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MAXIMUM PRINCIPLE FOR HYBRID SYSTEMS WITH POLYNOMIAL IMPULSES

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Abstract. We address variational problems with trajectories of bounded variation and polynomial impulses subject to mixed asymptotic constraints [14]. We provide first order necessary optimality conditions in the spirit of the impulsive Maximum principle [19]. The result is based on the problem transformation to a suitable nonimpulsive variational problem by a specific space-time reparameterization technique being a particular extension of the well-known discontinuous time changing method [20, 25, 26].

1. Introduction

We study impulsive hybrid control problems with dynamics being a BV-relaxation (a compactification of the trajectory tube in the weak* topology of the space BV of functions with bounded variation) of the following control system

$$\dot{x} = f_0(x,u) + \sum_{q\in Q} f_q(x,u)v^q, \tag{1.1}$$

$$x(0) = x_0, \tag{1.2}$$

$$u \in \mathcal{U}_T, \quad v \in \mathcal{V}_T. \tag{1.3}$$

The system's right-hand side is a polynomial of a given rational power $p \geq 1$ of the control variable $v \in \mathbb{R}$ with the coefficients $f_q(x,u)$, $q \in Q \cup \{0\}$, depending on the state variable $x \in \mathbb{R}^n$ and the extra control variable $u \in \mathbb{R}^m$. Here, Q is a given finite set of distinct positive rational numbers with $\max Q = p$ such that the maps $v \mapsto v^q$, $q \in Q$, are defined for negative values v; $x_0 \in \mathbb{R}^n$ is a fixed initial state, and the sets $\mathcal{U}_T$ and $\mathcal{V}_T$ of control functions are as follows:

$$\mathcal{U}_T \doteq \{u :\, [0,T] \to U|\ u \in L_\infty([0,T],\mathbb{R}^m)\}$$

$$\mathcal{V}_T \doteq \left\{v :\, [0,T] \to \mathbb{R}|\ v \in L_\infty([0,T],\mathbb{R}),\ \|v\|_{L_p([0,T],\mathbb{R})} \leq M^{1/p}\right\}$$

with given positive reals T, M, and a compact $U \subset \mathbb{R}^m$.

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In [13] we give a constructive description of an impulse-trajectory extension (relaxation) of system (1.1)–(1.3). Under some natural convexity assumptions, it is shown that such an extension can be described by a certain measure driven dynamical system, which is traditionally abbreviated to the following symbolic measure differential equation:

$$dx = f_0(x,u)dt + \sum_{q\in Q\setminus\{p\}} f_q(x,u)\, l^q\, dt + f_p(x,u)\, \vartheta(dt),\ x(0-) = x_0, \tag{1.4}$$

where ϑ is an *impulsive control* in the sense to be specified below, at this point it can be roughly thought of as a signed Borel measure.

Definition 1.1 ([13])**.** A function $\mathrm{x} \in BV$ is said to be a generalized solution to system (1.1)–(1.3) if and only if there exists a sequence $\{(u_k, v_k)|\ k \in \mathbb{N}\}$ of controls $u_k \in \mathcal{U}_T$, $v_k \in \mathcal{V}_T$ such that the respective sequence of Carathéodory solutions $x_k = x[u_k, v_k]$ of (1.1) converges to x in the weak* topology of BV (i.e., at all points of continuity, and at $t = T$).

Denote by $\mathbb{X}$ the set of generalized solutions to (1.1)–(1.3). In [13] we prove that $\mathbb{X}$ coincides with the set of solutions to the measure differential equation (1.4) in the sense of the concept specified below.

The model of our consideration is weighted with the mixed asymptotic constraints

$$x(t-) \in Z_-,\ x(t) \in Z_+ \quad |\vartheta|\text{-a.e. on } [0,T], \tag{1.5}$$

used in the formalization of hybrid dynamical systems by means of the impulsive control theory [11, 14]. Here $Z_\pm \subseteq \mathbb{R}^n$ are given closed sets playing the part of "jump permitting" and "jump destination" domains (in terms of the hybrid system theory, see, e.g., [3, 8]). With respect to the mathematical impulsive control theory, constraints (1.5) are of particular interest. Indeed, we note the following:

- For impulsive systems driven by measures, conditions of this sort represent a natural type of mixed constraints as constraints imposed on both trajectory and control measure.
- Inclusions (1.5) give a fruitful mathematical formalization of switching rules in a wide class of hybrid systems. The evaluation of models with such constraints contributes to the unification of the impulsive control and hybrid system theories [2, 3, 11, 16, 18, 20, 21].
- Finally, conditions (1.5) can be thought of as a certain form of impulsive feedback.

The theory of affine (linear) impulsive control and respective non-regular variational problems are relatively well studied (see, e.g., [9, 10, 15, 19, 20, 24–28] and the bibliography therein).

Our interest to dynamical systems with nonlinear impulses is inspired by the quadratic case ($p = 2$), which seems to be typical for some problems from Lagrangean mechanics [4–7]. A polynomial case was addressed (in different contexts) by [13, 22, 23], and impulse extensions of general nonlinear systems with unbounded control sets were studied in [15, 20, 25–27].

One of the simplest model of the sort (1.4), (1.5) with nonlinear impulses can be found in [12], while the case $p = 1$ was investigated in [14].

The purpose of the paper is to derive first order necessary optimality conditions in the form of impulsive maximum principle for control problems with dynamical systems (1.4), (1.5).

2. Optimal control problem for hybrid systems with polynomial impulses

In this section we give a formal statement of the optimization problem of our interest. Beforehand, we specify the notion of control (input signal) $\varrho \doteq (u, \vartheta)$ and introduce the space $\mathcal{P}$ of admissible controls.

Let λ denote the Lebesgue measure on $\mathbb{R}$, i.e., $\lambda(dt) = dt$.

1) The "usual" part u of control input ϱ is played by Borel measurable ($\mathcal{B}$-measurable) functions $u : [0,T] \to \mathbb{R}^m$ such that

$$u(t) \in U \ \lambda\text{-almost everywhere (a.e.) on } [0,T]. \tag{2.1}$$

We denote the set of all such controls by $\mathcal{U}_T$.

