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INVERSE PROBLEMS OF THE CALCULUS OF VARIATIONS FOR DISCRETE-TIME SYSTEMS

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Abstract. We obtain the solvability conditions for two inverse problems of the discrete calculus of variations dealing with systems of equations of second and fourth orders. Explicit expressions for the functional, for which the given system of discrete nonlinear equations coincides with the Euler-Lagrange equation for this functional, are also presented.

Illustrative examples are given.

1. Introduction

Most of the publications devoted to the calculus of variations deals with continuous-time systems. However, discrete-time systems very often appear in practical problems, for instance, in economics, sociology, and biology. Furthermore, for using numerical methods, continuous-time systems must be reduced to discrete-time ones. The study of sampled-data control systems and computer-based adaptive control systems leads in a natural way to the third source of discrete-time models.

The problem of the calculus of variations for discrete-time systems, discussed in [6], consists in the minimization of functionals of the form

$$J = \sum_{k=0}^{N-1} V_k(x_k, x_{k+1}),\ N > 1, \tag{1.1}$$

where $x_k \in I\!R$, a natural number N is fixed, boundary values x_0, x_N are given, and functions V_k are differentiable. In this case, the discrete-time version of the Euler-Lagrange equation is

$$\frac{\partial}{\partial x_k}(V_k(x_k, x_{k+1}) + V_{k-1}(x_{k-1}, x_k)) = 0,\ k = 1, ..., N-1 \tag{1.2}$$

([6], p. 195).

Equation (1.2) was obtained in [6] as the result of finding the first variation of functional (1.1). But it is not difficult to see that the discrete-time version of Euler-Lagrange equation (1.2) immediately follows from the well-known necessary condition of an extremum of a function of several variables (1.1), namely, partial derivatives of this function must be equal to zero. It is equivalent to the equality $grad\ J = 0$.

We will further assume that functions V_k are twice continuously differentiable.

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The problem of finding a functional for which a given differential equation is the Euler-Lagrange equation is called as the inverse problem of the calculus of variations. Note here paper [7], where the author consider several aspects of the inverse problem of the calculus of variations as they have developed since 1979, giving some of the principal results, listing significant primary sources and mentioning review articles for further references. The inverse problem of the discrete calculus of variations consists in the following. For a given system of discrete equations we want to know whether there is a functional for which the given system coincides with the Euler-Lagrange equation for this functional. If such a functional exists, then we need to find it. To our knowledge, inverse problems of the calculus of variations were studied only for continuous-time systems (see, for instance, [1]-[4], [7]-[9]).

The inverse problem of the calculus of variations arises when variational methods for equations solving are used. If for a given system, continuous or discrete, we found a functional such that the considered system is the Euler-Lagrange equation for a problem of minimization of this functional, then we can use packages of programs for solving optimization problems in order to find a solution of the considered system.

The present paper deals with two inverse problems of the calculus of variations for discrete-time systems. Namely, we consider systems of discrete equations of the second and fourth orders, for which solvability conditions and implicit expressions for a functional are obtained. We use here the idea of the method for solving classical inverse problems of the calculus of variations for continuous-time systems from [8].

2. Inverse problems

2.1. **Systems of discrete equations of the second order.**

Theorem 2.1. *If a system of the form*

$$\varphi_k(x_{k-1}, x_k, x_{k+1}) = 0,\ k = 1, ..., N-1, \tag{2.1}$$

where x_0, x_N are specified and φ_k are everywhere continuously differentiable functions, is the Euler-Lagrange equation for a functional of type (1.1) *with twice continuously differentiable functions V_k, then*

$$\frac{\partial}{\partial x_{k+1}}\varphi_k(x_{k-1}, x_k, x_{k+1}) = \frac{\partial}{\partial x_k}\varphi_{k+1}(x_k, x_{k+1}, x_{k+2}),\ k = 1, ..., N-2. \tag{2.2}$$

