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NECESSARY CONDITIONS FOR A CLASS OF BILEVEL OPTIMAL CONTROL PROBLEMS EXPLOITING THE VALUE FUNCTION

KONSTANTIN PALAGACHEV AND MATTHIAS GERDTS

Abstract. This paper presents first order necessary optimality conditions for a class of bilevel optimal control problems, in which both, the upper and lower level problems, are optimal control problems. It relies on the explicit computation of the value function of the lower level problem and reformulation of the original problem as a single level optimal control problem subject to a partially calmness constraint qualification. An example illustrates the theory and allows for an explicit evaluation of the value function for the lower level problem and the necessary conditions.

1. Introduction

In this paper, we consider a hierarchy of two optimal control problems, i.e. a *bilevel optimal control problem*. In the first problem, which will be referred as *upper level problem*, a certain control policy v has to be chosen, such that the states of the problem (x, y) minimize a cost function Φ. Furthermore, the state component x is constrained to be an optimal solution of another optimal control problem to which we refer as *lower level problem*. The lower level problem itself depends on the state y, which enters as a parameter in it. In this way, solving one of the two problems inevitably influences the solution of the other and vice versa. Hence, in order to find a solution of the bilevel problem, methods treating both, the upper and the lower level problem, have to be considered.

We consider the following upper level problem (ULP):

Problem 1.1 (ULP)**.**

(1.1) *Minimize* $\Phi(x(T), y(T))$

(1.2) *subject to* $\dot{y}(t) = F(x(t), y(t), v(t))$ *a.e. in* $(0, T)$,

(1.3) $v(t) \in V$ *a.e. in* $(0, T)$,

(1.4) $y(0) = y_0$,

(1.5) $(x, u) \in \mathcal{S}(y(T))$.

Herein, $\mathcal{S}$ maps a vector $y \in \mathbb{R}^{n_y}$ into the set of solutions of the following lower level problem (LLP(y)):

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Problem 1.2 (LLP(y)).

$$\text{(1.6)} \quad \textit{Minimize} \quad \varphi(x(T))$$

$$\text{(1.7)} \quad \textit{subject to} \quad \dot{x}(t) = f(x(t), u(t)) \quad a.e.\ in\ (0, T),$$

$$\text{(1.8)} \quad u(t) \in U \quad a.e.\ in\ (0, T),$$

$$\text{(1.9)} \quad x(0) = x_0,$$

$$\text{(1.10)} \quad \psi(x(T), y) = 0.$$

The functions in ULP and $LLP(y)$ are defined as follows:

$$\begin{array}{ll} \Phi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}, & \varphi : \mathbb{R}^{n_x} \to \mathbb{R}, \\ F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_v} \to \mathbb{R}^{n_y}, & f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}, \\ \psi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_\psi}. & \end{array}$$

Moreover, $V \subset \mathbb{R}^{n_v}$ and $U \subset \mathbb{R}^{n_u}$ are given control sets, the final time $T > 0$ is supposed to be fixed, and $y_0 \in \mathbb{R}^{n_y}$ and $x_0 \in \mathbb{R}^{n_x}$ are given vectors.

Bilevel optimal control problems occur in various applications, e.g. in locomotion and biomechanics, see [1, 2, 16, 20], in optimal control under safety constraints, see [13, 18, 19], or in Stackelberg dynamic games, compare [24, 12].

Bilevel optimization problems turn out to be very challenging with regard to both, the investigation of theoretical properties and numerical methods, compare [10]. Necessary conditions have been investigated, e.g., in [11, 27]. Typical solution approaches aim at reducing the bilevel structure into a single stage optimization problem, where the lower level problem is replaced by its first order necessary optimality conditions, compare [1, 4, 31]. The reduction is equivalent only when the lower level problem is convex, since in that case the first order necessary conditions are also sufficient. Furthermore, the reduction to a single stage problem leads to an increase of the dimensions of the problem, since minimization is performed on the original states as well as on the multipliers, and to complementarity constrains. Nevertheless, this approach is often used, especially for finite dimensional problems, owing to a well-established theory and the availability of numerical methods for mathematical programs with complementarity constraints.

In this paper, we focus on an equivalent transformation of the bilevel problem to a single level one. To this end we exploit the value function of the lower level problem, compare [9, 21, 27, 30]. It has been shown in [27, 30] that the reduced problem is equivalent to the original bilevel optimal control problem. The drawback is that the value function is nonsmooth, even in the case where the problem data is smooth. We prove the Lipschitz continuity of the value function and a representation of its subgradient by means of the Lagrange multipliers of the lower level problem (compare Theorem 2.2). This enables us to formulate a nonsmooth minimum principle, taylored to the reformulated single level problem (see Problem 3.1), together with conditions for normality of the solution (Theorem 3.3). Finally, we provide a numerical example in which the nonsmoothness effect of the value function is emphasized and the optimal solution is found by means of necessary optimality conditions.

Let us first define the value function for the lower level problem. For a given $y \in \mathbb{R}^{n_y}$, let $\mathcal{A}(y)$ denote the set of admissible controls for $LLP(y)$, i.e.

$$\mathcal{A}(y) := \left\{ u \in L^\infty([0,T], U) \;\middle|\; \begin{array}{c} \exists\, x_u \in W^{1,\infty}([0,T], \mathbb{R}^{n_x}) \textit{ such that} \\ (1.7)\text{-}(1.10) \textit{ are satisfied} \end{array} \right\}.$$

Then the value function, related to $LLP(y)$, is defined as

$$\mathcal{V}(y) := \inf_{u \in \mathcal{A}(y)} \varphi(x_u(T)) \tag{1.11}$$

with the convention $\inf \emptyset := +\infty$. The value function provides the best cost that can be achieved, given the parameter y. Throughout this paper, we will assume that the following assumptions hold:

- (A_1) The functions Φ, F, φ and f are continuously differentiable and ψ is twice continuously differentiable with respect to all arguments.
- (A_2) V and U are compact and convex subsets of $\mathbb{R}^{n_v}$ and $\mathbb{R}^{n_u}$ respectively.
- (A_3) There exists an integrable function $k : [0,T] \to \mathbb{R}$ such that

$$\|f(x,u)\| \le k(t)(1 + \|x\|) \qquad \forall\ (t,x,u) \in [0,T] \times \mathbb{R}^{n_x} \times U.$$

- (A_4) $f(x,U)$ is a convex subset of $\mathbb{R}^{n_x}$ for every $x \in \mathbb{R}^{n_x}$.
- (A_5) $\nabla_y \psi(x,y)$ has a full rank for every $(x,y) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$.

Finally, we say that (x,u) is a local minimum of $LLP(y)$ if (x,u) satisfies (1.7)-(1.10) and there exists $C > 0$, such that for any (x',u') satisfying (1.7)-(1.10) and $\max\left\{\|x - x'\|_{W^{1,\infty}}, \|u - u'\|_{L^\infty}\right\} \le C$ it holds $\varphi(x(T)) \le \varphi(x'(T))$. The global minimum principle, see [14, Theorem 7.1.6], implies that if (x,u) is a local solution of $LLP(y)$, then there exist nontrivial multipliers $\lambda_0 \ge 0$, $(\lambda, \sigma) \in W^{1,\infty}([0,T], \mathbb{R}^{n_x}) \times \mathbb{R}^{n_\psi}$ such that

$$\dot\lambda(t) = -\nabla_x f(x(t), u(t))^\top \lambda(t) \qquad \textit{a.e. in } (0,T) \tag{1.12}$$

$$\min_{u \in U} \left\{ \lambda(t)^\top f(x(t), u) \right\} = \lambda(t)^\top f(x(t), u(t)) \qquad \textit{a.e. in } (0,T) \tag{1.13}$$

$$\lambda(T) = \lambda_0 \nabla \varphi(x(T)) + \nabla_x \psi(x(T), y)^\top \sigma. \tag{1.14}$$

We will refer to $(\lambda_0, \lambda, \sigma)$ as *(Lagrange) multipliers* associated with (x,u). The multiplier λ_0 can be normalized to one, if a constraint qualification is satisfied. In the sequel we assume throughout that the lower level problem $LLP(y)$ satisfies a constraint qualification.

Remark 1.3. Note that in $LLP(y)$, we have assumed that the objective function in (1.6) is of *Mayer-type* (i.e. depends only of the state at the final time T). In fact, in case an additional Lagrange term $\int_0^T l(x(t), u(t))dt$ is present in (1.6), it is sufficient to introduce the new state variable ξ with $\dot\xi(t) = l(x(t), u(t))$ in $(0,T)$ and $\xi(0) = 0$, in order to obtain

$$\varphi(x(T)) + \int_0^T l(x(t), u(t))dt = \varphi(x(T)) + \xi(T).$$

The same observation holds also for ULP.

