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INTERNAL FEEDBACK STABILIZATION OF A CAHN-HILLIARD SYSTEM WITH VISCOSITY EFFECTS

GABRIELA MARINOSCHI

Abstract. This article addresses the internal feedback stabilization of a phase field system of Cahn-Hilliard type with viscosity effects in the equation for the phase field, by using two feedback controllers acting in a subset of the domain. The stabilization technique is applied here to an integro-differential system obtained by some transformations of the original Cahn-Hilliard system. The proof of the stabilization of the linearized system is done in a complexified space, since the linearized operator is no longer self-adjoint as it was in the nonviscous case studied in [7]. For the stationary solution aimed to be stabilized there are no longer conditions limiting the magnitude of its gradient and Laplacian, as in [7]. These stand as essential differences with respect to the nonviscous case. The technique is based on the design of the feedback controller as a linear combination of the unstable modes of the corresponding linearized system and the controller is represented in a feedback form by means of an optimization method. Results are provided both for a regular potential involved in the phase field equation (the double-well potential) and for a singular potential of logarithmic type.

1. Problem statement

We address the local stabilization of the Cahn-Hilliard system (see [11]) describing the evolution of the phase field φ and chemical potential μ, in the Caginalp approach, that is coupled with the equation for the dynamics of the temperature θ (see [9, 10]), with initial data and homogeneous Neumann boundary conditions

$$(\theta + l\varphi)_t - \Delta\theta = 0, \quad \text{in} \quad (0,\infty)\times\Omega, \tag{1.1}$$

$$\varphi_t - \Delta\mu = 0, \quad \text{in} \quad (0,\infty)\times\Omega, \tag{1.2}$$

$$\mu = \tau\varphi_t - \nu\Delta\varphi + F'(\varphi) - \gamma\theta, \quad \text{in} \quad (0,\infty)\times\Omega, \tag{1.3}$$

$$\theta(0) = \theta_0,\ \varphi(0) = \varphi_0, \quad \text{in} \quad \Omega, \tag{1.4}$$

$$\frac{\partial\theta}{\partial\nu} = \frac{\partial\varphi}{\partial\nu} = \frac{\partial\mu}{\partial\nu} = 0, \quad \text{on} \quad (0,\infty)\times\partial\Omega. \tag{1.5}$$

This system stands for a model of the instantaneous separation of a binary mixture in its components, by taking into account the possible effects of the viscosity τ of the mixture. The space domain Ω is an open bounded connected subset of $\mathbb{R}^d$,

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$d = 1, 2, 3$, enough regular, the time $t \in (0, \infty)$, ν is the outward normal vector to the boundary, l and γ are positive constants with some physical meaning. The function F' is the derivative of a potential coming from the physical model deduced by the Gingzburg-Landau theory. Standard functions for the potential F are polynomials of even degree with a strictly positive leading coefficient, as e.g., the double-well potential

$$F(r) = \frac{(r^2 - 1)^2}{4}, \tag{1.6}$$

the logarithmic potential

$$F(r) = (1 + r)\ln(1 + r) + (1 - r)\ln(1 - r) - ar^2, \text{ for } r \in (-1, 1), \tag{1.7}$$

where a is positive and large enough to prevent the convexity and subdifferentials of convex lower semicontinuous functions (see e.g., some explanations in [12]). We shall consider the cases with the double-well potential and the logarithmic potential.

The stabilization will be investigated around a stationary solution $(\theta_\infty, \varphi_\infty)$ to (1.1)–(1.5), and involves two controllers with the support in an open subset ω of Ω, acting on the right-hand sides of equations (1.1)–(1.2). By making the function transformation

$$\theta = \sigma - l\varphi, \tag{1.8}$$

and plugging the expression of μ into (1.2), the controlled system to be studied is

$$(1 - \tau\Delta)\varphi_t + \nu\Delta^2\varphi - \Delta F'(\varphi) - \gamma l\Delta\varphi + \gamma\Delta\sigma = (1 - \tau\Delta)(f_\omega v), \quad \text{in } (0, \infty) \times \Omega, \tag{1.9}$$

$$\sigma_t - \Delta\sigma + l\Delta\varphi = f_\omega u, \quad \text{in} \quad (0, \infty) \times \Omega, \tag{1.10}$$

$$\varphi(0) = \varphi_0, \ \sigma(0) = \sigma_0 := \theta_0 + l\varphi_0, \quad \text{in} \quad \Omega, \tag{1.11}$$

$$\frac{\partial\varphi}{\partial\nu} = \frac{\partial\Delta\varphi}{\partial\nu} = \frac{\partial\sigma}{\partial\nu} = 0, \quad \text{in} \quad (0, \infty) \times \partial\Omega, \tag{1.12}$$

where the second boundary condition in (1.12) follows by (1.5).

The function f_ω is taken such that

$$f_\omega \in C_0^\infty(\Omega), \quad \text{supp } f_\omega \subset \omega, \quad f_\omega > 0 \text{ on } \omega_0, \tag{1.13}$$

where ω_0 is an open subset of positive measure of ω. The form $(1 - \tau\Delta)(f_\omega v)$ was chosen in order to ensure that this controller has the support in ω_0. This choice will be more obvious later.

The aim is to stabilize exponentially this system around a stationary solution $(\varphi_\infty, \theta_\infty)$ using the controllers computed in a feedback form, namely to show that $\lim_{t\to\infty}(\varphi(t), \theta(t)) = (\varphi_\infty, \theta_\infty)$, with an exponential rate of convergence, whether the initial datum (φ_0, θ_0) is in a certain neighborhood of $(\varphi_\infty, \theta_\infty)$.

The stabilization of the Cahn-Hilliard system in the nonviscous case, that is for $\tau = 0$, was discussed in the paper [7] for the regular potential (1.6).

We specify that the set of stationary states of the uncontrolled system (1.1)–(1.5) is not empty, because it may have any constant solution θ_∞ with some constant or not constant solution φ_∞. A discussion concerning the solutions to this stationary system with F the double-well potential is presented in [7], Lemma A1 in Appendix. The result asserts that θ_∞ is constant and $\varphi_\infty \in H^4(\Omega) \subset C^2(\overline{\Omega})$. For the logarithmic case it is enough to observe that always there are constant stationary solutions θ_∞ and φ_∞. Anyway, the proof of the existence of the solutions to the stationary system is beyond our current aim.

The proof will consist in a sequence of intermediate results referring to: the well-posedness and stabilization of the linearized system by a finite dimensional control, in Propositions 2.1 and 2.2; the representation of the feedback controller and the determination of its properties in Propositions 2.3 and 2.4; the proof of the existence of a unique solution to the *nonlinear closed loop system* (with the feedback controller expressed in terms of the solution) and the stabilization of this solution, in Theorem 2.5. The technique we shall approach is that introduced first in [17] and used then in [3–6] for Navier-Stokes equations and nonlinear parabolic systems and relies on the construction of the feedback controller as a linear combination of the unstable modes of the corresponding linearized system. We mention that in this viscous case, due to some transformations of the system, the theory will refer to an integro-differential system. Moreover, in this case the linearized system has no longer a self-adjoint operator, as in the degenerate case ($\tau = 0$) studied in [7], so that the privileges offered by such an operator cannot be used. Instead, complex eigenvalues and eigenvectors should be taken into considerations as well as the stabilization in a complexified space. Working exactly with the linearized system and not with a modified one as in [7], it is no longer necessary to impose conditions limiting the magnitude of the gradient and Laplacian of the stationary solution aimed to be stabilized (as done in [7]). These stand as essential differences with respect to the case discussed in [7]. Results will be provided first in the three-dimensional case for the system with a regular potential F.

In a separate section, the case of a logarithmic potential will be treated. This last part will include the proofs of the results previously described, presented first for a regularization F_ε of the singular potential, in Theorem 3.1. These will imply the stabilization of the system with the singular function F, on the basis of a compactness result working in one-dimension, in Theorem 3.2.

We can specify that the stabilization theorem we shall obtain for the nonlinear system corresponding to F_ε can be seen as a stand-alone result which could also work for other models, as for example, for reaction-diffusion processes with nonlinear sources.

1.1. **Intermediate considerations.** Before performing some transformations of the system (1.9)–(1.12) we write the stationary system in terms of φ_∞ and σ_∞,

$$\begin{aligned} \nu\Delta^2\varphi_\infty - \Delta F'(\varphi_\infty) - \gamma l\Delta\varphi_\infty + \gamma\Delta\sigma_\infty &= 0, \text{ in } \Omega, \\ -\Delta\sigma_\infty + l\Delta\varphi_\infty &= 0, \text{ in } \Omega, \\ \frac{\partial\varphi_\infty}{\partial\nu} = \frac{\partial\Delta\varphi_\infty}{\partial\nu} = \frac{\partial\sigma_\infty}{\partial\nu} &= 0, \text{ on } \partial\Omega. \end{aligned} \tag{1.14}$$

We denote $y := \varphi - \varphi_\infty$, $z := \sigma - \sigma_\infty$, compute the difference between system (1.9)–(1.12) and (1.14), and get

$$\begin{aligned}(1-\tau\Delta)y_t + \nu\Delta^2 y - \Delta(F'(y+\varphi_\infty) - F'(\varphi_\infty)) - \gamma l\Delta y + \gamma\Delta z \\ = (1-\tau\Delta)(f_\omega v), \quad \text{in } (0,\infty)\times\Omega,\end{aligned} \tag{1.15}$$

$$z_t - \Delta z + l\Delta y = f_\omega u, \quad \text{in} \quad (0,\infty)\times\Omega, \tag{1.16}$$

$$y(0) = y_0 = \varphi_0 - \varphi_\infty,\ z(0) = z_0 = \sigma_0 - \sigma_\infty, \quad \text{in} \quad \Omega, \tag{1.17}$$

$$\frac{\partial y}{\partial \nu} = \frac{\partial \Delta y}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, \quad \text{on} \quad (0,\infty)\times\partial\Omega. \tag{1.18}$$

Next, we write in (1.15) the Taylor expansion for $F'(y+\varphi_\infty)$ around φ_∞, obtaining

$$\begin{aligned}(1-\tau\Delta)y_t + \nu\Delta^2 y - \Delta(F''(\varphi_\infty)y) - \gamma l\Delta y + \gamma\Delta z \\ = (1-\tau\Delta)(f_\omega v) + \Delta F_r(y),\ \text{in } (0,\infty)\times\Omega,\end{aligned} \tag{1.19}$$

where the rest of second order of the Taylor expansion, represented by the nonlinear part $F_r(y)$, was moved on the right-hand side. This expansion can be written for the functions considered before (polynomial and logarithmic), under certain hypotheses in the second case (these will be specified in Section 3).

Let us consider the standard space triplet $H = L^2(\Omega)$, $V = H^1(\Omega)$, $V' = (H^1(\Omega))'$, and introduce the linear operator, $A : D(A) \subset H \to H$,

$$A = I - \tau\Delta, \quad \text{with } D(A) = \left\{ w \in H^2(\Omega);\ \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}, \tag{1.20}$$

where I is the identity operator. The operator A is linear continuous, self-adjoint and m-accretive on H. One can define its fractional powers A^α, $\alpha \geq 0$ (see e.g., [16], p. 72), with the domain $D(A^\alpha) = \{w \in H; \|A^\alpha w\|_H < \infty\}$ and the norm $\|w\|_{D(A^\alpha)} = \|A^\alpha w\|_H$. Moreover, $D(A^\alpha) \subset H^{2\alpha}(\Omega)$, with equality if and only if $2\alpha < 3/2$.

We write (1.19), (1.16)–(1.18) in terms of A (by replacing $\Delta = \frac{1}{\tau}(I - A)$) and since A is surjective we can apply A^{-1} in (1.19). The stabilization for the system (1.15)–(1.18) is reduced thus to the stabilization of the equivalent integro-differential system

$$\begin{aligned}&y_t + \tfrac{\nu}{\tau^2}(A + A^{-1} - 2)y - \tfrac{1}{\tau}(A^{-1} - I)(F''(\varphi_\infty)y) \\ &+\tfrac{\gamma}{\tau}(A^{-1} - I)z - \tfrac{\gamma l}{\tau}(A^{-1} - I)y = f_\omega v + \tfrac{1}{\tau}(A^{-1} - I)F_r(y), \\ &\qquad\qquad \text{in } (0,\infty)\times\Omega,\end{aligned} \tag{1.21}$$

$$z_t + \frac{1}{\tau}(A - I)z + \frac{l}{\tau}(I - A)y = f_\omega u, \quad \text{in} \quad (0,\infty)\times\Omega, \tag{1.22}$$

$$y(0) = y_0,\ z(0) = z_0, \quad \text{in} \quad \Omega. \tag{1.23}$$

We shall study in fact the stabilization for this system around the state $(0,0)$, for the initial datum (y_0, z_0) lying in a neighborhood of $(0,0)$. It is obvious that by making the backward transformations we obtain the stabilization result for the initial system in (φ, θ), in Theorem 3.3, Section 3.

