$$

By (9.4), for all sufficiently large natural numbers k,

$$|x_k(t) - x(t)| \le \epsilon/2 \text{ for all } t \in [0, S_0].$$

This contradicts the property (i). The contradiction we have reached proves Lemma 9.2. □

Note that Lemma 9.2 can also be applied for the triplet $(f, -A, -B)$.

Assume that $S_1 \in R^1$, $S_2 > S_1$ and $g \in \mathfrak{M}$. For each $(x, u) \in R^n \times R^m$ and each $t \in [S_1, S_2]$ set

$$\mathcal{L}_{S_1,S_2}(g)(t, x, u) = g(S_2 - t + S_1, x, u), \tag{9.12}$$

for each $(t, x, u) \in (-\infty, S_1) \times R^n \times R^m$ set

$$\mathcal{L}_{S_1,S_2}(g)(t, x, u) = \mathcal{L}_{S_1,S_2}(g)(S_1, x, u)$$

and for each $(t, x, u) \in (S_2, \infty) \times R^n \times R^m$ set

$$\mathcal{L}_{S_1,S_2}(g)(t, x, u) = \mathcal{L}_{S_1,S_2}(g)(S_2, x, u).$$

It is clear that $\mathcal{L}_{S_1,S_2}(g) \in \mathfrak{M}$, if $g \in \mathfrak{M}_b$, then $\mathcal{L}_{S_1,S_2}(g) \in \mathfrak{M}_b$ and that $\mathcal{L}_{S_1,S_2}$ is a self-mapping of $\mathfrak{M}$ and of $\mathfrak{M}_b$.

It is not difficult to see that the following result holds.

Proposition 9.3 (Proposition 4.10 of [45])**.** *Let V be a neighborhood of F in $\mathfrak{M}$. Then there exists a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}$ such that $\mathcal{L}_{S_1,S_2}(g) \in V$ for each $g \in \mathcal{U}$, each $S_1 \in R^1$ and each $S_2 > S_1$.*

Let $S_1 \in R^1$, $S_2 > S_1$, $g \in \mathfrak{M}$ and

$$(x, u) \in X(A, B, S_1, S_2) \ (X(-A, -B, S_1, S_2) \text{ respectively}).$$

Recall that

$$\bar{x}(t) = x(S_2 - t + S_1), \ \bar{u}(t) = u(S_2 - t + S_1), \ t \in [S_1, S_2]. \tag{9.13}$$

In view of (9.12) and (9.13),

$$\begin{aligned} &\int_{S_1}^{S_2} \mathcal{L}_{S_1,S_2}(g)(t, \bar{x}(t), \bar{u}(t))dt \\ &= \int_{S_1}^{S_2} g(S_2 - t + S_1, x(S_2 - t + S_1), u(S_2 - t + S_1))dt \\ &= \int_{S_1}^{S_2} g(t, x(t), u(t))dt. \end{aligned} \tag{9.14}$$

Let $T_2 > T_1$ be a pair of real numbers, $y, z \in R^n$, $h, \xi \in \mathfrak{A}$ and $g \in \mathfrak{M}_b$. For each $(x, u) \in X(-A, -B, T_1, T_2)$ set

$$I^g(T_1, T_2, x, u) = \int_{T_1}^{T_2} g(t, x(t), u(t))dt$$

and set

$$\begin{aligned} &\sigma_-(g, h, \xi, y, z, T_1, T_2) = \inf\{I^g(T_1, T_2, x, u) + h(x(T_2)) + \xi(x(T_1)) : \\ &(x, u) \in X(-A, -B, T_1, T_2) \text{ and } x(T_1) = y, \ x(T_2) = z\}, \end{aligned} \tag{9.15}$$