Moreover, if functions φ_k in (2.1) *satisfy conditions* (2.2)*, then a solution of the inverse problem of the calculus of variations for system* (2.1) *can be found as follows*

$$J = \int_0^{x_1} \varphi_1(x_0, x_1, x_2)dx_1 + \sum_{k=2}^{N-1} \int_0^{x_k} \varphi_k(0, x_k, x_{k+1})dx_k. \tag{2.3}$$

Proof. Let system (2.1) be an Euler-Lagrange equation for a functional of type (1.1). Hence, there exist functions $V_j,\ j = 0, 1, \ldots, N-1$, such that the following equalities (see (1.2))

$$\varphi_k(x_{k-1}, x_k, x_{k+1}) = \frac{\partial}{\partial x_k}(V_k(x_k, x_{k+1}) + V_{k-1}(x_{k-1}, x_k)),\quad k = 1, ..., N-1, \tag{2.4}$$

are valid.

Differentiating the last equality with respect to x_{k+1}, we have

$$\frac{\partial}{\partial x_{k+1}}\varphi_k(x_{k-1},x_k,x_{k+1}) = \frac{\partial^2}{\partial x_{k+1}\partial x_k}V_k(x_k,x_{k+1}),\ k=1,...,N-1. \tag{2.5}$$

From (2.4), it follows

$$\varphi_{k+1}(x_k,x_{k+1},x_{k+2}) = \frac{\partial}{\partial x_{k+1}}(V_{k+1}(x_{k+1},x_{k+2}) + V_k(x_k,x_{k+1})).$$

Differentiating this equality with respect to x_k, we obtain

$$\frac{\partial}{\partial x_k}\varphi_{k+1}(x_k,x_{k+1},x_{k+2}) = \frac{\partial^2}{\partial x_k\partial x_{k+1}}V_k(x_k,x_{k+1}).$$

Comparing the last equality with (2.5) we get (2.2).

Further, suppose that functions from system (2.1) satisfy condition (2.2) and define the functional J by (2.3). Then we get

$$\frac{\partial J}{\partial x_1} = \varphi_1(x_0,x_1,x_2).$$

Taking into account (2.2), we have

$$\begin{aligned}\frac{\partial J}{\partial x_2} &= \int_0^{x_1}\frac{\partial}{\partial x_2}\varphi_1(x_0,x_1,x_2)dx_1 + \varphi_2(0,x_2,x_3)\\ &= \int_0^{x_1}\frac{\partial}{\partial x_1}\varphi_2(x_1,x_2,x_3)dx_1 + \varphi_2(0,x_2,x_3)\\ &=\varphi_2(x_1,x_2,x_3).\end{aligned}$$

Similarly, we obtain for $k>2$ in view of (2.2) the following

$$\begin{aligned}\frac{\partial J}{\partial x_k} =&\varphi_k(0,x_k,x_{k+1}) + \int_0^{x_{k-1}}\frac{\partial}{\partial x_k}\varphi_{k-1}(0,x_{k-1},x_k)dx_{k-1}\\ =&\varphi_k(0,x_k,x_{k+1}) + \int_0^{x_{k-1}}\frac{\partial}{\partial x_{k-1}}\varphi_k(x_{k-1},x_k,x_{k+1})dx_{k-1}\\ =&\varphi_k(x_{k-1},x_k,x_{k+1}).\end{aligned}$$

This completes the proof. □

Remark 2.2. Different methods are used for discretization of differential equations. Usually, for differential equations of the second order the first derivative $x'(kh)$ is approximated by the difference $(x_{k+1}-x_{k-1})/2h$ and the second derivative $x''(kh)$ is changed by $(x_{k+1}-2x_k+x_{k-1})/h^2$, where $x_k = x(kh)$ and h is a step of partition of the argument interval into equal parts. As a result, a system of form (2.1) is obtained.

Remark 2.3. The statement of Theorem 2.1 in this paper contains the revised last formula from the abstract in ([5], p. 10).