2. Stabilization of the viscous Cahn-Hilliard system with the double-well potential

In this section we consider system (1.21)–(1.23) for F the double-well potential given by (1.6) and assume that

$$\varphi_\infty \text{ is an analytic function in } \Omega. \tag{2.1}$$

We discuss first the stabilization of the linearized system by a finite dimensional controller and then the stabilization of the nonlinear system by a feedback controller which will be constructed in Section 2.2.

For simplicity we shall denote the norm in $L^\infty(\Omega)$ by $\|\cdot\|_\infty$.

2.1. **Stabilization of the linearized system for F the double-well potential.** The linearized system extracted from (1.21)-(1.23),

$$\begin{aligned} & y_t + \frac{\nu}{\tau^2}(A + A^{-1} - 2)y - \frac{1}{\tau}(A^{-1} - I)(F''(\varphi_\infty)y) \\ & \quad + \frac{\gamma}{\tau}(A^{-1} - I)z - \frac{\gamma l}{\tau}(A^{-1} - I)y = f_\omega v, \quad \text{in} \quad (0,\infty)\times\Omega, \end{aligned} \tag{2.2}$$

$$z_t + \frac{1}{\tau}(A - I)z + \frac{l}{\tau}(I - A)y = f_\omega u, \quad \text{in} \quad (0,\infty)\times\Omega, \tag{2.3}$$

$$y(0) = y_0,\ z(0) = z_0, \quad \text{in} \quad \Omega, \tag{2.4}$$

is rewritten in the abstract form

$$\frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) = f_\omega U(t),\ \text{a.e. } t \in (0,\infty), \tag{2.5}$$

$$(y(0), z(0)) = (y_0, z_0), \tag{2.6}$$

where $U(t) = (v(t), u(t))$.
The operator $\mathcal{A}$ is defined on $D(\mathcal{A}) \subset H \times H \to H \times H$, by

$$\mathcal{A}=\begin{bmatrix} \frac{\nu}{\tau^2}(A+A^{-1}-2)-\frac{\gamma l}{\tau}(A^{-1}-I)-\frac{1}{\tau}(A^{-1}-I)(F''(\varphi_\infty)\cdot) & \frac{\gamma}{\tau}(A^{-1}-I) \\ \frac{l}{\tau}(I-A) & \frac{1}{\tau}(A-I) \end{bmatrix} \tag{2.7}$$

and has the domain

$$D(\mathcal{A}) = \left\{ w = (y,z) \in L^2(\Omega) \times L^2(\Omega);\ \mathcal{A}w \in H \times H,\ \frac{\partial y}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0 \text{ on } \Gamma \right\}.$$

We notice that, under the assumption of a regular enough domain, we have $D(\mathcal{A}) = H^2(\Omega) \times H^2(\Omega)$.

We set $\mathcal{H} = H \times H$, $\mathcal{V} = D(A^{1/2}) \times D(A^{1/2})$, $\mathcal{V}' = (D(A^{1/2}) \times D(A^{1/2}))'$, and note that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ algebraically and topologically, with compact injections. We define the scalar products on $\mathcal{H}$ and $\mathcal{V}$ by

$$\begin{aligned} ((y,z),(\psi_1,\psi_2))_{\mathcal{H}} &= \int_\Omega \left(\frac{\tau l^2}{\nu} y\psi_1 + z\psi_2 \right) dx, \\ ((y,z),(\psi_1,\psi_2))_{\mathcal{V}} &= \int_\Omega \left(\frac{\tau l^2}{\nu} (\nabla y \cdot \nabla \psi_1 + y\psi_1) + \nabla z \cdot \nabla \psi_2 + z\psi_2 \right) dx. \end{aligned}$$

Proposition 2.1. *The operator $\mathcal{A}$ is quasi m-accretive on $\mathcal{H}$ and its resolvent is compact. Moreover, $-\mathcal{A}$ generates a C_0-analytic semigroup.*

Let $(y_0, z_0) \in \mathcal{H}$ and $(v, u) \in L^2(0, T; \mathcal{H})$. Then, problem (2.5)–(2.6) has a unique solution $(y, z) \in C([0, T]; \mathcal{H}) \cap L^2(0, T; \mathcal{V}) \cap W^{1,2}(0, T; \mathcal{V}')$, for all $T > 0$, and

$$\begin{aligned} &\|(y(t), z(t))\|^2_{\mathcal{H}} + \|(y, z)\|^2_{L^2(0,T;\mathcal{V})} \\ &\leq C_\infty \left(\|(y_0, z_0)\|^2_{\mathcal{H}} + \textstyle\int_0^T \|f_\omega U(s)\|^2_{\mathcal{H}}\, ds \right), \quad \text{for all } t \in [0, T]. \end{aligned} \tag{2.8}$$

In addition, $(y, z) \in C((0, T]; \mathcal{V})$ and we have the estimate

$$\begin{aligned} &t\,\|(y(t), z(t))\|^2_{\mathcal{V}} + t\,\|(Ay(t), Az(t)\|^2_{\mathcal{H}} \\ &\leq C_\infty \left(\|(y_0, z_0)\|^2_{\mathcal{H}} + \int_0^T \|f_\omega U(s)\|^2_{\mathcal{H}}\, ds \right), \text{ for all } t \in (0, T]. \end{aligned} \tag{2.9}$$

The constant C_∞ depends on Ω, T, the problem parameters and $\|F''(\varphi_\infty)\|_\infty$.

Proof. We consider the operator, denoted still by $\mathcal{A}$, from $\mathcal{V}$ to $\mathcal{V}'$ given by

$$\begin{aligned} &\langle \mathcal{A}(y, z), (\psi_1, \psi_2)\rangle_{\mathcal{V}',\mathcal{V}} \\ &= \int_\Omega \left(l^2 \nabla y \cdot \nabla \psi_1 + \frac{l^2}{\nu} y \psi_1 \left(l + F''(\varphi_\infty) - \frac{\nu}{\tau} \right) + \frac{l^2}{\nu} \psi_1 A^{-1} y \left(\frac{\nu}{\tau} - l \right) \right) dx \\ &\quad - \int_\Omega \frac{l^2}{\nu} \psi_1 A^{-1}(F''(\varphi_\infty) y) dx \\ &\quad + \int_\Omega \left(\nabla z \cdot \nabla \psi_2 - l \nabla y \cdot \nabla \psi_2 + \frac{l^2 \gamma}{\nu} \psi_1 (A^{-1} z - z) \right) dx \end{aligned} \tag{2.10}$$

for any $(\psi_1, \psi_2) \in \mathcal{V}$. Taking into account that by the regularity results of the elliptic equations $\|A^{-1} y\|_H \leq C \|y\|_H$ we deduce that this operator is bounded from $\mathcal{V}$ to $\mathcal{V}'$,

$$\begin{aligned} \|\mathcal{A}(y, z)\|_{\mathcal{V}'} &= \sup_{(\psi_1,\psi_2)\in\mathcal{V}, \|(\psi_1,\psi_2)\|_{\mathcal{V}} \leq 1} \left| \langle \mathcal{A}(y, z), (\psi_1, \psi_2)\rangle_{\mathcal{V}',\mathcal{V}} \right| \\ &\leq C \|(y, z)\|_{\mathcal{V}}, \end{aligned} \tag{2.11}$$

and that it satisfies

$$\langle \mathcal{A}(y, z), (y, z)\rangle_{\mathcal{V}',\mathcal{V}} \geq C_1 \|(y, z)\|^2_{\mathcal{V}} - C_2 \|(y, z)\|^2_{\mathcal{H}}, \text{ for all } (y, z) \in \mathcal{V}. \tag{2.12}$$

Then, problem (2.5)-(2.6) has a unique solution (2.8) which by a straightforward computation satisfies estimate (2.9), implying thus that $(y, z) \in C((0, T]; \mathcal{V})$. We do not show all computations because they are standard. For the reader convenience, the computation of (2.9) is done by a similar technique as for proving (2.43) in the further Proposition 2.3. Moreover, by (2.12), it follows that $\lambda I + \mathcal{A}$ is coercive for some $\lambda > 0$, and so its restriction to $\mathcal{H}$ is m-accretive. Also, $\mathcal{A}$ generates a C_0-analytic semigroup and this follows by (2.12) and by Theorem 5.2 in [16], p. 61. The constants C_1, C_2 depend on the problem parameters and the norm $\|F''(\varphi_\infty)\|_\infty$. In the polynomial case, the latter reduces to in fact $\|\varphi_\infty\|_\infty$. □

We denote by λ_i and $\{(\varphi_i, \psi_i)\}_{i\geq 1}$ the complex eigenvalues and eigenfunctions of $\mathcal{A}$, that is $\mathcal{A}(\varphi_i, \psi_i) = \lambda_i(\varphi_i, \psi_i)$, $i \geq 1$. Since the resolvent of $\mathcal{A}$ is compact, there exists a finite number of eigenvalues with the real part nonpositive, $Re\lambda_i \leq 0$. Each of these eigenvalues may have the order of multiplicity l_i, $i = 1, \dots, p$. We write the sequence $Re\lambda_1 \le Re\lambda_2 \le \cdots \le Re\lambda_N \le 0$, where each eigenvalue is counted according its corresponding order of multiplicity and $N = l_1 + l_2 + \cdots + l_p$. We denote by $\overline{\lambda_i}$ (conjugated) and $\{(\varphi_i^*, \psi_i^*)\}_{i\ge 1}$ the eigenvalues and eigenfunctions of the adjoint $\mathcal{A}^*$ of $\mathcal{A}$, that is $\mathcal{A}^*(\varphi_i^*, \psi_i^*) = \overline{\lambda_i}(\varphi_i^*, \psi_i^*)$, $i \ge 1$, where

$$\mathcal{A}^* = \begin{bmatrix} \frac{\nu}{\tau^2}(A + A^{-1} - 2) - \frac{\gamma l}{\tau}(A^{-1} - I) - \frac{1}{\tau}(A^{-1} - I)(F''(\varphi_\infty)\cdot) & \frac{l}{\tau}(I - A) \\ \frac{\gamma}{\tau}(A^{-1} - I) & \frac{1}{\tau}(A - I) \end{bmatrix}. \tag{2.13}$$

The controller aimed to stabilize the linear system is searched as a linear combination of the unstable eigenvectors of the adjoint operator $\mathcal{A}^*$ (see e.g., [17], [5]), namely

$$f_\omega U(t,x) = \sum_{j=1}^{N} f_\omega Re(\widetilde{w_j}(t)(\varphi_j^*(x), \psi_j^*(x))),\ t \ge 0,\ x \in \Omega, \tag{2.14}$$

where $\widetilde{w_j} \in C([0,\infty); \mathbb{C})$, $j = 1, \dots, N$. This form replaced in (2.5) provides the *open loop linear system*

$$\frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) = \sum_{j=1}^{N} f_\omega Re(\widetilde{w_j}(t)(\varphi_j^*(x), \psi_j^*(x))), \quad \text{a.e. } t \in (0,\infty). \tag{2.15}$$

We take here an arbitrary initial condition

$$(y(0), z(0)) = (y^0, z^0). \tag{2.16}$$

Proposition 2.2. *Let the eigenvalues λ_i be semi-simple and (2.1) hold. Then, there exist $w_j \in L^2(\mathbb{R}^+)$, $j = 1, \dots, 2N$, such that the controller (2.14) stabilizes exponentially system (2.15)-(2.16), that is, its solution (y, z) satisfies*

$$\|y(t)\|_H + \|z(t)\|_H \le C_\infty e^{-k_\infty t} \left(\|y^0\|_H + \|z^0\|_H\right), \ \textit{for all } t \ge 0. \tag{2.17}$$

Moreover, we have

$$\left(\sum_{j=1}^{2N} \int_0^\infty |w_j(t)|^2\, dt\right)^{1/2} \le C\left(\|y^0\|_H + \|z^0\|_H\right), \tag{2.18}$$

where C_∞, C and k_∞ depend on the problem parameters ν, γ, l and Ω and $\|F''(\varphi_\infty)\|_\infty$.

