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# OPTIMAL STRUCTURAL CONTROL OF SYSTEMS DETERMINED BY STRONGLY NONLINEAR OPERATOR VALUED MEASURES

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Abstract. In this paper we consider a class of infinite dimensional systems on Banach spaces (distributed parameter systems) determined by strongly nonlinear operator valued measures. We consider optimal structural control problems where the controls are operator valued measures perturbing the principal operator. We present existence, uniqueness and regularity properties of weak solutions and then consider optimal control problems. We introduce a class of operator valued measures representing structural controls. Characterization of weakly compact sets in the space of structural controls is presented. Using these results we prove existence of optimal structural controls. For illustration two examples are presented.

## 1. Introduction

In a series of papers [1–6, 8–12] we considered general evolution equations on Banach spaces determined by operator valued measures and controlled by vector measures. We studied the questions of existence and regularity properties of mild solutions for semilunar problems [2–4] and weak solutions for strongly nonlinear parabolic and hyperbolic problems [5, 6]. For semilunar and strongly nonlinear problems, determined by nonlinear operator valued measures substantial theory of optimal controls has been developed [1–6]. There are many physical systems which are amenable to structural control resulting in changes in the dynamic behavior of the system. There are both physical and theoretical motivations for study of this class of problems. For details see [12]. We mention here some physical motivations leading to such models. It is well known in aerospace engineering, that a change of physical configuration (or structure) of an aircraft can significantly alter the flight dynamics. This is done by appropriate maneuver of ailerons, rudders, elevators, wing flaps etc. These are structural controls. In material sciences, structural changes of molecules, for example polymerization, can produce new materials with desired properties. In the study of hemodynamics of artificial heart, the boundary

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of the heart chamber is subjected to periodic forces leading to periodic contraction and expansion of the chamber and thereby controlling blood flow. These are examples of structural control. Many more examples are given in [12]. However not much systematic control theory on structural control exists in the literature except those mentioned above. Here in this paper we consider structural control problem for a large class of systems where the principal operator is a strongly nonlinear operator valued measure. The structural control consists of linear operator valued measures. The major goal of this paper is to prove existence of optimal structural controls.

A closely related topic, yet very distinct, is the subject of relaxed controls. This has been studied extensively in the literature and well documented in the recent book of Santorini [19] where the reader will find extensive references. These systems are governed by differential equations on Banach spaces with controls which are probability measure valued functions, while the systems considered in this paper are determined by operator valued measures and controlled by vector or operator valued measures. The first distinction is in the structure of the system dynamics and the second is in the space of controls used.

The rest of the paper is organized as follows: We present relevant notations and terminologies in section 2. In section 3, a brief review of recent results on existence and regularity properties of weak solutions for strongly nonlinear parabolic systems determined by nonlinear operator valued functions and measures is presented. In section 4, we consider control systems determined by strongly nonlinear operator valued measures subject to structural control and study the questions of existence of solutions and their continuous dependence on controls. In section 5 we study structural control problems proving existence of optimal structural control. For illustration, in section 6, we present two examples one dealing with a parabolic problem and another dealing with a hyperbolic problem. The paper is concluded with some comments on open problems.

## 2. Notations and terminologies

**Some Function Spaces:** Let $H$ be a real separable Hilbert space with scalar product and norms denoted by $(v, w)$ and $|v| \equiv \sqrt{(v, v)}$ respectively for $v, w \in H$. Let $V$ be a linear subspace of the Hilbert space $H$ carrying the structure of a reflexive Banach space with $V^*$ denoting its topological dual. Identifying $H$ with its own dual and assuming that $V$ is dense in $H$, we have the inclusion

$$V \hookrightarrow H \hookrightarrow V^*$$

where the injections are continuous and dense. Collectively these Hilbert spaces $\{V, H, V^*\}$ are known as the Gelfand triple. The duality pairing between $v \in V$ and $w \in V^*$ is denoted by

$$\langle v, w\rangle \equiv \langle v, w\rangle_{V,V^*} \equiv \langle w, v\rangle_{V^*,V}.$$

In case $w \in H$, this reduces to the scalar product in $H$. We assume that there exists a complete system of basis vectors $\{v_i\} \subset V$ which is orthogonal in $V$ and $V^*$ and orth-normal in $H$ and that they span all the three spaces $\{V, H, V^*\}$ known as the Gelfand triple. For more details on these spaces see [6, 7] and the references therein.

Let $I \equiv [0,T]$ be an interval with $T < \infty$ and let $\Sigma \equiv \sigma(I)$ denote the sigma algebra of subsets of the set $I$. Let $B_\infty(I,H) \subset L_\infty(I,H)$ denote the vector space of bounded $\Sigma$ measurable functions on $I$ with values in $H$. Furnished with the sup norm topology, this is a Banach space. Let $\mu$ be any countably additive positive measure on $\Sigma$ having bounded total variation on $I$. For any of the spaces $X \equiv \{V,H,V^*\}$ and $1 \leq p < \infty$, we let $L_p(\mu,X)$ denote the Lebesgue-Bochner space of measurable functions on $I$ with values in $X$ satisfying

$$\int_I \parallel f(s) \parallel_X^p \mu(ds) < \infty.$$

Strictly speaking this is the equivalence class of $\mu$ measurable $X$ valued functions whose $X$-norms are $p-th$ power integrable.

By $BV(I,X)$ we denote the vector space of functions, defined on $I$ and taking values from the Banach space $X$, having bounded total variation. Furnished with total variation norm this is a Banach space. Similarly, for any Banach space $X$, if $\gamma$ is any finite positive measure on $I$, we let $L_\infty(\gamma,X)$ denote the space of $\Sigma$ measurable functions with values in $X$ having $\gamma$-essentially bounded norms.

**Some Vector Measures:** Let $F$ be a Banach space and $\mathcal{M}_c(\Sigma,F)$ the space of countably additive bounded vector measures defined on the sigma algebra $\Sigma$ with values in the Banach space $F$. Let $\mathcal{M}_{cabv}(\Sigma,F)$ be a proper subspace of the space $\mathcal{M}_c(\Sigma,F)$ consisting of countably additive $F$-valued vector measures having bounded total variation. Furnished with the topology induced by the total variation norm as defined below,

$$|\nu| \equiv |\nu|(I) \equiv \sup_\pi \left( \sum_{\sigma\in\pi} \parallel \nu(\sigma) \parallel_F \right), \tag{2.1}$$

it is a Banach space. Here the supremum is taken over all partitions $\pi$ of the interval $I$ into a finite number of disjoint members of $\Sigma$. For any $\sigma \in \Sigma$, denote the variation of $\nu$ on $\sigma$ by $|\nu|(\sigma)$. Since $\nu$ is countably additive and bounded, this defines a countably additive bounded positive measure on $\Sigma$ Diestel and Uhl [16, Proposition 9, p.3]. In case $F = R$, the real line, we have the space of real valued signed measures which we denote by $\mathcal{M}_c(\Sigma)$ and if they are nonnegative we use $\mathcal{M}_c^+(\Sigma)$.

We introduce two other topologies which are used later. Let $1 \leq q < \infty$, $\pi$ any finite partition of the interval $I$ by disjoint members of $\Sigma$ and $\nu \in \mathcal{M}_c(\Sigma,F)$. The vector measure $\nu$ is said to have $q$-variation if

$$\sup_\pi \left( \sum_{\sigma\in\pi} \parallel \nu(\sigma) \parallel_F^q \right)^{1/q} < \infty$$

where the supremum is taken over all such partitions $\pi$. We denote this vector space by $BV_q(\Sigma,F)$. It is easy to verify that this is a Banach space with respect to the norm topology

$$\parallel \nu \parallel_{BV_q(\Sigma,F)} \equiv \sup_\pi \left( \sum_{\sigma\in\pi} \parallel \nu(\sigma) \parallel_F^q \right)^{1/q}. \tag{2.2}$$

Clearly $BV_1(\Sigma,F) \equiv \mathcal{M}_{cabv}(\Sigma,F)$.

The second topology is dependent on a given countably additive bounded non-negative measure, say $\gamma$. Let $BV_q(\gamma, F)$ denote the class of vector measures $\mu \in \mathcal{M}_c(\Sigma, F)$ for which

$$\| \mu \|_{BV_q(\gamma,F)} \equiv \sup_{\pi} \left\{ \sum_{\sigma\in\pi} \left( \frac{\| \mu(\sigma) \|_F}{\gamma(\sigma)} \right)^q \gamma(\sigma) \right\}^{1/q} < \infty, \tag{2.3}$$

where we use the convention $0/0 = 0$. With respect to this norm topology, $BV_q(\gamma, F)$ is a Banach space. Since $\gamma$ is a countably additive bounded positive measure and $q \geq 1$, it is easy to verify that the embeddings

$$BV_q(\gamma, F) \hookrightarrow BV_q(\Sigma, F) \hookrightarrow \mathcal{M}_{cabv}(\Sigma, F),$$

are continuous. In case $F = E^*$, the dual of a Banach space $E$, the space of vector measures $BV_q(\gamma, E^*)$ is the topological dual of the Banach space $L_p(\gamma, E)$ for $1 \leq p < \infty$, $1/p + 1/q = 1$, see Diestel & Uhl [16, Notes and Remarks, p.115]. However, if $E^*$ has RNP (Radon-Nikodym property), then $BV_q(\gamma, E^*) \cong L_q(\gamma, E^*)$, and for each $m \in BV_q(\gamma, E^*)$ there exists a unique $g \in L_q(\gamma, E^*)$ such that

$$\begin{aligned} \langle m, f \rangle_{BV_q(\gamma,E^*),L_p(\gamma,E)} &= \int_I \langle m(ds), f(s) \rangle_{E^*,E} \\ &= \int_I \langle g(s), f(s) \rangle_{E^*,E} \gamma(ds) \end{aligned}$$

for all $f \in L_p(\gamma, E)$.

**Operator Valued Measures:** Let $E$ and $F$ be any pair of Banach spaces and $\mathcal{L}(E, F)$ the space of bounded linear operators from $E$ to $F$. A set function $\Phi$ mapping $\Sigma \times E$ to $F$ is said to be an operator valued measure if for each $\sigma \in \Sigma, e \in E$, $\Phi(\sigma, e) \in F$ and $\Phi(\emptyset, e) = 0$ the zero operator. The operator $\Phi$ is said to be weakly countably additive if for any family of pair wise disjoint sets $\sigma_i \in \Sigma$ and any pair $(e, f^*) \in E \times F^*$, we have

$$\langle \Phi(\bigcup \sigma_i, e), f^* \rangle_{F,F^*} = \sum \langle \Phi(\sigma_i, e), f^* \rangle_{F,F^*}.$$

If $e \longrightarrow \Phi(\sigma, e)$ is linear we may write $\Phi : \Sigma \longrightarrow \mathcal{L}(E, F)$. Further notations will be introduced as and when required.