2) The impulsive control ϑ is defined similarly to [1] as a collection

$$\vartheta \doteq \left(\nu, \mu, l, \{e_\tau, u_\tau\}_{\tau\in\Delta_\nu(T)}\right).$$

Here,

- $\nu, \mu \in C^*([0,T], \mathbb{R})$ are Lebesgue-Stieltjes measures (the Lebesgue-Stieltjes extensions of the Borel measures induced by functions of bounded variation; in what follows, we identify Borel measures with their unique Lebesgue-Stieltjes extensions and call them simply "measures") with

$$|\mu| \le \nu, \ |\mu|_c = \nu_c, \text{ and } \nu([0,T]) \le M \tag{2.2}$$

  (in respect of a measure, $|\cdot|$ makes sense of its total variation; by definition $|\vartheta| = \nu$; $\nu_c \doteq \nu_{ac} + \nu_{sc}$, where the summands represent the absolutely continuous and singular continuous components of the Lebesgue decomposition of ν).
- $l : [0,T] \to \mathbb{R}$ is a $\mathcal{B}$-measurable function with

$$\int_0^t l^p(\theta)d\theta = \mu_{ac}([0,t]) \text{ for all } t \in [0,T]. \tag{2.3}$$

- $\{e_\tau, u_\tau\}_{\tau\in\Delta_\nu(T)}$ is a family of $\mathcal{B}$-measurable functions

$$e_\tau : [0,T_\tau] \to \mathbb{R}, \quad u_\tau : [0,T_\tau] \to \mathbb{R}^m$$

  parameterized by atoms of ν and such that

$$|e_\tau(\theta)| = 1, \ u_\tau(\theta) \in U \ \lambda\text{-a.e. on } [0,T_\tau], \tag{2.4}$$

$$\int_0^{T_\tau} e_\tau(\theta)d\theta = \mu(\{\tau\}), \tag{2.5}$$

  with $\Delta_\nu(t) \doteq \{\tau \in [0,t] |\ \nu(\{\tau\}) > 0\}$, and $T_\tau \doteq \nu(\{\tau\})$.

Let Θ denote the set of impulsive controls satisfying constraints (2.1)–(2.5). Finally, we set

$$\mathcal{P} = \mathcal{U}_T \times \Theta.$$

Remark 2.1 ([13]). If $\upsilon \mapsto \upsilon^p$ is an even function, then $\mu = \nu$, and $e_\tau(\theta) = 1$ on $[0, T_\tau]$ for any atom τ of μ; and if the mapping $\upsilon \mapsto \upsilon^p$ is odd, then $l = \left(\dot{F}_{\mu_{ac}}\right)^{1/p}$, where $F_{\mu_{ac}}$ is the distribution function of μ_{ac}.

Assume that

- the functions f_q, $q \in Q \cup \{0\}$, are continuous in all variables, uniformly Lipschitz continuous in x, and satisfy the linear growth condition with respect to (wrt) x;
- the set $\mathcal{F}(x)$ of vectors
$$(a^p, a^p f_0(x,w) + \sum_{q \in Q \setminus \{p\}} a^{p-q} b^q f_q(x,w) + b^p f_p(x,w), |b|^p) \in \mathbb{R}^{n+2}$$
such that $(a,b) \in A$, and $w \in U$ is *convex* for any $x \in \mathbb{R}^n$. Here, $A \doteq \mathrm{co}\{(a,b) \in \mathbb{R}^2 |\ a \geq 0,\ a^p + |b|^p = 1\}$, and $\mathrm{co}\, A$ denotes the convex hull of a set A.

Remark 2.2. One can lift the latter convexity assumption and consider the impulsive dynamical system to be described below, regardless of its usual prototype.

Given a lower semicontinuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$, consider the following problem (P) of optimal impulsive control:

$$\text{Minimize } I = \varphi(x(T))$$

over the right continuous arcs $x : [0,T] \to \mathbb{R}^n$ of bounded variation ($x \in BV^+([0,T], \mathbb{R}^n)$) satisfying the integral relation

$$\begin{aligned} x(t) &= x_0 + \int_0^t f_0(x(\theta), u(\theta))\, d\theta + \\ &+ \sum_{q \in Q \setminus \{p\}} \int_0^t f_q(x(\theta), u(\theta))\, l^q(\theta)\, d\theta + \\ &+ \int_0^t f_p(x(\theta), u(\theta)) \mu_c(d\theta) + \sum_{\tau \in \Delta_\nu(t)} [\kappa_\tau(T_\tau) - x(\tau-)] \end{aligned} \tag{2.6}$$

together with condition (1.5), and with control inputs

$$\varrho \doteq (u, \vartheta) \in \mathcal{P}. \tag{2.7}$$

In (2.6), for each $\tau \in \Delta_\nu(T)$, κ_τ is a solution to the "limit" system [15]

$$\frac{d}{d\theta}\kappa = f_p(\kappa, u_\tau) e_\tau, \quad \kappa(0) = x(\tau-). \tag{2.8}$$

The functions $\varkappa_\tau$ present the behavior of the system's state in a "fast tempo" along jumps, and the intervals $[0, T_\tau]$ can be thought of as intervals of such "fast motions" [20].

The existence and uniqueness of a solution $x[\varrho]$ to (2.6)–(2.8) under any $\varrho \in \mathcal{P}$ are implied by the general result [20, Theorem 8.22]. One can refer to the measure differential equation (1.4) as to a symbolic form of (2.6), (2.8), or, equivalently, can consider relations (2.6), (2.8) as a concept of solution to (1.4).

Given a control ϱ, we call the family $\mathcal{X} = \mathcal{X}[\varrho] \doteq \{\kappa_\tau\}_{\tau\in\Delta_\nu(T)}$ of solutions to the limit system a *graph completion* of a trajectory $x[\varrho]$. Clearly, such a graph completion is generically not unique.

A couple $\sigma \doteq (x, \varrho) = (x[\varrho], \varrho)$ with $\varrho \in \mathcal{P}$ is said to be an *admissible control process*, and $\Sigma(P)$ denotes the set of all admissible processes. We assume $\Sigma(P) \neq \emptyset$.

3. Problem transformation

In this section we present a technique for equivalent transformation of problem (P) to a standard problem of dynamic optimization. This technique will serve us later to derive first order necessary optimality conditions in the form of the Maximum Principle.

On a time interval $[0, S]$, $S \geq T$, consider the following reduced problem (RP):

$$\text{Minimize } J = \varphi(y_+(S))$$

subject to the constraints

$$\begin{aligned}\frac{d}{ds}y_\pm = \alpha^p f_0(y_\pm,\omega) + \sum_{q\in Q,\ q<p} \gamma_\pm^{q/p}\alpha^{p-q}\beta^q f_q(y_\pm,\omega)\\ + \gamma_\pm\beta^p f_p(y_\pm,\omega), \quad y_\pm(0) = x_0,\end{aligned} \tag{3.1}$$

$$\frac{d}{ds}\xi = \alpha^p,\ \frac{d}{ds}(\eta,\zeta)_\pm = \gamma_\pm(\beta^p, |\beta|^p), \tag{3.2}$$

$$\xi(0) = \eta_\pm(0) = \zeta_\pm(0) = 0, \tag{3.3}$$

$$y_+(S) = y_-(S), \quad \eta_+(S) = \eta_-(S), \tag{3.4}$$

$$\xi(S) = T, \quad \zeta_+(S) = \zeta_-(S) \leq M, \tag{3.5}$$

$$\zeta_- - \zeta_+ \leq 0, \tag{3.6}$$

$$\int_0^S \Gamma(\alpha,\beta,\gamma,y,\eta,\zeta)ds = 0, \tag{3.7}$$

$$\omega \in \mathcal{U}_S, \quad (\alpha,\beta,\gamma) \in \mathcal{A}_S,\ \gamma = (\gamma_+,\gamma_-). \tag{3.8}$$