Proof. Since the eigenfunctions are complex we have to work in the complexified space $\widetilde{\mathcal{H}} = \mathcal{H} + i\mathcal{H}$, $i = \sqrt{-1}$. Let us introduce the following system for the complex functions $(\widetilde{y}, \widetilde{z}) = (y, z) + i(Y, Z)$,

$$\frac{d}{dt}(\widetilde{y}(t), \widetilde{z}(t)) + \mathcal{A}(\widetilde{y}(t), \widetilde{z}(t)) = \sum_{j=1}^{N} f_\omega(\widetilde{w_j}(t)(\varphi_j^*(x), \psi_j^*(x)), \quad \text{a.e. } t \in (0,\infty), \tag{2.19}$$

$$(\widetilde{y}(0), \widetilde{z}(0)) = (y^0, z^0). \tag{2.20}$$

At the end of this proof, we take $(y,z) = (Re\widetilde{y}, Re\widetilde{z})$, $(v,u) = (Re\widetilde{v}, Re\widetilde{u})$ and the pair (y,z) constructed in this way turns out to be the solution to the open loop system (2.15)-(2.16) corresponding to the controller (2.14). To this end we need to give some notions (see e.g., [5]).

We consider the linear space generated by the eigenfunctions $\{(\varphi_i,\psi_i)\}_{i=1,\dots,N}$ and denote it by $\widetilde{\mathcal{H}_N} = \text{lin span}\{(\varphi_1,\psi_1),\dots,(\varphi_N,\psi_N)\}$. Also, $\widetilde{\mathcal{H}_S} = \text{lin span}$ $\{(\varphi_{N+1},\psi_{N+1}),\dots\}$ and we have the unique algebraic decomposition $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}_N} \oplus \widetilde{\mathcal{H}_S}$, which is not orthogonal. Then, we have

$$\widetilde{\mathcal{H}} \ni (\widetilde{y},\widetilde{z}) = (y_N,z_N) + (y_S,z_S),\ (y_N,z_N) \in \widetilde{\mathcal{H}_N},\ (y_S,z_S) \in \widetilde{\mathcal{H}_S}. \tag{2.21}$$

Moreover, $\widetilde{\mathcal{H}_N} = P_N\mathcal{H}$, $\widetilde{\mathcal{H}_S} = (I-P_N)\mathcal{H}$, where $P_N : \mathcal{H} \to \mathcal{H}_N$ is the algebraic projector, namely $P_N(y_1,z_1) = \sum_{i=1}^N \chi_j(\varphi_j,\psi_j)$, $\chi_j \in \mathbb{C}$. Since $\mathcal{A}$ has a compact resolvent it lets invariant $\widetilde{\mathcal{H}_N}$ and $\widetilde{\mathcal{H}_S}$, that is $\mathcal{A}\widetilde{\mathcal{H}_N} \subset \widetilde{\mathcal{H}_N}$ and $\mathcal{A}\widetilde{\mathcal{H}_S} \subset \widetilde{\mathcal{H}_S}$. Moreover, if $\mathcal{A}$ has the eigenfunctions $\{\varphi_i,\psi_i\}_{i\geq 1}$, then $\mathcal{A}_N = \mathcal{A}|_{\widetilde{\mathcal{H}_N}}$ has the eigenfunctions $\{\varphi_i,\psi_i\}_{i=1,\dots,N}$ and $\mathcal{A}_S = \mathcal{A}|_{\widetilde{\mathcal{H}_S}} = (I-P_N)\mathcal{A}$ has the eigenfunctions $\{\varphi_i,\psi_i\}_{i\geq N+1}$.

On the invariant subspace $\widetilde{\mathcal{H}_S}$ the operator $-\mathcal{A}_S$ generates a C_0-analytic semigroup, that is

$$\left\| e^{-\mathcal{A}_S t} \right\|_{\mathcal{L}(\widetilde{\mathcal{H}_S}\times\widetilde{\mathcal{H}_S})} \leq Ce^{-\widetilde{k}t},\quad \widetilde{k} = Re(\lambda_{N+1}-\lambda_N). \tag{2.22}$$

Now we split system (2.19)–(2.20) in two systems

$$\begin{gathered}\tfrac{d}{dt}(y_N(t),z_N(t)) + \mathcal{A}_N(y_N(t),z_N(t)) = P_N\Big(\textstyle\sum_{j=1}^N f_\omega \widetilde{w_j}(t)(\varphi_j^*,\psi_j^*)\Big),\\ (y_N(0),z_N(0)) = P_N(y^0,z^0)\end{gathered} \tag{2.23}$$

and

$$\begin{gathered}\tfrac{d}{dt}(y_S(t),z_S(t)) + \mathcal{A}_S(y_S(t),z_S(t)) = (I-P_N)\textstyle\sum_{j=1}^N f_\omega \widetilde{w_j}(t)(\varphi_j^*,\psi_j^*),\\ (y_S(0),z_S(0)) = (I-P_N)(y^0,z^0).\end{gathered} \tag{2.24}$$

Let $T_0 > 0$ be arbitrary, fixed. We shall prove that system (2.23) is null controllable in T_0 and the solution to (2.24) decreases exponentially to 0, as $t\to\infty$.

We begin with the first system. We write the solution to (2.19)–(2.20) as

$$(\widetilde{y}(t,x),\widetilde{z}(t,x)) = \sum_{j=1}^\infty \xi_j(t)(\varphi_j(x),\psi_j(x)),\ (t,x)\in(0,\infty)\times\Omega, \tag{2.25}$$

with $\xi_j \in C([0,\infty);\mathbb{C})$, replace it in the system, and multiply scalarly the equation by $(\overline{\varphi_i^*},\overline{\psi_i^*})$, getting

$$\xi_i' + \lambda_i\xi_i = \sum_{j=1}^N \widetilde{w_j}d_{ij},\quad \xi_i(0) = \xi_{i0},\ \text{for } i\geq 1, \tag{2.26}$$

where

$$\xi_i(0) = \xi_{i0} := \int_\Omega (y^0\overline{\varphi_j^*}(x) + z^0\overline{\psi_j^*}(x))dx,\ i\geq 1, \tag{2.27}$$

and

$$d_{ij} = \int_\Omega f_\omega(\varphi_i^* \overline{\varphi_j^*} + \psi_i^* \overline{\psi_j^*})dx,\ j = 1, \dots, N,\ i \geq 1. \tag{2.28}$$

We specify that we used the assumption that λ_i are semi-simple, which implies that the system $\{(\varphi_i, \psi_i)\}_i$, $\{(\varphi_i^*, \psi_i^*)\}_i$ is bi-orthogonal, that is $((\varphi_i, \psi_i), (\overline{\varphi_j^*}, \overline{\psi_j^*}))_{H\times H} = \delta_{ij}$. Moreover, $|d_{ij}| \leq C_\infty$ where C_∞ depends on problem data and $\|F''(\varphi_\infty)\|_\infty$, which in the case with a polynomial potential reduces to $\|\varphi_\infty\|_\infty$.

By taking $i = 1, \dots, N$ in (2.26) we get the system corresponding to (2.23). It can be written in the form

$$X' + \mathcal{M}X = D\widetilde{W},\ X(0) = X_0, \tag{2.29}$$

where

$$\mathcal{M} = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \lambda_N \end{bmatrix},\quad X = \begin{bmatrix} \xi_1 \\ \dots \\ \xi_N \end{bmatrix},\quad X_0 = \begin{bmatrix} \xi_{10} \\ \dots \\ \xi_{N0} \end{bmatrix},$$

$$D = \begin{bmatrix} d_{11} & \dots & d_{1N} \\ \dots & \dots & \dots \\ d_{N1} & \dots & d_{NN} \end{bmatrix},\quad \widetilde{W} = \begin{bmatrix} \widetilde{w_1} \\ \dots \\ \widetilde{w_N} \end{bmatrix}.$$

In the matrix $\mathcal{M}$ each λ_j is repeated according to its order of multiplicity.
Next, we prove that, for every $T_0 > 0$, system (2.26), for $i = 1, \dots, N$, is null controllable on $[0, T_0]$. To do that, we show first that the system $\{\sqrt{f_\omega}\varphi_j^*, \sqrt{f_\omega}\psi_j^*\}_{j=1}^N$ is linearly independent in ω (since supp $f_\omega \subset \omega$). To this end we assume that $\sum_{j=1}^N \alpha_j(\sqrt{f_\omega}\varphi_j^*, \sqrt{f_\omega}\psi_j^*) = 0$ in ω and prove that $\alpha_j = 0$ for $j = 1, \dots, N$. Denoting $S^* := \sum_{j=1}^N \alpha_j(\varphi_j^*, \psi_j^*)$ we have $\sqrt{f_\omega}S^* = 0$ in ω and this implies that $S^* = 0$ in the open set ω_0 because $f_\omega > 0$ on ω_0. Let us study the system $\mathcal{A}^*(Y, Z) = \overline{\lambda}(Y, Z)$, (where (Y, Z) stands for each (φ_j^*, ψ_j^*) and $\overline{\lambda}$ for each $\overline{\lambda_j}$), that is

$$\begin{aligned} &\frac{\nu}{\tau^2}(AY + A^{-1}Y - 2Y) - \frac{\gamma l}{\tau}(A^{-1}Y - Y) - \frac{1}{\tau}(A^{-1} - I)(F''(\varphi_\infty)Y) \\ &\quad + \frac{l}{\tau}(Z - AZ) = \overline{\lambda}Y, \\ &\frac{\gamma}{\tau}(A^{-1}Y - Y) + \frac{1}{\tau}(AZ - Z) = \overline{\lambda}Z. \end{aligned}$$

We apply the operator A to both equations and obtain an elliptic system. Under our assumptions, φ_∞ is analytic, F'' is a second degree polynomial, hence $F''(\varphi_\infty)$ is analytic, and so the elliptic system has an analytic solution (Y, Z) (see [15]). Thus, S^* is analytic too, whence $S^* = 0$ in Ω and so $\alpha_j = 0$ for $j = 1, \dots, N$, since the system $\left\{(\varphi_j^*, \psi_j^*)\right\}_{j=1}^N$ is linearly independent in Ω.

In conclusion, the system $\{(\sqrt{f_\omega}\varphi_i^*, \sqrt{f_\omega}\psi_i^*)\}_i$ is linearly independent on ω and so, the determinant of $[d_{ij}]_{i,j}$ is not zero. This implies that any solution to the system

$$\sum_{i=1}^N d_{ij}p_i(t) = 0,\ t \in [0, T_0],\ j = 1, \dots, N, \tag{2.30}$$

must be zero, that is $p_i(t) = 0$ for all $i = 1, \dots, N$. Using Kalman's Lemma (see e.g., [13]) it follows that there are $\widetilde{w_i} \in C([0,\infty);\mathbb{C})$ such that $\xi_i(T_0) = 0$ for all $i = 1, \dots, N$, and

$$\left(\int_0^{T_0} \sum_{i=1}^N |\widetilde{w_i}(t)|^2 \, dt\right)^{1/2} \le C \sum_{i=1}^N |\xi_{i0}| \,. \tag{2.31}$$

Thus, this finite dimensional controller steers the solution $\{\xi_j\}_{j=1}^N$ to (2.26) into the origin, at $t = T_0$, and it follows that $(y_N(T_0), z_N(T_0)) = (0,0)$, too.

Since $(\widetilde{v}, \widetilde{u}) = \sum_{j=1}^N \widetilde{w_j}(t)(\varphi_j^*, \psi_j^*)$ we get by (2.31) and (2.27) that

$$\begin{aligned} \Big(\int_0^{T_0} (\|\widetilde{v}(t)\|_H^2 + \|\widetilde{u}(t)\|_H^2) dt\Big)^{1/2} &\le C\Big(\int_0^{T_0} \sum_{i=1}^N |\widetilde{w_i}(t)|^2 \, dt\Big)^{1/2} \\ &\le C \sum_{i=1}^N |\xi_{i0}| \le C \left(\left\|y^0\right\|_H + \left\|z^0\right\|_H\right). \end{aligned} \tag{2.32}$$

Finally, we extend $\widetilde{w_i}$ (and so ξ_i) by 0 at the right of $t = T_0$, and take as a new controller

$$\widetilde{U_{ext}}(t) = \begin{cases} (\widetilde{v}(t), \widetilde{u}(t)) & \text{for } t < T_0 \\ 0 & \text{for } t \ge T_0. \end{cases} \tag{2.33}$$

Using this controller in (2.31) and (2.32) they remain valid if we make $T_0 = +\infty$.