## 3. Brief review of some related work

In this section we present a brief review of some recent studies of systems governed by strongly nonlinear parabolic equations determined by operator valued functions coupled with scalar valued measures. Consider the system

$$dx + A(t, x)\alpha(dt) = f(t)\alpha(dt), x(0) = x_0, t \in I, \tag{3.1}$$

where $A : I \times V \longrightarrow V^*$ is an operator valued function, $\alpha$ is a countably additive bounded positive measure and $f$ is a $V^*$ valued function. Throughout the presentation, it is assumed that both $V$ and its dual $V^*$ have the structure of separable reflexive Banach spaces with the embeddings $V \hookrightarrow H \hookrightarrow V^*$ being continuous and dense. Further, it is assumed that the pair of numbers $\{p, q\}$ are conjugate satisfying

$$1 < q \leq 2 \leq p < \infty \text{ with } 1/p + 1/q = 1.$$

We assume that $\alpha$ is a countably additive bounded positive measure on $\Sigma$ having bounded total variation, and the operator $A$ satisfies the following properties:

(A1) $A(t,\cdot): V \longrightarrow V^*$ is monotone, hemi-continuous for $\alpha$-a.a $t \in I$; and for every $u,v \in V$, $t \to \langle A(t,u),v\rangle_{V^*,V}$ is continuous.

(A2) there exist $a > 0, b \geq 0$ so that $< A(t,v), v >_{V^*,V} + b|v|_H^2 \geq a \parallel v \parallel_V^p$ for $\alpha$-a.a $t \in I$.

(A3) there exist constants $c_1, c_2 \geq 0$ so that $|A(t,v)|_{V^*} \leq c_1 + c_2 \parallel v \parallel_V^{p/q}$ $\alpha$-a.a. $t \in I$.

Following result was proved in a recent paper of the author.

**Theorem 3.1.** *Consider the evolution equation (3.1) and suppose the operator valued function $A$ satisfies the assumptions (A1)-(A3) and $f \in L_q(\alpha, V^*)$. Then for each $x_0 \in H$, equation (3.1) has a unique weak solution $x \in L_\infty(I,H) \cap L_p(\alpha,V) \cap BV_q(\alpha, V^*)$.*

*Proof.* See [5, Theorem 4.4, p.475]. □

**Remark 3.2.** In case $\alpha$ is the Lebesgue measure or absolutely continuous with respect to Lebesgue measure, we recover the classical result [7, Theorem 4.1, p.96].

Note that in the above theorem, $A$ is assumed to be a nonlinear operator valued function mapping $I \times V$ to $V^*$. In recent years this result has been further extended covering nonlinear operator valued measures [6]. That is, $A : \Sigma \times V \longrightarrow V^*$ is an operator valued set function. The system model considered is given by

$$dx + A(dt, x(t)) = f(t)\gamma(dt), t \in I, x(0) = x_0. \tag{3.2}$$

Here the basic assumptions used are as follows:

(B1) The map $A : \Sigma \times V \longrightarrow V^*$ is maximal monotone and hemicontinuous in the second argument satisfying

$$\langle A(\sigma,u) - A(\sigma,v), u - v\rangle_{V^*,V} \geq 0, \forall\ \sigma \in \Sigma, \text{ and } \forall\ u,v \in V.$$

There exist two countably additive nonnegative measures $\gamma, \beta \in M_c^+(\Sigma)$ having bounded variations on $I$ with $\gamma$ being strictly positive (on non void sets); and two real numbers $c_1 \geq 0, c_2 \geq 0$, such that

(B2) $\langle A(\sigma,v),v\rangle_{V^*,V} + \beta(\sigma)|v|_H^2 \ \geq\ \gamma(\sigma) \parallel v \parallel_V^p\ \ \forall\ \sigma \in \Sigma,$

(B3) $\parallel A(\sigma,v) \parallel_{V^*} \leq \gamma(\sigma)\{c_1 + c_2 \parallel v \parallel_V^{p/q}\}\ \forall\ \sigma \in \Sigma.$

**Note:** We wish to point out that the measure $\gamma$ is not assumed to be nonatomic.

We are concerned with the question of existence of solutions for the system (3.2). By a solution, we mean a weak solution as defined below. Let $C_T^1(0,T)$ denote the class of $C^1$ functions on $I \equiv [0,T]$ vanishing at $T$.

**Definition 3.3.** An element $x \in B_\infty(I,H) \cap L_p(\gamma,V)$ is said to be a weak solution of the problem (3.2) if for every $v \in V$ and $\varphi \in C_T^1(0,T)$, it satisfies the following identity

$$-(x_0, \varphi(0)v)_H - \int_I (x(t), \dot{\varphi}(t)v)_H dt + \int_I \langle A(dt, x(t)), \varphi(t)v\rangle_{V^*,V}$$

$$= \int_I \langle f(t), \varphi(t)v\rangle_{V^*,V}\gamma(dt). \tag{3.3}$$

Now we present some recent results from [6] on the questions of existence of solutions and their regularity properties.

**Theorem 3.4.** *Suppose $\gamma$ is a countably additive bounded positive measure having bounded variation on $I$ and the operator valued measure $A$ satisfy the assumptions (B1)-(B3) and $f \in L_q(\gamma, V^*)$. Then for each $x_0 \in H$, the system (3.2) has a unique weak solution $x \in B_\infty(I, H) \cap L_p(\gamma, V)$ and further $x \in BV_q(\Sigma, V^*)$.*

*Proof.* See [6, Theorem 5.3, p.800]. □

In the system model (3.2), the same measure $\gamma(\cdot)$ has been used to represent both the external as well as the internal forces embodied in the fundamental operator $A$. A more general representation is given by

$$dx(t) + A(dt, x(t)) = \nu(dt), x(0) = x_0, t \in I, \tag{3.4}$$

where the operator valued measure $A$ satisfies the assumptions (B1)-(B3) involving the scalar measures $\gamma, \beta$; while $\nu$ is a $V^*$-valued vector measure. The following result shows that under a mild assumption this general model can be reduced to the one given by (3.2).

**Corollary 3.5.** *Consider the system (3.4) and suppose the operator valued measure $A$ and the measures $\gamma, \beta$ satisfy the assumptions of Theorem 3.4. Let $\nu$ be a countably additive $V^*$-valued vector measure having finite $q$-variation on $I$ and that it is $\gamma$ continuous. Then for each $x_0 \in H$, the system (3.4) has a unique weak solution $x \in B_\infty(I, H) \cap L_p(\gamma, V)$ and further $x \in BV_q(\Sigma, V^*)$.*

*Proof.* The proof follows from Theorem 3.4 by simply noting that $V^*$, being a reflexive Banach space, has the Radon-Nikodym property (RNP) and, $\nu$ being a $\gamma$-continuous vector measure having $q$-variation, there exists an $f \in L_q(I, V^*)$ such that $d\nu = f d\gamma$. □

Theorem 3.4 was also extended in reference [6] to cover systems of the form

$$dx(t) + A(dt, x(t)) = f(x(t))\gamma(dt), x(0) = x_0, t \in I, \tag{3.5}$$

where $f$ is a suitable nonlinear map from $V$ to $V^*$ or from $H$ to $H$. For details see ref [6].

In reference [1] optimal control of systems given by the model (3.5) was considered. In particular, the associated control system is described by the following nonlinear evolution equation,

$$dx + A(dt, x) = f(x)\gamma(dt) + B(t)u(dt), t \in I, x(0) = \xi. \tag{3.6}$$

where $B \in L_\infty(\gamma, \mathcal{L}(F^*, V^*))$ and $u \in BV_q(\gamma, F^*)$.

**Theorem 3.6.** *Consider the Gelfand triple $\{V, H, V^*\}$ with the injection $V \hookrightarrow H$ being compact. Suppose the operator valued measure $A$, along with the scalar measures $\{\gamma, \beta\}$, satisfies the assumptions (B1)-(B3) and the operator $f : H \longrightarrow H$ is continuous satisfying the growth condition*

$$|(f(h), h)| \leq K(1+ \parallel h \parallel^2)$$

*for some $K > 0$ finite. Let $B \in L_\infty(\gamma, \mathcal{L}(F^*, V^*))$ so that*

$$B^* \in L_\infty(\gamma, \mathcal{L}(V, F)) \subset L_\infty(\gamma, \mathcal{L}(V, F^{**})).$$

*Then, for each $x_0 = \xi \in H$ and $u \in \mathcal{U}_{ad} \subset BV_q(\gamma, F^*)$, the system (3.6) has at least one weak solution $x \in B_\infty(I, H) \cap L_p(\gamma, V)$. The vector measure $\mu_x$, given by the relation*

$$\begin{aligned} \mu_x(\psi) &\equiv \int_I \langle \psi(t), \mu_x(dt) \rangle_{V,V^*} \\ &= \int_I \langle \psi(t), dx(t) \rangle_{V,V^*} \ \forall \ \psi \in L_p(\gamma, V), \end{aligned} \tag{3.7}$$

*is an element of $BV_q(\gamma, V^*)$. Further, if $-f$ is monotone, the solution is unique.*

*Proof.* For detailed proof see [1, Theorem 4.3, p.175]. □

In fact, the proof of the above theorem is based on similar approach as detailed in the proof of Theorem 4.2. We use a-priori bounds, finite dimensional projection, maximal monotonicity of the operator valued measure $A$, Crandall-Liggett generation theorem for nonlinear semigroups [7, Theorem 4.7, p.120], and the following compact embedding [8, Theorem 3.2, p.911]

$$M_{p,q} \hookrightarrow L_p(\gamma, H)$$

where

$$M_{p,q} \equiv \{ x : x \in L_p(\gamma, V) \ \& \ \mu_x \in BV_q(\Sigma, V^*) \ \}. \tag{3.8}$$

For the class of control systems of the form (3.6), the questions of continuous dependence of weak solutions with respect to controls from the class of vector measures $BV_q(\gamma, F^*)$, and the questions of existence of optimal controls from the same class are extensively studied in the papers [1, Corollary 4.4, Theorem 5.1, Theorem 5.2].

In this paper we are interested in structural control of systems like (3.5). This is considered in the next section.

## 4. Existence and regularity of solutions

We consider the following control system

$$dx + A(dt, x) = B(dt)x + f(x)\gamma(dt), t \in I, x(0) = \xi, \tag{4.1}$$

where $B$ is the structural control, belonging to the class $\mathcal{M}_{cabsv}(\Sigma, \mathcal{L}(V, V^*))$ consisting of countably additive set functions defined on $\Sigma$ and taking values from the space of bounded linear operators $\mathcal{L}(V, V^*)$ and having bounded semi-variation. We assume that the operator valued measure $B$ satisfies the following condition:

(B4) The operator valued measure $B$ has Radon-Nikodym derivative with respect to the scalar measure $\gamma$, that is, $B(dt) = \Lambda(t)\gamma(dt)$ with $\Lambda \in L_r(\gamma, \mathcal{L}(V, V^*))$ for a suitable $r > 1$.

For characterization of operator valued measures having Radon-Nikodym derivative with respect to scalar measures see reference [2].

For proof of existence of weak solutions we need the following a-priori estimate.