2.2. **Systems of discrete equations of the fourth order.** Consider the problem of minimizing the functional

$$J = \sum_{k=0}^{N-2} V_k(x_k, x_{k+1}, x_{k+2}),\ N > 2, \tag{2.6}$$

where boundary conditions $x_0,\ x_1,\ x_{N-1},\ x_N$ are specified and functions V_k are differentiable.

Using the necessary condition of an extremum of a function of several variables we obtain the Euler-Lagrange equation for functional (2.6) in the form

$$\frac{\partial}{\partial x_k}(V_k(x_k, x_{k+1}, x_{k+2}) + V_{k-1}(x_{k-1}, x_k, x_{k+1}) + V_{k-2}(x_{k-2}, x_{k-1}, x_k)) = 0, \quad k = 2, ..., N-2. \tag{2.7}$$

Theorem 2.4. *If a system of the form*

$$\varphi_k(x_{k-2}, x_{k-1}, x_k, x_{k+1}, x_{k+2}) = 0,\ k = 2, ..., N-2, \tag{2.8}$$

where $x_0,\ x_1,\ x_{N-1},\ x_N$ are specified and φ_k are continuously differentiable functions, is the Euler-Lagrange equation for a functional of type (2.6) *with twice continuously differentiable functions V_k, then*

$$\frac{\partial}{\partial x_{k+1}}\varphi_k(x_{k-2}, x_{k-1}, x_k, x_{k+1}, x_{k+2}) = \frac{\partial}{\partial x_k}\varphi_{k+1}(x_{k-1}, x_k, x_{k+1}, x_{k+2}, x_{k+3}), \quad k = 2, ..., N-3, \tag{2.9}$$

and

$$\frac{\partial}{\partial x_{k+2}}\varphi_k(x_{k-2}, x_{k-1}, x_k, x_{k+1}, x_{k+2}) = \frac{\partial}{\partial x_k}\varphi_{k+2}(x_k, x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}), \quad k = 2, ..., N-4. \tag{2.10}$$

Moreover, if functions φ_k in (2.8) *satisfy conditions* (2.9) *and* (2.10)*, then a solution of the inverse problem of the calculus of variations for system* (2.8) *can be found as follows*

$$\begin{aligned} J = &\int_0^{x_2} \varphi_2(x_0, x_1, x_2, x_3, x_4)dx_2 + \int_0^{x_3} \varphi_3(x_1, 0, x_3, x_4, x_5)dx_3 \\ &+ \sum_{k=4}^{N-2}\int_0^{x_k} \varphi_k(0, 0, x_k, x_{k+1}, x_{k+2})dx_k. \end{aligned} \tag{2.11}$$

Proof. Let system (2.8) be an Euler-Lagrange equation for a functional of type (2.6). Hence, there exist functions $V_j,\ j = 0, 1, ..., N-2$, such that the following equalities (see (2.7))

$$\begin{aligned} \varphi_k(x_{k-2}, x_{k-1}, x_k, x_{k+1}, x_{k+2}) = &\frac{\partial}{\partial x_k}(V_k(x_k, x_{k+1}, x_{k+2}) \\ &+ V_{k-1}(x_{k-1}, x_k, x_{k+1}) \\ &+ V_{k-2}(x_{k-2}, x_{k-1}, x_k)),\ k = 2, ..., N-2, \end{aligned} \tag{2.12}$$

are valid.