From (2.26), by the formula of variation of constants, we have

$$\xi_i(t) = e^{-\lambda_i t}\xi_{i0} + \sum_{j=1}^N d_{ij} \int_0^t e^{-\lambda_i(t-s)} \widetilde{w_j}(s) ds, \text{ for } t \ge 0.$$

Using (2.31) and recalling that $Re\lambda_i \le 0$ for $i = 1, \dots, N$, we deduce the estimate

$$|\xi_i(t)| \le C_2 e^{-k_N t} \left(\left\|y^0\right\|_H + \left\|z^0\right\|_H\right) \le C_3 \left(\left\|y^0\right\|_H + \left\|z^0\right\|_H\right), \tag{2.34}$$

for $t \in [0, T_0]$ and $i = 1, \dots, N$. Therefore, we have

$$\|(y_N, z_N)(t)\|_{\widetilde{\mathcal{H}}} \le C_N \left\|(y^0, z^0)\right\|_{\widetilde{\mathcal{H}}}, \text{ for } t \in [0, T_0) \tag{2.35}$$

and $(y_N(t), z_N(t)) = (0,0)$ for $t \ge T_0$. All constants C, C_2, C_3, C_N in (2.32)–(2.35) depend on problem data and $\|F''(\varphi_\infty)\|_\infty$.

Now we study system (2.24). Since $-\mathcal{A}_S$ generates a C_0-analytic semigroup we have

$$(y_S, z_S)(t) = e^{-\mathcal{A}_S t}(I - P_N)(y^0, z^0) + \int_0^t e^{-\mathcal{A}_S(t-s)} \sum_{j=1}^N \widetilde{w_j}(s)(I - P_N)(\varphi_j^*, \psi_j^*) ds$$

and then, by (2.22),

$$\|(y_S, z_S)(t)\|_{\widetilde{\mathcal{H}}} \le C_4 e^{-\widetilde{k}t} \left\|(y^0, z^0)\right\|_{\widetilde{\mathcal{H}}} + \sum_{j=1}^N \int_0^{T_0} e^{-\widetilde{k}(t-s)} \left|\widetilde{w_j}(s)\right| \left\|(\varphi_j^*, \psi_j^*)\right\|_{\widetilde{\mathcal{H}}}$$

$$\leq C_4 e^{-\widetilde{k}t} \left\|(y^0, z^0)\right\|_{\widetilde{\mathcal{H}}} + C_5 e^{-\widetilde{k}t} \sum_{j=1}^{N} \int_0^{T_0} |\widetilde{w_j}(s)|\, e^{\widetilde{k}s} ds$$

$$\leq C_4 e^{-\widetilde{k}t} \left\|(y^0, z^0)\right\|_{\widetilde{\mathcal{H}}}$$

$$+ C_5 e^{-\widetilde{k}t} \sum_{j=1}^{N} \left(\int_0^{T_0} |\widetilde{w_j}(s)|^2 \, ds \right)^{1/2} \left(\int_0^{T_0} e^{2\widetilde{k}s} ds \right)^{1/2}.$$

In conclusion, by (2.31) we get

$$\|(y_S, z_S)(t)\|_{\widetilde{\mathcal{H}}} \leq C_S e^{-\widetilde{k}t} \left\|(y^0, z^0)\right\|_{\widetilde{\mathcal{H}}} \text{ for } t \geq 0, \tag{2.36}$$

and $\|(y_S, z_S)(t)\|_{\widetilde{\mathcal{H}}} \to 0$ as $t \to \infty$.

Recalling (2.21), we get by (2.35) and (2.36) that

$$\|(\widetilde{y}, \widetilde{z})(t)\|_{\widetilde{\mathcal{H}}} \leq C e^{-kt} \left\|(y^0, z^0)\right\|_{\widetilde{\mathcal{H}}}, \text{ for } t > 0 \tag{2.37}$$

and $\|(\widetilde{y}, \widetilde{z})(t)\|_{\widetilde{\mathcal{H}}} \to 0$ as $t \to \infty$. The constants C and k depend on the problem data and $\|\varphi_\infty\|_\infty$.

Now, $f_\omega U = f_\omega(v, u) = f_\omega(Re\widetilde{v}, Re\widetilde{u}) = \sum_{j=1}^N f_\omega Re(\widetilde{w_j}(t)(\varphi_j^*(x), \psi_j^*(x)))$ and so

$$v(t,x) = \sum_{j=1}^{N} (Re\widetilde{w_j}(t) Re\varphi_j^*(x) - Im\widetilde{w_j}(t) Im\varphi_j^*(x)), \tag{2.38}$$

and a similar expression takes place for $u(t, x)$ with ψ_j^* instead of φ_j^*. We observe now that in fact we have a controller consisting in a sequence of $2N$ terms, obtained by setting

$$w_j := Re\widetilde{w_j}, \text{ for } j = 1, \dots, N, \quad w_{j+N} := Im\widetilde{w_j}, \text{ for } j = 1, \dots, N. \tag{2.39}$$

Then, by (2.31) we get (2.18). Moreover,

$$\begin{aligned} v(t,x) &= \sum_{j=1}^{N} w_j(t) Re\varphi_j^*(x) - \sum_{j=1}^{N} w_{j+N}(t) Im\varphi_j^*(x), \\ u(t,x) &= \sum_{j=1}^{N} w_j(t) Re\psi_j^*(x) - \sum_{j=1}^{N} w_{j+N}(t) Im\psi_j^*(x). \end{aligned} \tag{2.40}$$

At the end we take $(y, z) = (Re\widetilde{y}, Re\widetilde{z})$ and we get by (2.37) the stabilization inequality (2.17), as claimed. □

2.2. **Construction of the feedback controller.** The feedback controller (depending on the solution (y, z)) which stabilizes exponentially the solution to (2.15)-(2.16) will be found in relation with the solution to the minimization problem

$$\Phi(y^0, z^0) = \underset{W \in L^2(0,\infty;\mathbb{R}^{2N})}{\mathrm{Min}} \left\{ \frac{1}{2} \int_0^\infty (\|Ay(t)\|_H^2 + \|Az(t)\|_H^2 + \|W(t)\|_{\mathbb{R}^{2N}}^2) dt \right\} \tag{2.41}$$

subject to (2.15)–(2.16). Here $W = (w_1, \dots, w_N, w_{N+1}, \dots, w_{2N}) \in L^2(0, \infty; \mathbb{R}^{2N})$ defined in (2.39). We note that $D(\Phi) = \{(y^0, z^0) \in H \times H;\ \Phi(y^0, z^0) < \infty\}$. Let $\mathbb{R}^+ = (0, +\infty)$.

Proposition 2.3. *For each pair* $(y^0, z^0) \in D(A^{1/2}) \times D(A^{1/2})$, *problem (2.41) has a unique optimal solution*

$$(2.42) \qquad (\{w_j^*\}_{j=1}^{2N}, y^*, z^*) \in L^2(\mathbb{R}^+; \mathbb{R}^{2N}) \times L^2(\mathbb{R}^+; D(A^{1/2})) \times L^2(\mathbb{R}^+; D(A^{1/2}))$$

and it satisfies

$$(2.43) \qquad c_1(\|A^{1/2}y^0\|_H^2 + \|A^{1/1}z^0\|_H^2) \leq \Phi(y^0, z^0) \leq c_2(\|A^{1/2}y^0\|_H^2 + \|A^{1/2}z^0\|_H^2).$$

If $(y^0, z^0) \in D(A) \times D(A)$, *then*

$$(2.44) \qquad \begin{aligned} (\|Ay^*(t)\|_H^2 + \|Az^*(t)\|_H^2) + \textstyle\int_0^t (\|A^{3/2}y^*(s)\|_H^2 + \|A^{3/2}z^*(s)\|_H^2) ds \\ \leq c_3(\|Ay^0\|_H^2 + \|A^{1/2}z^0\|_H^2), \text{ for all } t \geq 0, \end{aligned}$$

where c_1, c_2, c_3 *are positive constants (depending on* Ω, *the problem parameters and* $\|F''(\varphi_\infty)\|_\infty$*).*

Proof. For all $(y^0, z^0) \in H \times H$, it follows by Proposition 2.2, that there exist $w_j \in L^2(\mathbb{R}^+; \mathbb{R})$, $j = 1, \ldots, 2N$, such that (2.15)–(2.16) has a solution with the property (2.17) and $\{w_j\}_j$ satisfies (2.18). Since the functional in (2.41) is nonnegative, its infimum d exists and it is nonnegative. We take in (2.41) a minimizing sequence $\{W^n\}_{n\geq 1}$, $W^n = (w_1^n, \ldots, w_{2N}^n)$ such that $(u_n(t), v_n(t))$ is given by (2.40) corresponding to W^n. We have

$$(2.45) \qquad d \leq J(W^n) = \frac{1}{2}\int_0^\infty (\|Ay_n(t)\|_H^2 + \|Az_n(t)\|_H^2 + \|W^n(t)\|_{\mathbb{R}^{2N}}^2) dt \leq d + \frac{1}{n},$$

where (y_n, z_n) is the solution to (2.15)-(2.16) corresponding to W^n. By (2.45) we have on a subsequence $\{n \to \infty\}$ that $w_j^n \to w_j^*$ weakly in $L^2(\mathbb{R}^+; \mathbb{R})$, $j = 1, \ldots, 2N$, $(y_n, z_n) \to (y^*, z^*)$ weakly in $L^2(\mathbb{R}^+; D(A) \times D(A))$, and by (2.15), $\frac{d}{dt}(y_n, z_n) \to \frac{d}{dt}(y, z)$ weakly in $L^2(\mathbb{R}^+; \mathcal{H})$. Since $(u_n(t), v_n(t))$ is given by (2.40) it follows that

$$(u_n, v_n) \to (u^*, v^*) \text{ weakly in } L^2(\mathbb{R}^+; H \times H).$$

Thus, (y^*, z^*) is the solution to (2.15)–(2.16) corresponding to $W^* := (w_1^*, \ldots, w_{2N}^*)$. Moreover, passing to the limit in (2.45) we get on the basis of the weakly lower semicontinuity of J that $J(W^*) = d$.

The uniqueness follows by the fact that J is strictly convex and the state system is linear.

Next, we show (2.43). Let $(y^0, z^0) \in D(A^{1/2}) \times D(A^{1/2})$ and let α_0 a positive constant that will be determined later. We multiply (2.15), by $(Ay(t), \alpha_0 Az(t))$ scalarly in $\mathcal{H}$, keeping for simplicity on the right-hand side the form $(f_\omega v, f_\omega u)$, and obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|A^{1/2}y(t)\|_H^2+\alpha_0\|A^{1/2}z(t)\|_H^2)+\frac{\nu}{\tau^2}\|Ay(t)\|_H^2+\frac{\alpha_0}{\tau}\|Az(t)\|_H^2\\
&\quad=-\frac{\nu-\tau\gamma l}{\tau^2}\|y(t)\|_H^2-\frac{1}{\tau}(\gamma l+F''(\varphi_\infty)-\frac{2\nu}{\tau})\|A^{1/2}y(t)\|_H^2\\
&\qquad+\frac{1}{\tau}(A^{-1}(F''(\varphi_\infty)y(t)),Ay(t))_{\mathcal{H}}-\frac{\gamma}{\tau}(A^{-1}z(t),Ay(t))_{\mathcal{H}}\\
&\qquad+\frac{\gamma}{\tau}(z(t),Ay(t))_{\mathcal{H}}-\frac{\alpha_0 l}{\tau}(y(t),Az(t))_{\mathcal{H}}+\frac{\alpha_0 l}{\tau}(Ay(t),Az(t))_{\mathcal{H}}\\
&\qquad+\frac{\alpha_0}{\tau}\|A^{1/2}z(t)\|_H^2+(f_\omega v(t),Ay(t))_{\mathcal{H}}+(\alpha_0 f_\omega u(t),Az(t))_{\mathcal{H}},\\
&\hspace{20em}\text{a.e. } t>0.
\end{aligned}\tag{2.46}
$$