**Lemma 4.1.** *Suppose the operator valued measure $A$ along with the scalar measures $\{\gamma, \beta\}$ satisfies the conditions (B1)-(B3) for $\{p, q\}$ conjugate satisfying*

$$1 < q \leq 2 < p < \infty \text{ with } 1/p + 1/q = 1.$$

*The operator valued measure $B \in \mathcal{M}_{cabsv}(\Sigma, \mathcal{L}(V, V^*))$ satisfying the assumption (B4) with $r = p/(p-2)$ and the operator $f$ satisfies the assumptions of Theorem 3.6 and $x_0 = \xi \in H$. Then, if $x$ is any solution of the system (4.1), it must be an element of $L_p(\gamma, V) \cap B_\infty(I, H)$ where $\nu$ is a positive measure with finite total variation on $I$.*

*Proof.* Let $x$ be any solution of equation (4.1) corresponding to the initial state $x_0 = \xi \in H$ and $B \in \mathcal{M}_{cabsv}(\Sigma, \mathcal{L}(V, V^*))$. Scalar multiplying equation (4.1) by $x$ and then integrating by parts over the interval $I_t \equiv [0, t]$ and using (B2), it is easy to verify that for each $t \in I$,

$$\begin{aligned} |x(t)|_H^2 + 2\int_0^t \| x(s) \|_V^p \gamma(ds) \leq |\xi|_H^2 &+ 2\int_0^t |x(s)|_H^2 \beta(ds) \\ &+ 2\int_0^t (f(x(s)), x(s))_H \gamma(ds) \\ &+ 2\int_0^t \langle B(ds)x(s), x(s)\rangle_{V^*,V}. \end{aligned} \tag{4.2}$$

It follows from our assumption (B4) with respect to the operator valued measure $B$ that

$$\Big| \int_0^t \langle B(ds)x(s), x(s)\rangle_{V^*,V} \Big| \leq \int_0^t \| \Lambda(s) \|_{\mathcal{L}(V,V^*)} \| x(s) \|_V^2 \gamma(ds). \tag{4.3}$$

Since $\gamma(I)$ is finite, it is clear that $L_p(\gamma, V) \subset L_2(\gamma, V)$ for any $p > 2$. Therefore, it follows from Hölder inequality, applied to the inequality (4.3), that

$$\begin{aligned} &\int_0^t \| \Lambda(s) \| \| x(s) \|_V^2 \gamma(ds) \\ &\quad \leq \Big( \int_0^t \| \Lambda(s) \|^r \gamma(ds) \Big)^{(1/r)} \Big( \int_0^t \| x(s) \|_V^p \gamma(ds) \Big)^{2/p} \end{aligned} \tag{4.4}$$

for all $t \in I$. Now using Cauchy-Young inequality, it follows from (4.4) that, for any $\varepsilon > 0$, we have

$$\begin{aligned} &\int_0^t \| \Lambda(s) \| \| x(s) \|_V^2 \gamma(ds) \\ &\quad \leq \frac{2\varepsilon^{p/2}}{p} \int_0^t \| x(s) \|_V^p \gamma(ds) + \frac{(p-2)}{p\varepsilon^{p/p-2}} \Big( \int_0^t \| \Lambda(s) \|^r \gamma(ds) \Big). \end{aligned} \tag{4.5}$$

Choosing $\varepsilon = (p/4)^{2/p}$ in the expression (4.5) and then substituting in (4.3) we obtain

$$\begin{aligned} \Big| \int \langle B(ds)x(s), x(s)\rangle_{V^*,V} \Big| &\leq \int_0^t \| \Lambda(s) \| \| x(s) \|_V^2 \gamma(ds) \\ &\leq (1/2) \int_0^t \| x(s) \|_V^p \gamma(ds) + C_\Lambda. \end{aligned} \tag{4.6}$$

where

$$C_\Lambda \equiv \frac{(p-2)4^{2/p-2}}{p^{(p/p-2)}} \int_0^T \| \Lambda(s) \|^r \gamma(ds), \text{ and } r = (p/p-2).$$

Clearly, it follows from the growth property of $f$, as assumed in Theorem 3.6, that

$$\Big| \int_0^t (f(x(s)), x(s))_H \gamma(ds) \Big| \leq K\gamma(I) + K \int_0^t |x(s)|_H^2 \gamma(ds). \tag{4.7}$$

Using the estimates (4.6) and (4.7) in the inequality (4.2) we arrive at the following inequality

$$|x(t)|_H^2 + \int_0^t \| x(s) \|_V^p \ \gamma(ds) \leq C_1 + \int_0^t |x(s)|_H^2 \ \nu(ds), t \in I, \tag{4.8}$$

where

$$C_1 = |\xi|_H^2 + 2K\gamma(I) + 2C_\Lambda$$

and the measure $\nu$ is given by

$$\nu(\sigma) = 2\beta(\sigma) + 2K\gamma(\sigma), \forall \ \sigma \in \Sigma.$$

Since the measures $\beta$ and $\gamma$ are positive having bounded total variation, so is the measure $\nu$. Thus by virtue of generalized Gronwall inequality [9, Lemma 5, p268] it follows from (4.8) that

$$|x(t)|_H^2 \leq C_1\big\{1 + \nu(\{0\})\big\} \exp \nu(I), \text{ for all } t \in I. \tag{4.9}$$

If $\{0\}$ is not an atom of the measure $\nu$ one can omit $\nu(\{0\})$ in the above expression. In view of the above estimate, which holds for all $t \in I$, it is clear that $x \in B_\infty(I, H)$, the space of norm-bounded $H$ valued functions. Further, using the estimate (4.9) in the inequality (4.8) we arrive at the following inequality

$$\int_0^t \| x(s) \|_V^p \gamma(ds) \leq C_1\big\{1 + [1 + \nu(\{0\})]\nu(I) \exp \nu(I)\big\}, \ \forall \ t \in I, \tag{4.10}$$

which shows that $x \in L_p(\gamma, V)$. Clearly $B_\infty(I, H) \subset L_\infty(I, H)$. Thus it follows from the above analysis that

$$x \in L_p(\gamma, V) \cap B_\infty(I, H) \subset L_p(\gamma, V) \cap L_\infty(I, H).$$

This completes the proof. □

Now we are prepared to consider the question of existence of solution of the evolution equation (4.1).

**Theorem 4.2.** *Suppose the operator valued measure $A$, the scalar measures $\{\gamma, \beta\}$ and the operator $f$ satisfy the assumptions of Lemma 4.1 with the injection $V \hookrightarrow H$ being compact. Then, for each $x_0 = \xi \in H$ and $B \in \mathcal{M}_{cabv}(\Sigma, \mathcal{L}(V, V^*))$ satisfying the assumption of Lemma 4.1, the evolution equation (4.1) has at least one weak solution $x \in B_\infty(I, H) \cap L_p(\gamma, V)$. The vector measure $\mu_x$, given by the relation*

$$\mu_x(\psi) \equiv \int_I \langle \psi(t), \mu_x(dt) \rangle_{V,V^*} = \int_I \langle \psi(t), dx(t) \rangle_{V,V^*} \ \forall \ \psi \in L_p(\gamma, V),$$

*is an element of $BV_q(\gamma, V^*)$.*

*Proof.* We use similar technique as in [1, Theorem 4.3]. So for convenience of the readers we present it briefly. It is based on a-priori bounds, finite dimensional projection to an increasing family of (finite dimensional) subspaces determined by $X_n \equiv$ lin.span $\{v_i, 1 \leq i \leq n\}$, maximal monotonicity of the operator valued measure $A$, Crandall-Liggett generation theorem for nonlinear semigroups corresponding to maximal monotone operators [7, Theorem 4.7, pp.120-121], see also [14, p.115], and continuity of $f$ and, most importantly, compact embedding [8, Theorem 3.2, p911] of $M_{p,q} \hookrightarrow L_p(\gamma, H)$ where

$$M_{p,q} \equiv \{x : x \in L_p(\gamma, V) \ \& \ \mu_x \in BV_q(\Sigma, V^*) \ \}. \tag{4.11}$$

Let $\{v_i\}$ be a complete basis for the Gelfand triple $V \subset H \subset V^*$ so that they are orthogonal in $V$ and $V^*$, and orthonormal in $H$. Now we use finite dimensional projection of the system (4.1) to $X_n$ and denote the corresponding solutions (if they exist) by $x_n \equiv \sum_{i=1}^n z_i^n v_i$ giving the family of finite dimensional systems,

$$\begin{aligned} \langle dx_n, v_i\rangle + \langle A(dt, \sum_{j=1}^n z_j^n v_j), v_i\rangle_{V^*,V} \\ = (f(\sum_{j=1}^n z_j^n v_j), v_i)_H \gamma(dt) + \sum_{j=1}^n z_j^n \langle B(dt) v_j, v_i\rangle_{V^*,V}, \end{aligned} \tag{4.12}$$

$$i = 1, 2, \ldots, n; n \in N,$$

with the initial condition given by $x_n(0) \equiv \sum_{i=1}^n (\xi, v_i) v_i$. Define the maps

$$G(\sigma, z) \equiv col\{G_i(\sigma, z), 1 \leq i \leq n\}, \tilde{f}(z) \equiv col\{f_i(z), 1 \leq i \leq n\}, \tag{4.13}$$

$$M_B(\sigma) \equiv \text{matrix}\{\mu_{i,j}^B(\sigma), 1 \leq i, j \leq n\}, \sigma \in \Sigma, z \in R^n, \tag{4.14}$$

where $z = (z_1, z_2, \ldots, z_n)'$ and

$$\begin{aligned} G_i(\sigma, z) \equiv \langle A(\sigma, \sum_{j=1}^n z_j v_j), v_i\rangle_{V^*,V}, \ f_i(z) \equiv (f(\sum_{j=1}^n z_j v_j), v_i)_H, \\ \mu_{i,j}^B(\sigma) \equiv \langle B(\sigma) v_j, v_i\rangle, 1 \leq i, j \leq n, \sigma \in \Sigma, z \in R^n. \end{aligned} \tag{4.15}$$

Let $M_B$ denote the matrix valued measure with entries $\{\mu_{ij}^B\}$. It is clear from (4.15) and our assumption on $B$ that $M_B \in \mathcal{M}_{cabv}(\Sigma, \mathcal{L}(R^n, R^n))$. Using these notations, for any $n \in N$ the system (4.12) can be written in the form,

$$\begin{aligned} dz + G(dt, z) = \tilde{f}(z)\gamma(dt) + M_B(dt) z, \\ z(0) = col\{(\xi, v_i), i = 1, 2, \ldots, n\}, n \in N. \end{aligned} \tag{4.16}$$