Here, $\mathcal{U}_S$ is defined similarly to $\mathcal{U}_T$ in Section 1, $\mathcal{A}_S$ denotes the set of control functions (α,β,γ) with $\mathcal{B}$-measurable components $\alpha,\beta,\gamma_\pm : [0,S] \to \mathbb{R}$ such that $(\alpha,\beta,\gamma)(s) \in \widetilde{A}$ for λ-a.a. $s \in [0,S]$, where $\widetilde{A}$ denotes the set of vectors $\widetilde{a} = (a, b, c_+, c_-) \in \mathbb{R}^4$ such that

$$a,\ c_\pm \geq 0,\ a^p + |b|^p \leq 1, \text{ and } c_+ + c_- = 1. \tag{3.9}$$

The state variables are y, ξ, η, and ζ with $y = (y_+, y_-)$, $\eta = (\eta_+, \eta_-)$, $\zeta = (\zeta_+, \zeta_-)$, where $\xi, \eta_\pm, \zeta_\pm \in \mathbb{R}_+$, and $y_\pm \in \mathbb{R}^n$ ($\mathbb{R}_+$ is the set of nonnegative reals).

We denote $\Delta\zeta \doteq \zeta_+ - \zeta_-$ and take similar notations for the other state components. The function Γ in (3.7) takes the form

$$\Gamma = \alpha^p\big\{\Delta\zeta + W_{\{0\}}^{\mathbb{R}^{n+2}}(\Delta(y,\eta))\big\} + |\beta|^p\,\big\{\gamma_+ W_{Z_-}^{\mathbb{R}^n}(y_-) + \gamma_- W_{Z_+}^{\mathbb{R}^n}(y_+)\big\}.$$

Here $W_Y^X : X \to \mathbb{R}_+$ is a given (conveniently chosen) continuous function which vanishes only on a given subset Y of a finite-dimensional space X (such a function, even a smooth one, does exist for any closed subset of $\mathbb{R}^n$ and can be defined, say, as in classical partition of unity).

As is easily observed, the reduced problem (RP) implies the minimization over absolutely continuous arcs (elements of $AC([0,S],\mathbb{R}^{2n+5})$) satisfying a "usual" (non-impulsive) dynamical system under pointwise state, terminal and functional constraints.

A collection $\varsigma = (y,\xi,\eta,\zeta,\alpha,\beta,\gamma,\omega;S)$ enjoying (3.1)–(3.8) is said to be an admissible process for (RP). By $\Sigma(RP)$ we denote the set of all admissible processes.

Remark 3.1. In general, the discontinuous time changing method is a way to put the system's fast- and usual-tempo motions in a common time scale by extending instants of impulses into intervals of the length proportional to the intensity of the applied impulsive control. The original idea of the proposed space-time reparameterization consists in the following: In order to evaluate constraint (1.5), the left and right one-sided limits are regarded as different state trajectories, and the lengths of the fast time intervals are duplicated. The state trajectory marked with "+" — corresponding to the right limit — evolves over the first half of an interval of fast motion while the "minus"-trajectory, associated with the left limit, stays fixed in Z_-. Once the "plus"-branch reaches Z_+, we keep it fixed during the second half of the interval and let the "minus"-component catch up the "plus" one. So, both the branches coincide with each other by the end of the interval of fast motion.

The realization of this idea assumes an extension of the dimension of the state space: The states $y_\pm$ correspond to x, $\eta_\pm$ are associated with μ, and $\zeta_\pm$ correspond to ν (the measures are formally considered as extra trajectories). Constraints (3.4)–(3.7) also serve this idea.

Given $\varrho = (u,\vartheta) \in \mathcal{P}$ with $\vartheta = \big(\nu,\mu,l,\{e_\tau,u_\tau\}_{\tau\in\Delta_\nu(T)}\big)$, define a function Υ : $[0,T] \to [0,T+2\nu([0,T])]$ by

$$\Upsilon(t) = t + 2\nu([0,t]),\ t\in[0,T), \quad \Upsilon(T) = T + 2\nu([0,T]),$$

and denote by $\upsilon : [0,T+2\nu([0,T])] \to [0,T]$ the inverse of Υ.

Given $S \in [T,T+2M]$, and $(\alpha,\beta,\gamma) \in \mathcal{A}_S$ such that the respective solution ξ of (3.2), (3.3) meets constraint (3.5), and $\omega \in \mathcal{U}_S$, we introduce the following map $\Xi :\ [0,T] \to [0,S]$,

$$\Xi(t) = \inf\{s\in[0,T]|\ \xi(s) > t\},\ t\in[0,T), \quad \Xi(T) = S. \tag{3.10}$$

Problems (P) and (RP) are equivalent to each other in the following sense.

Proposition 3.2. *1) For any control process $\sigma \in \Sigma(P)$, there exists a process $\varsigma = (y,\xi,\eta,\zeta,\alpha,\beta,\gamma,\omega;S) \in \Sigma(RP)$, $y = (y_+,y_-)$, $\eta = (\eta_+,\eta_-)$, $\zeta = (\zeta_+,\zeta_-)$, $\gamma = (\gamma_+,\gamma_-)$ such that*

$$\upsilon = \xi \quad on\ [0,S]; \tag{3.11}$$

$$x = y_\pm\circ\Upsilon,\ F_\mu = \eta_\pm\circ\Upsilon,\ F_\nu = \zeta_\pm\circ\Upsilon\ on\ [0,T]. \tag{3.12}$$

Here, F_μ, F_ν are the distribution functions of the measures, and "$\circ$" denotes the composition of functions.

2) For any process $\varsigma \in \Sigma(RP)$, there exists a process $\sigma = (x,\varrho) \in \Sigma(P)$, $\varrho = (u,\vartheta) \in \mathcal{P}$, $\vartheta = \big(\nu,\mu,l,\{e_\tau,u_\tau\}_{\tau\in\Delta_\nu(T)}\big)$, such that

$$y_\pm\circ\Xi = x,\ \eta_\pm\circ\Xi = F_\mu,\ \zeta_\pm\circ\Xi = F_\nu\ on\ [0,T]. \tag{3.13}$$

3) Solutions for problems (P) *and* (RP) *can exist only simultaneously. For optimal processes* $\sigma^* \in \sigma(P)$ *and* $\varsigma^* \in \Sigma(RP)$ *we get*

$$I(\sigma^*) = J(\varsigma^*). \tag{3.14}$$

The *proof* is similar to [14], and is based on the following considerations.