In the following computations we shall account on the following interpolation and embedding inequalities involving the powers of A:

$$\|A^\alpha w\|_H\le C\,\|A^{\alpha_1}w\|_H^\lambda\,\|A^{\alpha_2}w\|_H^{1-\lambda},\ \text{ for } \alpha=\lambda\alpha_1+(1-\lambda)\alpha_2,\ \lambda\in[0,1],\tag{2.47}$$

$$\|A^\alpha w\|_H\ \le\ C\|A^\beta w\|_H,\ \text{ if } \alpha<\beta,\tag{2.48}$$

$$\|A^\alpha w\|_{H^\beta(\Omega)}^2\ \le\ C\|A^{\alpha+\beta/2}w\|_H^2,\tag{2.49}$$

with C depending on the domain and the exponents. For example we have

$$c\|A^{1/2}y(t)\|_H^2\le C_\infty\|Ay(t)\|_H\|y(t)\|_H\le\delta\|Ay(t)\|_H^2+CC_\infty\|y(t)\|_H^2,$$

where $c=\frac{1}{\tau}(\frac{2\nu}{\tau}-F''(\varphi_\infty)-\gamma l)$ and C_∞ represents a constant linearly depending on $\|F''(\varphi_\infty)\|_\infty$. In a similar way there are treated all the other terms and collecting them in (2.46) we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|A^{1/2}y(t)\|_H^2+\|A^{1/2}z(t)\|_H^2)+\Big(\frac{\nu}{\tau^2}-5\delta\Big)\|Ay(t)\|_H^2+\frac{\alpha_0}{2\tau}\|Az(t)\|_H^2\\
&\quad\le C_\delta\frac{l^2\alpha_0^2}{\tau^2}\|Az(t)\|_H^2+C_{\infty,\alpha_0,\delta}(\|y(t)\|_H^2+\|z(t)\|_H^2+\|u(t)\|_H^2+\|v(t)\|_H^2)\\
&\quad\le C_\delta\frac{l^2\alpha_0^2}{\tau^2}\|Az(t)\|_H^2+C\left\{e^{-kt}(\|y^0\|_H^2+\|z^0\|_H^2)+\|u(t)\|_H^2+\|v(t)\|_H^2\right\},
\end{aligned}\tag{2.50}
$$

where $C_{\infty,\alpha_0,\delta}$ depends on C_∞, α_0 and δ. Choosing δ and α_0 small enough, integrating in time and recalling (2.17), (2.18) and (2.40) we obtain by some calculations

$$
\begin{aligned}
&\int_0^\infty\left(\|Ay(t)\|_H^2+\|Az(t)\|_H^2+\|u(t)\|_H^2+\|v(t)\|_H^2\right)dt\\
&\quad\le C_\infty(\|A^{1/2}y^0\|_H^2+\|A^{1/2}z^0\|_H^2)\le c_2(\|A^{1/2}y^0\|_H^2+\|A^{1/4}z^0\|_H^2),
\end{aligned}\tag{2.51}
$$

where c_2 depends on the problem parameters and $\|F''(\varphi_\infty)\|_\infty$ (i.e., $\|\varphi_\infty\|_\infty$ in the polynomial case).

To prove the left-hand side relation in (2.43) we write by (2.46)

$$
\begin{aligned}
&\int_0^t\Big(\frac{\nu}{\tau^2}\|Ay(s)\|_H^2+\frac{\alpha_0}{\tau}\|Az(s)\|_H^2\Big)\,ds\\
&=\frac{1}{2}\Big(\|A^{1/2}y^0\|_H^2+\|A^{1/2}z^0\|_H^2\Big)-\frac{1}{2}\Big(\|A^{1/2}y(t)\|_H^2+\|A^{1/2}z(t)\|_H^2\Big)+S,
\end{aligned}
$$

where S is the right-hand side in (2.46). We shall estimate the terms as before, but changing δ and C_δ when using the Young inequality, e.g.,

$$c\|A^{1/2}y(t)\|_H^2 \leq C_\infty \|Ay(t)\|_H \|y(t)\|_H \leq C_\delta C_\infty \|Ay(t)\|_H^2 + \delta \|y(t)\|_H^2,$$

which implies

$$c\|A^{1/2}y(t)\|_H^2 \geq -C_\delta C_\infty \|Ay(t)\|_H^2 - \delta \|y(t)\|_H^2.$$

By treating all the terms in the same way we arrive at

$$\begin{aligned}
&\int_0^t \left(\frac{\nu}{\tau^2}\|Ay(s)\|_H^2 + \frac{\alpha_0}{\tau}\|Az(s)\|_H^2\right) ds \geq \frac{1}{2}\left(\|A^{1/2}y^0\|_H^2 + \|A^{1/2}z^0\|_H^2\right) \\
&-\frac{1}{2}\left(\|A^{1/2}y(t)\|_H^2 + \|A^{1/2}z(t)\|_H^2\right) - C_\delta \int_0^t \left(\|Ay(s)\|_H^2 + \|Az(s)\|_H^2\right) ds \\
&-C_1\delta \int_0^t (\|y(s)\|_H^2 + \|z(s)\|_H^2) ds - \delta \int_0^t (\|u(s)\|_H^2 + \|v(s)\|_H^2) ds,
\end{aligned}$$

where $C_1 > 0$ follows by some algebra and δ is taken small enough. By relying on some computations involving (2.17), (2.18), for treating the last two terms on the right-hand side, and (2.48), we obtain

$$\begin{aligned}
(2.52) \qquad & C\int_0^t \left(\|Ay(s)\|_H^2 + \|Az(s)\|_H^2\right) ds \\
&\geq \frac{1}{4}\left(\|A^{1/2}y^0\|_H^2 + \|A^{1/2}z^0\|_H^2\right) - \frac{1}{2}\left(\|A^{1/2}y(t)\|_H^2 + \|A^{1/2}z(t)\|_H^2\right).
\end{aligned}$$

Since the last term on the right-hand side is a continuous L^1 function, one can take a sequence $t_j \nearrow \infty$ such that $\|A^{1/2}y(t_j)\|_H^2 + \|A^{1/2}z(t_j)\|_H^2 \to 0$. Passing to the limit in (2.52) along such a sequence we obtain

$$\int_0^\infty \left(\|Ay(s)\|_H^2 + \|Az(s)\|_H^2\right) ds \geq c_1 \left(\|A^{1/2}y^0\|_H^2 + \|A^{1/2}z^0\|_H^2\right),$$

This relation written for the optimal pair $(W^*, (y^*, z^*))$ yields the left inequality in (2.43).

Relation (2.51), valid also for the optimal pair, leads to the right-hand side of (2.43).

For proving (2.44) we multiply (2.15) by $(A^2y(t), \alpha_0 A^2 z(t))$ scalarly in $H \times H$. The computations are done in a similar way and we do no longer present them, except for the terms involving u and v and $F''(\varphi_\infty)y(t)$. We use the representation (2.40) and observe that $f_\omega v(t) \in C^\infty(\Omega)$, hence

$$\begin{aligned}
\left(f_\omega v(t), A^2 y(t)\right)_{\mathcal{H}} &= (A(f_\omega v(t)), Ay(t))_{\mathcal{H}} \\
&\leq \delta \|A^{3/2}y(t)\|_H^2 + C_\delta \|y(t)\|_H^2 + C\left(\|y^0\|_H^2 + \|z^0\|_H^2\right),
\end{aligned}$$

where

$$\|A(f_\omega v(t))\|_H^2 \leq C \sum_{j=1}^{2N} \int_0^\infty |w_j(t)|^2 \, dt \leq C\left(\|y^0\|_H^2 + \|z^0\|_H^2\right).$$

For the other term we write

$$\begin{aligned}
&\left(\tfrac{1}{\tau}(A^{-1} - I)(F''(\varphi_\infty)y(t)), A^2y(t)\right)_{\mathcal{H}} = \left(\tfrac{1}{\tau}(I - A)(F''(\varphi_\infty)y(t)), Ay(t)\right)_{\mathcal{H}} \\
&= (F''(\varphi_\infty)\Delta y(t) + 2\nabla F''(\varphi_\infty) \cdot \nabla y(t) + y\Delta F''(\varphi_\infty), Ay(t))_{\mathcal{H}} \leq C\|Ay(t)\|_H^2.
\end{aligned}$$

Finally, we are led to

$$\|Ay(t)\|_H^2 + \|Az(t)\|_H^2 + \int_0^t \left(\|A^2 y(s)\|_H^2 + \|A^2 z(s)\|_H^2\right) ds$$
$$\leq C_1 \left(\|Ay^0\|_H^2 + \|Az^0\|_H^2\right) + C_2(\|y^0\|_H^2 + \|z^0\|_H^2),$$

which written for the optimal pair implies (2.44), as claimed. □

An immediate consequence of this result is that there exists a functional $R : \mathcal{V} \to \mathcal{V}'$ such that

$$\Phi(y^0, z^0) = \frac{1}{2} \left\langle R(y^0, z^0), (y^0, z^0)\right\rangle_{\mathcal{V}',\mathcal{V}}, \tag{2.53}$$

for all $(y^0, z^0) \in \mathcal{V} = D(A^{1/2}) \times D(A^{1/2})$. As a matter of fact $R(y^0, z^0)$ is the Gâteaux derivative of the function Φ at (y^0, z^0),

$$\Phi'(y^0, z^0) = R(y^0, z^0), \text{ for all } (y^0, z^0) \in \mathcal{V}. \tag{2.54}$$

Since Φ is coercive by (2.43) we can define the restriction of R to $H \times H$ (denoted still by R) having the domain $D(R) = \{(y^0, z^0) \in \mathcal{V};\ R(y^0, z^0) \in H \times H\}$ and we have that R is self-adjoint.

In the next proposition a representation for the optimal solution to (2.41) is constructed. Before that let us introduce now the operators $B : \mathbb{R}^{2N} \to H \times H$ and $B^* : H \times H \to \mathbb{R}^{2N}$, by

$$Bp = \begin{bmatrix} f_\omega \left(\sum_{j=1}^N (p_j Re\varphi_j^* - \sum_{j=N+1}^{2N} p_j Im\varphi_j^*\right) \\ f_\omega \left(\sum_{j=1}^N (p_j Re\psi_j^* - \sum_{j=N+1}^{2N} p_j Im\psi_j^*\right) \end{bmatrix} \tag{2.55}$$
$$\text{for all } p = \begin{bmatrix} p_1 \\ \dots \\ p_{2N} \end{bmatrix} \in \mathbb{R}^{2N}$$

and

$$B^* q = \begin{bmatrix} \int_\Omega f_\omega (q_1 Re\varphi_1^* + q_2 Re\psi_1^*) dx \\ \dots \\ \int_\Omega f_\omega (q_1 Re\varphi_N^* + q_2 Re\psi_N^*) dx \\ -\int_\Omega f_\omega (q_1 Im\varphi_1^* + q_2 Im\psi_1^*) dx \\ \dots \\ -\int_\Omega f_\omega (q_1 Im\varphi_N^* + q_2 Im\psi_N^*) dx \end{bmatrix}, \tag{2.56}$$
$$\text{for all } q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in H \times H.$$

Then, (2.15)–(2.16) can be rewritten as

$$\begin{aligned} \frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) &= BW(t), \text{ a.e. } t > 0, \\ (y(0), z(0)) &= (y^0, z^0). \end{aligned} \tag{2.57}$$

Proposition 2.4. *Let* $W^*=\{w_i^*\}_{i=1}^{2N}$ *and* (y^*,z^*) *be optimal for problem (2.41), corresponding to* $(y^0,z^0)\in D(A^{1/2})\times D(A^{1/2})$. *Then* W^* *is expressed as*

$$W^*(t) = -B^*R(y^*(t),z^*(t)),\ \textit{for all}\ t>0. \tag{2.58}$$

Moreover, R has the following properties

$$2c_1\|(y^0,z^0)\|^2_{\mathcal{V}} \le \left\langle R(y^0,z^0),(y^0,z^0)\right\rangle_{\mathcal{V}'\times\mathcal{V}} \le 2c_2\|(y^0,z^0)\|^2_{\mathcal{V}}, \quad \textit{for all}\ (y^0,z^0)\in\mathcal{V}=D(A^{1/2})\times D(A^{1/2}), \tag{2.59}$$

$$\|R(y^0,z^0)\|_{H\times H} \le C_R\|(y^0,z^0)\|_{D(A)\times D(A)},\ \textit{for all}\ (y^0,z^0)\in D(A)\times D(A), \tag{2.60}$$

and R satisfies the Riccati algebraic equation

$$2\left(R(\overline{y},\overline{z}),\mathcal{A}(\overline{y},\overline{z})\right)_{H\times H} + \|B^*R(\overline{y},\overline{z})\|^2_{\mathbb{R}^{2N}} = \|A\overline{y}\|^2_H + \|Az\|^2_H, \quad \textit{for all}\ (\overline{y},\overline{z})\in D(A)\times D(A). \tag{2.61}$$

Here, c_1, c_2, C_R *are constants depending on the problem parameters,* Ω *and* $\|F''(\varphi_\infty)\|_\infty$.