$$\sigma_-(g,h,y,T_1,T_2) = \inf\{I^g(T_1,T_2,x,u) + h(x(T_2)) :$$

$$(x,u) \in X(-A,-B,T_1,T_2) \text{ and } x(T_1) = y\}, \tag{9.16}$$

$$\widehat{\sigma}_-(g,\xi,z,T_1,T_2) = \inf\{I^g(T_1,T_2,x,u) + \xi(x(T_1)) :$$

$$(x,u) \in X(-A,-B,T_1,T_2) \text{ and } x(T_2) = z\}, \tag{9.17}$$

$$\sigma_-(g,h,\xi,T_1,T_2) = \inf\{I^g(T_1,T_2,x,u) + h(x(T_2)) + \xi(x(T_1)) :$$

$$(x,u) \in X(-A,-B,T_1,T_2)\}. \tag{9.18}$$

Relations (9.14) implies the following result.

Proposition 9.4. *Let $S_2 > S_1$ be real numbers, $M \geq 0$, $g \in \mathfrak{M}_b$ and that $(x_i,u_i) \in X(A,B,S_1,S_2)$, $i = 1,2$. Then*

$$I^g(S_1,S_2,x_1,u_1) \geq I^g(S_1,S_2,x_2,u_2) - M$$

if and only if

$$I^{\bar{g}}(S_1,S_2,\bar{x}_1,\bar{u}_1) \geq I^{\bar{g}}(S_1,S_2,\bar{x}_2,\bar{u}_2) - M,$$

where $\bar{g} = \mathcal{L}_{S_1,S_2}(g)$.

Proposition 9.4 implies the following result.

Proposition 9.5. *Let $S_2 > S_1$ be real numbers, $M \geq 0$, $g \in \mathfrak{M}_b$, $\bar{g} = \mathcal{L}_{S_1,S_2}(g)$, $h,\xi \in \mathfrak{A}$ and*

$$(x,u) \in X(A,B,S_1,S_2).$$

Then the following assertions hold:

$$I^g(S_1,S_2,x,u) + h(x(S_2)) + \xi(x(S_1)) \leq \sigma(g,h,\xi,S_1,S_2) + M$$

if and only if

$$I^{\bar{g}}(S_1,S_2,\bar{x},\bar{u}) + h(\bar{x}(S_1)) + \xi(\bar{x}(S_2)) \leq \sigma_-(\bar{g},\xi,h,S_1,S_2) + M;$$

$$I^g(S_1,S_2,x,u) + h(x(S_2)) \leq \sigma(g,h,x(S_1),S_1,S_2) + M$$

if and only if

$$I^{\bar{g}}(S_1,S_2,\bar{x},\bar{u}) + h(\bar{x}(S_1)) \leq \widehat{\sigma}_-(\bar{g},\bar{x}(S_2),S_1,S_2) + M;$$

$$I^g(S_1,S_2,x,u) + h(x(S_1)) \leq \widehat{\sigma}(g,h,x(S_2),S_1,S_2) + M$$

if and only if

$$I^{\bar{g}}(S_1,S_2,\bar{x},\bar{u}) + h(\bar{x}(S_2)) \leq \sigma_-(\bar{g},h,\bar{x}(S_1),S_1,S_2) + M.$$

10. Proof of Theorem 5.3

Lemma 10.1. *Let $L_0 > 0$, $\gamma \in (0,1)$, $M > 0$ and $h \in \mathfrak{A}$. Then there exist $\delta > 0$, a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$, a neighborhood $\mathcal{V}$ of h in $\mathfrak{A}$ and $L_1 > L_0$ such that for each $T \geq L_1$, each $g \in \mathcal{U}$, each $\xi \in \mathcal{V}$ and each $(x,u) \in X(A,B,0,T)$ which satisfies*

$$|x(0)| \leq M,$$
$$I^g(0,T,x,u) + \xi(x(T)) \leq \sigma(g,\xi,x(0),0,T) + \delta$$

the pair of functions

$$\tilde{x}(t) = x(T-t),\ \tilde{u}(t) = u(T-t),\ t \in [0,T] \tag{10.1}$$

satisfies

$$(\tilde{x},\tilde{u}) \in X(-A,-B,0,T),$$
$$(\pi_-^f + h)(\tilde{x}(0)) \leq \inf(\pi_-^f + h) + \gamma$$

and

$$I^f(0,L_0,\tilde{x},\tilde{u}) - L_0\mu(f) - \pi_-^f(\tilde{x}(0)) + \pi_-^f(\tilde{x}(L_0)) \leq \gamma.$$