1) *Direct transform.* Set $S = T + 2\nu([0,T])$ and define the functions

$$\omega(s) = \begin{cases} (u_\tau \circ \theta_{\tau\pm})(s), & \text{if } \exists\, \tau \in D_\Upsilon \text{ such that (s.t.) } s \in \Upsilon_{\tau\pm}, \\ (u \circ \upsilon)(s), & \text{otherwise}, \end{cases} \tag{3.15}$$

$$\alpha(s) = \begin{cases} (m_1^{1/p} \circ \upsilon)(s), & \text{if } \upsilon(s) \in \operatorname{supp} \nu_{ac}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.16}$$

$$\beta(s) = \begin{cases} (e_\tau \circ \theta_{\tau\pm})(s), & \text{if } \exists\, \tau \in D_\Upsilon \text{ s.t. } s \in \Upsilon_{\tau\pm}, \\ 2^{1/p}\,(l \circ \upsilon)(s) \cdot \alpha(s), & \text{if } \upsilon(s) \in \operatorname{supp} \nu_{ac}, \\ 2^{1/p}\,(m_2^{1/p} \circ \upsilon)(s), & \text{if } \upsilon(s) \in \operatorname{supp} \nu_{sc}. \end{cases} \tag{3.17}$$

$$\gamma_+(s) = \begin{cases} 1, & \text{if } \exists\, \tau \in D_\Upsilon \text{ s.t. } s \in \Upsilon_{\tau+}, \\ 0, & \text{if } \exists\, \tau \in D_\Upsilon \text{ s.t. } s \in \Upsilon_{\tau-}, \\ 1/2, & \text{otherwise}, \end{cases} \tag{3.18}$$

and $\gamma_-(s) = 1 - \gamma_+(s)$, $s \in [0,S]$. Here,

$$m_1 = \frac{d\lambda}{d(\lambda + 2\nu)}, \quad \text{and } m_2 = \frac{d\mu_{sc}}{2\nu},$$

where the fractions denote the Radon-Nikodym derivatives of the measures;

$$\theta_{\tau+}(s) = s - \Upsilon(\tau-),\ \theta_{\tau-}(s) = \theta_{\tau+}(s) - \nu(\{\tau\}) \text{ for } s \in \Upsilon_\tau,$$
$$\Upsilon_\tau \doteq [\Upsilon(\tau-), \Upsilon(\tau)],\ \Upsilon_{\tau+} = \Upsilon(\tau-) + [0, T_\tau),\ \Upsilon_{\tau-} = \Upsilon_\tau \setminus \Upsilon_{\tau+},$$
$$D_\Upsilon = \{\tau \in [0,T] |\ \Upsilon(\tau) - \Upsilon(\tau-) > 0\},$$

and "$\operatorname{supp} \nu$" is the set where a measure ν is concentrated. As above, $T_\tau \doteq \nu(\{\tau\})$.

The defined control $(\omega, \alpha, \beta, \gamma)$ is of the class $\mathcal{U}_S \times \mathcal{A}_S$. Indeed, one can observe the following:

i) The component ω is $\mathcal{B}$-measurable as a composition of $\mathcal{B}$-measurable functions and takes values in U.

ii) For the just defined (α, β), the following implications hold:

– If $\upsilon(s) \in \operatorname{supp} \nu_{ac}$, then $|l|^p \doteq \frac{d|\mu_{ac}|}{d\lambda} = \frac{d\nu_{ac}}{d\lambda}$, and

$$\alpha^p + |\beta|^p = m_1(1 + 2|l|^p) \circ \upsilon = \frac{d\lambda}{d(\lambda + 2\nu_{ac})} \left(1 + 2\frac{d\nu_{ac}}{d\lambda}\right) \circ \upsilon = 1.$$

– If $\upsilon(s) \in \operatorname{supp} \nu_{sc}$, then $\alpha(s) = 0$, and

$$\alpha^p + |\beta|^p = 2(|m_2| \circ \upsilon), \text{ and } |m_2| = \frac{d|\mu_{sc}|}{2d\nu} = \frac{1}{2}.$$

Thus,

$$\alpha^p + |\beta|^p = 1 \text{ over } \{s \in [0,S] |\ \upsilon(s) \in \operatorname{supp} \nu_{sc}\}.$$

– Assume that $s \in \Upsilon_{\tau\pm}$ for some $\tau \in D_\Upsilon$. Then $\alpha(s) = 0$, and from (2.4) it follows that

$$\alpha^p + |\beta|^p = |(e_\tau \circ \theta_{\tau\pm})| = 1.$$

Thus, $\alpha^2 + \beta^2 = 1$ λ-a.e. over $[0,S]$.

iii) Finally, $\alpha, \gamma_\pm$ are nonnegative and $\gamma_+ + \gamma_- = 1$ by their definitions.

2) *Inverse transform.*
Define a desired control $\varrho = (u, \vartheta) \in \mathcal{P}$, $\vartheta = \big(\nu, \mu, l, \{e_\tau, u_\tau\}_{\tau \in \Delta_\nu(T)}\big)$ through the formulas

$$u \doteq \omega \circ \Xi; \tag{3.19}$$

$$\mu = dF_\mu, \quad \nu = dF_\nu, \tag{3.20}$$

where the functions F_μ, F_ν of bounded variation are introduced as

$$F_\mu(t) \doteq (\eta_+ \circ \Xi)(t), \quad F_\nu(t) \doteq (\zeta_+ \circ \Xi)(t),\ t \in [0,T],$$

with $F_\mu(0-) = 0,\ F_\nu(0-) = 0;$

$$l = (\gamma_+^{1/p} \alpha^{\oplus} \beta) \circ \Xi, \tag{3.21}$$

where $^\oplus$ denotes the operation of pseudoinversion, i.e., $\alpha^\oplus = 0$, if $\alpha = 0$, and $\alpha^\oplus = \alpha^{-1}$, otherwise.

For each $\tau \in \Delta_\nu(T)$, we can set

$$e_\tau = \beta \circ s_\tau, \quad u_\tau = \omega \circ s_\tau, \tag{3.22}$$

Here, $s_\tau(\theta) \doteq \inf\{s \in [\Xi(\tau-), \Xi(\tau)] : \theta_\tau(s) > \theta\}$ for $\theta \in [0, T_\tau)$ with $\theta_\tau(s) \doteq \zeta_+(s) - \nu([0,\tau))$, $s_\tau(T_\tau) \doteq \Xi(\tau)$, and $T_\tau \doteq \nu(\{\tau\})$.

The control $\varrho = (u, \vartheta)$ with u and $\vartheta \doteq \big(\nu, \mu, l, \{e_\tau, u_\tau\}_{\tau \in \Delta_\nu(T)}\big)$ defined by formulas (3.19)–(3.22) is of the class $\mathcal{P}$:

1) It is clear that $u \in \mathcal{U}_T$.