Proof. Let T be positive and arbitrary. By the dynamic programming principle (see e.g., [1], p. 104), the minimization problem (2.41) is equivalent to the following problem

$$\underset{W\in L^2(0,T;\mathbb{R}^{2N})}{\mathrm{Min}}\left\{\frac{1}{2}\int_0^T(\|Ay(t)\|^2_H+\|Az(t)\|^2_H+\|W(t)\|^2_{\mathbb{R}^{2N}})dt+\Phi(y(T),z(T))\right\} \tag{2.62}$$

subject to (2.15)–(2.16), equivalently (2.57). We introduce the adjoint system

$$\begin{aligned} \tfrac{d}{dt}(p^T,q^T)(t)-\mathcal{A}^*(p^T(t),q^T(t)) &= (A^2y^*(t),A^2z^*(t)),\ \text{in}\ (0,T)\times\Omega,\\ (p^T(T),q^T(T)) &= -R(y^*(T),z^*(T)),\ \text{in}\ \Omega. \end{aligned} \tag{2.63}$$

We have used (2.54) for writing the final condition at $t=T$. We shall prove that the solution to (2.63) is independent of T. By the maximum principle in (2.62), we have that

$$W^*(t) = B^*(p^T(t),q^T(t)),\ \text{a.e.}\ t\in(0,T) \tag{2.64}$$

(see [14], p. 114; see also [1], p. 190). For proving (2.60), let $(y^0,z^0)\in D(A)\times D(A)$.

We shall prove that (p^T,q^T) is in $C([0,T];H\times H)$. For the reader's convenience we give the argument, following the idea from [6] and [7], adapted to the current problem. We define $(\widetilde{p},\widetilde{q})=\widetilde{A}(p^T,q^T)$ where $\widetilde{A}$ is the operator

$$\widetilde{A} = \begin{bmatrix} A^{-1/2} & 0\\ 0 & A^{-1/2}\end{bmatrix}.$$

By recalling (2.13) we see that $\mathcal{A}^*$ and $\widetilde{A}$ commute and so we obtain the system

$$\begin{aligned} \tfrac{d}{dt}(\widetilde{p},\widetilde{q})(t)-\mathcal{A}^*(\widetilde{p}(t),\widetilde{q}(t)) &= (A^{3/2}y^*(t),A^{3/2}z^*(t)),\ \text{in}\ (0,T)\times\Omega,\\ (\widetilde{p}(T),\widetilde{q}(T)) &= -\widetilde{A}R(y^*(T),z^*(T)),\ \text{in}\ \Omega. \end{aligned} \tag{2.65}$$

According to (2.44), we have $(A^{3/2}y^*,A^{3/2}z^*)\in L^2(0,T;H\times H)$. Since $R(y^*(T),z^*(T))\in V'\times V'$ we get $\widetilde{A}R(y^*(T),z^*(T))\in H\times H$. By applying a backward version of Proposition 2.1, formula (2.9) we see that system (2.65) has a unique solution $(\widetilde{p},\widetilde{q}) \in C([0,T);\mathcal{V})$ and so $(p^T,q^T) \in C([0,T);H \times H)$. Next, we prove the relation

$$R(y^0,z^0) = -(p^T(0),q^T(0)). \tag{2.66}$$

To this end, let us consider two solutions to (2.62), (W^*,y^*,z^*) and (W_1^*,y_1^*,z_1^*), corresponding to (y^0,z^0) and (y^1,z^1), respectively, both belonging to $D(A)\times D(A)$ and get by a straightforward computation that

$$\Phi(y^0,z^0) - \Phi(y^1,z^1) \leq -((p^T(0),q^T(0)),(y^0-y^1,z^0-z^1))_{H\times H}. \tag{2.67}$$

This implies that $-(p^T(0),q^T(0)) \in \partial\Phi(y^0,z^0)$. Since, Φ is differentiable on $D(A^{1/2})\times D(A^{1/2})$ it follows that $-(p^T(0),q^T(0)) = \Phi'(y^0,z^0) = R(y^0,z^0)$, as claimed in (2.66). Since $(p^T,q^T) \in C([0,\infty);H\times H)$, this implies that $(p^T(0),q^T(0)) \in H\times H$ and so

$$R(y^0,z^0) \in H\times H \text{ for all } (y^0,z^0) \in D(A)\times D(A). \tag{2.68}$$

On the other hand, one can easily see that R is a linear closed operator from $D(A)\times D(A)$ to $H\times H$, and so by the closed graph theorem we conclude that it is continuous (see e.g., [8], Th. 2.9, p. 37), that is $R \in \mathcal{L}(D(A)\times D(A);H\times H)$, as claimed by (2.60).

We define the restriction of R to $H\times H$, still denoted by R. Thus, its domain contains $D(A)\times D(A)$. Next, resuming (2.64) which extends by continuity at $t=T$, in V' we get

$$W^*(T) = B^*(p^T(T),q^T(T)). \tag{2.69}$$

Moreover, since $(y^*(t),z^*(t)) \in D(A)\times D(A)$ for all $t\geq 0$, by (2.44), we have by (2.68) that $R(y^*(t),z^*(t)) \in H\times H$ for all $t\geq 0$. In particular, this is true for $t=T$ and so using the final condition in (2.63) we get

$$(p^T(T),q^T(T)) = -R(y^*(T),z^*(T)) \in H\times H. \tag{2.70}$$

This relation combined with (2.69) implies

$$W^*(T) = -B^*R(y^*(T),z^*(T))$$

where T is arbitrary. Therefore, it can be written for any t, as in (2.58), as claimed.

Inequalities (2.59) follow immediately by (2.53) and (2.43).

By (2.58), we also remark that

$$f_\omega U(t) = f_\omega(v^*(t),u^*(t)) = -BB^*R(y^*(t),z^*(t)), \tag{2.71}$$

that can be used to give the expressions of u^* and v^*.

To prove (2.61) we consider $(y^0,z^0) \in D(A)\times D(A)$. By (2.41) and (2.62) written with $T=t$ we get

$$\Phi(y^*(t),z^*(t)) = \frac{1}{2}\int_t^\infty \left(\|Ay^*(s)\|_H^2 + \|Az^*(s)\|_H^2 + \|W^*(s)\|_{\mathbb{R}^{2N}}^2\right) ds, \tag{2.72}$$

for any $t\geq 0$. We note that

$$\begin{aligned}\|BB^*R(y^*(t),z^*(t))\|_{H\times H} &\leq C_1\|R(y^*(t),z^*(t))\|_{H\times H}\\ &\leq C_2\|(y^*(t),z^*(t))\|_{D(A)\times D(A)},\end{aligned}$$

since $(Ay^*(t),Az^*(t)) \in H\times H$ and $R(y^*(t),z^*(t)) \in \mathcal{H}$, a.e. $t>0$.

System (2.57) in which the right-hand side is replaced by (2.71) becomes a *closed loop* system with the right-hand side $-BB^*R(y^*(t), z^*(t))$. Since $\mathcal{A}$ satisfies (2.11) and (2.12) and BB^*R is continuous from $V \times V \to V' \times V'$, then $-(\mathcal{A} + BB^*R)$ generates a C_0-semigroup on $H \times H$ (see also Lemma A3 in [7], Appendix).

Hence, the closed loop system (2.57) has, for $(y^0, z^0) \in D(A) \times D(A) = D(\mathcal{A})$, a unique weak solution $(y^*(t), z^*(t)) \in C([0,\infty); H \times H)$ (see [2], p. 141), such that

$$\mathcal{A}(y^*(t), z^*(t)) + BB^*R(y^*(t), z^*(t)) \in L^\infty(0,\infty;\mathcal{H}),$$
$$\frac{d}{dt}(y^*(t), z^*(t)) \in L^\infty(0,\infty;\mathcal{H}).$$

But $BB^*R(y^*(t), z^*(t)) \in L^2(0,\infty;\mathcal{H})$ and so $\mathcal{A}(y^*(t), z^*(t)) \in L^2(0,\infty;\mathcal{H})$.

Now, we differentiate (2.72) with respect to t, recalling (2.53) and that R is symmetric. We get

$$\left(R(y^*(t), z^*(t)), \frac{d}{dt}(y^*(t), z^*(t))\right)_{H\times H} + \frac{1}{2}\left(\|Ay^*(t)\|_H^2 + \|Az^*(t)\|_H^2\right) + \frac{1}{2}\|B^*R(y^*(t), z^*(t))\|_{\mathbb{R}^{2N}}^2 = 0, \text{ a.e. } t > 0. \tag{2.73}$$

Replacing $\frac{d}{dt}(y^*(t), z^*(t))$ from (2.57) in (2.73) and taking into account (2.58) we have

$$(R(y^*(t), z^*(t)), -\mathcal{A}(y^*(t), z^*(t)))_{H\times H} + \frac{1}{2}\left(\|Ay^*(t)\|_H^2 + \|Az^*(t)\|_H^2\right)$$
$$+\frac{1}{2}\|B^*R(y^*(t), z^*(t))\|_{\mathbb{R}^{2N}}^2 = (R(y^*(t), z^*(t)), BB^*R(y^*(t), z^*(t))_{H\times H}, \ t \geq 0$$

which implies (2.61). □

2.3. **Feedback stabilization of the nonlinear system.** In this section we shall deal the nonlinear system (1.21)-(1.23) in which the right-hand side $(f_\omega v, f_\omega u)$ is replaced by the feedback controller determined in the previous section, that is

$$f_\omega U(t) = -BB^*R(y(t), z(t)). \tag{2.74}$$

In the abstract form the closed loop system reads

$$\frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) = \mathcal{G}(y(t)) - BB^*R(y(t), z(t)), \text{ a.e. } t > 0, \quad (y(0), z(0)) = (y_0, z_0), \tag{2.75}$$

where (y_0, z_0) is fixed now by (1.17), $\mathcal{G}(y(t)) = (G(y(t)), 0)$ and

$$G(y) = \frac{1}{\tau}(A^{-1} - I)F_r(y). \tag{2.76}$$

We recall that F_r is the rest of second order of the Taylor expansion of $F'(y + \varphi_\infty)$, which is expressed here in the integral form

$$F_r(y) = y^2 \int_0^1 (1-s)F'''(\varphi_\infty + sy)dy = y^3 + 3\varphi_\infty y^2. \tag{2.77}$$

Theorem 2.5. *Let $(y_0, z_0) \in D(A^{1/2}) \times D(A^{1/2})$. There exists ρ such that if*

$$\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/2})} \leq \rho, \tag{2.78}$$

the closed loop system (2.75) *has a unique solution*

$$(y,z) \in C([0,\infty);H\times H)\cap L^2(0,\infty;D(A)\times D(A)) \cap W^{1,2}(0,\infty;(D(A^{1/2})\times D(A^{1/2}))'), \tag{2.79}$$

which is exponentially stable, namely

$$\|y(t)\|_{D(A^{1/2})}+\|z(t)\|_{D(A^{1/2})}\le C_\infty e^{-k_\infty t}(\|y_0\|_{D(A^{1/2})}+\|z_0\|_{D(A^{1/2})}), \tag{2.80}$$

for some positive constants k_∞ *and* C_∞, *which depend on* Ω, *the problem parameters and* $\|\varphi_\infty\|_\infty$.

Proof. The proof of this theorem will address the existence and uniqueness of the solution to (2.75) and the stabilization result. The arguments are as in the proof of Theorem 3.1 in [7], but relevant modifications due to the new form of the operator $\mathcal{A}$ do impose.

First, existence and uniqueness are proved on every interval $[0,T]$ by the Schauder fixed point theorem and then they will be extended to the whole $[0,\infty)$.

Let r be a positive constant which will be specified later. For $T>0$ arbitrary fixed, we introduce the set

$$S_T=\Big\{(y,z)\in L^2(0,T;H\times H);\ \sup_{t\in(0,T)}\big(\|y(t)\|^2_{D(A^{1/2})}+\|z(t)\|^2_{D(A^{1/2})}\big) + \int_0^T(\|Ay(t)\|_H^2+\|Az(t)\|_H^2)dt\le r^2\Big\} \tag{2.81}$$

which is a convex closed subset of $L^2(0,T;D(A^{1/2})\times D(A^{1/2}))$.

We fix $(\overline{y},\overline{z})\in S_T$ and consider the Cauchy problem

$$\frac{d}{dt}(y(t),z(t))+\mathcal{A}(y(t),z(t))+BB^*R(y(t),z(t))=\mathcal{G}(\overline{y}(t)),\ \text{a.e. } t>0, \quad (y(0),z(0))=(y_0,z_0). \tag{2.82}$$

We prove that the solution to this problem exists and it is unique and define $\Psi_T: S_T\to L^2(0,T;\mathcal{V})$ by $\Psi_T(\overline{y},\overline{z})=(y,z)$ the solution to (2.82).