2. Differentiability of the value function

Let us now focus on the differentiability of the value function, defined in (1.11). First, we need to introduce the notion of *proximal*, *limiting* and *generalized subgradients*, compare [5, 6]. Let X be a general Hilbert space, let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous functional and let $x \in X$ be a point, where f is finite. A vector $\zeta \in X$ is called *proximal subgradient* of f at x if and only if there exist $M,\ \delta > 0$ such that

$$f(x') - f(x) + M\|x' - x\|^2 \geq \langle \zeta, x' - x\rangle \qquad \forall\ x' \in B_\delta(x).$$

The set of all proximal subgradients of f at x is denoted with $\partial_P f(x)$. The *limiting subgradient* of f at x is the set

$$\partial_L f(x) := \left\{ \text{weak} \lim_{i\to+\infty} \zeta_i \;\middle|\; \zeta_i \in \partial_P f(x_i),\ x_i \to x,\ f(x_i) \to f(x) \right\}.$$

Assume now that f is Lipschitz continuous in x and let v be any other vector in X. The *generalized directional derivative* of f at x in the direction v is defined as

$$f^\circ(x; v) := \limsup_{x'\to x\ \ t\searrow 0} \frac{f(x' + tv) - f(x')}{t}.$$

Finally, the *generalized gradient* of f at x is the subset of X given by

$$\partial f(x) := \left\{ \zeta \in X \;\middle|\; f^\circ(x; v) \geq \langle \zeta, v\rangle \textit{ for all } v \textit{ in } X \right\}.$$

For properties of the previously stated sets, we refer the readers to [5] and [6]

Before stating the main result in this section, we need the following theorem, a proof can be found in [6, Theorem 23.2]:

Theorem 2.1. *Let (f, U) be a control system on the interval $[a, b]$ for which:*

- (a) *$f(t, x, u)$ is measurable in t and continuous in (x, u);*
- (b) *$U(\cdot)$ is measurable and compact valued;*
- (c) *f has linear growth: there is an integrable function M such that*

$$(t, x) \in [a, b] \times \mathbb{R}^n,\ u \in U(t) \implies \|f(t, x, u)\| \leq M(t)(1 + \|x\|);$$

- (d) *The set $f(t, x, U)$ is convex for each (t, x).*

Let (x_i, u_i) be a sequence of processes for the control system (f, U) such that the set $\{x_i(a)\}_{i\in\mathbb{N}}$ is bounded. Then there exists a subsequence of $\{x_i\}$ converging uniformly to a state trajectory x^* of the system.

Theorem 2.2. *Let $(A_1) - (A_5)$ hold and let $y \in \mathbb{R}^{n_y}$ be such that there exists a neighborhood $\mathcal{I}_y$ of y and a constant $C_y > 0$, such that for every $y' \in \mathcal{I}_y$ with non-empty $\mathcal{S}(y)$ and every solution (x', u') of $LLP(y')$ with associated multipliers $(\lambda'_0, \lambda', \sigma')$, it holds $\lambda'_0 = 1$ and $\|\sigma'\| \leq C_y$. Then $\mathcal{V}$ is Lipschitz continuous in y and*

$$\partial\mathcal{V}(y) \subseteq co \bigcup_{(x,u)\in\mathcal{S}(y)} \left\{ \zeta \in \mathbb{R}^{n_y} \;\middle|\; \begin{array}{l} \exists\ \lambda \in W^{1,\infty}([0,T], \mathbb{R}^{n_x}),\ \sigma \in \mathbb{R}^{n_\psi}: \\ \dot\lambda(t) = -\nabla_x f(x(t), u(t))^\top \lambda(t) \\ \lambda(T) = \nabla\varphi(x(T)) + \nabla_x \psi(x(T), y)^\top \sigma \\ \zeta = \nabla_y \psi(x(T), y)^\top \sigma \end{array} \right\}. \tag{2.1}$$

Proof. Let us first prove that the value function $\mathcal{V}$ is lower semicontinuous in y. Let $\{y_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathbb{R}^{n_y}$ converging to y. For each $i \in \mathbb{N}$, let $(x_i, u_i) \in \mathcal{S}(y_i)$, hence $\mathcal{V}(y_i) = \varphi(x_i(T))$ and

$$\dot{x}_i(t) = f(x_i(t), u_i(t)) \qquad a.e.\ in\ (0,T), \tag{2.2}$$

$$u_i(t) \in U \qquad a.e.\ in\ (0,T), \tag{2.3}$$

$$x_i(0) = x_0, \tag{2.4}$$

$$\psi(x_i(T), y_i) = 0. \tag{2.5}$$

We observe that the control system (2.2)-(2.4) satisfies the hypothesis of Theorem 2.1. In fact, assumption $(A_1) - (A_4)$ imply $(a) - (d)$ respectively. Hence there exist (x^*, u^*) such that $x_i \to x^*$ uniformly in $[0,T]$ and

$$\begin{aligned} &\dot{x}^*(t) = f(x^*(t), u^*(t)) && a.e.\ in\ (0,T),\\ &u^*(t) \in U && a.e.\ in\ (0,T),\\ &x^*(0) = x_0. \end{aligned}$$

Furthermore, by the uniform convergence of x_i to x^* and by the continuity of ψ we get

$$0 = \lim_{i\to\infty} \psi(x_i(T), y_i) = \psi(x^*(T), y).$$

Hence we obtain

$$\liminf_{i\to+\infty} \mathcal{V}(y_i) = \liminf_{i\to+\infty} \varphi(x_i(T)) = \varphi(x^*(T)) \geq \inf_{u\in\mathcal{A}(y)} \{\varphi(x_u(T))\} = \mathcal{V}(y)$$

which implies that $\mathcal{V}$ is lower semicontinuous in y.

We now focus on computing the proximal subgradient of $\mathcal{V}$. Let $\zeta \in \partial_P \mathcal{V}(y)$, by definition there exist $M,\ \delta > 0$ such that

$$\mathcal{V}(y') - \mathcal{V}(y) + M\|y' - y\|^2 \geq \langle \zeta, y' - y\rangle \qquad \forall\ y' \in B_\delta(y)$$

or equivalently

$$\mathcal{V}(y') - \langle \zeta, y'\rangle + M\|y' - y\|^2 \geq \mathcal{V}(y) - \langle \zeta, y\rangle \qquad \forall\ y' \in B_\delta(y). \tag{2.6}$$

Let $(x,u) \in \mathcal{S}(y)$. For every feasible point (x', u') of $LLP(y')$, inequality (2.6) becomes

$$\varphi(x'(T)) - \zeta^\top y' + M\|y' - y\|^2 \geq \varphi(x(T)) - \zeta^\top y.$$

It follows that $(x,u,y) \in W^{1,\infty}([0,T], \mathbb{R}^{n_x}) \times L^\infty([0,T], \mathbb{R}^{n_u}) \times \mathbb{R}^{n_y}$ is a solution of the following optimal control problem:

$$min \quad \varphi(x'(T)) - \zeta^\top y'(T) + M\|y'(T) - y\|^2 \tag{2.7}$$

$$s.t. \quad \dot{x}'(t) = f(x'(t), u'(t)) \qquad a.e.\ in\ (0,T), \tag{2.8}$$

$$\dot{y}'(t) = 0 \qquad a.e.\ in\ (0,T), \tag{2.9}$$

$$u'(t) \in U \qquad a.e.\ in\ (0,T), \tag{2.10}$$

$$x'(0) = x_0, \tag{2.11}$$

$$\psi(x'(T), y'(T)) = 0. \tag{2.12}$$

Note that y' appears as parameter in (2.7)-(2.12), due to equation (2.9). From the global minimum principle (with $\lambda_0 = 1$ as assumed), see [14, Theorem 7.1.6], it follows that there exist $\lambda \in W^{1,\infty}([0,T], \mathbb{R}^{n_x})$ and $\sigma \in \mathbb{R}^{n_\psi}$ such that

$$\dot{\lambda}(t) = -\nabla_x f(x(t), u(t))^\top \lambda(t) \qquad a.e.\ in\ (0,T), \tag{2.13}$$

$$\min_{u \in U} \left\{ \lambda(t)^\top f(x(t), u) \right\} = \lambda(t)^\top f(x(t), u(t)) \qquad a.e.\ in\ (0,T), \tag{2.14}$$

$$\lambda(T) = \nabla \varphi(x(T)) + \nabla_x \psi(x(T), y)^\top \sigma, \tag{2.15}$$

$$\zeta = \nabla_y \psi(x(T), y)^\top \sigma. \tag{2.16}$$

Let us now consider $\zeta \in \partial_L \mathcal{V}(y)$. By the definition of the limiting subgradient, there exist sequences $\{y_i\}$ and $\{\zeta_i\}$ in $\mathbb{R}^{n_y}$, such that $y_i \to y$, $\mathcal{V}(y_i) \to \mathcal{V}(y)$, $\zeta_i \in \partial_P \mathcal{V}(y_i)$ and $\zeta_i \to \zeta$. From (2.13)-(2.16) it follows that for each $i \in \mathbb{N}$ there exist a solution (x_i, u_i) of $LLP(y_i)$, $\lambda_i \in W^{1\infty}([0,T], \mathbb{R}^{n_x})$ and $\sigma_i \in \mathbb{R}^{n_\psi}$ such that