This is a measure driven n-dimensional system. For any $m \in N$, partition the interval $I$ into $m$ disjoint subintervals giving $I = \cup_{i=0}^{m-1} \sigma_i$ where $\sigma_i \equiv (t_i, t_{i+1}], 0 \leq i \leq m-1$, with $t_0 = 0, t_m = T$. Define the nonlinear operator valued set function $\hat{G}(\sigma)(z) \equiv G(\sigma, z)$ from $R^n$ to $R^n$ and $\sigma \in \Sigma$. Since for each $\sigma \in \Sigma$, $A(\sigma, \cdot)$ is maximal monotone (from $V$ to $V^*$), $\hat{G}(\sigma)$ is maximal monotone on $R^n$. Hence the range of the operator $(I + \hat{G}(\sigma))$ is all of $R^n$, that is, $\mathcal{R}(I + \hat{G}(\sigma)) = R^n$. Thus, by use of implicit difference scheme, one can construct the sequence $\{z_m(t), t \in I\}$ by linear interpolation of the nodes given by

$$z_m(t_{i+1}) \equiv (I+\hat{G}(\sigma_i))^{-1}\big(z_m(t_i)+\tilde{f}(z_m(t_i))\gamma(\sigma_i)+M_B(\sigma_i)z_m(t_i)\big),\ i=0,1,\ldots,m-1.$$

It follows from Crandall-Ligget generation theorem for nonlinear semigroups [7, Theorem 4.7, p120-121], that $z_m(t) \longrightarrow z(t)$ uniformly on $I$ and that $z$ solves equation (4.16). We denote this solution by $z = z^n$. Then, $x_n \equiv \sum_{i=1}^{n} z_i^n v_i$ solves equation (4.12). By virtue of the a-priori bounds given by Lemma 4.1, $\{x_n\}$ is contained in a bounded subset of $L_p(\gamma, V) \cap B_\infty(I, H) \subset L_p(\gamma, V) \cap L_\infty(I, H)$. Since the spaces $(V, H, V^*)$ are all reflexive there exists a subsequence, relabeled as the original sequence, and an element $x \in L_p(\gamma, V) \cap B_\infty(I, H)$ such that

$$x_n \xrightarrow{w^*} x \text{ in } L_\infty(I, H) \tag{4.17}$$

$$x_n \xrightarrow{w} x \text{ in } L_p(\gamma, V). \tag{4.18}$$

Let $C_T^1[0, T]$ denote the class of $C^1$ functions vanishing at $T$. Multiplying equation (4.12) by $\varphi \in C_T^1[0, T]$ and integrating by parts we obtain

$$\begin{aligned} &-(x_n(0), \varphi(0)v_i) - \int_I (x_n(t), \dot{\varphi}(t)v_i)_H dt + \int_I \langle A(dt, x_n(t)), \varphi(t)v_i\rangle_{V^*,V} \\ &= \int_I (f(x_n(t)), \varphi(t)v_i)_H \gamma(dt) + \int_I \langle x_n(t), B^*(dt)\varphi(t)v_i\rangle_{V,V^*}. \end{aligned} \tag{4.19}$$

Since $V$ is dense in $H$, it is clear that $x_n(0) \xrightarrow{s} \xi$ in $H$. Letting $n \to \infty$, it follows from this and (4.17) that

$$\begin{aligned} &-(x_n(0), \varphi(0)v_i)_H - \int_I (x_n(t), \dot{\varphi}(t)v_i)_H dt \\ &\qquad \longrightarrow -(\xi, \varphi(0)v_i)_H - \int_I (x(t), \dot{\varphi}(t)v_i)_H dt. \end{aligned} \tag{4.20}$$

It follows from the assumption (B4) and the fact that $r > q > 1$, the measure $\mu$ given by $\mu(\sigma) \equiv \int_\sigma B^*(dt)\varphi(t)v_i, \sigma \in \Sigma_\gamma$, is an element of $BV_q(\gamma, V^*)$. In other words, $\mu$ is a $V^*$ valued $\gamma$ continuous vector measure having finite $q$ variation. Using this fact and (4.18) and the natural duality pairing between the spaces $L_p(\gamma, V)$ and $BV_q(\Sigma_\gamma, V^*)$ we conclude that

$$\int_I \langle x_n(t), B^*(dt)\varphi(t)v_i\rangle_{V,V^*} \longrightarrow \int_I \langle x(t), B^*(dt)\varphi(t)v_i\rangle_{V,V^*}, \tag{4.21}$$

as $n \to \infty$. Considering the third term on the left of equation (4.19) and following similar approach as in [1, Theorem 4.3, p175] we prove that the operator valued measure $A(\cdot, x_n(\cdot)) \longrightarrow A(\cdot, x(\cdot))$ in $BV_q(\Sigma_\gamma, V^*)$ in the weak star topology. We sketch the proof briefly. Define the sequence of $V^*$-valued vector measures,

$$a_n(\sigma) \equiv \int_\sigma A(ds, x_n(s)), \sigma \in \Sigma. \tag{4.22}$$

Clearly, it follows from the assumption (B3) and countable additivity of the measure $\gamma$ that for each $v \in V$ and $\sigma \in \Sigma$,

$$\langle a_n(\sigma), v\rangle_{V^*,V} \equiv \int_\sigma \langle A(ds, x_n(s)), v\rangle_{V^*,V}$$

is well defined and that the set function $\sigma \longrightarrow \langle a_n(\sigma), v\rangle_{V^*,V}$ is countably additive and $\gamma$-continuous. Thus $\{a_n\}$ is a sequence of weakly countably additive $V^*$ valued vector measures and that it vanishes on $\gamma$ null sets. Hence it follows from Pettis theorem [Dunford-Schwartz, 18, Theorem 1V.10.1, p318] that it is countably additive and $\gamma$ continuous. These facts along with the a-priori bounds of $\{x_n\}$ (see Lemma 4.1) imply that the set $\{a_n\}$ is contained in a bounded subset of $\mathcal{M}_c(\Sigma, V^*)$, and that

$$\lim_{\gamma(\sigma)\to 0} |a_n|(\sigma) = 0, \text{ uniformly in } n \in N. \tag{4.23}$$

Since $V^*$ is a reflexive Banach space, it follows from boundedness of the set $\{a_n\}$ that for each $\sigma \in \Sigma$, $\{a_n(\sigma), n \in N\}$ is a conditionally weakly compact subset of $V^*$. As the dual pair of Banach spaces $\{V, V^*\}$ are reflexive, they have Radon-Nikodym property (RNP). Thus it follows from Bartle-Dunford-Schwartz compactness theorem for vector measures [16, Theorem 5, p105] that there exists an $a \in \mathcal{M}_c(\Sigma, V^*)$ such that, along a subsequence if necessary, $a_n \longrightarrow a$ weakly. Hence, for the third term on the left of the expression (4.19), we have

$$\begin{aligned} &\int_I \langle A(dt, x_n(t)), \varphi(t)v_i\rangle_{V^*,V} \\ &\qquad \equiv \int_I \langle a_n(dt), \varphi(t)v_i\rangle_{V^*,V} \longrightarrow \int_I \langle a(dt), \varphi(t)v_i\rangle_{V^*,V}. \end{aligned} \tag{4.24}$$

Then, by using the monotonicity and hemicontinuity assumption (B1), one can easily verify that

$$a(\sigma) = \int_\sigma A(ds, x(s)) \;\forall\; \sigma \in \Sigma. \tag{4.25}$$

For details see [1, Theorem 4.3]. Thus we have proved that, along a subsequence if necessary,

$$\begin{aligned} &\int_I \langle A(dt, x_n(t)), \varphi(t)v_i\rangle_{V^*,V} \;\equiv\; \int_I \langle a_n(dt), \varphi(t)v_i\rangle_{V^*,V} \\ &\longrightarrow \int_I \langle a(dt), \varphi(t)v_i\rangle_{V^*,V} = \int_I \langle A(dt, x(t)), \varphi(t)v_i\rangle_{V^*,V}. \end{aligned} \tag{4.26}$$

Next, we verify that the first term on the right hand side of equation (4.19) converges to the desired limit, that is,

$$\int_I (f(x_n(t)), \varphi(t)v_i)_H \gamma(dt) \longrightarrow \int_I (f(x(t)), \varphi(t)v_i)_H \gamma(dt). \tag{4.27}$$

Here we use a compact embedding theorem [8, Theorem 3.2, p.911]. Define the sequence of $V^*$ valued vector measure $\mu_n$ by

$$\mu_n(\psi) \equiv \int_I \langle \psi(t), \mu_n(dt) \rangle_{V,V^*} \equiv \int_I \langle \psi(t), dx_n(t) \rangle_{V,V^*}, \psi \in L_p(\gamma, V).$$

Since $\{\gamma, \beta\}$ are countably additive bounded positive measures and $\{x_n\}$ is contained in a bounded subset of $B_\infty(I, H) \cap L_p(\gamma, V)$, it follows from straightforward computation, using the above identity and the assumptions (B3), that $\{\mu_n\} \subset BV_q(\gamma, V^*)$. Then it follows from the embedding $BV_q(\gamma, V^*) \hookrightarrow BV_q(\Sigma, V^*)$, as seen in section 2, that $\{\mu_n\} \subset BV_q(\Sigma, V^*)$. Thus by definition of $M_{p,q}$ (see the expression (4.11)) we have $\{x_n\} \subset M_{p,q}$. Since the embedding $M_{p,q} \hookrightarrow L_p(\gamma, H)$ is compact [8, Theorem 3.2, p911] we can extract a subsequence of the sequence $\{x_n\}$, relabeled as $\{x_n\}$, so that $x_n \xrightarrow{s} x$ in $L_p(\gamma, H)$. Hence there exists a further subsequence of the sequence $\{x_n\}$, relabeled as $\{x_n\}$, such that

$$x_n(t) \xrightarrow{s} x(t) \text{ in } H \text{ for } \gamma - a.a\ t \in I.$$

By our assumption, $f : H \longrightarrow H$ is continuous and bounded on bounded sets and hence

$$f(x_n(t)) \xrightarrow{s} f(x(t)) \text{ in } H \text{ for } \gamma - a.a\ t \in I.$$

From the growth assumption for $f$ (see the statement in Theorem 3.6) and the fact that the family $\{x_n, x\}$ satisfies the same bounds as stated in Lemma 4.1, $\{f(x_n(\cdot))\}$ is dominated by an element from $L_p(\gamma, H)$. Thus by Lebesgue dominated convergence theorem, $f(x_n(\cdot)) \xrightarrow{s} f(x(\cdot))$ in $L_p(\gamma, H)$ and, since $\varphi v_i \in C(I, H) \subset B_\infty(I, H)$, we have

$$\int_I (f(x_n(t)), \varphi(t) v_i)_H \gamma(dt) \longrightarrow \int_I (f(x(t)), \varphi(t) v_i)_H \gamma(dt) \tag{4.28}$$

proving (4.27). Collecting the above results and letting $n \to \infty$ in equation (4.19) we arrive at the following identity

$$\begin{aligned} &-(\xi, \varphi(0) v_i)_H - \int_I (x(t), \dot{\varphi} v_i)_H dt + \int_I \langle A(dt, x(t)), \varphi(t) v_i \rangle_{V^*,V} \\ &= \int_I (f(x(t)), \varphi(t) v_i)_H \gamma(dt) + \int_I \langle x(t), B^*(dt) \varphi(t) v_i \rangle_{V,V^*} \end{aligned} \tag{4.29}$$

which holds for all $\varphi \in C_T^1(I)$ and for all $i \in N$. Since $\{v_i\}$ is a basis for $V$, this identity holds for all $v \in V$. Hence $x$ is a weak solution of equation (4.1) satisfying all the properties as stated. This completes the proof. □

For uniqueness of weak solutions we need some stronger assumptions. This is given in the following proposition.