Proof. In view of Propositions 3.3, 3.8 and 6.5, there exists $\delta_1 \in (0,\gamma/4)$ such that:

for each $z \in R^n$ satisfying $|z - x_f| \leq 2\delta_1$,

$$|\pi_-^f(z)| = |\pi_-^f(z) - \pi_-^f(x_f)| \leq \gamma/8; \tag{10.2}$$

for each $y, z \in R^n$ satisfying

$$|y - x_f| \leq 2\delta_1,\ |z - x_f| \leq 2\delta_1$$

we have

$$|\sigma(f,y,z,1) - \mu(f)| \leq \gamma/8. \tag{10.3}$$

By Theorem 5.2, there exist $l_0 > 0$, $\delta_2 \in (0,\delta_1/8)$, a neighborhood $\mathcal{U}_1$ of F in $\mathfrak{M}_b$ and a neighborhood $\mathcal{V}_1$ of h in $\mathfrak{A}$ such that the following property holds:

(P5) for each $T > 2l_0$, each $g \in \mathcal{U}_1$, each $\xi \in \mathcal{V}_1$ and each

$$(x,u) \in X(A,B,0,T)$$

such that

$$|x(0)| \leq M,$$
$$I^g(0,T,x,u) + \xi(x(T)) \leq \sigma(g,\xi,x(0),0,T) + \delta_2$$

we have

$$|x(t) - x_f| \leq \delta_1 \text{ for all } t \in [l_0, T - l_0]. \tag{10.4}$$

By Theorem 2.2, there exists an $(f,-A,-B)$-overtaking optimal pair

$$(\bar{x}_*,\bar{u}_*) \in X(-A,-B,0,\infty)$$

such that

$$(\pi_-^f + h)(\bar{x}_*(0)) = \inf(\pi_-^f + h). \tag{10.5}$$

Assumption (A3) implies that there exists $l_1 > 0$ such that

$$|\bar{x}_*(t) - x_f| \leq \delta_1 \text{ for all } t \geq l_1. \tag{10.6}$$

By Proposition 6.3, there exists a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of F in $\mathfrak{M}_b$ such that the following property holds:

(P6) for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \in [T_1 + 1, T_1 + 2L_0 + 2l_0 + 2l_1 + 4]$ and each trajectory-control pair $(x, u) \in X(A, B, T_1, T_2)$ which satisfies

$$\min\{I^f(T_1, T_2, x, u), I^g(T_1, T_2, x, u)\} \le (|\mu(f)| + 2)(2L_0 + 2l_0 + 2l_1 + 6) + 2 + |\pi_-^f(\bar{x}_*(0))| + |h(\bar{x}_*(0))| + a_1$$

the inequality

$$|I^f(T_1, T_2, x, u) - I^g(T_1, T_2, x, u)| \le \delta_2/8$$

holds.

By Proposition 6.6, there exists $\Delta_0 > 0$ such that such that the following property holds:

(P7) for each $T_1 \in R^1$, each $T_2 \in [T_1 + 1, T_1 + 2L_0 + 2l_0 + 2l_1 + 4]$ and each $(x, u) \in X(A, B, T_1, T_2)$ satisfying

$$I^f(T_1, T_2, x, u) \le (|\mu(f)| + 2)(2L_0 + 2l_0 + 2l_1 + 6) + 2|\pi_-^f(\bar{x}_*(0))| + |h(\bar{x}_*(0))| + a_1 + 4$$

the inequality $|x(t)| \le \Delta_0$ holds for all $t \in [T_1, T_2]$.