$$\dot{x}_i(t) = f(x_i(t), u_i(t)) \qquad a.e.\ in\ (0,T), \tag{2.17}$$

$$\dot{\lambda}_i(t) = -\nabla_x f(x_i(t), u_i(t))^\top \lambda_i(t) \qquad a.e.\ in\ (0,T), \tag{2.18}$$

$$u_i(t) \in U \qquad a.e.\ in\ (0,T), \tag{2.19}$$

$$\lambda_i(T) = \nabla \varphi(x_i(T)) + \nabla_x \psi(x_i(T), y_i)^\top \sigma_i, \tag{2.20}$$

$$\zeta_i = \nabla_y \psi(x_i(T), y_i)^\top \sigma_i. \tag{2.21}$$

We now apply Theorem 2.1 to the control system (2.17)-(2.21). Note that points (a) and (b) are satisfied due to assumption (A_1) and (A_2), while (d) holds due to (A_4). Observe now that due to (A_3) and the Gronwall's lemma, for each $i \in \mathbb{N}$ and $t \in [0,T]$ it holds

$$\begin{aligned} \|x_i(t) - x_0\| &\le \int_0^t \exp\left(\int_s^t k(\tau) d\tau \right) k(s) \|x_0\| ds \\ &\le \int_0^T \exp\left(\int_s^T k(\tau) d\tau \right) k(s) \|x_0\| ds \le \|x_0\| \|k\|_1 \exp(\|k\|_1). \end{aligned}$$

With $\delta_1 := (1 + \|x_0\|)\|k\|_1 \exp(\|k\|_1)$ we get

$$x_i(t) \in B_{\delta_1}(x_0) \qquad \forall\ t \in [0,T],\ i \in \mathbb{N}, \tag{2.22}$$

and the linear growth requirement (c) is satisfied, since

$$\|f(x_i(t), u_i(t))\| \le k(t) \|x_i(t)\|$$

and

$$\|\nabla_x f(x_i(t), u_i(t))^\top \lambda_i(t)\| \le \sup_{(x,u) \in B_{\delta_1}(x_0) \times U} \|\nabla_x f(x,u)\| \|\lambda_i(t)\|.$$

Let us now show that $\{\lambda_i(0)\}$ is bounded. Note that by (A_5), $\nabla_y \psi(x_i(T), y_i)$ is of full rank. Hence, if we denote with $A_i = \nabla_y \psi(x_i(T), y_i)^\top$ and with $A_i^+ := (A_i^\top A_i)^{-1} A_i^\top$ its pseudoinverse, by (2.21) we obtain $\sigma_i = A_i^+ \zeta_i$, which substituted in (2.20) leads to

$$\lambda_i(T) = \nabla \varphi(x_i(T)) + \nabla_x \psi(x_i(T), y_i)^\top A_i^+ \zeta_i. \tag{2.23}$$

Note that all the terms in the right hands side of (2.23) are converging as $i \to +\infty$, hence $\{\lambda_i(T)\}$ is convergent and in particular it remains bounded. Then it is easy to show by means of Gronwall's lemma that also $\{\lambda_i(0)\}$ is bounded and application of Theorem 2.1 leads to the existence of a converging subsequence of $\{(x_i, \lambda_i, \sigma_i)\}$, functions $x, \lambda \in W^{1,\infty}([0,T], \mathbb{R}^{n_x})$ and $\sigma \in \mathbb{R}^{n_\psi}$, such that $x_i \to x$ and $\lambda_i \to \lambda$ uniformly in $[0,T]$ and $\sigma_i \to \sigma$. Hence we conclude that

$$(2.24) \quad \partial_L \mathcal{V}(y) \subseteq \bigcup_{(x,u)\in\mathcal{S}(y)} \left\{ \zeta \in \mathbb{R}^{n_y} \;\middle|\; \begin{array}{l} \exists\, \lambda \in W^{1,\infty}([0,T], \mathbb{R}^{n_x}),\ \sigma \in \mathbb{R}^{n_\psi}: \\ \dot\lambda(t) = -\nabla_x f(x(t), u(t))^\top \lambda(t) \\ \lambda(T) = \nabla\varphi(x(T)) + \nabla_x \psi(x(T), y)^\top \sigma \\ \zeta = \nabla_y \psi(x(T), y)^\top \sigma \end{array} \right\}.$$

It remains to prove that the value function $\mathcal{V}$ is Lipschitz continuous in y. We will use the following characterization for Lipschitz continuity in y (the proof can be found in [8, Theorem 3.6]): $\mathcal{V}$ is Lipschitz continuous of rank C in y if and only if

$$\sup\left\{\|\zeta\| \;\middle|\; \zeta \in \partial_P \mathcal{V}(y')\right\} \le C \qquad \forall\, y' \textit{in a neighborhood of } y.$$

Let $y' \in \mathcal{I}_y$ and let $\zeta' \in \partial_P \mathcal{V}(y')$, then according to (2.13)-(2.16) there exist a solution (x', u') of $LLP(y')$ with associated multipliers (λ', σ'), such that $\zeta' = \nabla_y \psi(x'(T), y')^\top \sigma'$. Since σ' is bounded, it follows that

$$\|\zeta'\| = \|\nabla_y \psi(x'(T), y')^\top \sigma'\| \le \sup_{(x',y')\in B_{\delta_1}(x_0)\times\mathcal{I}_y} \|\nabla_y \psi(x', y')\| \cdot C_y.$$

Hence

$$\sup\left\{\|\zeta'\| \;\middle|\; \zeta' \in \partial_P \mathcal{V}(y')\right\} \le \sup_{(x',y')\in B_{\delta_1}(x_0)\times\mathcal{I}_y} \|\nabla_y \psi(x', y')\| \cdot C_y \qquad \forall\, y' \in \mathcal{I}_y$$

which leads to the Lipschitz continuity of $\mathcal{V}$ in y. Finally, we note that $\partial\mathcal{V}(y) = \mathrm{co}\, \partial_L \mathcal{V}(y)$, according to [6, Proposition 11.23]. Hence inclusion (2.24) leads to

$$\partial \mathcal{V}(y) \subseteq \mathrm{co} \bigcup_{(x,u)\in\mathcal{S}(y)} \left\{ \zeta \in \mathbb{R}^{n_y} \;\middle|\; \begin{array}{l} \exists\, \lambda \in W^{1,\infty}([0,T], \mathbb{R}^{n_x}),\ \sigma \in \mathbb{R}^{n_\psi} \\ \dot\lambda(t) = -\nabla_x f(x(t), u(t))^\top \lambda(t) \\ \lambda(T) = \nabla\varphi(x(T)) + \nabla_x \psi(x(T), y)^\top \sigma \\ \zeta = \nabla_y \psi(x(T), y)^\top \sigma \end{array} \right\}.$$

□

3. Single level transformation

In this section, we provide a single level reformulation of the initial ULP via the value function $\mathcal{V}$ of $LLP(y)$. This method has been already investigated in [27] and [30], where the equivalence between the initial and the reformulated problem has been proven. The main drawback is the loss of regularity (by Theorem 2.2 we know that the value function of $LLP(y)$ is only Lipschitz continuous in y), hence results from nonsmooth analysis have to be used.