For $k = 2, \ldots, N-3$, differentiating the last equality with respect to x_{k+1}, we have

$$\frac{\partial}{\partial x_{k+1}}\varphi_k(x_{k-2}, x_{k-1}, x_k, x_{k+1}, x_{k+2}) = \frac{\partial^2}{\partial x_{k+1}\partial x_k}(V_k(x_k, x_{k+1}, x_{k+2}) + V_{k-1}(x_{k-1}, x_k, x_{k+1})). \tag{2.13}$$

From (2.12), it follows

$$\varphi_{k+1}(x_{k-1}, x_k, x_{k+1}, x_{k+2}, x_{k+3}) = \frac{\partial}{\partial x_{k+1}}(V_{k+1}(x_{k+1}, x_{k+2}, x_{k+3}) + V_k(x_k, x_{k+1}, x_{k+2}) + V_{k-1}(x_{k-1}, x_k, x_{k+1})).$$

Differentiating this equality with respect to x_k, we obtain

$$\frac{\partial}{\partial x_k}\varphi_{k+1}(x_{k-1}, x_k, x_{k+1}, x_{k+2}, x_{k+3}) = \frac{\partial^2}{\partial x_k \partial x_{k+1}}(V_k(x_k, x_{k+1}, x_{k+2}) + V_{k-1}(x_{k-1}, x_k, x_{k+1})).$$

Comparing the last equality with (2.13), we get (2.9).

For $k = 2, \ldots, N-4$, differentiating (2.12) with respect to x_{k+2}, we have

$$\frac{\partial}{\partial x_{k+2}}\varphi_k(x_{k-2}, x_{k-1}, x_k, x_{k+1}, x_{k+2}) = \frac{\partial^2}{\partial x_{k+2}\partial x_k}V_k(x_k, x_{k+1}, x_{k+2}). \tag{2.14}$$

From (2.12), it follows

$$\varphi_{k+2}(x_k, x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}) = \frac{\partial}{\partial x_{k+2}}(V_{k+2}(x_{k+2}, x_{k+3}, x_{k+4}) + V_{k+1}(x_{k+1}, x_{k+2}, x_{k+3}) + V_k(x_k, x_{k+1}, x_{k+2})).$$

Differentiating this equality with respect to x_k, we obtain

$$\frac{\partial}{\partial x_k}\varphi_{k+2}(x_k, x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}) = \frac{\partial^2}{\partial x_k \partial x_{k+2}}V_k(x_k, x_{k+1}, x_{k+2}).$$

Comparing the last equality with (2.14), we get (2.10).

Further, suppose that functions from system (2.8) satisfy conditions (2.9) and (2.10) and define the functional by equality (2.11). Then we get

$$\frac{\partial J}{\partial x_2} = \varphi_2(x_0, x_1, x_2, x_3, x_4).$$

Taking into account (2.9), we have

$$\begin{aligned}\frac{\partial J}{\partial x_3} &= \int_0^{x_2} \frac{\partial}{\partial x_3}\varphi_2(x_0, x_1, x_2, x_3, x_4)dx_2 + \varphi_3(x_1, 0, x_3, x_4, x_5) \\ &= \int_0^{x_2} \frac{\partial}{\partial x_2}\varphi_3(x_1, x_2, x_3, x_4, x_5)dx_2 + \varphi_3(x_1, 0, x_3, x_4, x_5) \\ &= \varphi_3(x_1, x_2, x_3, x_4, x_5).\end{aligned}$$

Using immediate differentiation, in view of (2.9) and (2.10) we get

$$\frac{\partial J}{\partial x_k} = \varphi_k(x_{k-2}, x_{k-1}, x_k, x_{k+1}, x_{k+2}),\ k = 4, 5.$$

Similarly, we obtain for $k > 5$ the following

$$\begin{aligned}\frac{\partial J}{\partial x_k} =&\varphi_k(0,0,x_k,x_{k+1},x_{k+2}) + \int_0^{x_{k-1}} \frac{\partial}{\partial x_k}\varphi_{k-1}(0,0,x_{k-1},x_k,x_{k+1})dx_{k-1}\\ &+ \int_0^{x_{k-2}} \frac{\partial}{\partial x_k}\varphi_{k-2}(0,0,x_{k-2},x_{k-1},x_k)dx_{k-2}\\ =&\varphi_k(0,0,x_k,x_{k+1},x_{k+2}) + \int_0^{x_{k-1}} \frac{\partial}{\partial x_{k-1}}\varphi_k(0,x_{k-1},x_k,x_{k+1},x_{k+2})dx_{k-1}\\ &+ \int_0^{x_{k-2}} \frac{\partial}{\partial x_{k-2}}\varphi_k(x_{k-2},x_{k-1},x_k,x_{k+1},x_{k+2})dx_{k-2}\\ =&\varphi_k(x_{k-2},x_{k-1},x_k,x_{k+1},x_{k+2}).\end{aligned}$$