Let

$$\mathcal{V} = \{\xi \in \mathcal{V}_1 : \ |\xi(z) - h(z)| \le \delta_1/16 \text{ for all } z \in R^n \text{ satisfying } |z| \le 2 + |\bar{x}_*(0)| + \Delta_0\}. \tag{10.6}$$

Choose $\delta > 0$ and $L_1 > 0$ such that

$$\delta \le \delta_2/4, \ L_1 \ge 2L_0 + 2l_0 + 2l_1 + 4. \tag{10.7}$$

Assume that

$$T \ge L_1, \ g \in \mathcal{U}, \ \xi \in \mathcal{V}, \ (x, u) \in X(A, B, 0, T), \tag{10.8}$$

$$|x(0)| \le M, \tag{10.9}$$

$$I^g(0, T, x, u) + \xi(x(T)) \le \sigma(g, \xi, x(0), 0, T) + \delta \tag{10.10}$$

and that $(\tilde{x}, \tilde{u})$ is defined by (10.1).

It follows from property (P5), (10.10) and (10.6)-(10.8) that relation (10.4) is true. By (10.7) and (10.8),

$$[T - l_0 - l_1 - L_0 - 4, T - l_0 - l_1 - L_0] \subset [l_0, T - l_0 - l_1 - L_0]. \tag{10.11}$$

Relations (10.4) and (10.11) imply that

$$|x(t) - x_f| \le \delta_1 \text{ for all } t \in [T - l_0 - l_1 - L_0 - 4, T - l_0 - l_1 - L_0]. \tag{10.12}$$

By Proposition 6.1, there exists a trajectory-control pair

$$(x_1, u_1) \in X(A, B, 0, T)$$

such that

$$x_1(t) = x(t), \ u_1(t) = u(t), \ t \in [0, T - l_0 - l_1 - L_0 - 4],$$

$$x_1(t) = \bar{x}_*(T - t), \ u_1(t) = \bar{u}_*(T - t), \ t \in [T - l_0 - l_1 - L_0 - 3, T],$$

$$I^f(T - l_0 - l_1 - L_0 - 4, T - l_0 - l_1 - L_0 - 3, x_1, u_1) = \sigma(f, x(T - l_0 - l_1 - L_0 - 4), \bar{x}_*(l_0 + l_1 + L_0 + 3), 1). \tag{10.13}$$

It follows from (10.10) and (10.13) that

$$\begin{aligned} -\delta &\le I^g(0,T,x_1,u_1)+\xi(x_1(T))-(I^g(0,T,x,u)+\xi(x(T))) \\ &= I^g(T-l_0-l_1-L_0-4,T-l_0-l_1-L_0-3,x_1,u_1) \\ &\quad + I^g(T-l_0-l_1-L_0-3,T,x_1,u_1)+\xi(x_1(T)) \\ &\quad - I^g(T-l_0-l_1-L_0-4,T-l_0-l_1-L_0-3,x,u) \\ &\quad - I^g(T-l_0-l_1-L_0-3,T,x,u)-\xi(x(T)). \end{aligned} \tag{10.14}$$

It follows from (10.3), (10.6), (10.12) and (10.13) that

$$\begin{aligned} &I^f(T-l_0-l_1-L_0-4,T-l_0-l_1-L_0-3,x_1,u_1) \\ &= \sigma(f,x(T-l_0-l_1-L_0-4),\bar{x}_*(l_0+l_1+L_0+3),1) \\ &\le \mu(f)+\gamma/8. \end{aligned} \tag{10.15}$$