Exploiting the value function $\mathcal{V}$ defined in (1.11), we can reformulate ULP as the following single-level optimal control problem:

Problem 3.1 (SLOCP)**.**

$$
\begin{array}{llll}
(3.1) & \textit{Minimize} & \Phi(x(T), y(T)) & \\
(3.2) & \textit{subject to} & \dot{x}(t) = f(x(t), u(t)) & \textit{a.e. in } (0,T), \\
(3.3) & & \dot{y}(t) = F(x(t), y(t), v(t)) & \textit{a.e. in } (0,T), \\
(3.4) & & u(t) \in U, \quad v(t) \in V & \textit{a.e. in } (0,T), \\
(3.5) & & x(0) = x_0, \quad y(0) = y_0, & \\
(3.6) & & \psi(x(T), y(T)) = 0, & \\
(3.7) & & \varphi(x(T)) - \mathcal{V}(y(T)) \leq 0. &
\end{array}
$$

We derive now necessary optimality conditions for $SLOCP$. It has been shown in [28] and [29] that the equivalent single level optimal control problem $SLOCP$ has always nontrivial abnormal multipliers, i.e. functions $p_x \in W^{1,\infty}([0,T], \mathbb{R}^{n_x})$, $p_y \in W^{1,\infty}([0,T], \mathbb{R}^{n_y})$, and $\xi \in \mathbb{R}^{n_\psi}$ such that equations (3.8)-(3.13) are satisfied with $\lambda_0 = 0$. Furthermore, standard constraint qualifications such as *linear independence constraint qualification* and *Mangasarian-Fromowitz constraint qualification* are not sufficient to guarantee the existence of normal multipliers. For the bilevel optimal control problem, the right constraint qualification to assume is the *calmness*-type constraint qualification, compare [28, 29]:

Definition 3.2. Let $(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ be an optimal solution for ULP (equivalently $SLOCP$). $SLOCP$ is said to be *partially calm* in $(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ with modulus $\mu \geq 0$ if for every (x, y, u, v) satisfying

$$
\begin{array}{ll}
\dot{x}(t) = f(x(t), u(t)) & \textit{a.e. in } (0,T), \\
\dot{y}(t) = F(x(t), y(t), v(t)) & \textit{a.e. in } (0,T), \\
u(t) \in U, \quad v(t) \in V & \textit{a.e. in } (0,T), \\
x(0) = x_0, \quad y(0) = y_0, & \\
\psi(x(T), y(T)) = 0, &
\end{array}
$$

we have

$$
\Phi(x(T), y(T)) - \Phi(\hat{x}(T), \hat{y}(T)) + \mu \left(\varphi(x(T)) - \mathcal{V}(\hat{y}(T)) \right) \geq 0.
$$

Theorem 3.3. *Let (A_1)–(A_5) hold and let $(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ be a local solution of $SLOCP$, such that it is partially calm in $(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ with modulus $\mu \geq 0$. Then there exist $\lambda_0 \geq 0$, $p_x \in W^{1,\infty}([0,T], \mathbb{R}^{n_x})$, $p_y \in W^{1,\infty}([0,T], \mathbb{R}^{n_y})$, $\xi \in \mathbb{R}^{n_\psi}$, and $h \in \mathbb{R}$, such* *that*

$$\dot{p}_x(t) = -\nabla_x f(\hat{x}(t),\hat{u}(t))^\top p_x(t) - \nabla_x F(\hat{x}(t),\hat{y}(t),\hat{v}(t))^\top p_y(t), \tag{3.8}$$

$$\dot{p}_y(t) = -\nabla_y F(\hat{x}(t),\hat{y}(t),\hat{v}(t))^\top p_y(t), \tag{3.9}$$

$$\min_{u\in U}\left\{f(\hat{x}(t),u)^\top p_x(t)\right\} = f(\hat{x}(t),\hat{u}(t))^\top p_x(t), \tag{3.10}$$

$$\min_{v\in V}\left\{F(\hat{x}(t),\hat{y}(t),v)^\top p_y(t)\right\} = F(\hat{x}(t),\hat{y}(t),\hat{v}(t))^\top p_y(t), \tag{3.11}$$

$$\begin{aligned} p_x(T) &= \lambda_0\nabla_x\Phi(\hat{x}(T),\hat{y}(T)) + \lambda_0\mu\nabla_x\varphi(\hat{x}(T)) \\ &\qquad + \nabla_x\psi(\hat{x}(T),\hat{y}(T))^\top\xi, \end{aligned} \tag{3.12}$$

$$\begin{aligned} p_y(T) &\in \lambda_0\nabla_y\Phi(\hat{x}(T),\hat{y}(T)) - \lambda_0\mu\partial\mathcal{V}(\hat{y}(T)) \\ &\qquad + \nabla_y\psi(\hat{x}(T),\hat{y}(T))^\top\xi, \end{aligned} \tag{3.13}$$

$$f(\hat{x}(t),\hat{u}(t))^\top p_x(t) + F(\hat{x}(t),\hat{y}(t),\hat{v}(t))^\top p_y(t) = h. \tag{3.14}$$

Furthermore, if the hypothesis from Theorem 2.2 hold in $\hat{y}(T)$*, then* $\partial\mathcal{V}(\hat{y}(T))$ *is given by* (2.1)*.*

In addition, if the matrix

$$\begin{bmatrix} \nabla_u f(\hat{x}(t),\hat{u}(t)) & 0 \\ 0 & \nabla_v F(\hat{x}(t),\hat{y}(t),\hat{v}(t)) \end{bmatrix}$$

is of full rank almost everywhere in $(0,T)$ *and there exist a solution* $\hat{d} = (\hat{d}_x,\hat{d}_y,\hat{d}_u,\hat{d}_v) \in W^{1,\infty}([0,T],\mathbb{R}^{n_x}) \times W^{1,\infty}([0,T],\mathbb{R}^{n_y}) \times L^\infty([0,T],\mathbb{R}^{n_u}) \times L^\infty([0,T],\mathbb{R}^{n_v})$ *of the system*

$$\begin{aligned} &\dot{d}_x(t) = \nabla_x f(\hat{x},\hat{u})d_x(t) + \nabla_u f(\hat{x},\hat{u})d_u(t), \\ &\dot{d}_y(t) = \nabla_x F(\hat{x},\hat{y},\hat{v})d_x(t) + \nabla_y F(\hat{x},\hat{y},\hat{v})d_y(t) + \nabla_v F(\hat{x},\hat{y},\hat{v})d_v(t), \\ &d_x(0) = 0, \qquad d_y(0) = 0 \\ &\nabla_x\psi(\hat{x}(T),\hat{y}(T))d_x(T) + \nabla_y\psi(\hat{x}(T),\hat{y}(T))d_y(T) = 0, \end{aligned}$$

such that $\hat{d}_u(t)+\hat{u}(t) \in int(U)$ *and* $\hat{d}_v(t)+\hat{v}(t) \in int(V)$ *almost everywhere in* $(0,T)$*, equation* (3.8)-(3.14) *hold with* $\lambda_0 = 1$*.*

Proof. Since $SLOCP$ is partially calm in $(\hat{x},\hat{y},\hat{u},\hat{v})$ with modulus $\mu \geq 0$, it is easy to see that $(\hat{x},\hat{y},\hat{u},\hat{v})$ is also solution of the following optimal control problem

$$\min \quad \Phi(x(T),y(T)) + \mu(\varphi(x(T)) - \mathcal{V}(y(T))) \tag{3.15}$$

$$\text{s.t.} \quad \dot{x}(t) = f(x(t),u(t)) \qquad a.e.\ in\ (0,T), \tag{3.16}$$

$$\dot{y}(t) = F(x(t),y(t),v(t)) \qquad a.e.\ in\ (0,T), \tag{3.17}$$

$$u(t) \in U, \quad v(t) \in V \qquad a.e.\ in\ (0,T), \tag{3.18}$$

$$x(0) = x_0, \quad y(0) = y_0, \tag{3.19}$$

$$\psi(x(T),y(T)) = 0. \tag{3.20}$$

The Hamiltonian function, related to problem (3.15)-(3.20) is defined as $\mathcal{H} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$ such that

$$\mathcal{H}(x,y,u,v,p_x,p_y) := f(x,u)^\top p_x + F(x,y,v)^\top p_y.$$

Since the objective function of the problem is non differentiable (due to the presence of the value function $\mathcal{V}$), we apply results from nonsmooth analysis, in order to derive necessary optimality conditions for (3.15)-(3.20). Extensive studies in this direction can be found in [5],[7], [8], and [26]. According to [26, Theorem 6.2.3], there exist $\lambda_0 \geq 0$, $p_x \in W^{1,1}([0,T],\mathbb{R}^{n_x})$, $p_y \in W^{1,1}([0,T],\mathbb{R}^{n_y})$, $\xi_x \in \mathbb{R}^{n_x}$, $\xi_y \in \mathbb{R}^{n_y}$, $\xi \in \mathbb{R}^{n_\psi}$, and $h \in \mathbb{R}$,

$$\lambda_0 + \|p_x\|_{L^\infty} + \|p_y\|_{L^\infty} + \|\xi_x\| + \|\xi_y\| + \|\xi\| > 0,$$

such that

$$\dot{p}_x(t) = -\nabla_x \mathcal{H}(\hat{x}(t),\hat{y}(t),\hat{u}(t),\hat{v}(t),p_x(t),p_y(t)), \tag{3.21}$$

$$\dot{p}_y(t) = -\nabla_y \mathcal{H}(\hat{x}(t),\hat{y}(t),\hat{u}(t),\hat{v}(t),p_x(t),p_y(t)), \tag{3.22}$$

$$\begin{aligned} &\min_{u\in U, v\in V} \mathcal{H}(\hat{x}(t),\hat{y}(t),u,v,p_x(t),p_y(t)) \\ &\quad = \mathcal{H}(\hat{x}(t),\hat{y}(t),\hat{u}(t),\hat{v}(t),p_x(t),p_y(t)) \end{aligned} \tag{3.23}$$

$$(p_x,p_y)(0) = -(\xi_x,\xi_y), \tag{3.24}$$

$$\begin{aligned} &(p_x,p_y)(T) \in \partial\Big\{\lambda_0\big[\Phi(\hat{x}(T),\hat{y}(T)), \\ &\qquad + \mu(\varphi(\hat{x}(T)) - \mathcal{V}(\hat{y}(T)))\big] + \xi^\top \psi(\hat{x}(T),\hat{y}(T))\Big\}, \end{aligned} \tag{3.25}$$

$$\mathcal{H}(\hat{x}(t),\hat{y}(t),\hat{u}(t),\hat{v}(t),p_x(t),p_y(t)) = h. \tag{3.26}$$