2) For $t \in (0,T]$, we have

$$\begin{aligned}
\int_0^t l^p(\theta) d\theta &= \int_0^{\Xi(t)} l^p(\xi(s))\, d\xi(s) \\
&= \int_0^{\Xi(t)} \big[\alpha(s)^\oplus \alpha(s)\big]^p \gamma_+(s) \beta^p(s)\ ds \\
&= \int_0^{\Xi(t)} I_{\{s \in [0,S] |\ \alpha(s) > 0\}}(s)\, d\eta_+(s) \doteq \int_0^{\Xi(t)} dF_{\mu_{ac}}(\xi(s)) \\
&= \mu_{ac}([0,t]).
\end{aligned}$$

Here, $I_A : [0,S] \to \{0,1\}$ denotes the characteristic function of a set A, i.e. $I_A(s) = 1$ if $s \in A$, and $I_A(s) = 0$, otherwise.

3) By definition of the function Ξ, $\alpha(s) = 0$ λ-a.e. on $[\Xi(\tau-), \Xi(\tau)]$. Then $|\beta(s)| = 1$ λ-a.e. on this interval, and therefore $|e_\tau| = |\beta \circ s_\tau| = 1$. Furthermore,

$$\begin{aligned}\int_0^{T_\tau} e_\tau(\theta)d\theta &= \int_{\theta_\tau(\Xi(\tau-))}^{\theta_\tau(\Xi(\tau))} \beta^p(s_\tau(\theta))d\theta \\ &= \int_{\Xi(\tau-)}^{\Xi(\tau)} \beta^p(s)d\theta_\tau(s) = \int_{\Xi(\tau-)}^{\Xi(\tau)} \beta^p(s)\ d\zeta_+(s) \\ &= \int_{\Xi(\tau-)}^{\Xi(\tau)} d\eta_+(s) = (\eta_+ \circ \Xi)(\tau) - (\eta_+ \circ \Xi)(\tau-) \\ &\doteq F_\mu(\tau) - F_\mu(\tau-) = \mu(\{\tau\}),\end{aligned}$$

which proves (2.5).

Under the defined controls, this is a rather standard exercise to prove equalities (3.11)–(3.13) by virtue of change of variable under the sign of Lebesgue-Stieltjes integral, see, e.g., [20]. Finally, one can ensure the inclusions (1.5) hold by the arguments similar to [14].

4. Necessary optimality condition

For simplicity, let problem (P) be free of "ordinary" control u. It implies that the reduced problem (RP) gets rid of dependence upon ω.

Assume that the function φ is continuously differentiable, f_q, $q \in Q \cup \{0\}$, are continuously differentiable in x, and all the functions $W^X_{\{0\}}$ that appear in this section are continuously differentiable and such that $\nabla W^X_{\{0\}}(0) = 0$.

Introduce some necessary objects related to the formalism of the Maximum Principle:

- The vector $\psi = (\psi_t, \psi_x, \psi_\nu)$ of variables dual to (t, x, ν) (t and ν are formally regarded as state variables).
- The "partial" Hamiltonians $H_q(x, \psi_x) = \langle \psi_x, f_q(x)\rangle,\ q \in Q \cup \{0\}$.

Theorem 4.1 (Maximum Principle)**.** *Let $\sigma = (x, \mathcal{X}, \varrho) \in \Sigma$ with $x = x[\varrho] \in BV^+([0,T], \mathbb{R}^n)$, $\mathcal{X} = \mathcal{X}[\rho] = \{\varkappa_\tau\}_{\tau\in\Delta_\nu(T)}$, $\varkappa_\tau \in AC([0,T_\tau], \mathbb{R}^n)$, $T_\tau = \nu(\{\tau\})$, and $\varrho = (u, \vartheta) \in \mathcal{P}$, $\vartheta = (\nu, \mu, l, \{e_\tau, u_\tau\})$, be an optimal control process for problem (P). Then, there exists a collection*

$$\Lambda = (\varpi_I, \varpi_\nu; \psi_t, (\psi_x, \Psi_x), \psi_\nu; \pi)$$

of "Lagrange multipliers": $\varpi_I, \varpi_\nu \geq 0$, $\psi_t \in \mathbb{R}$, $\psi_x \in BV^+([0,T], \mathbb{R}^n)$, $\Psi_x = \{\phi_\tau\}_{\tau\in\Delta_\nu(T)}$, $\phi_\tau \in AC([0,T_\tau], \mathbb{R}^n)$, $\psi_\nu = (\psi^+, \psi^-)_\nu$ with $\psi^\pm_\nu \in BV^+([0,T], \mathbb{R})$, and a nondecreasing $\pi \in BV^+([0,T], \mathbb{R})$ with $\pi(0-) = 0$, such that the following conditions (C_1)–(C_4) hold true.

(C_1) ***Nontriviality:*** $\varpi_I + \varpi_\nu + |\psi_t| + \pi(T) > 0$.

(C_2) ***Complementary slackness condition, associated with the constraint on the total impulse of control:*** $\varpi_\nu(\nu([0,T]) - M) = 0$.

(C_3) ***Adjoint equations and transversality conditions*** *(below, ∇_x denotes the partial derivative wrt $x \in \mathbb{R}^n$, and $d\pi$ is the Lebesgue-Stieltjes measure induced by the function π): The functions $\psi_\nu^\pm$ are of the form*

$$\psi_\nu^\pm(t) = \pm\,(d\pi - \psi_t\lambda)\,([t,T]) \mp \psi_t - \varpi_\nu, \tag{4.1}$$

and ψ_x is a solution to the following Cauchy problem for the measure differential equation (in the integral form)

$$\begin{aligned}\psi_x(t) = -\varpi_I \nabla\varphi(x(T)) + \int_t^T \nabla_x H_0\,dt + \sum_{q\in Q\setminus\{p\}} \int_t^T \nabla_x H_q\, l^q\,dt\\ + \int_t^T \nabla_x H_p\,\mu_c(dt) + \sum_{\tau\in\Delta_\nu(t)} [\psi(\tau) - \phi_\tau(T_\tau)]\end{aligned} \tag{4.2}$$

(the derivatives of the functions H_q are computed along respective trajectories (x, ψ_x)) with the graph completion $\Psi_x = \{\phi_\tau\}_{\tau\in\Delta_\nu(T)}$. For every $\tau \in \Delta_\nu(T)$, the functions ϕ_τ satisfy the "adjoint limit system"

$$-\frac{d}{d\theta}\phi = \nabla_x H_p(\varkappa_\tau, \phi)\,e_\tau, \quad \phi(T_\tau) = \psi_x(\tau), \tag{4.3}$$

$$\theta \in [0, T_\tau],\ \ \tau \in \Delta_\nu(T).$$

(C_4) ***Optimality conditions:***

– ***Optimality beyond the support of the control measure.***
For λ-almost all (a.a.) $t \in [0,T] \setminus \operatorname{supp}\nu_c$, it holds

$$H_0(x, \psi_x) = \mathcal{H}(x, \psi_x, \psi_\nu), \tag{4.4}$$

where

$$\mathcal{H} = \max_{\widetilde{a}\in\widetilde{A}} \hat{H},$$

$$\begin{aligned}\hat{H} \doteq a^p H_0 + \frac{1}{2}\sum_{q\in Q,\ q<p} a^{p-q} b^q [c_+^{q/p} + c_-^{q/p}] H_q + \frac{1}{2}\, b^p [c_+ + c_-] H_p +\\ + |b|^p\ \left[c_+\ (\psi_\nu^+ + \psi_t W_{Z_-}^{\mathbb{R}^n}(x)) + c_-\ (\psi_\nu^- + \psi_t W_{Z_+}^{\mathbb{R}^n}(x))\right]\big\},\end{aligned}$$

$\widetilde{a} = (a,\, b,\, c_+,\, c_-)$, and $\widetilde{A}$ is defined in (3.9).