Together with (10.8) and property (P6) this implies that

$$\begin{aligned} &I^g(T-l_0-l_1-L_0-4,T-l_0-l_1-L_0-3,x_1,u_1) \\ &\le I^f(T-l_0-l_1-L_0-4,T-l_0-l_1-L_0-3,x_1,u_1)+\delta_2/8 \\ &\le \mu(f)+\gamma/8+\delta_2/8. \end{aligned} \tag{10.16}$$

By (10.3) and (10.12),

$$\begin{aligned} &I^f(T-l_0-l_1-L_0-4,T-l_0-l_1-L_0-3,x,u) \\ &\ge \sigma(f,x(T-l_0-l_1-L_0-4),x(T-l_0-l_1-L_0-3),1) \\ &\ge \mu(f)-\gamma/8. \end{aligned} \tag{10.17}$$

We show that

$$I^g(T-l_0-l_1-L_0-4,T-l_0-l_1-L_0-3,x,u) \ge \mu(f)-\gamma/2. \tag{10.18}$$

Assume the contrary. Then

$$I^g(T-l_0-l_1-L_0-4,T-l_0-l_1-L_0-3,x,u) < \mu(f)-\gamma/2. \tag{10.19}$$

By property (P6), (10.8) and (10.19),

$$\begin{aligned} &I^f(T-l_0-l_1-L_0-4,T-l_0-l_1-L_0-3,x,u) \\ &\le I^g(T-l_0-l_1-L_0-4,T-l_0-l_1-L_0-3,x,u)+\delta_2/8 \\ &< \mu(f)-\gamma/2+\delta_2/8 < \mu(f)-3\gamma/8. \end{aligned}$$

This contradicts (10.17). The contradiction we have reached proves (10.18).

By (10.14), (10.16) and (10.18),

$$\begin{aligned} &I^g(T-l_0-l_1-L_0-3,T,x_1,u_1)+\xi(x_1(T)) \\ &\quad - I^g(T-l_0-l_1-L_0-3,T,x,u)-\xi(x(T)) \\ &\ge -\delta - I^g(T-l_0-l_1-L_0-4,T-l_0-l_1-L_0-3,x_1,u_1) \\ &\quad + I^g(T-l_0-l_1-L_0-4,T-l_0-l_1-L_0-3,x,u) \\ &\ge -\delta+\mu(f)-\gamma/2-\mu(f)-\gamma/8-\delta_2/8 \\ &\ge -\delta-\gamma/2-\gamma/8-\delta_2/8. \end{aligned} \tag{10.20}$$

In view of (10.6) and the choice of δ_1 (see (10.2)),

$$|\pi_-^f(\bar{x}_*(l_0 + l_1 + L_0 + 3))| \le \gamma/8. \tag{10.21}$$

Since $(\bar{x}_*, \bar{u}_*)$ is an $(f, -A, -B)$-overtaking optimal pair it follows from (3.6), (10.13) and Proposition 3.2 that

$$\begin{aligned} I^f(T - l_0 - l_1 - L_0 - 3, T, x_1, u_1) &= I^f(0, l_0 + l_1 + L_0 + 3, \bar{x}_*, \bar{u}_*) \\ &= \mu(f)(l_0 + l_1 + L_0 + 3) \\ &\quad + \pi_-^f(\bar{x}_*(0)) \\ &\quad - \pi_-^f(\bar{x}_*(l_0 + l_1 + L_0 + 3)). \end{aligned} \tag{10.22}$$

Combined with (10.21) this implies that

$$I^f(T - l_0 - l_1 - L_0 - 3, T, x_1, u_1) \le \pi_-^f(\bar{x}_*(0)) + \mu(f)(l_0 + l_1 + L_0 + 3) + \gamma/8. \tag{10.23}$$

By (P6), (10.8) and (10.23),

$$\begin{aligned} I^g(T - l_0 - l_1 - L_0 - 3, T, x_1, u_1) &\le I^f(T - l_0 - l_1 - L_0 - 3, T, x_1, u_1) \\ &\quad + \delta_2/8 \\ &\le \pi_-^f(\bar{x}_*(0)) + \mu(f)(l_0 + l_1 + L_0 + 3) \\ &\quad + \gamma/8 + \delta_2/8. \end{aligned} \tag{10.24}$$