Note that equation (3.24) does not imply any constraints on $p_x(0)$ and $p_y(0)$. This is reasonable since the values of x and y are fixed at $t = 0$. If we denote with $\hat{f}[\cdot]$, $\hat{F}[\cdot]$, $\hat{\varphi}$, $\hat{\Phi}$, $\hat{\psi}$ and $\hat{\mathcal{V}}$ the values of f, F, φ, Φ, ψ and $\mathcal{V}$ in $(\hat{x},\hat{y},\hat{u},\hat{v})$, equations (3.21), (3.22), (3.23), (3.25), and (3.26) become

$$\dot{p}_x(t) = -\nabla_x \hat{f}[t]^\top p_x(t) - \nabla_x \hat{F}[t]^\top p_y(t) \qquad a.e.\ in\ (0,T), \tag{3.27}$$

$$\dot{p}_y(t) = -\nabla_y \hat{F}[t]^\top p_y(t) \qquad a.e.\ in\ (0,T), \tag{3.28}$$

$$\min_{u\in U}\big\{f(\hat{x}(t),u)^\top p_x(t)\big\} = \hat{f}[t]^\top p_x(t) \qquad a.e.\ in\ (0,T), \tag{3.29}$$

$$\min_{v\in V}\big\{F(\hat{x}(t),\hat{y}(t),v)^\top p_y(t)\big\} = \hat{F}[t]^\top p_y(t) \qquad a.e.\ in\ (0,T), \tag{3.30}$$

$$p_x(T) = \lambda_0 \nabla_x \hat{\Phi} + \lambda_0 \mu \nabla_x \hat{\varphi} + (\nabla_x \hat{\psi})^\top \xi, \tag{3.31}$$

$$p_y(T) \in \lambda_0 \nabla_y \hat{\Phi} - \lambda_0 \mu \nabla_y \hat{\mathcal{V}} + (\nabla_y \hat{\psi})^\top \xi, \tag{3.32}$$

$$\hat{f}[t]^\top p_x(t) + \hat{F}[t]^\top p_y(t) = h \qquad a.e.\ in\ (0,T). \tag{3.33}$$

Note that (3.27)-(3.33) are exactly equations (3.8)-(3.14). Furthermore, by (3.27)-(3.28) follows that p_x and p_y are functions in $W^{1,\infty}$.

We prove now the existence of normal multipliers. Let us assume that equations (3.27)-(3.33) are satisfied with $\lambda_0 = 0$. Define $z := (x,y,u,v)$ and the spaces $X := W^{1,\infty}([0,T],\mathbb{R}^{n_x}) \times W^{1,\infty}([0,T],\mathbb{R}^{n_y}) \times L^\infty([0,T],\mathbb{R}^{n_u}) \times L^\infty([0,T],\mathbb{R}^{n_v})$, $Y := L^\infty([0,T],\mathbb{R}^{n_x}) \times L^\infty([0,T],\mathbb{R}^{n_y}) \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_\psi}$, $S := \big\{z \in X \;\big|\; u(t) \in U,\ v(t) \in V\ a.e.\ in\ (0,T)\big\}$ and the operator $H : X \to Y$ defined as

$$H(z) := \begin{bmatrix} \dot{x} - f(x,u) \\ \dot{y} - F(x,u,v) \\ x(0) - x_0 \\ y(0) - y_0 \\ \psi(x(T), y(T)) \end{bmatrix}.$$

Let us first prove that $H'(\hat{z})$ is surjective. This is equivalent of finding a solution $(d_x, d_y, d_u, d_v) \in X$ of the following system

$$\dot{d}_x(t) = \nabla_x \hat{f}[t] d_x(t) + \nabla_u \hat{f}[t] d_u(t) + \alpha_x(t), \tag{3.34}$$

$$\dot{d}_y(t) = \nabla_x \hat{F}[t] d_x(t) + \nabla_y \hat{F}[t] d_y(t) + \nabla_v \hat{F}[t] d_v(t) + \alpha_y(t), \tag{3.35}$$

$$d_x(0) = \beta_x, \tag{3.36}$$

$$d_y(0) = \beta_y, \tag{3.37}$$

$$\nabla_x \hat{\psi} d_x(T) + \nabla_y \hat{\psi} d_y(T) = \gamma \tag{3.38}$$

for any $(\alpha_x, \alpha_y, \beta_x, \beta_y, \gamma) \in Y$. If we denote with

$$\hat{A}(t) := \begin{bmatrix} \nabla_x \hat{f}[t] & 0 \\ \nabla_x \hat{F}[t] & \nabla_y \hat{F}[t] \end{bmatrix}, \quad \hat{B}(t) := \begin{bmatrix} \nabla_u \hat{f}[t] & 0 \\ 0 & \nabla_v \hat{F}[t] \end{bmatrix}$$

and with $d_{xy} := (d_x, d_y)$, $d_{uv} := (d_u, d_v)$, $\alpha := (\alpha_x, \alpha_y)$, $\beta := (\beta_x, \beta_y)$, equations (3.34)-(3.38) become

$$\dot{d}_{xy}(t) = \hat{A}(t) d_{xy}(t) + \hat{B}(t) d_{uv}(t) + \alpha, \tag{3.39}$$

$$d_{xy}(0) = \beta, \tag{3.40}$$

$$\nabla \hat{\psi} d_{xy}(T) = \gamma. \tag{3.41}$$

Let $\Gamma(t)$ be the solution of the Cauchy problem

$$\begin{cases} \dot{\Gamma}(t) = \hat{A}(t)\Gamma(t) & a.e. \ in \ (0,T), \\ \Gamma(0) = I_{n_x + n_y}. \end{cases}$$

Then the solution of (3.39)-(3.40) is given by

$$d_{xy}(t) = \Gamma(t) \left\{ \beta + \int_0^t \Gamma(\tau)^{-1} \left[\hat{B}(\tau) d_{uv}(\tau) + \alpha(\tau) \right] d\tau \right\},$$

which, as substituted in (3.41), gives

$$\nabla \hat{\psi} \Gamma(T) \left\{ \beta + \int_0^T \Gamma(\tau)^{-1} \left[\hat{B}(\tau) d_{uv}(\tau) + \alpha(\tau) \right] d\tau \right\} = \gamma$$

or

$$\nabla \hat{\psi} \Gamma(T) \int_0^T \Gamma(\tau)^{-1} \hat{B}(\tau) d_{uv}(\tau) d\tau = \tilde{\gamma} \tag{3.42}$$

where

$$\tilde{\gamma} := \gamma - \nabla \hat{\psi} \Gamma(T) \left\{ \beta + \int_0^T \Gamma(\tau)^{-1} \alpha(\tau) d\tau \right\}.$$

Hence, for every $\tilde{\gamma} \in \mathbb{R}^{n_\psi}$, we have to find $d_{uv} \in L^\infty([0,T], \mathbb{R}^{n_u+n_v})$ such that (3.42) is satisfied. Let is define for each $\tilde{\gamma} \in \mathbb{R}^{n_\psi}$

$$d_{uv}(\tau) := \frac{1}{T}\hat{B}(\tau)^+\Gamma(\tau)\Gamma(T)^{-1}\nabla\hat{\psi}^+\tilde{\gamma} \qquad \textit{for a.e. } \tau \in (0,T). \tag{3.43}$$

We have denoted with $\hat{B}(\tau)^+$ and $\nabla\hat{\psi}^+$ the pseudoinverse matrices of $\hat{B}(\tau)$ and $\nabla\hat{\psi}$, defined as

$$\hat{B}(\tau)^+ = \hat{B}(\tau)^\top(\hat{B}(\tau)\hat{B}(\tau)^\top)^{-1} \quad \textit{and} \quad \nabla\hat{\psi}^+ = \nabla\hat{\psi}^\top(\nabla\hat{\psi}\nabla\hat{\psi}^\top)^{-1}.$$

Note that d_{uv} defined in (3.43) satisfies (3.42).