– ***Optimality with respect to the support of the absolutely continuous part of the control measure.***
For ν_{ac}-a.a. $t \in \operatorname{supp}\nu_{ac}$, the function l satisfies

$$l = \tfrac{1}{2^{1/p}}\,\bar{a}^{\oplus}\bar{b} \tag{4.5}$$

where $\bar{a}$ and $\bar{b}$ are the first two components of an argument $\widetilde{a}$ of $\max_{\widetilde{a}\in\widetilde{A}} \hat{H}(x, \psi_x, \psi_\nu; a, b)$ (note that $W_{Z_-}^{\mathbb{R}^n}(x) = W_{Z_+}^{\mathbb{R}^n}(x) = 0$ over $\operatorname{supp}\nu_{ac}$); $\mu_{ac} = l^p\lambda$, and $\nu_{ac} = |\mu_{ac}|$.

– ***Optimality of the graph completion.***

For all instants $\tau \in \Delta_\nu(T)$ *of impulses, we have*

$$e_\tau \in \operatorname{Sign} H_p(\varkappa_\tau, \phi_\tau), \;\; and$$
$$|H_p(\varkappa_\tau, \phi_\tau)| - \varpi_\nu = \max_{\widetilde{a}\in\widetilde{A}} \left[H^+ + H^-\right]$$

λ*-a.e. over the intervals* $[0, T_\tau]$ *of fast motions. The multifunction* Sign *denotes the signature with* $\operatorname{Sign}(0) = [-1, 1]$. *Above,*

$$\begin{aligned} H^+(\theta, x, \phi; \widetilde{a}) \doteq a^p/2 \Big\{ & H_0(x, \phi) + H_0(x(\tau-), \phi(\tau-)) \\ & + \psi_t \Big[\theta + W^{\mathbb{R}}_{\{0\}}\left(\int_0^\theta e_\tau(\epsilon)d\epsilon\right) + W^{\mathbb{R}^n}_{\{0\}}(\varkappa_\tau(\theta) - x(\tau-))\Big]\Big\} \\ & + 1/2 \sum_{q\in Q,\, q<p} a^{p-q} b^q \left[c_+^{q/p} H_q(x, \phi) + c_-^{q/p} H_q(x(\tau-), \phi(\tau-))\right] \\ & + \; b^p/2\left[c_+ H_p(x, \phi) + c_- H_p(x(\tau-), \phi(\tau-))\right] \\ & + \; |b|^p\left[c_+\psi_\nu^+(\tau) + c_-[\psi_\nu^-(\tau) + \psi_t W^{\mathbb{R}^n}_{Z_+}(\varkappa_\tau(\theta))]\right], \end{aligned}$$

and

$$\begin{aligned} H^- \doteq a^p/2 \Big\{ & H_0(x, \phi) + H_0(x(\tau), \phi(\tau)) + \psi_t\big[\nu(\{\tau\}) - \theta \\ & + \; W^{\mathbb{R}}_{\{0\}}\left(\mu(\{\tau\}) - \int_0^\theta e_\tau(\epsilon)d\epsilon\right) + W^{\mathbb{R}^n}_{\{0\}}(x(\tau) - \varkappa_\tau(\theta))\big]\Big\} \\ & + \; 1/2 \sum_{q\in Q,\, q<p} a^{p-q} b^q \left[c_+^{q/p} H_q(x, \phi) + c_-^{q/p} H_q(x(\tau), \phi(\tau))\right] \\ & + b^p/2 \;\left[c_+ H_p(x, \phi) + c_- H_p(x(\tau), \phi(\tau))\right] \\ & + \; |b|^p \;\left[c_+[\psi_\nu^+(\tau) + \psi_t W^{\mathbb{R}^n}_{Z_-}(\varkappa_\tau(\theta))] + c_-\psi_\nu^-(\tau)\right]. \end{aligned}$$

– ***Optimality with respect to the support of the singular continuous part of the control measure.***
For ν_{sc}*-a.a.* $t \in \operatorname{supp}\nu_{sc}$*, it holds*

$$\begin{aligned} m_{sc} \in \operatorname{Sign} H_p(x, \psi_x), \\ |H_p(x, \psi_x)| - \varpi_\nu = \mathcal{H}(x, \psi_x, \psi_\nu), \end{aligned} \tag{4.6}$$

where m_{sc} *stands for the Radon-Nikodym derivative of the measure* μ_{sc} *wrt* ν*, and we stress that* $W^{\mathbb{R}^n}_{Z_-}(x) = W^{\mathbb{R}^n}_{Z_+}(x) = 0$ *over* $\operatorname{supp}\nu_{sc}$.

Remark 4.2. A rather general form of the right-hand side of system (3.1)–(3.8) does not let us go deeper in the representation of conditions (C_4). For the linear and square cases, the conditions can be further detalized similarly to [14] or [13].

The main technical difference of the presented necessary optimality conditions compared to the impulsive maximum principles for different classes of state and mixed constrained impulsive and hybrid control problems [10, 20, 21] consists in "doubling" the space of dual trajectories $\psi_\nu^\pm$. The reason is the way we handle constraint (1.5) in the process of the reduction, when applied the time reparameterization. In fact, the one-sided limits of solutions x to the measure differential equations are considered as different state trajectories (see Remark 3.1).

Theorem 4.1 is an interpretation of the Maximum Principle [17] for problem (RP) by virtue of the proposed direct time-spatial transform. To ease the exposition of the proof, let us preliminarily make the following modifications of (RP):

- Reduce the functional constraint to a terminal one by introducing an extra trajectory $z:\ [0,S]\to\mathbb{R}_+$ as a solution to

$$\frac{d}{ds}z=\Gamma,\quad z(0)=0. \tag{4.7}$$

  Then constraint (3.7), obviously, equals

$$z(S)=0. \tag{4.8}$$

- Rewrite the terminal block (3.5) of the problem together with (4.8) as follows:

$$\mathcal{T}((\xi,y,\eta,\zeta)(S))=0, \tag{4.9}$$

$$\zeta_+(S)+\zeta_-(S)-2M\le 0, \tag{4.10}$$

  with

$$\mathcal{T}(\xi,y,\eta,\zeta)\doteq\Delta\zeta+W^{\mathbb{R}^{n+1}}_{\{0\}}(\Delta(y,\eta))+W^{\mathbb{R}}_{\{0\}}(\xi-T)+z.$$

- In order to bring a symmetry to the problem we also replace the original cost with the functional

$$\widetilde{J}=\varphi\left(\frac{y_+(S)+y_-(S)}{2}\right),$$

  which equals J over $\Sigma(RP)$.