We assert that this mapping has the properties:

(i) $\Psi_T(S_T)\subset S_T$ provided that r is well chosen;
(ii) $\Psi_T(S_T)$ is relatively compact in $L^2(0,T;D(A^{1/2})\times D(A^{1/2}))$;
(iii) Ψ_T is continuous in the $L^2(0,T;D(A^{1/2})\times D(A^{1/2}))$ norm.

To show all these we give next the proof.

One can observe, by (2.55) and (2.56), that BB^* is continuous also from $\mathcal{V}'$ to $\mathcal{H}$,

$$\|BB^*R(y(t),z(t))\|_{\mathcal{H}}\le C\|R(y(t),z(t))\|_{\mathcal{V}'}$$

with C depending on $\|F(\varphi_\infty)\|_\infty$. Using the definition of R one can check that $\mathcal{A}+BB^*R$ satisfies similar relations as in (2.11) and (2.12). Since $(y_0,z_0)\in\mathcal{V}=D(A^{1/2})\times D(A^{1/2})$ the Cauchy problem (2.82) has a unique solution

$$(y,z)\in C([0,T];\mathcal{V})\cap W^{1,2}(0,T;\mathcal{H})\cap L^2(0,T;D(\mathcal{A})) \tag{2.83}$$

provided that $\mathcal{G}(\overline{y}) \in L^2(0,T;H \times H)$. Moreover, relation (2.83) implies that $(y(t), z(t)) \in D(A) \times D(A)$ a.e. $t \in (0,T)$ and so $R(y(t), z(t)) \in H \times H$ a.e. $t \in (0,T)$.

Recalling that $\mathcal{G}(\overline{y})$ has the first component $G(\overline{y})$ and the second zero, it remains to show that $G(\overline{y}) \in L^2(0,T;H)$. We have

$$\begin{aligned} \|G(\overline{y}(t))\|_H^2 &= \left\| \frac{1}{\tau}(A^{-1} - I) F_r(\overline{y}(t)) \right\|_H^2 \\ &\leq C_G \|F_r(\overline{y}(t))\|_H^2 \\ &\leq C_G \|\overline{y}^3(s) + 3\varphi_\infty \overline{y}^2(s)\|_H^2 \\ &\leq C_G \left(\|\overline{y}(t)\|_{D(A^{1/2})}^6 + \|\varphi_\infty\|_\infty^2 \|\overline{y}(t)\|_{D(A^{1/2})}^4 \right) \end{aligned} \tag{2.84}$$

with C_G depending on Ω and the problem parameters. Since $(\overline{y}, \overline{z}) \in S_T$ it follows

$$\begin{aligned} \int_0^t \|G(\overline{y}(s))\|_H^2 ds &\leq C_G \int_0^t \|A\overline{y}(s)\|_H^2 \|A^{1/2}\overline{y}(s)\|_H^2 \left(\|A^{1/2}\overline{y}(t)\|_H^2 + \|\varphi_\infty\|_\infty^2 \right) ds \\ &\leq C_G (r^4 + \|\varphi_\infty\|_\infty^2 r^2) \int_0^t \|A\overline{y}(s)\|_H^2 ds \\ &\leq C_G (r^6 + \|\varphi_\infty\|_\infty^2 r^4). \end{aligned} \tag{2.85}$$

To prove that $(y, z) \in S_T$ we multiply (2.82) by $R(y(t), z(t)) \in \mathcal{H}$ scalarly in $\mathcal{H}$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(R(y(t), z(t)), (y(t), z(t)))_{H\times H} + (\mathcal{A}(y(t), z(t)), R(y(t), z(t)))_{H\times H} \\ &= -\|B^* R(y(t), z(t))\|_{\mathbb{R}^{2N}}^2 + (\mathcal{G}(\overline{y}(t)), R(y(t), z(t)))_{H\times H}, \text{ a.e. } t > 0, \end{aligned}$$

and use then the Riccati equation (2.61). Recalling (2.60) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(R(y(t), z(t)), (y(t), z(t)))_{H\times H} \\ &+\frac{1}{2}\left(\|Ay(t)\|_H^2 + \|Az(t)\|_H^2 + \|B^* R(y(t), z(t))\|_{\mathbb{R}^{2N}}^2 \right) \\ &\leq \|\mathcal{G}(\overline{y}(t))\|_{H\times H} \|R(y(t), z(t))\|_{H\times H} \\ &\leq C_R \|G(\overline{y}(t))\|_H \left(\|Ay(t)\|_H^2 + \|Az(t)\|_H^2 \right)^{1/2} \\ &\leq \frac{1}{4}\left(\|Ay(t)\|_H^2 + \|Az(t)\|_H^2 \right) + 4C_R^2 \|G(\overline{y}(t))\|_H^2, \text{ a.e. } t \in (0,T). \end{aligned}$$

Integrating over $(0,t)$ and using (2.59) we get

$$\begin{aligned} &2c_1 \|y(t)\|_{D(A^{1/2})}^2 + \|z(t)\|_{D(A^{1/2})}^2 + \frac{1}{2}\int_0^t \left(\|Ay(s)\|_H^2 + \|Az(s)\|_H^2 \right) ds \\ &\leq 2c_2 \left(\|y_0\|_{D(A^{1/2})}^2 + \|z_0\|_{D(A^{1/2})}^2 \right) + 8C_R^2 \int_0^t \|G(\overline{y}(s))\|_H^2 ds, \end{aligned}$$

and further

$$\begin{aligned} &\|y(t)\|_{D(A^{1/2})}^2 + \|z(t)\|_{D(A^{1/2})}^2 + \frac{1}{4c_1}\int_0^t \left(\|Ay(s)\|_H^2 + \|Az(s)\|_H^2 \right) ds \\ &\leq \frac{c_2}{c_1}\left(\|y_0\|_{D(A^{1/2})}^2 + \|z_0\|_{D(A^{1/2})}^2 \right) + \frac{4C_R^2}{c_1}\int_0^t \|G(\overline{y}(s))\|_H^2 ds. \end{aligned} \tag{2.86}$$

We have to impose that the right-hand side is less than r^2 and using (2.85) we write

$$\frac{c_2}{c_1}\rho^2 + \overline{C_1}(r^6 + \|\varphi_\infty\|_\infty^2 r^4)) \leq r^2, \tag{2.87}$$

where $\overline{C_1} = \frac{4C_R^2 C_G}{c_1}$ depends on the problem parameters, Ω and $\|F(\varphi_\infty)\|_\infty$ (which, in this case, is proportional with $\|\varphi_\infty\|_\infty$). Relation (2.87) is satisfied e.g., by the choice $\frac{c_2}{c_1}\rho^2 = \frac{r^2}{2}$, that is,

$$\rho = r\sqrt{\frac{c_1}{2c_2}} \tag{2.88}$$

and by setting the appropriate r from the inequality

$$2\overline{C_1}r^4 + 2\overline{C_1}\|\varphi_\infty\|_\infty^2 r^2 - 1 \leq 0.$$

We take

$$0 < r \leq r_1 := \sqrt{\frac{-\overline{C_1}\|\varphi_\infty\|_\infty^2 + \sqrt{\overline{C_1}^2\|\varphi_\infty\|_\infty^4 + 2\overline{C_1}}}{2\overline{C_1}}}. \tag{2.89}$$

(ii) The fact that $\Psi_T(S_T)$ is relatively compact in $L^2(0,T;D(A^{1/2}) \times D(A^{1/2}))$ follows because by (2.83), $\frac{d}{dt}(y,z) \in L^2(0,T;H\times H)$, $(y,z) \in L^2(0,T;D(A)\times D(A))$ and $D(A) \times D(A)$ is compactly embedded in $D(A^{1/2}) \times D(A^{1/2})$.

(iii) Let $(\overline{y_n}, \overline{z_n}) \in S_T$, $(\overline{y_n}, \overline{z_n}) \to (\overline{y}, \overline{z})$ strongly in $L^2(0,T;D(A^{1/2}) \times D(A^{1/2}))$, as $n \to \infty$. We have to prove that the corresponding solution $(y_n, z_n) = \Psi_T(\overline{y_n}, \overline{z_n})$ to (2.82) converges strongly to $(y,z) = \Psi_T(\overline{y}, \overline{z})$ in $L^2(0,T;D(A^{1/2}) \times D(A^{1/2}))$. The solution (y_n, z_n) to (2.82) corresponding to $(\overline{y_n}, \overline{z_n})$ is bounded in the spaces (2.83), due to the estimate (2.86). Hence, on a subsequence $\{n \to \infty\}$ it follows that

$$(y_n, z_n) \to (y,z) \text{ weakly in } L^2(0,T;D(A) \times D(A)),$$
$$\left(\frac{dy_n}{dt}, \frac{dz_n}{dt}\right) \to \left(\frac{dy}{dt}, \frac{dz}{dt}\right) \text{ weakly in } L^2(0,T;H \times H),$$

and by the Aubin-Lions lemma

$$(y_n, z_n) \to (y,z) \text{ strongly in } L^2(0,T;D(A^{1/2}) \times D(A^{1/2}).$$

Let us to show that $G(\overline{y_n}) \to G(\overline{y})$ weakly in $L^2(0,T;H)$, by treating the terms in (2.85). We want to show that

$$\begin{aligned}\int_0^T \int_\Omega G(\overline{y_n}) - G(\overline{y}))\psi dx dt &= \int_0^T \left(\frac{1}{\tau}(A^{-1} - I)(F_r(\overline{y_n}(t) - F_r(\overline{y}(t)), \psi(t)\right)_H dt \\ &= \frac{1}{\tau}\int_0^T \int_\Omega (F_r(\overline{y_n}) - F_r(\overline{y}))(A^{-1} - I)\psi(t) dx dt \to 0, \\ &\qquad \text{for all } \psi \in L^2(0,T;H).\end{aligned}$$

Because $\{\overline{y_n}(t)\}_n$ is bounded in V for all $t \in [0,T]$ and $\overline{y_n} \to \overline{y}$ strongly in $L^2(0,T;V)$, we have $\overline{y_n}^3 \to \overline{y}^3$ and $\overline{y_n}^2 \to \overline{y}^2$ weakly in $L^2(0,T;H)$ implying $F_r(\overline{y_n}) \to F_r(\overline{y})$ weakly in $L^2(0,T;H)$.

Now, writing the weak form of (2.82) corresponding to $(\overline{y_n}, \overline{z_n})$ and passing to the limit we get that $(y,z) = \Psi_T(\overline{y}, \overline{z})$. As the same holds for any subsequence this ends the proof of the continuity of Ψ_T.

In conclusion Ψ satisfies the conditions of Schauder's theorem and has a fixed point, $\Psi(y) = y$.

The uniqueness is proved using system (1.21)–(1.23) before expanding $F'(y+\varphi_\infty)$. We consider two solutions (y_1, z_1), (y_2, z_2) and denote $y = y_1 - y_2$, $z = z_1 - z_2$. We write the equations corresponding to these solutions and subtract them

$$\begin{aligned} y_t + \frac{\nu}{\tau^2}(A + A^{-1} - 2)y - \frac{1}{\tau}(A^{-1} - I)(F'(y_1 + \varphi_\infty) - F'(y_2 + \varphi_\infty)) \\ + \frac{\gamma}{\tau}(A^{-1} - I)z - \frac{\gamma l}{\tau}(A^{-1} - I)y = f_\omega v, \text{ in } (0,\infty) \times \Omega, \\ z_t + \frac{1}{\tau}(A - I)z + \frac{l}{\tau}(I - A)y = f_\omega u, \text{ in } (0,\infty) \times \Omega, \\ y(0) = 0, \ z(0) = 0, \text{ in } \Omega, \end{aligned}$$

where $F'(r) = r^3 - r$. Next, we multiply the first equation by y and the second by λz, with λ a value to be specified later, getting

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \frac{\lambda}{2}\frac{d}{dt}\|z(t)\|_H^2 + \frac{\nu}{\tau^2}\|A^{1/2}y(t)\|_H^2 + \frac{\lambda}{\tau}\|A^{1/2}z(t)\|_H^2 \\ &\quad \le C_1(A^{-1}y(t), y(t))_H + C_2\|y(t)\|_H^2 + \frac{1}{\tau}\left((I - A^{-1})(F'(Y_1) - F'(Y_2)), Y\right)_{\mathcal{H}} \\ &\quad + C_3((A^{-1} - I)z(t), y(t))_H + C_4\|z(t)\|_H^2 \\ &\quad + C_5(y(t), z(t))_H + C_6(Ay(t), z(t))_H + (f_\omega v(t), y(t))_H + \lambda(f_\omega u(t), z(t))_H \end{aligned}$$

and proceed by estimating the terms in a similar way as in the previous calculations. Here, $Y_i := y_i(t) + \varphi_\infty$, $i = 1, 2$, $Y := Y_1 - Y_2 = y(t)$.