Observe now that by definition $\hat{d} \in \operatorname{int}(S - \{\hat{z}\})$. From the Open Map Theorem follows that there exist $\delta, \varepsilon > 0$ such that $B_\delta(\hat{d}) \subseteq \operatorname{int}(S - \{\hat{z}\})$ and $B_\varepsilon(0_Y) \subseteq H'(\hat{z})(B_\delta(\hat{d}))$, which implies that

$$0_Y \in \operatorname{int}\left\{H'(\hat{z})(z - \hat{z}) \;\middle|\; z \in S\right\}. \tag{3.44}$$

Consider now $\lambda^* = (p_x, p_y, p_x(0), p_y(0), -\xi) \in Y^*$. We observe that $\langle \lambda^*, H'(\hat{z})(z - \hat{z})\rangle \le 0$ for each $z \in S$. In fact, by denoting with $d_x = x - \hat{x}$ and $d_y = y - \hat{y}$, we get

$$\begin{aligned}
\langle \lambda^*, H'(\hat{z})(z-\hat{z})\rangle &= \int_0^T p_x(t)^\top \left\{\dot{d}_x(t) - \nabla_x\hat{f}[t]d_x(t) - \nabla_u\hat{f}[t](u-\hat{u})(t)\right\} dt \\
&\quad + \int_0^T p_y(t)^\top \left\{\dot{d}_y(t) - \nabla_x\hat{F}[t]d_x(t) - \nabla_y\hat{F}[t]d_y(t) - \nabla_v\hat{F}[t](v-\hat{v})(t)\right\} dt \\
&\quad + p_x(0)^\top d_x(0) + p_y(0)^\top d_y(0) - \xi^\top \left\{\nabla_x\hat{\psi}d_x(T) + \nabla_y\hat{\psi}d_y(T)\right\} \\
&= \left\{p_x(T)^\top - \xi^\top\nabla_x\hat{\psi}\right\} d_x(T) + \left\{p_y(T)^\top - \xi^\top\nabla_y\hat{\psi}\right\} d_y(T) \\
&\quad - \int_0^T \left\{\dot{p}_x(t)^\top + p_x(t)^\top\nabla_x\hat{f}[t] + p_y(t)^\top\nabla_x\hat{F}[t]\right\} d_x(t)dt \\
&\quad - \int_0^T \left\{\dot{p}_y(t)^\top + p_y(t)^\top\nabla_y\hat{F}[t]\right\} d_y(t)dt \\
&\quad - \int_0^T \left\{p_x(t)^\top\nabla_u\hat{f}[t](u(t)-\hat{u}(t)) + p_y(t)^\top\nabla_v\hat{F}[t](v(t)-\hat{v}(t))\right\} dt \\
&= -\int_0^T \left\{p_x(t)^\top\nabla_u\hat{f}[t](u(t)-\hat{u}(t)) + p_y(t)^\top\nabla_v\hat{F}[t](v(t)-\hat{v}(t))\right\} dt \le 0.
\end{aligned}$$

Note that we have used (3.27)-(3.33) in order to obtain the last estimate. It follows that the functional λ^* separates 0_Y from $\left\{H'(\hat{z})(z - \hat{z}) \;\middle|\; z \in S\right\}$ what contradicts (3.44). Hence, the assumption $\lambda_0 = 0$ has to be wrong. □

Remark 3.4. It remains an open question and a subject of future research to investigate the meaning of $\partial\mathcal{V}(y)$, if the assumptions in Theorem 3.3 do not hold.

4. Application to a Pursuer-Evader problem

In order to illustrate the method we developed in the previous sections, we consider a pursuer-evader scenario in the two dimensional plane (the generalization to the three dimensional case is straightforward). Pursuer-evader problems have been widely studied in literature (compare [3, 15, 17, 25]) by means of different techniques. For instance, in [17] a differential game approach was adopted, where the key idea is to solve the Isaacs equations, which provides necessary and sufficient optimality conditions. Different techniques, based on probability analysis and Bayesian reasoning was used in [25]. In this paper, we consider the bilevel optimization approach, in order to solve the problem.

The idea behind the problem is the following: The pursuer (P) aims to reach the position of the evader (E) in a minimum time T, while the evader aims to maximize the final time T reduced by a term representing its control effort. The problem can be formulated as a bilevel optimal control problem, where the solution of the lower level problem describes the pursuer's optimal strategy, while the solution of the upper level problem describes the evader's optimal strategy. The upper level problem reads as follows:

$$
\begin{array}{llll}
(PE_u) & min & -T+\displaystyle\int_0^T \frac{1}{2}u^E(t)^2dt & \\
 & s.t. & \dot{x}^E(t)=v^E(t) & \forall\ t\in(0,T),\\
 & & \dot{v}^E(t)=u^E(t) & \forall\ t\in(0,T),\\
 & & u_i^E(t)\in[-u_{max}^E,u_{max}^E] & \forall\ t\in(0,T),\ i\in\{1,2\},\\
 & & x^E(0)=x_0^E,\ v^E(0)=0, & \\
 & & T\in\mathcal{S}(x^E(T)), &
\end{array}
$$

where $\mathcal{S}(x^E(T))$ is the set of optimal objective function values of the lower level problem:

$$
\begin{array}{llll}
(PE_l) & min & T & \\
 & s.t. & \dot{x}^P(t)=v^P(t) & \forall\ t\in(0,T),\\
 & & \dot{v}^P(t)=u^P(t) & \forall\ t\in(0,T),\\
 & & u_i^P(t)\in[-u_{max}^P,u_{max}^P] & \forall\ t\in(0,T),\ i\in\{1,2\},\\
 & & x^P(0)=x_0^P,\ x^P(T)=x^E(T), & \\
 & & v^P(0)=v^P(T)=0. &
\end{array}
$$

Note that the final time T can be considered as a state of the lower level problem with $\dot{T}=0$ and free initial value.

Using similar techniques as in [14, Example 7.1.15] it can be shown that the value function of (PE_l) is given by

$$
\mathcal{V}(y)=\max\left\{2\sqrt{\frac{|y_1-x_{0,1}^P|}{u_{max}^P}},2\sqrt{\frac{|y_2-x_{0,2}^P|}{u_{max}^P}}\right\}, \tag{4.1}
$$

where $y=(y_1,y_2)^\top\in\mathbb{R}^2$ and $x_0^P=(x_{0,1}^P,x_{0,2}^P)^\top$. Note that $\mathcal{V}$ is not Lipschitz-continuous at $y=x_0^P$. However, in this case the minimum time T of the pursuer is zero since the pursuer can capture the evader immediately. Hence, only the case $y \neq x_0^P$ is relevant and $\mathcal{V}$ is locally Lipschitz in this case.

4.1. **Reformulated single stage problem.** In this section, we exploit the value function of (PE_l), provided by (4.1) in order to reformulate the original bilevel optimal control problem (PE_u) as a single level optimal control problem. Notice that in our settings, it is not required to include the differential equations and constraints of the lower level problem, since the coupling is only present through the final time T. Hence, the reformulated problem reads as

$$
\begin{array}{llll}
(PE) & min & \displaystyle\int_0^T -1 + \frac{1}{2}u^E(t)^2 dt & \\
& s.t. & \dot{x}^E(t) = v^E(t) & \forall\, t \in (0,T), \\
& & \dot{v}^E(t) = u^E(t) & \forall\, t \in (0,T), \\
& & u_i^E(t) \in [-u_{max}^E, u_{max}^E] & \forall\, t \in (0,T),\ i \in \{1,2\}, \\
& & x^E(0) = x_0^E,\ v^E(0) = 0, & \\
& & T - \mathcal{V}(x^E(T)) \leq 0. &
\end{array}
$$

Instead of evaluating the necessary conditions of Theorem 3.3 we apply the necessary optimality conditions in [26, Theorem 6.2.3] directly to (PE) as it was done in the proof of Theorem 3.3 in the general case. The Hamiltonian function, relative to (PE) is given by

$$\mathcal{H}(x,u,v,\lambda_x,\lambda_v) = 1 - \frac{1}{2}u^2 + \lambda_x^\top v + \lambda_v^\top u.$$

The necessary optimality conditions for a minimum $(\hat{x},\hat{v},\hat{u})$ of (PE) yield the existence of multipliers (λ_x, λ_v) and $h \in \mathbb{R}$, such that

$$\dot{\lambda}_x(t) = -\mathcal{H}_x'[t] = 0_{\mathbb{R}^2} \tag{4.2}$$

$$\dot{\lambda}_v(t) = -\mathcal{H}_v'[t] = -\lambda_x(t) \tag{4.3}$$

$$
\begin{aligned}
\hat{u}_i(t) &= \arg \max_{u_i \in [-u_{max}^E, u_{max}^E]} \left\{ 1 - \frac{1}{2}u_i^2 + \lambda_{x,i}(t)\hat{v}_i(t) + \lambda_{v,i}(t)u_i \right\} \\
&= \arg \max_{u_i \in [-u_{max}^E, u_{max}^E]} \left\{ -\frac{1}{2}u_i^2 + \lambda_{v,i}(t)u_i \right\}, \quad i \in \{1,2\}
\end{aligned} \tag{4.4}
$$

$$h = \mathcal{H}(\hat{x}(t), \hat{v}(t), \hat{u}(t), \lambda_x(t), \lambda_v(t)) \tag{4.5}$$

$$(h, -\lambda_x(T)) \in N_S^L(T, \hat{x}(T)) \tag{4.6}$$

$$\lambda_v(T) = 0_{\mathbb{R}^2}. \tag{4.7}$$

Note that in (4.6), we have introduced the set $S := \big\{(t,x) \bigm| t - \mathcal{V}(x) \leq 0\big\}$ and denoted with $N_S^L(T,\hat{x}(T))$ its limiting normal cone in the point $(T, \hat{x}(T))$.