Proof. Introduce ψ_ξ, $\psi_y^\pm$, $\psi_\eta^\pm$, $\psi_\zeta^\pm$, and ψ_z as dual to the respective state variables ξ, $y_\pm$, $\eta_\pm$, $\zeta_\pm$, and z. Let h denote the standard Pontryagin function for (RP):

$$\begin{aligned} h\doteq{}&\alpha^p\big[H_0^++H_0^-+\psi_z[\Delta\zeta+W^{\mathbb{R}^{n+1}}_{\{0\}}(\Delta(y,\eta))]\big]\\ &+\sum_{q\in Q,\ q<p}\alpha^{p-q}\beta^q\left[\gamma_+^{q/p}H_q^++\gamma_-^{q/p}H_q^-\right]\\ &+\beta^p\ \left[\gamma_+[H_p^++\psi_\eta^+]+\gamma_-[H_p^-+\psi_\eta^-]\right]\\ &+|\beta|^p\ \left[\gamma_+[\psi_\zeta^++\psi_zW^{\mathbb{R}^n}_{Z_-}(y_-)]+\gamma_-[\psi_\zeta^-+\psi_zW^{\mathbb{R}^n}_{Z_+}(y_+)]\right]\end{aligned}$$

and $\widetilde{h}$ be the (maximized) Hamiltonian:

$$\widetilde{h}\doteq\max_{\widetilde{a}\doteq(a,b,c_+,c_-)\in\widetilde{A}}h$$

(we use the abbreviation $H_q^\pm\doteq H_q(y_\pm,\psi_y^\pm)$ for $q\in Q\cup\{0\}$). The terminal Lagrangian is of the form

$$\mathcal{L}(\xi,y,\eta,\zeta)=\varpi_J\,\varphi\left(\frac{y_++y_-}{2}\right)+\varpi_{\mathcal{T}}\mathcal{T}(\xi,y,\eta,\zeta)+\varpi_\zeta(\zeta_++\zeta_--2M).$$

Let $\varsigma=(\xi,y,\eta,\zeta,\alpha,\beta,\gamma;S)\in\Sigma(RP)$, $y=(y_+,y_-)$, $\eta=(\eta_+,\eta_-)$, $\zeta=(\zeta_+,\zeta_-)$, $\gamma=(\gamma_+,\gamma_-)$, be the process of the reduced system obtained by formulas (3.16)–(3.18) from an optimal for (P) process σ. By Proposition 3.2, γ is optimal for (RP) and satisfies (3.14). Then, γ satisfies the Maximum Principle [17] for control problems under pointwise state and terminal constraints with a collection of Lagrange multipliers ϖ_J, ϖ_ζ and b such that $\varpi_J, \varpi_\zeta \geq 0$, $\varpi_{\mathcal{T}} \in \mathbb{R}$, $\rho \in BV_+([0,T],\mathbb{R})$, $\rho(0-) = 0$, ρ is nondecreasing, and the respective differential measure $d\rho$ is concentrated on $\{s \in [0,S] |\ \Delta\zeta(s) = 0\}$.

1) Set

$$\varpi_I \doteq \varpi_J,\ \psi_t \doteq -\varpi_{\mathcal{T}},\ \varpi_\nu \doteq \varpi_\zeta,\ \text{and } \pi = \rho \circ \Xi \tag{4.11}$$

with $\Xi(= \Upsilon$ for $\varsigma)$ defined by (3.10).

The nontriviality of the collection $(\varpi_J, \varpi_\zeta, \varpi_{\mathcal{T}}, \rho)$ then gives (C_1).

2) The complementary slackness condition

$$0 = \varpi_\zeta\left(\zeta_+(S) - M\right) = \varpi_\zeta\left(\zeta_+(\Xi(T)) - M\right) = \varpi_\nu\left(\nu([0,T]) - M\right),$$

leads to (C_2). Observe that the measure da assumes no restriction on its support due to the definition.

3) Clearly, $\psi_z \equiv -\varpi_{\mathcal{T}}$, and therefore $\psi_t = \psi_z$.

The function ψ_ξ meets the relations:

$$-\frac{d}{ds}\psi_\xi = \nabla_\xi h = 0, \quad -\psi_\xi(S) = \varpi_{\mathcal{T}} \nabla_\xi \mathcal{T}((\xi, y, \eta, \zeta)(S)).$$

By the assumption $\nabla W^{\mathbb{R}}_{\{0\}}(0) = 0$, made in the beginning of the section, we obtain that $\psi_\xi \equiv 0$.

The functions $\psi^\pm_\eta$ solve the respective systems

$$-\frac{d}{ds}\psi^\pm_\eta = \alpha^p\, \nabla W^{\mathbb{R}}_{\{0\}}(\Delta\eta) = 0, \quad -\psi^\pm_\eta(S) = \nabla W^{\mathbb{R}}_{\{0\}}(\Delta\eta(S)).$$

Since $\Delta\eta = 0$ for s such that $\alpha(s) > 0$, and, again, by the assumption $\nabla W^{\mathbb{R}}_{\{0\}}(0) = 0$, we get $\psi^\pm_\eta \equiv 0$.

The functions $\psi^\pm_\zeta$ satisfy

$$-d\psi^\pm_\zeta = \nabla_{\zeta_\pm} h ds \pm d\rho = -\varpi_{\mathcal{T}} \nabla_{\zeta_\pm} \Gamma ds \pm d\rho = \mp(\varpi_{\mathcal{T}} \cdot \alpha^p ds) \pm d\rho,$$

and $-\psi^\pm_\zeta(S) = \nabla_{\zeta_\pm} \mathcal{L}((\xi, y, \eta, \zeta)(S)) = \pm\varpi_{\mathcal{T}} + \varpi_\zeta$. Thus,

$$\psi^\pm_\zeta(s) = \pm\left[\rho(S) - \rho(s) - \varpi_{\mathcal{T}}(\xi(S) - \xi(s))\right] \mp \varpi_{\mathcal{T}} - \varpi_\zeta,$$

and the substitution $\psi^\pm_\nu \doteq \psi^\pm_\zeta \circ \Xi$ combined with (4.11) brings us to (4.1).