**Proposition 4.3.** *Consider the system (4.1) and suppose the assumptions of Theorem 4.2 hold, and further the operator valued measure $A$ is strictly monotone satisfying*

$$\langle A(\sigma, x) - A(\sigma, y), x - y \rangle_{V^*,V} \geq \gamma(\sigma) \parallel x - y \parallel_V^2, \forall\ \sigma \in \Sigma.$$

*There exists a constant $M \geq 0$ such that*

$$|(f(x) - f(y), x - y)_H| \leq M |x - y|_H^2, \forall x, y \in H.$$

*The structural control $B : \Sigma \longrightarrow \mathcal{L}(V,H)$ is an operator valued measure having Radon-Nikodym derivative with respect to the scalar measure $\gamma$, that is, $B(dt) = \Lambda(t)\gamma(dt), t \in I$, with $\Lambda \in L_2(\gamma, \mathcal{L}(V,H))$. Then, for every $x_0 = \xi \in H$, the weak solution of the evolution equation (4.1) is unique.*

*Proof.* Suppose there are two weak solutions $x, y \in B_\infty(I,H) \cap L_p(\gamma, V)$ starting from the same initial state $\xi \in H$. Subtracting equation (4.1) with $x$ as the weak solution, from the same equation with $y$ as the weak solution, and scalar multiplying this equation by $(x-y)$ and integrating by parts one can easily verify that, for all $t \in I$,

$$\begin{aligned}|x(t)-y(t)|_H^2 &+ 2\int_0^t \langle A(ds, x(s)) - A(ds, y(s)), x(s)-y(s)\rangle_{V^*,V} \\ &= 2\int_0^t (f(x(s)) - f(y(s)), x(s)-y(s))_H \gamma(ds) \\ &\quad + 2\int_0^t (B(ds)(x(s)-y(s)), x(s)-y(s))_H.\end{aligned}$$

It follows from the assumptions on $\{A, f, B\}$ and the above identity that

$$\begin{aligned}|x(t)-y(t)|_H^2 + 2\int_0^t \| x(s)-y(s) \|_V^2 \gamma(ds) &\le 2M\int_0^t |x(s)-y(s)|_H^2 \gamma(ds) \\ &+ 2\int_0^t \| \Lambda(s) \|_{\mathcal{L}(V,H)} \| x(s)-y(s) \|_V |x(s)-y(s)|_H \gamma(ds).\end{aligned}$$

Considering the last term of the above expression and using Hölder and Cauchy inequalities, it is easy to verify that, for any $\varepsilon > 0$,

$$\begin{aligned}&2\int_0^t \| \Lambda(s) \|_{\mathcal{L}(V,H)} \| x(s)-y(s) \|_V |x(s)-y(s)|_H \gamma(ds) \\ &\quad \le \varepsilon \int_0^t \| x(s)-y(s) \|_V^2 \gamma(ds) + (1/\varepsilon)\int_0^t \| \Lambda(s) \|_{\mathcal{L}(V,H)}^2 |x(s)-y(s)|_H^2 \gamma(ds).\end{aligned}$$

Choosing $\varepsilon = 1$, it follows from the preceding two inequalities that

$$\begin{aligned}|x(t)-y(t)|_H^2 &+ \int_0^t \| x(s)-y(s) \|_V^2 \gamma(ds) \\ &\le \int_0^t \big(2M + \| \Lambda(s) \|_{\mathcal{L}(V,H)}^2\big)|x(s)-y(s)|_H^2 \gamma(ds), t \in I.\end{aligned}$$

By Theorem 4.2, $x, y \in L_p(\gamma, V)$; and, since $p > 2$ and $\gamma$ is a finite positive measure, it is evident that $L_p(\gamma, V) \hookrightarrow L_2(\gamma, V)$. Thus the expression on the left hand side of the above inequality is well defined. Since $\Lambda \in L_2(\gamma, \mathcal{L}(V,H))$ and $\gamma(I) < \infty$ and $x, y \in B_\infty(I,H) \subset L_\infty(I,H)$, the expression on the right hand side is also well defined. Define

$$\kappa(t) \equiv 2M + \| \Lambda(t) \|_{\mathcal{L}(V,H)}^2$$

and $\varphi(t) \equiv |x(t) - y(t)|_H^2$ and $\psi(t) \equiv \| x(t) - y(t) \|_V^2$ for $t \in I$. With this notation, the above inequality can be rewritten as follows:

$$\varphi(t) + \int_0^t \psi(s)\gamma(ds) \leq \int_0^t \kappa(s)\varphi(s)\gamma(ds), t \in I.$$

It follows from the assumption on $B$ and the fact that $I$ is a finite interval, that $\kappa \in L_1^+(\gamma)$, and, since both $x, y \in L_2(\gamma, V)$, $\psi \in L_1^+(\gamma)$ also. Now by virtue of generalized Gronwall inequality, it follows from the above expression that $\varphi(t) \equiv 0$ $\gamma - a.e$ and hence $\psi(t) \equiv 0$ $\gamma - a.e.$ This means $x = y$ as elements of $L_\infty(I, H) \cap L_2(\gamma, V)$. Hence, under the given assumptions, the evolution equation (4.1) has a unique weak solution. This completes the proof. □

**Remark 4.4.** If the operator valued measure $B : \Sigma \longrightarrow \mathcal{L}(H)$ has the RND (Radon-Nikodym derivative) $\Lambda \in L_2(\gamma, \mathcal{L}(H))$ then the assumption on strict monotonicity of the operator $A$ can be relaxed to simple monotonicity, that is, $< A(\sigma, x) - A(\sigma, y), x - y >_{V^*,V} \geq 0$ for all $\sigma \in \Sigma$ and all $x, y \in V$.

## 5. Existence of optimal structural controls

In this section we introduce the class of admissible structural controls and prove a result on continuous dependence of solutions on controls. Using these results we prove the existence of optimal controls for certain typical cost functionals.

5.1. **Continuous dependence of Solutions:** Let $\mathcal{L}(V, H)$ denote the space of bounded linear operators from $V$ to $H$, and $\mathcal{M}_{fa}(\Sigma, \mathcal{L}(V, H))$ the space of finitely additive bounded operator valued measures. It is well known that it is a Banach space with respect to the variation norm. Let $\mathcal{M}_{ad}$ be a bounded subset of $\mathcal{M}_{fa}(\Sigma, \mathcal{L}(V, H))$ and suppose it is uniformly $\gamma$-continuous satisfying the assumption (B4) and countably additive in the weak operator topology. Further characterization of admissible controls is given later in this section. To consider optimal control problems we need continuous dependence of solutions on controls. This is given in the following theorem.

**Theorem 5.1.** *Consider the system (4.1) and suppose the assumptions of Theorem 4.2 hold. Then the map $B \longrightarrow x$ is continuous in the sense that whenever $B_n \xrightarrow{w} B_o$ in $\mathcal{M}_{ad}$, $x_n \xrightarrow{w^*} x^o$ in $L_\infty(I, H)$ and $x_n \xrightarrow{w} x^o$ in $L_p(\gamma, V)$, where $x^o$ is the weak solution of equation (4.1) corresponding to $B_o$.*

*Proof.* Let $\{B_n\} \subset \mathcal{M}_{ad}$ and $\{x_n\}$ the corresponding weak solutions of the system (4.1). Then it follows from the definition of weak solution that, for every $v \in V$ and $\varphi \in C_T^1(I)$, the following identity holds

$$\begin{aligned} -(\xi, \varphi(0)v)_H - \int_I (x_n(t), \dot{\varphi}v)_H dt + \int_I \langle A(dt, x_n(t)), \varphi(t)v \rangle_{V^*,V} \\ = \int_I (f(x_n(t)), \varphi(t)v)_H \gamma(dt) + \int_I \langle B_n^*(dt)\varphi(t)v, x_n(t) \rangle_{V^*,V}. \end{aligned} \tag{5.1}$$

Suppose $B_n \xrightarrow{w} B_o$ in $\mathcal{M}_{ad}$. Since the set $\mathcal{M}_{ad}$ is a bounded subset of $\mathcal{M}_{fa}(\Sigma, \mathcal{L}(V, H))$, the solution set

$$X \equiv \{x \in L_\infty(I, H) \cap L_p(\gamma, V) : x = x(B), B \in \mathcal{M}_{ad}\}$$

is a bounded subset of $B_\infty(I,H) \cap L_p(\gamma,V) \subset L_\infty(I,H) \cap L_p(\gamma,V)$. Thus the sequence of solutions $\{x_n\}$ corresponding to the sequence $\{B_n\}$ is contained in a bounded subset of $B_\infty(I,H) \cap L_p(\gamma,V) \subset L_p(\gamma,V) \cap L_\infty(I,H)$. Since $V$ has the structure of a reflexive Banach space and $H$ is a Hilbert space there exists a subsequence of the sequence $\{x_n\}$, relabeled as $\{x_n\}$, and an element $x^o \in L_\infty(I,H) \cap L_p(\gamma,V)$ such that

$$x_n \xrightarrow{w^*} x^o \text{ in } L_\infty(I,H) \tag{5.2}$$

$$x_n \xrightarrow{w} x^o \text{ in } L_p(\gamma,V). \tag{5.3}$$

Then, following similar arguments as used in the proof of Theorem 4.2, we conclude that as $n \to \infty$, along a subsequence if necessary, we have

$$\int_I (x_n(t), \dot\varphi(t)v)_H dt \longrightarrow \int_I (x^o(t), \dot\varphi(t)v)_H dt \tag{5.4}$$

$$\int_I \langle A(dt, x_n(t)), \varphi(t)v\rangle_{V^*,V} \longrightarrow \int_I \langle A(dt, x^o(t)), \varphi(t)v\rangle_{V^*,V} \tag{5.5}$$

$$\int_I (f(x_n(t)), \varphi(t)v)_H \gamma(dt) \longrightarrow \int_I (f(x^o(t)), \varphi(t)v)_H \gamma(dt) \tag{5.6}$$

for every $v \in V$ and $\varphi \in C_T^1(I)$. Let us now consider the last term in the expression (5.1). Since $B_n \xrightarrow{w} B_o$ in $\mathcal{M}_{ad}$, the restriction of $B_n^*$ to $V$ denoted by $B_n^*|_V$ converges weakly to $B_o^*$ in $\mathcal{M}_{ad}$. By assumption, the embedding $V \overset{i}{\hookrightarrow} H$ is compact. Thus it's adjoint is compact and hence the embedding $H^* = H \overset{i^*}{\hookrightarrow} V^*$ is also compact. Therefore, along a subsequence if necessary (relabeled as the original sequence),

$$B_n^*(\sigma)|_V \varphi(t)v = B_n^*(\sigma)\varphi(t)v \xrightarrow{s} B_o^*(\sigma)\varphi(t)v$$

in $V^*$ for every $\sigma \in \Sigma$ and $t \in I$. By assumption, the family of operator valued measures $\mathcal{M}_{ad}$ is uniformly $\gamma$-continuous. Thus the $V^*$-valued measures $\{\mu_n, \mu_o\}$, given by $\mu_n(\sigma) \equiv \int_\sigma B_n^*(ds)\varphi(s)v$ and $\mu_o(\sigma) \equiv \int_\sigma B_o^*(ds)\varphi(s)v$, $\sigma \in \Sigma$, are uniformly $\gamma$-continuous. In our assumption (B4), $r = (p/p-2)$ and therefore $q < r$, where $q$ is the conjugate (number) of $p$ and consequently $BV_r(\gamma, V^*) \subset BV_q(\gamma, V^*)$. Thus the sequence of measures $\{\mu_n, \mu_o\}$ belong to $BV_q(\gamma, V^*)$ and $\mu_n \xrightarrow{s} \mu_o$ in $BV_q(\gamma, V^*)$. On the other hand we have already seen that $x_n \xrightarrow{w} x^o$ in $L_p(\gamma, V)$. Hence letting $n \to \infty$, we have the convergence of the duality pairing

$$\langle \mu_n, x_n\rangle_{BV_q(\gamma,V^*),L_p(\gamma,V)} \longrightarrow \langle \mu_o, x^o\rangle_{BV_q(\gamma,V^*),L_p(\gamma,V)}.$$