The adjoint equations (4.2)-(4.3) imply that there exist $c_x, c_v \in \mathbb{R}^2$, such that

$$\lambda_x(t) = c_x \quad and \quad \lambda_v(t) = -c_x t + c_v.$$

Furthermore, by the transversality condition (4.7), we have $c_v = c_x T$ and so $\lambda_v(t) = c_x(T-t)$. Let us now investigate (4.6). Note that S is the set of all $(t,x) \in \mathbb{R}\times\mathbb{R}^2$, for which $\varphi(t,x) \le 0$ where

$$\varphi(t,x) := t - \mathcal{V}(x) = t - \max\left\{2\sqrt{\frac{|x_1 - x^P_{0,1}|}{u^P_{max}}}, 2\sqrt{\frac{|x_2 - x^P_{0,2}|}{u^P_{max}}}\right\}$$

Since φ is locally Lipschitz function in $\mathbb{R} \times \left(\mathbb{R}^2 \setminus \{x^P_0\}\right)$, assuming that $\hat{x}(T) \neq x^P_0$, its subdifferential in $(T, \hat{x}(T))$ is given by

$$\partial\varphi(T,\hat{x}(T)) = \left\{(1, -\omega_1\hat{k}_1, \omega_2\hat{k}_2) \;\middle|\; \begin{array}{c}\omega_1,\omega_2 \in [0,1]\\ \omega_1+\omega_2 = 1\end{array}\right\} \tag{4.8}$$

where

$$\hat{k}_1 := \frac{\hat{x}_1(T) - x^P_{0,1}}{\sqrt{u^P_{max}|\hat{x}_1(T) - x^P_{0,1}|^3}} \quad \textit{and} \quad \hat{k}_2 := \frac{\hat{x}_2(T) - x^P_{0,2}}{\sqrt{u^P_{max}|\hat{x}_2(T) - x^P_{0,2}|^3}}.$$

Note that in case $|\hat{x}_1(T) - x^P_{0,1}| > |\hat{x}_2(T) - x^P_{0,2}|$, the set $\partial\varphi(T,\hat{x}(T))$ reduces to the single element $\left\{(1,\hat{k}_1,0)\right\}$ which is captured by (4.8) by setting $\omega_1 = 1$ and $\omega_2 = 0$. In the same way, when $|\hat{x}_1(T) - x^P_{0,1}| < |\hat{x}_2(T) - x^P_{0,2}|$ we obtain $\partial\varphi(T,\hat{x}(T)) = \left\{(1,0,-\hat{k}_2)\right.$ for $\omega_1 = 0$ and $\omega_2 = 1$. By (4.8) it follows that the limiting normal cone $N^L_S(T,\hat{x}(T))$ is given by the set

$$\left\{\alpha\big(1, -\omega_1\hat{k}_1, -\omega_2\hat{k}_2\big) \;\middle|\; \begin{array}{l}\alpha \ge 0\\ \varphi(T,\hat{x}(T)) \le 0\\ \alpha\varphi(T,\hat{x}(T)) = 0\\ \omega_1,\omega_2 \in [0,1],\ \omega_1+\omega_2 = 1\\ \omega_2 = 0 \textit{ if } |\hat{x}_1(T) - x^P_{0,1}| > |\hat{x}_2(T) - x^P_{0,2}|\\ \omega_1 = 0 \textit{ if } |\hat{x}_1(T) - x^P_{0,1}| < |\hat{x}_2(T) - x^P_{0,2}|\end{array}\right\}.$$

Hence by (4.6) it follows that there exist $\alpha \ge 0$, $\alpha\varphi(T,\hat{x}(T)) = 0$ and $\omega_1,\omega_2 \in [0,1]$, $\omega_1 + \omega_2 = 1$ such that for $i \in \{1,2\}$ we have

$$\lambda_{x,i}(t) = \alpha\omega_i\hat{k}_i \quad \textit{and} \quad \lambda_{v,i}(t) = \alpha\omega_i\hat{k}_i(T-t).$$

Exploiting (4.5) we can derive the optimal control $\hat{u}_i$:

$$\hat{u}_i(t) = \begin{cases} -u^E_{max} & \textit{if } \lambda_{v,i}(t) < -u^E_{max}\\ \alpha\omega_i\hat{k}_i(T-t) & \textit{if } \lambda_{v,i}(t) \in [-u^E_{max}, u^E_{max}]\\ u^E_{max} & \textit{if } \lambda_{v,i}(t) > u^E_{max}\end{cases}.$$

Since $\lambda_{v,i}(t)$ is a linear function of t and $\lambda_{v,i}(T) = 0$, it follows that $\lambda_{v,i}$ and thus $\hat{u}_i$ have constant sign in $(0,T)$, depending on the sign of $\hat{x}_i(T) - x^P_{0,i}$. Let us suppose that $\hat{x}_i(T) - x^P_{0,i} > 0$ (the other case can be treated in the same way) and let us define

$$t_{s,i} = \begin{cases}\max\left\{T - \frac{u^E_{max}}{\alpha\omega_i\hat{k}_i}, 0\right\} & \textit{if } \omega_i > 0\\ 0 & \textit{if } \omega_i = 0\end{cases} \tag{4.9}$$

i.e. the projection of the instance at which $\lambda_{v,i}(t) = u^E_{max}$ on the interval $[0, T]$. The optimal control $\hat{u}_i$ is given by

$$\hat{u}_i(t) = \begin{cases} u^E_{max} & t \in (0, t_{s,i}) \\ \alpha\omega_i \hat{k}_i (T - t) & t \in [t_{s,i}, T) \end{cases}.$$

We obtain $\hat{v}_i$ by integrating $\hat{u}_i$ over $[0, T]$:

$$\hat{v}_i(t) = \begin{cases} u^E_{max} t & t \in [0, t_{s,i}) \\ u^E_{max} t_{s,i} + \frac{1}{2}\alpha\omega_i \hat{k}_i \left\{ (T - t_{s,i})^2 - (T - t)^2 \right\} & t \in [t_{s,i}, T] \end{cases}$$

and $\hat{x}_i$ by integrating $\hat{v}_i$:

$$\hat{x}_i(t) = \begin{cases} x^E_{0,i} + \frac{1}{2} u^E_{max} t^2 & t \in [0, t_{s,i}) \\ x^E_{0,i} + u^E_{max} t_{s,i}(t - \frac{1}{2} t_{s,i}) & \\ +\frac{1}{6}\alpha\omega_i \hat{k}_i (T - t_{s,i})^2 (3t - 2t_{s,i} - T) & \\ +\frac{1}{6}\alpha\omega_i \hat{k}_i (T - t)^3 & t \in [t_{s,i}, T] \end{cases}.$$

Let us now consider the Hamiltonian function $\mathcal{H}$. Note that it can be written as

$$\mathcal{H}[t] = 1 + \mathcal{H}_1[t] + \mathcal{H}_2[t]$$

where

$$\mathcal{H}_i[t] := -\frac{1}{2} (\hat{u}_i(t))^2 + \lambda_{x,i}(t)\hat{v}_i(t) + \lambda_{v,i}(t)\hat{u}_i(t) \quad \forall\, t \in [0, T],\ i \in \{1, 2\}.$$

We observe that when $\omega_i = 0$ we have $\mathcal{H}_i \equiv 0$. Hence we suppose that $\omega_i > 0$, we can distinguish two cases: $t_{s,i} = 0$ and $t_{s,i} > 0$.