It follows from (10.6)-(10.8), (10.13), (10.20) and (10.24) that

$$\begin{aligned} & I^g(T - l_0 - l_1 - L_0 - 3, T, x, u) + \xi(x(T)) \\ & \le I^g(T - l_0 - l_1 - L_0 - 3, T, x_1, u_1) \\ & \quad + \xi(x_1(T)) + \delta + \delta_2/8 + (5/8)\gamma \\ & \le \pi_-^f(\bar{x}_*(0)) + \mu(f)(l_0 + l_1 + L_0 + 3) \\ & \quad + \xi(x_1(T)) + \delta_2/8 + \delta + \delta_2/8 + (3/4)\gamma \\ & \le \mu(f)(l_0 + l_1 + L_0 + 3) + \pi_-^f(\bar{x}_*(0)) \\ & \quad + \xi(\bar{x}_*(0)) + \delta_1/16 + (3/4)\gamma \\ & \le \mu(f)(l_0 + l_1 + L_0 + 3) + \pi_-^f(\bar{x}_*(0)) \\ & \quad + h(\bar{x}_*(0)) + \delta_1/8 + (3/4)\gamma. \end{aligned} \tag{10.25}$$

By (5.1), (10.8) and (10.25),

$$\begin{aligned} I^g(T - l_0 - l_1 - L_0 - 3, T, x, u) &\le \pi_-^f(\bar{x}_*(0)) + \mu(f)(l_0 + l_1 + L_0 + 3) \\ &\quad + h(\bar{x}_*(0)) \\ &\quad + \delta_1/8 + (3/4)\gamma + a_1. \end{aligned} \tag{10.26}$$

Property (P6), (10.8) and (10.26) imply that

$$|I^g(T - l_0 - l_1 - L_0 - 3, T, x, u) - I^f(T - l_0 - l_1 - L_0 - 3, T, x, u)| \le \delta_2/8. \tag{10.27}$$

In view of (10.26) and (10.27),

$$\begin{aligned} I^f(T - l_0 - l_1 - L_0 - 3, T, x, u) &\le \pi_-^f(\bar{x}_*(0)) + \mu(f)(l_0 + l_1 + L_0 + 3) \\ &\quad + h(\bar{x}_*(0)) + a_1 + 4. \end{aligned} \tag{10.28}$$

Property (P7) and (10.28) imply that

$$|x(t)| \le \Delta_0,\ t \in [T - l_0 - l_1 - L_0 - 3, T]. \tag{10.29}$$

By (10.6), (10.8) and (10.29),

$$|h(x(T)) - \xi(x(T))| \le \delta_1/16. \tag{10.30}$$

It follows from (10.25), (10.27) and (10.30) that

$$\begin{aligned} &I^f(T - l_0 - l_1 - L_0 - 3, T, x, u) + h(x(T)) \\ &\le I^g(T - l_0 - l_1 - L_0 - 3, T, x, u) \\ &\quad + \xi(x(T)) + \delta_2/8 + \delta_1/16 \\ &\le \pi_-^f(\bar{x}_*(0)) + \mu(f)(l_0 + l_1 + L_0 + 3) \\ &\quad + h(\bar{x}_*(0)) + (3/4)\gamma + \delta_1/2. \end{aligned} \tag{10.31}$$

It is clear that $(\tilde{x}, \tilde{u}) \in X(-A, -B, 0, T)$ and by (3.6), (10.1) and (10.31),

$$\begin{aligned} &I^f(0, l_0 + l_1 + L_0 + 3, \tilde{x}, \tilde{u}) + h(\tilde{x}(0)) \\ &= I^f(T - l_0 - l_1 - L_0 - 3, T, x, u) + h(x(T)) \\ &\le \pi_-^f(\bar{x}_*(0)) + \mu(f)(l_0 + l_1 + L_0 + 3) \\ &+ h(\bar{x}_*(0)) + (3/4)\gamma + \delta_1/2. \end{aligned} \tag{10.32}$$