Clearly, this is equivalent to

$$\int_I \langle B_n^*(dt)\varphi(t)v, x_n(t)\rangle_{V^*,V} \longrightarrow \int_I \langle B_o^*(dt)\varphi(t)v, x^o(t)\rangle_{V^*,V} \text{ as } n \to \infty. \tag{5.7}$$

Now using (5.4)-(5.7) and letting $n \longrightarrow \infty$ in (5.1), we arrive at the following identity

$$-(\xi, \varphi(0)v)_H - \int_I (x^o(t), \dot\varphi v)_H dt + \int_I \langle A(dt, x^o(t)), \varphi(t)v\rangle_{V^*,V} \tag{5.8}$$

$$= \int_I (f(x^o(t)), \varphi(t)v)_H \gamma(dt) + \int_I \langle B_o^*(dt)\varphi(t)v, x^o(t)\rangle_{V^*,V}.$$

Since the above identity holds for arbitrary $v \in V$ and $\varphi \in C_T^1(I)$ it follows from this (identity) that $x^o$ is a weak solution of the system

$$\begin{aligned} dx + A(dt,x) &= B_o(dt)x + f(x)\gamma(dt), t \in I \\ x(0) &= \xi, \end{aligned} \tag{5.9}$$

and by Lemma 4.1, $x^o \in B_\infty(I,H) \cap L_p(\gamma, V)$. Thus we conclude that the control to solution map, $B \longrightarrow x = x(B)$, is (sequentially) continuous in the sense as stated in the theorem. This completes the proof. □

5.2. **Optimal Controls:** In this section we wish to consider control problems. Before we can do so we need further characterization of the set of admissible structural controls.

In order to include in the objective functional the cost of control representing the semivariation of the operator valued measure one may consider the following class of operator valued measures. Let $B_\infty(\gamma, V)$ denote the class of $\gamma$-essentially bounded measurable $V$ valued functions and let $H$ be the Hilbert space. Let $\mathcal{L}(B_\infty(\gamma, V), H)$ denote the space of bounded linear operators from the Banach space $B_\infty(\gamma, V)$ to the Hilbert space $H$. Suppose this is endowed with the weak operator topology $\tau_{wo}$. We denote this topological space by

$$(\mathcal{L}(B_\infty(\gamma, V), H), \tau_{wo}) = \mathcal{L}_{wo}(B_\infty(\gamma, V), H)).$$

This is a locally convex sequentially complete topological vector space. Let

$$\mathcal{M}_{fabsv}(\Sigma_\gamma, \mathcal{L}(V,H)) \subset \mathcal{M}_{fa}(\Sigma, \mathcal{L}(V,H))$$

denote the space of finitely additive $\gamma$ continuous $\mathcal{L}(V,H)$ valued vector measures having bounded semivariation contained in the space of finitely additive measures with values in $\mathcal{L}(V,H)$. Let $\mathcal{S}(\gamma, V)$ denote the class of $\gamma$-measurable simple functions from $I$ to $V$. Note that the semivariation of an operator valued measure $B$ on $\sigma \in \Sigma$ is also given by

$$\parallel B \parallel_{sv} (\sigma) \equiv \sup\Big\{ \Big| \int_\sigma B(dt)f(t)\Big|_H : f \in S(\gamma, V) \subset B_\infty(\gamma, V), \parallel f \parallel_\infty \leq 1 \Big\}, \tag{5.10}$$

with $\parallel B \parallel_{sv} \equiv \parallel B \parallel_{sv} (I) \equiv \sup\{\parallel B \parallel_{sv} (\sigma), \sigma \in \Sigma\}$. The integral in the expression (5.10) is understood in the sense of Dobrakov [17, 20]. Recall that the Dobrakov integral of $f \in B_\infty(\gamma, V)$ with respect to an operator valued measure $B \in \mathcal{M}_{fabsv}(\Sigma_\gamma, \mathcal{L}(V,H))$ is given by the limit

$$\lim_{n\to\infty} \int_I B(ds) f_n(s)$$

where $f_n \in \mathcal{S}(\gamma, V)$ and $f_n(t) \longrightarrow f(t)$ $\gamma$ *a.e.*

The following result is fundamental in the proof of existence of optimal structural controls and it is also of independent interest.

**Proposition 5.2.** *The class of operator valued measures* $\mathcal{M}_{fabsv}(\Sigma_\gamma, \mathcal{L}(V, H))$ *is isometrically isomorphic to* $\mathcal{L}_{wo}(B_\infty(\gamma, V), H))$ *and this is signified by*

$$\mathcal{M}_{fabsv}(\Sigma_\gamma, \mathcal{L}(V, H)) \cong \mathcal{L}_{wo}(B_\infty(\gamma, V), H)).$$

*Proof.* The proof is based on representation theorem due to [Brooks and Lewis, 15, Lemma 4.1, Theorem 4.4, Corollary 4.4.1, p154-155]. Let $B \in \mathcal{M}_{fabsv}(\Sigma_\gamma, \mathcal{L}(V, H))$ and define the operator $L_B$ on $B_\infty(\gamma, V)$ by

$$L_B f \equiv \int_I B(ds)f(s), \tag{5.11}$$

where, again the integration is understood in the sense of Dobrakov [17, 20]. Then, since $B$ has finite semivariation, it follows from the definition that

$$\parallel L_B f \parallel_H \leq \parallel B \parallel_{sv} \parallel f \parallel_{B_\infty(\gamma,V)} < \infty. \tag{5.12}$$

Thus, if $B$ describes a bounded subset (in the sense of semivariation) of $\mathcal{M}_{fabsv}(\Sigma_\gamma, \mathcal{L}(V, H))$, the corresponding operator $L_B$ describes a bounded subset of $\mathcal{L}(B_\infty(\gamma, V), H)$. Since $B_\infty(\gamma, V)$ is a Banach space and $H$ is a Hilbert space, it is well known that any closed bounded convex subset of $\mathcal{L}(B_\infty(\gamma, V), H)$ is compact in the weak operator topology $\tau_{wo} \equiv wo$. Thus $L_B \in \mathcal{L}_{wo}(B_\infty(\gamma, V), H)$ and it follows from (5.12) that its norm is dominated by the semivariation of $B$, that is,

$$\parallel L_B \parallel_{\mathcal{L}(B_\infty(\gamma,V),H)} \leq \parallel B \parallel_{sv} .$$

Conversely, let $L \in \mathcal{L}(B_\infty(\gamma, V), H)$. Since the range space is a Hilbert space $H$, the operator $L$ maps every bounded set of $B_\infty(\gamma, V)$ into a relatively weakly compact set in $H$ and so the operator $L$ is weakly compact. Then it follows from generalized Riesz representation theorem due to Brooks and Lewis [15, Corollary 4.4.1] that there exists a unique $B \in \mathcal{M}_{fabsv}(\Sigma_\gamma, \mathcal{L}(V, H))$, determined by $L$ alone, such that

$$Lf = \int_I B(ds)f(s)$$

for every $f \in B_\infty(\gamma, V)$, and $\parallel L \parallel = \parallel B \parallel_{sv}$ . This means that the operator norm of $L$ coincides with the semivariation of the corresponding representing measure $B$. Thus we have shown that every $B \in \mathcal{M}_{fabsv}(\Sigma_\gamma, \mathcal{L}(V, H))$ determines a unique bounded linear operator $L \in \mathcal{L}(B_\infty(\gamma, V), H)$ and conversely, to every $L \in \mathcal{L}(B_\infty(\gamma, V), H)$ there corresponds a unique $B \in \mathcal{M}_{fabsv}(\Sigma_\gamma, \mathcal{L}(V, H))$. Thus we have the isometric isomorphism

$$\mathcal{M}_{fabsv}(\Sigma_\gamma, \mathcal{L}(V, H)) \cong \mathcal{L}_{wo}(B_\infty(\gamma, V), H)).$$

This completes the proof. □

Using the Proposition 5.2 we can now characterize our admissible (structural) controls precisely as follows.

**Admissible Structural Controls:** We take any closed bounded convex subset $\mathcal{C}_{ad} \subset \mathcal{L}_{wo}(B_\infty(\gamma, V), H)$ containing the origin. Clearly $\mathcal{C}_{ad}$ is compact in the weak operator topology $\tau_{wo}$. Then let $\mathcal{M}_{ad} \subset \mathcal{M}_{fabsv}(\Sigma_\gamma, \mathcal{L}(V, H))$ denote the isomorphic image of the set $\mathcal{C}_{ad}$. Let $\tau_m$ denote the relative topology on $\mathcal{M}_{ad}$ induced, under the isomorphism, by the weak operator topology $\tau_{wo}$. Since, under isomorphism, compactness is preserved, we conclude that $\mathcal{M}_{ad}$ is $\tau_m$ compact. We consider

$\mathcal{M}_{ad}$, furnished with this relative topology $\tau_m$, as the class of admissible structural controls.