Let us first suppose that $t_{s,i} = 0$. Then for every $t \in [0, T]$ $\mathcal{H}_i$ becomes

$$\begin{aligned} \mathcal{H}_i[t] = &-\frac{1}{2} \left(\alpha\omega_i \hat{k}_i (T - t) \right)^2 + \alpha\omega_i \hat{k}_i \left\{ \frac{1}{2}\alpha\omega_i \hat{k}_i \left[T^2 - (T - t)^2 \right] \right\} \\ &+ \alpha^2\omega_i^2 \hat{k}_i^2 T^2 = \frac{1}{2}\alpha^2\omega_i^2 \hat{k}_i^2 T^2. \end{aligned}$$

Suppose now that $t_{s,i} > 0$. For each $t \in [0, t_{s,i})$ $\mathcal{H}_i$ becomes

$$\begin{aligned} \mathcal{H}_i(t) &= -\frac{1}{2} \left(u^E_{max} \right)^2 + \alpha\omega_i \hat{k}_i u^E_{max} t + \alpha\omega_i \hat{k}_i (T - t) u^E_{max} \\ &= -\frac{1}{2} \left(u^E_{max} \right)^2 + \alpha\omega_i \hat{k}_i u^E_{max} T \end{aligned}$$

while in $[t_{s,i}, T]$ we have

$$\begin{aligned}
\mathcal{H}_i[t] =& -\frac{1}{2}\left(\alpha\omega_i\hat{k}_i(T-t)\right)^2 + \alpha^2\omega_i^2\hat{k}_i^2(T-t)^2 \\
&+ \alpha\omega_i\hat{k}_i\left\{u^E_{max}t_{s,i} + \frac{1}{2}\alpha\omega_i\hat{k}_i\left[(T-t_{s,i})^2 - (T-t)^2\right]\right\} \\
=& \frac{1}{2}\alpha^2\omega_i^2\hat{k}_i^2(T-t_{s,i})^2 + \alpha\omega_i\hat{k}_i u^E_{max}t_{s,i} \\
=& \frac{1}{2}\alpha^2\omega_i^2\hat{k}_i^2\left(\frac{u^E_{max}}{\alpha\omega_i\hat{k}_i}\right)^2 + \alpha\omega_i\hat{k}_i u^E_{max}\left(T - \frac{u^E_{max}}{\alpha\omega_i\hat{k}_i}\right) \qquad (4.10) \\
=& -\frac{1}{2}\left(u^E_{max}\right)^2 + \alpha\omega_i\hat{k}_i u^E_{max}T.
\end{aligned}$$

Note that in (4.10) we have substituted the value of $t_{s,i}$ from (4.9). Summarizing

$$\mathcal{H}_i[t] = \begin{cases} 0 & \text{if } \omega_i = 0 \\ \frac{1}{2}\alpha^2\omega_i^2\hat{k}_i^2T^2 & \text{if } \omega_i > 0 \text{ and } t_{s,i} = 0 \\ -\frac{1}{2}\left(u^E_{max}\right)^2 + \alpha\omega_i\hat{k}_i u^E_{max}T & \text{if } \omega_i > 0 \text{ and } t_{s,i} > 0 \end{cases}. \qquad (4.11)$$

Thus in order to compute the optimal solution of (PE) we have to solve the (possibly underdetermined) system:

$$1 + \mathcal{H}_1[t] + \mathcal{H}_2[t] = \alpha \qquad (4.12)$$

$$\omega_1, \omega_2 \geq 0 \qquad i = 1, 2 \qquad (4.13)$$

$$\omega_1 + \omega_2 = 1 \qquad (4.14)$$

$$t_{s,i} = \begin{cases} \max\left\{T - \frac{u^E_{max}}{\alpha\omega_i\hat{k}_i}, 0\right\} & \text{if } \omega_i > 0 \\ 0 & \text{if } \omega_i = 0 \end{cases} \qquad i = 1, 2 \qquad (4.15)$$

$$\varphi(T, \hat{x}(T)) \leq 0 \qquad (4.16)$$

$$\alpha \geq 0 \qquad (4.17)$$

$$\alpha\varphi(T, \hat{x}(T)) = 0 \qquad (4.18)$$

with respect to α, T, $t_{s,1}$, $t_{s,2}$, ω_1 and ω_2. Once solution of (4.12)-(4.18) is found, we compute the optimal solution of (PE) by substituting the respective values in the expressions of $\hat{x}$, $\hat{v}$ and $\hat{u}$.

4.2. **Numerical Results.** Let us now provide a concrete numerical scenario. We set the starting position of the pursuer (P) at the origin $x^P_0 = (0, 0)$ and $u^P_{max} = 4$. Furthermore, let the evader (E) start from the point $x^E_0 = (3, 2)$ with initial velocity $v^E_0 = (0, 0)$ and $u^E_{max} = 1$. Applying the same reasoning as in the previous subsection, leads to the results in Figure 1.

5. Conclusions and Outlook

This paper suggests a general approach for solving a class of bilevel optimal control problems, exploiting the value function of the lower level problem and necessary optimality conditions. Necessary optimality conditions are obtained via single level reformulation and generalized subgradient of the value function. If it is impossible

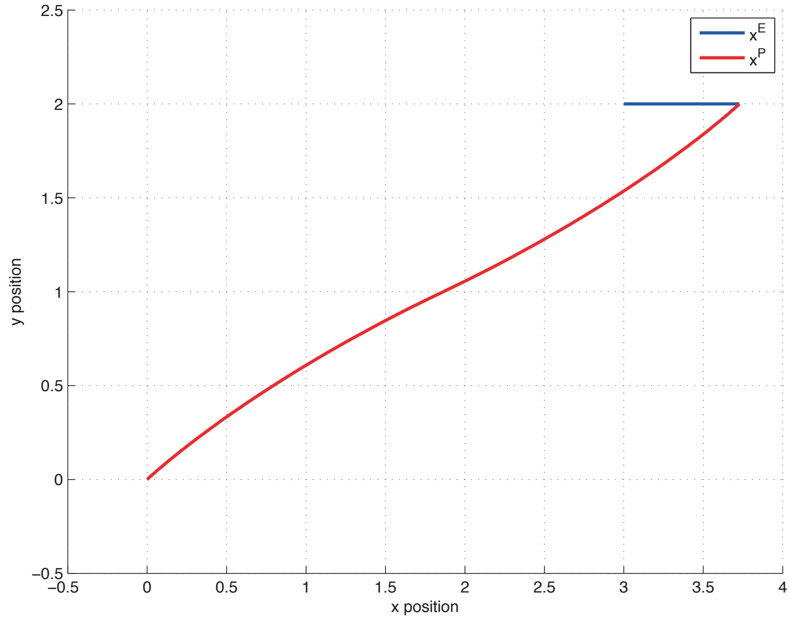
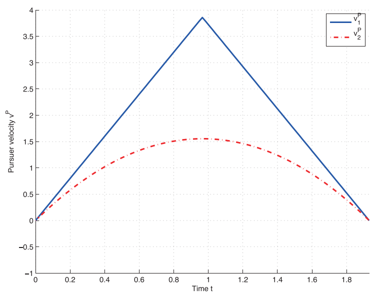
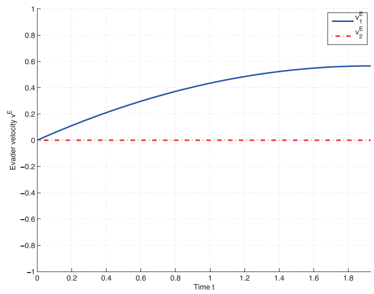


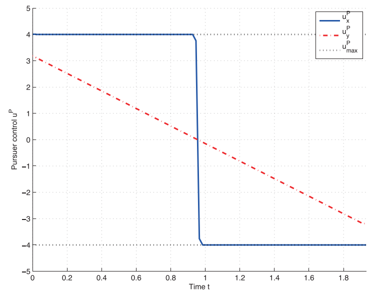
FIGURE 1. Trajectories of the pursuer and evader



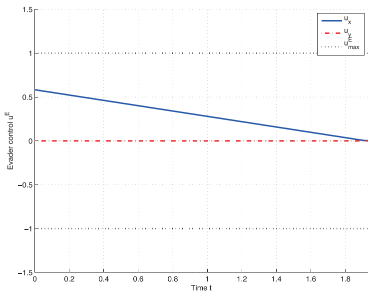
(a) Pursuer's speed v^E



(b) Evader's speed v^E



(c) Pursuer's control u^E



(d) Evader's control u^E

to obtain an explicit formulation of the value function, its generalized subgradient is characterized in terms of Lagrange multipliers of the lower level problem.

Naturally several issues have to be addressed in further studies, for instance the treatment of state constraints in the lower level problem or the incorporation of a general (numerical) approach for computing the value function as a viscosity solution of a Hamilton-Jacobi-Bellman equation. We leave these topics open for future research.

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K. Palagachev
Institut für Mathematik und Rechneranwendung, Fakultät für Luft- und Raumfahrttechnik, Universität der Bundeswehr München, Werner-Heisenberg-Weg 39, D-85577 Neubiberg/München, Germany
E-mail address: `konstantin.palagachev@unibw.de`

M. Gerdts
Institut für Mathematik und Rechneranwendung, Fakultät für Luft- und Raumfahrttechnik, Universität der Bundeswehr München, Werner-Heisenberg-Weg 39, D-85577 Neubiberg/München, Germany
E-mail address: `matthias.gerdts@unibw.de`