In view of (10.1) and (10.12),

$$|\tilde{x}(l_0 + l_1 + L_0 + 3) - x_f| \le \delta_1.$$

By the relation above and the choice of δ_1 (see (10.2)),

$$|\pi_-^f(\tilde{x}(l_0 + l_1 + L_0 + 3))| \le \gamma/8. \tag{10.33}$$

By (10.32), (10.33), $(f, -A, -B)$-overtaking optimality of $(\bar{x}_*, \bar{u}_*)$ and Proposition 3.1, which implies that the function

$$I^f(0, s, \tilde{x}, \tilde{u}) - s\mu(f) - \pi_-^f(\tilde{x}(0)) + \pi_-^f(\tilde{x}(s)),\ s \in (0, \infty)$$

is increasing, we have

$$\begin{aligned} &(\pi_-^f + h)(\tilde{x}(0)) - (\pi_-^f + h)(\bar{x}_*(0)) + I^f(0, L_0, \tilde{x}, \tilde{u}) \\ &\quad - L_0\mu(f) - \pi_-^f(\tilde{x}(0)) + \pi_-^f(\tilde{x}(L_0)) \\ &\le (\pi_-^f + h)(\tilde{x}(0)) - (\pi_-^f + h)(\bar{x}_*(0)) + I^f(0, l_0 + l_1 + L_0 + 3, \tilde{x}, \tilde{u}) \\ &\quad - \mu(f)(l_0 + l_1 + L_0 + 3) - \pi_-^f(\tilde{x}(0)) + \pi_-^f(\tilde{x}(l_0 + l_1 + L_0 + 3)) \\ &\le \pi_-^f(\tilde{x}(0)) - \pi_-^f(\bar{x}_*(0)) - h(\bar{x}_*(0)) + \mu(f)(l_0 + l_1 + L_0 + 3) \\ &\quad + \pi_-^f(\bar{x}_*(0)) + h(\bar{x}_*(0)) + (3/4)\gamma + \delta_1/2 - \mu(f)(l_0 + l_1 + L_0 + 3) \\ &\quad - \pi_-^f(\tilde{x}(0)) + \pi_-^f(\tilde{x}(l_0 + l_1 + L_0 + 3)) \\ &\le (3/4)\gamma + \delta_1/2 + \gamma/8 \le \gamma. \end{aligned}$$

It follows from the relation above, (10.5) and Proposition 3.1 that

$$(\pi_-^f + h)(\tilde{x}(0)) \le (\pi_-^f + h)(\bar{x}_*(0)) + \gamma = \inf(\pi_-^f + h) + \gamma,$$

$$I^f(0, L_0, \tilde{x}, \tilde{u}) - L_0\mu(f) + \pi_-^f(\tilde{x}(0)) + \pi_-^f(\tilde{x}(L_0)) \le \gamma.$$

Lemma 10.1 is proved. □

Completion of the proof of Theorem 5.3

By Lemma 9.2 applied to the triplet $(f, -A - B)$ there exist

$$\gamma \in (0, \epsilon/4)$$

such that the following property holds:

(P8) for each $(x, u) \in X(-A, -B, 0, L_0)$ which satisfies

$$(\pi_-^f + h)(x(0)) \le \inf(\pi_-^f + h)) + \gamma,$$

$$I^f(0, L_0, x, u) - L_0\mu(f) - \pi_-^f(x(0)) + \pi_-^f(x(L_0)) \le \gamma$$

there exists an $(f, -A, -B)$-overtaking optimal pair

$$(x_*, u_*) \in X(-A, -B, 0, \infty)$$

such that

$$(\pi_-^f + h)(x_*(0)) = \inf(\pi_-^f + h), \tag{10.34}$$

$$|x(t) - x_*(t)| \le \epsilon \text{ holds for all } t \in [0, L_0]. \tag{10.35}$$