**Lemma 5.3.** *Consider the following pair of spaces of operator valued measures*

$$\mathcal{M}_1 \equiv \mathcal{M}_{fabsv}(\Sigma_\gamma, \mathcal{L}(V,H)) \text{ and } \mathcal{M}_2 \equiv \mathcal{M}_{fabsv}(\Sigma_\gamma, \mathcal{L}(V,V^*)) \tag{5.13}$$

*and let the corresponding semivariations be denoted by* $\| \cdot \|_{sv1}$ *and* $\| \cdot \|_{sv2}$ *respectively. Consider the set* $\mathcal{M}_{ad} \subset \mathcal{M}_1$*, endowed with the* $\tau_m$ *topology as described above. Then the functional* $\Psi$ *given by* $\Psi(B) \equiv \| B \|_{sv2}$ *is lower semicontinuous on* $\mathcal{M}_1$ *with respect to the* $\tau_m$ *topology.*

*Proof.* Clearly, as vector spaces, $\mathcal{M}_1 \subset \mathcal{M}_2$. Let $B \in \mathcal{M}_{ad} \subset \mathcal{M}_1$. Let us compute its semivariation in $\mathcal{M}_2$ which is given by

$$\| B \|_{sv2} = \sup\Big\{ \| \int_I B(ds)f(s) \|_{V^*}, f \in B_\infty(\gamma, V), \| f \|_\infty \leq 1 \Big\}. \tag{5.14}$$

Let $\{B_n\} \in \mathcal{M}_{ad} \subset \mathcal{M}_1$ and suppose $B_n \xrightarrow{\tau_m} B_o$ in $\mathcal{M}_1$. Since $\mathcal{M}_{ad}$ is compact in the $\tau_m$ topology, $B_o \in \mathcal{M}_{ad}$. Then by definition, the semivariation of $B_o$, considered as an element of $\mathcal{M}_2$, is given by

$$\| B_o \|_{sv2} = \sup\Big\{ \| \int_I B_o(ds)f(s) \|_{V^*}, f \in B_\infty(\gamma, V), \| f \|_\infty \leq 1 \Big\}. \tag{5.15}$$

So, for every $\varepsilon > 0$, there exists an $f_\varepsilon \in B_\infty(\gamma, V)$ with $\| f_\varepsilon \|_\infty \leq 1$, such that

$$\begin{aligned} \| B_o \|_{sv2} &\leq (\varepsilon/2) + \| \int_I B_o(ds)f_\varepsilon(s) \|_{V^*} \\ &\leq (\varepsilon/2) + \| \int_I [B_o(ds) - B_n(ds)]f_\varepsilon(s) \|_{V^*} + \| \int_I B_n(ds)f_\varepsilon(s) \|_{V^*} \\ &\leq (\varepsilon/2) + \| \int_I [B_o(ds) - B_n(ds)]f_\varepsilon(s) \|_{V^*} + \| B_n \|_{sv2} \end{aligned} \tag{5.16}$$

for all $n \in N$. Since $B_n \xrightarrow{\tau_m} B_o$ in $\mathcal{M}_{ad} \subset \mathcal{M}_1$, it is clear that

$$\int_I [B_o(ds) - B_n(ds)]f_\varepsilon(s) \xrightarrow{w} 0 \text{ in } H \text{ as } n \to \infty.$$

Thus it follows from the compact embedding $H \hookrightarrow V^*$ that

$$\int_I [B_o(ds) - B_n(ds)]f_\varepsilon(s) \xrightarrow{s} 0 \text{ in } V^* \text{ as } n \to \infty.$$

Hence it follows from the expression (5.16) that, for any given $\varepsilon > 0$, there exists a natural number $n_\varepsilon$ such that

$$\| B_o \|_{sv2} \leq (\varepsilon/2) + (\varepsilon/2) + \| B_n \|_{sv2} \ \ \forall\, n \geq n_\varepsilon.$$

Since this is true for every $\varepsilon > 0$, we conclude that

$$\Psi(B_o) \equiv \| B_o \|_{sv2} \leq \underline{\lim} \| B_n \|_{sv2} \equiv \underline{\lim}\, \Psi(B_n).$$

This proves that the functional $\Psi$, as defined above, is lower semicontinuous on $\mathcal{M}_1$ with respect to the $\tau_m$ topology. □

**Remark 5.4.** It follows from the above result that a weakly compact subset of $\mathcal{M}_1$ is always (strongly) compact as a subset of $\mathcal{M}_2$. This implies that the embedding $\mathcal{M}_1 \hookrightarrow \mathcal{M}_2$ is compact.

In the case of operator valued measures as controls, the cost of controls should be an increasing function of either the total variation norm or the semivariation. In general, while the semivariation may be finite the total variation may be infinite. Thus it is logical to consider semivariation as the measure of control cost. However, if one wishes to impose a stronger constraint, one can always choose the total variation as the measure of cost. In that case the class of admissible structural controls is a smaller subset of $\mathcal{M}_1$.

**Theorem 5.5.** *Consider the system (4.1) with the following cost functional,*

$$J(B) \equiv \int_I \| x(B)(t) - x_d(t) \|_V^p \gamma(dt) + \int_I |x(B)(t) - x_d(t)|_H^p \gamma(dt) + \Psi(B), \tag{5.17}$$

*and suppose Theorem 5.1 holds. Let $\mathcal{M}_{ad}$, furnished with the topology $\tau_m$, denote the set of admissible controls and suppose the functional $\Psi$ is given by $\Psi(B) \equiv \| B \|_{sv2}$ . Then there exists a control $B_o \in \mathcal{M}_{ad}$ at which the functional $J$ attains its minimum.*

*Proof.* Since $\mathcal{M}_{ad}$ is a compact subset of $\mathcal{M}_{fabsv}(\Sigma_\gamma, \mathcal{L}(V,H))$ in the $\tau_m$ topology, it is bounded in semivariation. Thus, as seen in the proof of Theorem 5.1, the solution set $X \equiv \{x(B), B \in \mathcal{M}_{ad}\}$ is a bounded subset of $B_\infty(I,H) \cap L_p(\gamma, V)$. As the embedding $V \hookrightarrow H$ is continuous, it is clear that the embedding $L_p(\gamma, V) \hookrightarrow L_p(\gamma, H)$ is also continuous. Now let $B_n \xrightarrow{\tau_m} B_o$ in $\mathcal{M}_{ad}$ and let $\{x_n, x_o\}$ be the corresponding set of weak solutions of the evolution equation (4.1). Then, in view of Theorem 5.1, along a subsequence if necessary, $x_n \xrightarrow{w} x_o$ in $L_p(\gamma, V)$. Since, by virtue of Hahn-Banach theorem, the norm in any Banach space is weakly lower semi-continuous, it is easy to verify that the first term in the cost functional (5.17) is weakly lower semicontinuous. Further, it follows from the compact embedding $M_{p,q} \hookrightarrow L_p(\gamma, H)$, see [8, Theorem 3.1, p911], that along a subsequence, if necessary, $x_n \xrightarrow{s} x_o$ in $L_p(\gamma, H)$. Clearly, strong convergence of $x_n$ to $x_o$ as elements of $L_p(\gamma, H)$ implies continuity of the second term of the cost functional (5.17). Thus the sum of the first two terms, as a functional on $\mathcal{M}_{ad}$, is lower semi-continuous in the $\tau_m$ topology. It follows from Lemma 5.3 that the functional $\Psi$ is lower semicontinuous on $\mathcal{M}_{ad}$ in this topology. Thus $B \longrightarrow J(B)$, given by a finite sum of lower semicontinuous functionals, is lower semicontinuous in the $\tau_m$ topology in the sense that, as $B_n \xrightarrow{\tau_m} B_o$, we have $J(B_o) \leq \underline{\lim}\, J(B_n)$. Further, $\inf\{J(B), B \in \mathcal{M}_{ad}\} \geq 0$. Since $\mathcal{M}_{ad}$ is compact in the $\tau_m$ topology and $J$ is lower semicontinuous in this topology, it attains its minimum on it. Hence there exists a control at which $J$ attains its minimum proving the existence of an optimal control. □

**Remark 5.6.** It is interesting to note that one can use the control cost as any continuous nondecreasing function $\Phi$ of the semivariation $\| \cdot \|_{sv2}$ such as $\Psi(B) \equiv \Phi(\| B \|_{sv2})$.

## 6. Some examples

**(E1): Parabolic Problem.** A classical example of a strongly nonlinear parabolic problem representing nonlinear diffusion (for example, flow through porous media, temperature dependent conductivity in heat flow, nonlinear diffusion in plasma etc.) with homogeneous Dirichlet boundary condition is given by

$$\partial\psi(t,\xi)/\partial t - div\ \Phi(t,\nabla\psi) + a(t,\xi)\psi = (b(t,\xi),\nabla\psi), (t,\xi) \in I \times \Omega,$$
$$\psi|_{\partial\Omega}(t,\xi) = 0, (t,\xi) \in I \times \partial\Omega, \psi(0,\xi) = \phi(\xi), \xi \in \Omega,$$

where $\Omega$ is a bounded open connected domain in $R^n$ with smooth boundary $\partial\Omega$. We are interested in the measure driven version of this example including structural controls as presented below:

$$\begin{aligned}&\partial\psi(t,\xi) - div\ \Phi(dt,\nabla\psi) + a(dt,\xi)\psi = (b(dt,\xi),\nabla\psi(t,\xi)), (t,\xi) \in I \times \Omega,\\ &\psi|_{\partial\Omega}(t,\xi) = 0, (t,\xi) \in I \times \partial\Omega, \psi(0,\xi) = \phi(\xi), \xi \in \Omega,\end{aligned} \tag{6.1}$$

where $a : \Sigma \times \Omega \longrightarrow R$, and $b : \Sigma \times \Omega \longrightarrow R^n$ are set functions in the first argument defined on $\Sigma \equiv \sigma(I) \equiv$ (sigma algebra of subsets of the interval $I$) and measurable in the second. The operator $\Phi : \Sigma \times R^n \longrightarrow R^n$ is a set function with respect to the first argument and a point function in the second argument and continuous from $R^n$ to $R^n$ satisfying the following properties:

There exist two countably additive bounded nonnegative measures $\gamma(\cdot), \beta(\cdot)$ (not necessarily nonatomic) and nonnegative constants $c_1, c_2$ such that

(1) $(\Phi(\sigma,\zeta),\zeta) + \beta(\sigma)|\zeta|^2 \geq \gamma(\sigma)|\zeta|^p$ for all $\sigma \in \Sigma, \zeta \in R^n$
(2) $|\Phi(\sigma,\zeta)| \leq \gamma(\sigma)\{c_1 + c_2|\zeta|^{p-1}\}$, for all $\sigma \in \Sigma, \zeta \in R^n$
(3) $(\Phi(\sigma,\zeta) - \Phi(\sigma,\eta), \zeta - \eta) \geq 0$, for all $\sigma \in \Sigma, \zeta, \eta \in R^n$.