Now consider the functions $\psi^\pm_y$. They solve the following Cauchy problems:

$$\begin{aligned} -\frac{d}{ds}\psi^\pm_y = \nabla_{y_\pm} h &= \alpha^p\left[\nabla_x H_0(y_\pm, \psi^\pm_y) \mp \varpi_{\mathcal{T}}\, \nabla W^{\mathbb{R}^n}_{\{0\}}(\Delta y)\right] \\ &+ \sum_{q \in Q,\ q<p} \gamma_\pm^{p/q} \alpha^{p-q} \beta^q\, \nabla_x H_q(y_\pm, \psi^\pm_y) \\ &+ \gamma_\pm \beta^p\, \nabla_x H_p(y_\pm, \psi^\pm_y) - \gamma_\mp |\beta|^p\, \varpi_{\mathcal{T}}\, \nabla W^{\mathbb{R}^n}_{Z_\pm}(y_\pm), \end{aligned}$$

with

$$-\psi^\pm_y(S) = 1/2\, \varpi_J\, \nabla\varphi\left(\frac{y_+ + y_-}{2}(S)\right) + \varpi_{\mathcal{T}} \nabla W^{\mathbb{R}^n}_{\{0\}}(\Delta y(S)).$$

Remind that $\Delta y(s) = 0$ over s such that $\alpha(s) > 0$, and $y_+(s) = y_-(s) \in Z_+ \cap Z_-$ along s such that $\beta(s) \in (0,1)$. On the other hand, for each interval $\Upsilon_\tau \doteq$

$[\Upsilon(\tau-), \Upsilon(\tau)]$, $\tau \in \Delta_\nu(T)$, we have $\alpha(s) = 0$ and $|\beta(s)| = 1$, while $\gamma_+(s) = 1$ on $\Upsilon^+_\tau \doteq \Upsilon(\tau-) + [0, \nu(\{\tau\}))$, and $\gamma_+(s) = 0$ on the remainder Υ^-_τ. Then, for any atom τ of ν, we have

$$-\frac{d}{ds}\psi^+_y = \left\{\begin{array}{cc} \beta^p \nabla_x H_p(y_+, \psi^+_y), & s \in \Upsilon^+_\tau, \\ -\varpi_{\mathcal{T}} \nabla W^{\mathbb{R}^n}_{Z_+}(y_+(\Upsilon(\tau))), & s \in \Upsilon^-_\tau, \end{array}\right\}$$
$$= \left\{\begin{array}{cc} \beta^p \nabla_x H_p(y_+, \psi^+_y), & s \in \Upsilon^+_\tau, \\ 0, & s \in \Upsilon^-_\tau, \end{array}\right.$$

since $y_+(\Upsilon(\tau)) \doteq x(\tau) \in Z_+$. The "negative" branch ψ^-_y enjoys the symmetric property:

$$-\frac{d}{ds}\psi^-_y = \left\{\begin{array}{cc} 0, & s \in \Upsilon^+_\tau, \\ \beta^p \nabla_x H_p(y_-, \psi^-_y), & s \in \Upsilon^-_\tau. \end{array}\right.$$

Thus, ψ^+_y and ψ^-_y coincide with each other beyond the intervals Υ_τ, and $\psi^+_y(S) = \psi^-_y(S)$. Then, $\psi^+_y \circ \Xi = \psi^-_y \circ \Xi$. Setting $\psi \doteq (\psi^+_y + \psi^-_y) \circ \Xi$ and $\phi_\tau \doteq \psi^+_y \circ s^+_\tau + \psi^-_y \circ s^-_\tau$, $\tau \in \Delta_\nu(T)$, the integration now leads to (4.2), (4.3) thanks to (3.13). Here, $s^\pm_\tau(\theta) \doteq \inf\{s \in [\Xi(\tau-), \Xi(\tau)] : \theta^\pm_\tau(s) > \theta\}$ for $\theta \in [0, \nu(\{\tau\}))$, $\theta^\pm_\tau(s) \doteq \zeta_\pm(s) - \nu([0, \tau))$, and $s^\pm_\tau(\nu(\{\tau\})) \doteq \Xi(\tau)$. Note that $\psi = (\psi^+_y + \psi^-_y) \circ \Xi = 2\psi^+_y \circ \Xi = 2\psi^-_y \circ \Xi$, and $\phi_\tau = 2\psi^\pm_y \circ s^\pm_\tau$.

4) The optimality conditions (C_4) are an interpretation of the standard maximum condition for (RP): $h = \widetilde{h}$.

– For $s \in [0, S]$ such that $\upsilon(s) \in [0, T] \setminus \operatorname{supp} \mu_c$, we have $\alpha(s) = 1$, $\beta(s) = 0$, and $\Delta y(s) = 0$. Then, the maximum condition gives

$$\begin{aligned} \widetilde{h} = h &\doteq H_0(y_+, \psi^+_y) + H_0(y_-, \psi^-_y) + \psi_z[\Delta\zeta + W^{\mathbb{R}^{n+1}}_{\{0\}}(\Delta(y, \eta))] \\ &= H_0(y_+, \psi^+_y) + H_0(y_-, \psi^-_y) \\ &= H_0(y_+, \psi^+_y + \psi^-_y) = H_0(y_-, \psi^+_y + \psi^-_y), \end{aligned}$$

which is (4.4) up to the time reparameterization $s = \Xi(t)$.

– Consider an interval $\Upsilon_\tau \doteq \Upsilon^+_\tau \cup \Upsilon^-_\tau$, $\tau \in \Delta_\nu(T)$. By (3.16), (3.17), $\alpha(s) = 0$ and $|\beta(s)| = 1$ λ-a.e. on Υ_τ. Hence, the maximum condition yields

$$\begin{aligned} \widetilde{h} = h &\doteq \big[[H_p(y_\pm, \psi^\pm_y) + \psi^\pm_\eta]\beta + \psi^\pm_\zeta + \psi_z W^{\mathbb{R}^n}_{Z_\mp}(y_\mp)\big] \\ &= H_p(y_\pm, \psi^\pm_y)\beta + \psi^\pm_\zeta \end{aligned}$$

on $\Upsilon^\pm_\tau$. Then, the expression

$$h \circ s^+_\tau + h \circ s^-_\tau = \widetilde{h} \circ s^+_\tau + \widetilde{h} \circ s^-_\tau$$

coincides with (4.6) on $[0, T_\tau]$.

– Finally, for $\{s |\ \upsilon(s) \in \operatorname{supp} \nu_c\}$, it holds $\beta(s) > 0$ and $\Delta(y, \eta, \zeta)(s) = 0$. In particular, $\alpha(s) = 0$ over $\{s |\ \upsilon(s) \in \operatorname{supp} \nu_{sc}\}$. Then, the time change $\circ\Xi$ in the maximum condition achieves (4.6), and the optimality of control (α, β, γ) leads to (4.5) due to the definition $l \doteq (\gamma_+^{1/p} \alpha^\oplus \beta) \circ \Xi$. □

5. Conclusion

The paper exhibits a, to some extent, straightforward generalization of results [14] for the case of mixed constrained polynomial impulses. A practically more interesting situation appears, when a system dynamics involves impulsive signals of various powers, and some of them are subject to conditions like (1.5), say, there are mixed constrained affine impulses and state-free square ones. Models of this sort can be met in control of mechanical systems driven by both active state constraints and blocking/releasing certain degrees of freedom. Models of this kind will be the subject of our next study.

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