This completes the proof. □

Remark 2.5. The form of system (2.8) follows, for instance, from a differential equation of the second order if the first derivative $x'(kh)$ is approximated by the difference $(x_{k+1} - x_{k-1})/2h$ and the second derivative $x''(kh)$ is changed by $(x_{k+2} - 2x_k + x_{k-2})/4h^2$.

Remark 2.6. If it turns out that one of the integrals in the statement of Theorems 2.1, 2.4 does not exist, then we can replace the lower limit of integration by any number such that this integral does exist.

3. Examples

3.1. Consider the system

$$\begin{aligned}&k^2(x_{k-1}+1)^2 exp(k^2(x_{k-1}+1)^2 x_k)\\ &+2(k+1)^2(x_k+1)x_{k+1} exp((k+1)^2(x_k+1)^2 x_{k+1}) = 0,\\ &k = 1, ..., N-1,\ N > 1,\end{aligned} \tag{3.1}$$

with fixed end points, i.e. x_0 and x_N are specified.

We want to know whether there is a discrete functional for which this system is a discrete Euler-Lagrange equation.

The considered system (3.1) is a system of form (2.1). It is easy to see that condition (2.2) is satisfied. Hence, in view of Theorem 2.1 system (3.1) is the Euler-Lagrange equation for a functional. Using formula (2.3) we find one of such functionals

$$J = \sum_{k=0}^{N-1} exp((k+1)^2(x_k+1)^2 x_{k+1}). \tag{3.2}$$

We can immediately verify that system (3.1) is really the Euler-Lagrange equation for the found functional (3.2).

3.2. Consider the system

$$x_0 + 2x_1 + 2x_2 = 0,\quad x_1 + 2x_2 + x_3 = 0 \tag{3.3}$$

with a fixed-end points, i.e. x_0 and x_3 are specified.

The considered system is a system of form (2.1), where

$$\varphi_1(x_0, x_1, x_2) = x_0 + 2x_1 + 2x_2,\ \varphi_2(x_1, x_2, x_3) = x_1 + 2x_2 + x_3.$$

Condition (2.2) is not satisfied for these functions. Hence, in view of Theorem 2.1 system (3.3) is not the Euler-Lagrange equation for a functional of type (1.1).

If we multiply the first equation of system (3.3) by $\frac{1}{2}$, namely, we will consider the system

$$\frac{1}{2}x_0 + x_1 + x_2 = 0,\quad x_1 + 2x_2 + x_3 = 0,$$

then condition (2.2) is satisfied. In view of Theorem 2.1, the obtained system is the Euler-Lagrange equation for a functional. One of such functionals can be found by formula (2.3)

$$J = \frac{1}{2}x_0x_1 + \frac{1}{2}x_1^2 + x_1x_2 + x_2^2 + x_2x_3.$$

The method of multiplying the equation by the factor for obtaining the Euler-Lagrange equation for a functional was used earlier for continuous systems (see, for instance, [8]). It is called in this book as the method of the integrating multiplier. See about the multiplier problem, for instance, in [7].

4. Relation to continuous-time case

Firstly we present a result from ([8], p. 53).

An equation is the Euler-Lagrange equation for a some integral functional if and only if it has the form

$$A(t, x, x')x'' + B(t, x, x') = 0,$$

where the functions A and B satisfy the following equality

$$B_{x'} - A_t - x'A_x = 0. \tag{4.1}$$

4.1. Consider an equation from ([8], p. 58) of the form

$$-2t^2x'' - 4tx' + 24x = 0. \tag{4.2}$$

It is evident that this equation satisfy equality (4.1). Hence (4.2) is the Euler-Lagrange equation for a some integral functional.