Let $\{p, q\}$ be the conjugate pairs as defined in section 4 and $W_0^{1,p}(\Omega), p > 2$, denote the standard $L_p$-Sobolev space with the dual $W^{-1,q}(\Omega)$. For this example the appropriate vector spaces are $V \equiv W_0^{1,p}(\Omega)$ and $V^* \equiv W^{-1,q}(\Omega)$. Since $p \geq 2$ we can take $H \equiv L_2(\Omega)$. Thus we have the required Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ with continuous, dense and compact embeddings. By use of integration by parts, it is easy to verify that the operator $A$, subject to the homogeneous Dirichlet boundary condition, defined by

$$A(\sigma,\psi) \equiv -div\ \Phi(\sigma,\nabla\psi) + a(\sigma,\cdot)\psi \tag{6.2}$$

satisfies the following properties

(a1) $A : \Sigma \times V \longrightarrow V^*$,
(a2) $\langle A(\sigma,w), w\rangle_{V^*,V} + 2\beta(\sigma)|w|_H^2 \geq \gamma(\sigma) \parallel w \parallel_V^p, \forall\ w \in V, \sigma \in \Sigma$,
(a3) there exists a nonnegative constant $c$, dependent on $c_1, c_2$ and the Lebesgue measure of the set $\Omega$, and the embedding constants $V \hookrightarrow H$ such that

$$\parallel A(\sigma,w) \parallel_{V^*} \leq c\gamma(\sigma)\{1 + \parallel w \parallel_V^{p/q}\}.$$

(a4) $\langle A(\sigma,w) - A(\sigma,v), w - v\rangle_{V^*,V} \geq 0\ \forall\ w, v \in V$.

Further, the reader can verify that $A$ is hemicontinuous from $V$ to $V^*$. Note that the operator $A$ defined above contains $p$-Laplacian as a special case. The control operator is defined as follows:

$$B(\sigma,\cdot)v(\cdot) \equiv (b(\sigma,\cdot),\nabla v(\cdot)), \sigma \in \Sigma.$$

We assume that $b : \Sigma \times \Omega \longrightarrow R^n$, is a bounded set function (signed measure) and $\gamma$ continuous in the first argument uniformly with respect to $\xi \in \Omega$, and measurable in the second argument and that there exists a finite number $\tilde{b} > 0$ such that

$$esssup\ \{|b(\sigma,\xi)|_{R^n}, \xi \in \Omega\} \leq \tilde{b}\ \gamma(\sigma)\ \forall\ \sigma \in \Sigma \text{ and } \xi \in \Omega.$$

From this it follows that

$$|\langle B(\sigma)v, h\rangle_{V^*,V}| = |(B(\sigma)v, h)_H| \leq \tilde{b}\gamma(\sigma) \parallel v \parallel_V |h|_H,\ \forall\ v \in V,\ h \in H$$

and hence $\parallel B(\sigma)v \parallel_H \leq \tilde{b}\gamma(\sigma) \parallel v \parallel_V\ \ \forall\ \sigma \in \Sigma$. Introducing the vector valued function $x(t) \equiv \psi(t,\cdot)$, the system (6.1) can be reformulated as an abstract differential equation as follows,

$$dx + A(dt, x) = B(dt)x, x_0 \equiv \phi(\cdot). \tag{6.3}$$

For the cost functional one may choose the following expression

$$J(B) \equiv \int_I \parallel \psi(t,\cdot) - \psi_d(t,\cdot) \parallel^p_{W_0^{1,p}} \gamma(dt) + \parallel \psi(t,\cdot) - \psi_d(t,\cdot) \parallel^p_{L_2(\Omega)} dt + \eta(\parallel B \parallel_{sv}) \tag{6.4}$$

where $\eta$ is a real valued nondecreasing continuous function from $[0,\infty)$ to $[0,\infty]$. The problem is to find a structural control that minimizes this functional. Since both $A$ and $B$ of this example satisfy all the properties (B1)-B(4), and the cost functional satisfies the assumptions of Theorem 5.3, it follows from the results presented in section 5, that this problem has a solution.

**(E2): Hyperbolic Problem.** Here we consider the following second order differential equation,

$$\begin{aligned} d\dot{z} + Azdt + K(dt)z + D(dt)\dot{z} = g(t,z,\dot{z})\gamma(dt), \\ z(0) = z_0, \dot{z}(0) = z_1, t \in I. \end{aligned} \tag{6.5}$$

This equation represents a large class of semilinear mechanical structures such as bridges, tall buildings, aircraft, space station etc. The operator $A \in \mathcal{L}(V,V^*)$ $K : \Sigma \longrightarrow \mathcal{L}(V,H)$ and $D : \Sigma \longrightarrow \mathcal{L}(H)$ and $g : I \times V \times H \longrightarrow V^*$. Here $A$ can be interpreted as the principal elasticity operator of the structure. For example, the standard Euler beam operator with simply supported (or cantilever or mixed) boundary conditions. The operator $K$ determines stiffness of the structure and $D$ determines its viscous damping. Defining $x \equiv (z,\dot{z})'$, we can reformulate this second order evolution equation as a first order evolution in the product space $X \equiv V \times H$ as given below,

$$dx = \mathcal{A}xdt + B(dt)x + f(t,x)\gamma(dt), x(0) = x_0, t \in I. \tag{6.6}$$

Here the operators $\{\mathcal{A}, B, f\}$ are given by

$$\mathcal{A} = \begin{bmatrix} 0 & I_H \\ -A & 0 \end{bmatrix}, B(\cdot) = \begin{bmatrix} 0 & 0 \\ -K(\cdot) & -D(\cdot) \end{bmatrix} \text{ and } f(t,x) = \begin{bmatrix} 0 \\ g(t,x_1,x_2) \end{bmatrix}.$$

Without loss of generality we assume that $A$ is positive self adjoint. As shown in details in [3, Example E1], the state space for this abstract system is the Hilbert space $X = V \times H$ with the scalar product

$$(x, y)_X = (\sqrt{A}x_1, \sqrt{A}y_1)_H + (x_2, y_2)_H$$

and the norm $\| x \|_X$ given by

$$\| x \|_X = \sqrt{\| \sqrt{A}x_1 \|_H^2 + \| x_2 \|_H^2}$$

where $\sqrt{A}$ denotes the positive square root of the positive self adjoint operator $A$. The first term of the norm represents elastic potential energy and the second term represents the Kinetic energy. It is not difficult to show (see [3]) that the operator $\mathcal{A}$ is skew adjoint and hence $i\mathcal{A}$ is self adjoint. Thus it follows from Stones theorem [13, Theorem 3.1.4, p71] that $\mathcal{A}$ generates a $C_0$-unitary group $\{U(t), t \geq 0\}$ on $X$. Then using the variation of constants formula the (mild) solution $x$ of the system (6.6) is given by the solution of the following nonlinear integral equation (on $X$)

$$x(t) = U(t)x_0 + \int_0^t U(t-s)B(ds)x(s) + \int_0^t U(t-s)f(s, x(s))\gamma(ds), t \in I. \tag{6.7}$$

For this problem, we choose the class of operator valued measures which are $\gamma$-continuous and countably additive in the strong operator topology having bounded variation denoted by $\mathcal{M}_{casbv}(\Sigma_\gamma, \mathcal{L}(X))$. Clearly, this is a closed subspace of the space $\mathcal{M}_{casbsv}(\Sigma, \mathcal{L}(X))$. Then we choose for admissible structural controls a set $\mathcal{M}_{ad}$ satisfying the following properties:

(a) It is a bounded subset of $\mathcal{M}_{casbv}(\Sigma_\gamma, \mathcal{L}(X))$,

(b) It is uniformly $\gamma$ continuous in the sense that $|B|(\cdot) << \gamma(\cdot)$ uniformly with respect to $B \in \mathcal{M}_{ad}$. Since $X$ is a Hilbert space, it has the Radon-Nikodym property (RNP). Therefore it follows from Radon-Nikodym theorem for operator valued measures [Ahmed, 2, Theorem 3.1, p288] that, for each $B \in \mathcal{M}_{ad}$, there exists a unique strongly measurable operator valued function $L$ such that

$$B(\sigma)x = \int_\sigma L(s)x \; \gamma(ds) \text{ for every } x \in X, \text{ and } \sigma \in \Sigma.$$

In view of this, the integral equation (6.7) is equivalent to the following integral equation

$$x(t) = U(t)x_0 + \int_0^t U(t-s)L(s)x(s)\gamma(ds) + \int_0^t U(t-s)f(s, x(s))\gamma(ds), t \in I. \tag{6.8}$$

Assuming that $f$ is locally Lipschitz having at most linear growth, and the operator valued function $L$ is measurable in the uniform operator topology and Bochner integrable with respect to the measure $\gamma$, one can easily prove the existence of a unique mild solution $x \in B_\infty(I, X) \subset L_\infty(I, X)$. The cost functional can be chosen as

$$J(B) = \int_I \| x(t) - x_d(t) \|_X^2 \gamma(dt) + \Psi(B), \tag{6.9}$$

where $x_d \in L_2(\gamma, X)$ is the given (desired) trajectory and $x \in L_\infty(I, X) \subset L_2(\gamma, X)$ is the mild solution of the integral equation (6.8) corresponding to $B$ and hence $L$, and $\Psi$ is given by the total variation norm of $B \in \mathcal{M}_{ad}$. It follows from standard properties of vector measures that, with respect to the total variation norm, $\mathcal{M}_{casbv}(\Sigma_\gamma, \mathcal{L}(X))$ is a Banach space. Hence the reader can easily verify that the map $B \longrightarrow x$ is continuous with respect to this topology on $\mathcal{M}_{casbv}(\Sigma_\gamma, \mathcal{L}(X))$ and the norm topology on $L_\infty(I, X)$. Thus the first term of the above cost functional is continuously dependent on $B \in \mathcal{M}_{casbv}(\Sigma_\gamma, \mathcal{L}(X))$. Considering the second term, since $\Psi(B)$ is given by the variation norm, again it follows from Hahn-Banach theorem that it is weakly lower semicontinuous. Thus, as a special case, it follows from Theorem 5.5 that this problem has an optimal structural control in the admissible class $\mathcal{M}_{ad}$.

**Remark 6.1.** If one chooses $x_d \equiv 0$ in the expression (6.9), the optimal structural control is the one that tries to minimize (or damp out) the elastic and kinetic energies in the (mechanical) system. This is specially important if the system suddenly encounters unexpected aerodynamic disturbance.

**Remark 6.2.** In case the measure $\gamma$ is nonatomic, the solution $x(B) \in C(I, X)$. In this case one may consider time optimal control problems. For example, given a target set $K \subset X$ which is closed and convex with $x(0) = \xi \notin K$, one is interested to find a control $B \in \mathcal{M}_{ad}$ that minimizes the first hitting time of the target $K$. In this case the cost functional is given by

$$J(B) = \inf\{t \in [0, \infty) : d(x(B)(t), K) = 0\},$$

where $d$ is the metric (distance) induced by the norm in $X$. This functional is also lower semicontinuous and hence attains its minimum on $\mathcal{M}_{ad}$.

**Open Problems** (P1): In order to determine the optimal structural control, one must develop necessary conditions of optimality. We leave this as an open problem. (P2): It will be interesting to extend the results of this paper to stochastic systems of the form

$$dx + A(dt, x) = B(dt)x + f(x)\gamma(dt) + \sigma(t, x)dW,$$

where $W$ is an $E$ (separable Hilbert space) valued cylindrical Wiener process and $\sigma$ is a suitable map (possibly Hilbert-Schmidt) so that $\sigma : I \times H \longrightarrow \mathcal{L}(E, H)$.

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