By Lemma 10.1, there exist $\delta > 0$, a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$, a neighborhood $\mathcal{V}$ of h in $\mathfrak{A}$ and a number $L_1 > L_0$ such that the following property holds:

(P9) for each $T \ge L_1$, each $g \in \mathcal{U}$, each $\xi \in \mathcal{V}$ and each

$$(x, u) \in X(A, B, 0, T)$$

which satisfies

$$|x(0)| \le M,\ I^g(0, T, x, u) + \xi(x(T)) \le \sigma(g, \xi, x(0), 0, T) + \delta, \tag{10.36}$$

the pair of functions

$$\tilde{x}(t) = x(T - t),\ \tilde{u}(t) = u(T - t),\ t \in [0, T] \tag{10.37}$$

satisfies

$$(\tilde{x}, \tilde{u}) \in X(-A, -B, 0, T), \tag{10.38}$$

$$(\pi_-^f + h)(\tilde{x}(0)) \le \inf(\pi_-^f + h) + \gamma \tag{10.39}$$

and

$$I^f(0, L_0, \tilde{x}, \tilde{u}) - L_0\mu(f) - \pi_-^f(\tilde{x}(0)) + \pi_-^f(\tilde{x}(L_0)) \le \gamma. \tag{10.40}$$

Let

$$T \ge L_1,\ g \in \mathcal{U},\ \xi \in \mathcal{V},\ (x, u) \in X(A, B, 0, T)$$

satisfy (10.36) and let $\tilde{x}, \tilde{u}$ be defined by (10.37). Property (P9) imply (10.38)-(10.40). By relations (10.38)-(10.40) and property (P8), there exists an $(f, -A, -B)$-overtaking optimal pair

$$(x_*, u_*) \in X(-A, -B, 0, \infty)$$

such that (10.34) holds and

$$|\tilde{x}(t) - x_*(t)| \le \epsilon \text{ holds for all } t \in [0, L_0].$$

Together with (10.37) this implies that

$$|x(T-t) - x_*(t)| \leq \epsilon \text{ holds for all } t \in [0, L_0].$$

Theorem 5.3 is proved.

11. Proof of Theorem 5.4

Theorems 5.2 and 5.3 imply the following result.

Theorem 11.1. *Let $L_0 > 0$, $\epsilon > 0$, $h_1, h_2 \in \mathfrak{A}$. Then there exist $\delta > 0$, a neighborhood $\mathcal{U}$ of F in $\mathfrak{M}_b$, a neighborhood $\mathcal{V}_i$ of h_i, $i = 1, 2$ in $\mathfrak{A}$ and $L_1 > L_0$ such that for each $T \geq L_1$, each $g \in \mathcal{U}$, each $\xi_i \in \mathcal{V}_i$, $i = 1, 2$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies*

$$I^g(0, T, x, u) + \xi_1(x(T)) + \xi_2(x(0)) \leq \sigma(g, \xi_1, \xi_2, 0, T) + \delta$$

there exists an $(f, -A, -B)$-overtaking optimal pair

$$(\bar{x}_*, \bar{u}_*) \in X(-A, -B, 0, \infty)$$

such that

$$(\pi_-^f + h_1)(\bar{x}_*(0)) = \inf(\pi_-^f + h_1)$$

and for all $t \in [0, L_0]$,

$$|x(T-t) - \bar{x}_*(t)| \leq \epsilon.$$

Note that Theorem 5.3 is applied to the restriction of (x, u) to the interval $[L, T]$ with L as in Theorem 5.2.

Theorem 5.4 easily follows from Theorem 11.1 and Proposition 9.3.

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