If we use the discretization scheme from Remark 2.2 with $h = 1$ for second order differential equation (4.2) we obtain discrete system of form (2.1)

$$-2k(k+1)x_{k+1} + 4(k^2 + 6)x_k + 2k(1-k)x_{k-1} = 0. \tag{4.3}$$

It is not difficult to verify that condition (2.2) is satisfied for the last system. Therefore, in view of Theorem 2.1 system (4.3) is the Euler-Lagrange equation for a functional of type (1.1).

If we use another discretization scheme for equation (4.2), replacing $x'(k)$ by the difference $x_{k+1} - x_k$ and $x''(k)$ by the same expression as before, we obtain the discrete system

$$-2k(k+2)x_{k+1} + 4(k^2+k+6)x_k - 2k^2x_{k-1} = 0. \tag{4.4}$$

Condition (2.2) is not satisfied for this system. Taking into account Theorem 2.1 we conclude that system (4.4) can not be the Euler-Lagrange equation for a functional of type (1.1).

Hence, solvability of the inverse problem of the calculus of variations for discretized equation depends on a discretization method.

4.2. Condition (4.1) for the Bessel equation

$$t^2x'' + tx' + (t^2 - \nu^2)x = 0 \tag{4.5}$$

is not satisfied. Therefore, the inverse problem of the calculus of variations for this equation is not solvable.

Using the discretization method from Remark 2.2 with $h = 1$ we obtain from (4.5) the discrete system

$$k(k+\frac{1}{2})x_{k+1} - (k^2+\nu^2)x_k + k(k-\frac{1}{2})x_{k-1} = 0,$$

which is not the Euler-Lagrange equation for a functional of type (1.1) in view of Theorem 2.1.

4.3. Consider the equation of the form

$$t^3x'' + (3t^2 + a)x' + bx = 0, \tag{4.6}$$

where a and b are constants.

If we use the discretization method from Remark 2.2 with $h = 1$ for equation (4.6) we obtain the discrete system

$$(k^3 + \frac{3k^2+a}{2})x_{k+1} + (-2k^3+b)x_k + (k^3 - \frac{3k^2+a}{2})x_{k-1} = 0. \tag{4.7}$$

Let us assume that $a = 0$. Then condition (4.1) is satisfied and (4.6) is the Euler-Lagrange equation for an integral functional. In this case, condition (2.2) is not satisfied for corresponding discretized equation (4.7). Hence, this equation is not the Euler-Lagrange equation for a functional of type (1.1).

4.4. If $a = -\frac{1}{2}$ then condition (4.1) for equation (4.6) is not satisfied and (4.6) is not the Euler-Lagrange equation for a some integral functional. Condition (2.2) for such a is satisfied for discretized equation (4.7). Hence, this equation is the Euler-Lagrange equation for a functional of type (1.1) in view of Theorem 2.1.

Thus all four situations concerning solvability of the inverse problem of the calculus of variations for a differential equation of the second order and a corresponding discretized system are possible.

5. Conclusion

In this paper, the inverse problem of the calculus of variations for two discrete systems, where the first of them consists of second order discrete equations and the second system consists of fourth order equations, has been solved.

It is planed to obtain analogous results in future for systems of discrete equations of the 2r-th order of the form

$$\varphi_k(x_{k-r}, x_{k-r+1}, \ldots, x_{k+r}) = 0,\ k = r, ..., N - r,$$

where $x_0, \ldots, x_{r-1}$ and $x_{N-r+1}, \ldots, x_N$ are specified.

The problem with some free variables from the last two lists can be also considered. Such situation in the discrete calculus of variations, where the Euler-Lagrange equation is a system of second order equations, has been researched, for instance, in [6].

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