We present only some less evident estimates. We recall that by (2.55), (2.56), that BB^* is continuous from $\mathcal{V}'$ to $\mathcal{H}$ and write

$$\begin{aligned} (f_\omega u(t), \lambda z(t))_H &\le \lambda\|f_\omega u(t)\|_H\|z(t)\|_H \le \lambda\|f_\omega U(t)\|_{\mathcal{H}}\|z(t)\|_H \\ &= \lambda\|BB^*R(y(t), z(t))\|_{\mathcal{H}}\|z(t)\|_H \le C\lambda\|R(y(t), z(t))\|_{\mathcal{V}'}\|z(t)\|_H \\ &\le C\lambda\|(y(t), z(t))\|_{\mathcal{V}}\|z(t)\|_H \\ &\le \delta\|A^{1/2}y(t)\|_H^2 + C_\delta\lambda^2\|z(t)\|_H^2 + \frac{\lambda}{4\tau}\|A^{1/2}z(t)\|_H^2 + C\lambda\|z(t)\|_H^2, \end{aligned}$$

where C_δ denotes several constants depending on δ. Then, we have

$$\begin{aligned} \left((I - A^{-1})(F'(Y_1) - F'(Y_2)), Y\right)_H &= \left(F'(Y_1) - F'(Y_2), (I - A^{-1})Y\right)_H \\ &= \left(Y_1^3 - Y_2^3, Y\right)_{\mathcal{H}} - (Y_1 - Y_2, Y)_H \\ &\quad - \left(Y_1^3 - Y_2^3, A^{-1}Y\right)_H + \left(Y_1 - Y_2, A^{-1}Y\right)_H \end{aligned}$$

and note that the first term on the right-hand side is nonnegative. For the third scalar product we evaluate only one term, the others obeying a similar estimate:

$$\begin{aligned} \int_\Omega YY_1^2A^{-1}Y dx &\le \|Y\|_{L^4(\Omega)}\|Y_1^2\|_{L^2(\Omega)}\|A^{-1}Y\|_{L^4(\Omega)} \\ &\le \|Y\|_V\|Y_1\|_V^2\|Y\|_{V'} \le r^2\|A^{1/2}y(t)\|_H\|y(t)\|_H. \end{aligned}$$

The last term we evaluate is

$$\lambda(Ay(t), z(t))_{\mathcal{H}} = \lambda(A^{1/2}y(t), A^{1/2}z(t))_{\mathcal{H}} \leq \delta\|A^{1/2}y(t)\|_H^2 + C_\delta\lambda^2\|z(t)\|_H^2.$$

We obtain

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \frac{\lambda}{2}\frac{d}{dt}\|z(t)\|_H^2 + \left(\frac{\nu}{\tau^2} - k_0\delta\right)\|A^{1/2}y(t)\|_H^2 + \left(\frac{\lambda}{2\tau} - C_\delta\lambda^2\right)\|A^{1/2}z(t)\|_H^2 \leq C_3\|y(t)\|_H^2 + C_4\|z(t)\|_H^2,$$

where k_0 is a positive integer. For δ and λ chosen small enough, this relation implies the uniqueness.

In the proof of the well-posedness on $[0,T]$ the value r_1 does not depend on T. Thus, we can extend (y,z) from $[0,\infty)$ to $D(A^{1/2}) \times D(A^{1/2})$, by $(y(t), z(t)) = (y_T(t), z_T(t))$, for any $t \in [0,T]$, where $(y_T(t), z_T(t))$ denotes here the solution on $[0,T]$ constructed before. By the uniqueness proof, $(y_T(t), z_T(t)) = (y_{T'}(t), z_{T'}(t))$ on $[0,T] \subset [0,T']$ and so (y,z) is well defined. Moreover, by the first part of the proof, under the assumption (2.78) it follows that $(y,z) \in S_\infty$ which is S_T with $T = \infty$.

Finally, to prove the stabilization result we multiply equation (2.75) by $R(y(t), z(t))$ scalarly in $H \times H$. We get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(R(y(t), z(t)), (y(t), z(t)))_{H\times H} \\ &\quad + \frac{1}{2}\left(\|Ay(t)\|_H^2 + \|Az(t)\|_H^2 + \|B^*R(y(t), z(t))\|_{\mathbb{R}^{2N}}^2\right) \\ &\leq \|\mathcal{G}(y(t))\|_{H\times H}\|R(y(t), z(t))\|_{H\times H} \\ &\leq C_R\|G(y(t))\|_H\left(\|Ay(t)\|_H + \|Az(t)\|_H\right), \end{aligned} \tag{2.90}$$

a.e. $t \in (0,T)$. Here we used (2.60). We recall (2.84) and (2.48) and compute the right-hand side

$$\begin{aligned} I &= C_R\sqrt{C_G}\left(\|A^{1/2}y(t)\|_H^3 + \|\varphi_\infty\|_\infty^2\|A^{1/2}y(t)\|_H^2\right)\left(\|Ay(t)\|_H + \|Az(t)\|_H\right) \\ &\leq C\|Ay(t)\|_H^2\|A^{1/2}y(t)\|_H^2 + C\|A^{1/2}y(t)\|_H^2\|A^{1/2}y(t)\|_H\|Az(t)\|_H \\ &\quad + C\|\varphi_\infty\|_\infty^2\|Ay(t)\|_H^2\|A^{1/2}y(t)\|_H \\ &\quad + C\|\varphi_\infty\|_\infty^2\|Ay(t)\|_H\|A^{1/2}y(t)\|_H^{1/2}\|A^{1/2}y(t)\|_H^{1/2}\|Az(t)\|_H \\ &\leq C\|Ay(t)\|_H^2\|A^{1/2}y(t)\|_H^2 + C\|A^{1/2}y(t)\|_H^4 + C\|A^{1/2}y(t)\|_H^2\|Az(t)\|_H^2 \\ &\quad + C\|\varphi_\infty\|_\infty^2\|Ay(t)\|_H^2\|A^{1/2}y(t)\|_H \\ &\quad + C\|\varphi_\infty\|_\infty^2\|Ay(t)\|_H^2\|A^{1/2}y(t)\|_H + C\|\varphi_\infty\|_\infty^2\|A^{1/2}y(t)\|_H\|Az(t)\|_H^2. \end{aligned}$$

Since $(y,z) \in S_\infty$, we have $\|A^{1/2}y(t)\|_H \leq r$ and so

$$I \leq C_1(\|Ay(t)\|_H^2 + \|Az(t)\|_H^2)(r^2 + \|\varphi_\infty\|_\infty^2 r),$$

where C_1 depends on the problem parameters, Ω and $\|F(\varphi_\infty)\|_\infty$. Replacing in (2.90), we conclude that for a.e. t we get

$$\begin{aligned}\frac{d}{dt}(R(y(t),z(t)),(y(t),z(t)))_{H\times H} + \|Ay\|_H^2 + \|Az\|_H^2 \\ \leq C_1\left(\|Ay\|_H^2 + \|Az\|_H^2\right)\left(r^2 + \|\varphi_\infty\|_\infty^2 r\right).\end{aligned} \tag{2.91}$$

Now, we impose

$$\overline{C_2} := 1 - C_1\left(r^2 + \|\varphi_\infty\|_\infty^2 r\right) > 0$$

and so we get

$$r \leq r_2 := \frac{-\overline{C_2}\|\varphi_\infty\|_\infty^2 + \sqrt{\overline{C_2}^2\|\varphi_\infty\|_\infty^4 + 4\overline{C_2}}}{2\overline{C_2}}. \tag{2.92}$$

We fix ρ by (2.88) where

$$r \leq r_0 := \min\{r_1, r_2\}. \tag{2.93}$$

In conclusion, we have got that

$$\frac{d}{dt}(R(y(t),z(t)),(y(t),z(t)))_{H\times H} + C_2\left(\|Ay(t)\|_H^2 + \|Az(t)\|_H^2\right) \leq 0 \tag{2.94}$$

a.e. $t \in (0,\infty)$. Recalling (2.48) and (2.59), we deduce that

$$\begin{aligned}\frac{d}{dt}(R(y(t),z(t)),(y(t),z(t)))_{H\times H} \\ + C_2c_0(R(y(t),z(t)),(y(t),z(t)))_{H\times H} \leq 0, \text{ a.e. } t \in (0,\infty).\end{aligned} \tag{2.95}$$

This implies

$$(R(y(t),z(t)),(y(t),z(t)))_{H\times H} \leq e^{-2kt}(R(y_0,z_0),(y_0,z_0))_{H\times H} \tag{2.96}$$

where $k := \frac{C_2c_0}{2}$ and, owing on (2.59), we deduce that

$$c_1\|(y(t),z(t))\|^2_{D(A^{1/2})\times D(A^{1/2})} \leq c_2e^{-2kt}\|(y_0,z_0)\|^2_{D(A^{1/2})\times D(A^{1/2})}, \text{ a.e. } t>0,$$

which leads to (2.80). As seen along the calculations, the constants k, c_1, c_2 in the relation before depend on on the problem parameters and $\|F(\varphi_\infty)\|_\infty$ which reduces here to $\|\varphi_\infty\|_\infty$.

Thus, one can fix ρ by (2.88) depending on the problem parameters and on $\|\varphi_\infty\|_\infty$ such that the stationary solution is exponentially stabilized. This concludes the proof. □

3. Stabilization of the viscous Cahn-Hilliard system with the logarithmic potential

In this section we discuss the stabilization of system (1.21)–(1.23) in which F is the logarithmic potential (1.7). For this singular function we cannot follow the computations as provided before, but we need to work first with a regular potential which will be obtained by applying a cut-off function to F.

Let ε be positive fixed, $\varepsilon \in (0,1)$ and assume that

$$\varphi_\infty \text{ is analytic in } \Omega, \quad |\varphi_\infty(x)| < 1-\varepsilon \text{ for } x \in \overline{\Omega}. \tag{3.1}$$

We define $\chi_\varepsilon \in C_0^\infty(\mathbb{R})$ such that

$$\chi_\varepsilon(r) = \begin{cases} 1, \text{ for } |r| \le 1-\varepsilon \\ 0, \text{ for } |r| \ge 1-\frac{\varepsilon}{2}, \end{cases}$$

and

$$0 < \chi_\varepsilon(r) \le 1 \quad \text{for} \quad r \in (-1+\tfrac{\varepsilon}{2}, -1+\varepsilon] \cup [1-\varepsilon, 1-\tfrac{\varepsilon}{2}).$$

We also define the regularized potential

$$F_\varepsilon(r) = \begin{cases} F(r), & \text{for } r \in [1-\varepsilon, 1+\varepsilon] \\ F(r)\chi_\varepsilon(r), & \text{for } r \in (-1+\frac{\varepsilon}{2}, -1+\varepsilon] \cup [1-\varepsilon, 1-\frac{\varepsilon}{2}) \\ 0, & \text{for } |r| \ge 1-\frac{\varepsilon}{2}, \end{cases}$$

which is of class $C_0^\infty(\mathbb{R})$. The singular function F in (1.21) will be replaced by the regular function $F_\varepsilon(r)$ and similar results as those presented in Section 2 will be first proved for this new function.

We mention that, due to (3.1), F_ε and its derivatives computed at φ_∞ coincide with the derivatives of F at φ_∞, and so we can omit for them the subscript ε (that is why in (1.21) we can keep the notation $F''(\varphi_\infty)$). Moreover, F_ε and its derivatives are continuous (and bounded) on $[-1+\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}]$ and they are zero outside this interval. We also specify that the derivatives of F that will be involved in the next computations are continuous on $\left\{r; |r| \le 1-\frac{\varepsilon}{2}\right\}$. Let us denote

$$C_F'' = \|F''\|_{L^\infty(-1+\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2})}, \; C_F''' = \|F'''\|_{L^\infty(-1+\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2})},$$

and set

$$C_F = \max\{C_F'', C_F'''\}. \tag{3.2}$$

Due to the definition of F_ε it follows that

$$\|F_\varepsilon''\|_{L^\infty(-1+\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2})} \le CC_F'' \le CC_F, \quad \|F_\varepsilon'''\|_{L^\infty(-1+\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2})} \le CC_F.$$

The value C_F is a constant but we let it written as it is in order to recall its connection with the function F. In this case the nonlinear system (1.21)-(1.23) is

$$y_t + \frac{\nu}{\tau^2}(A + A^{-1} - 2)y - \frac{1}{\tau}(A^{-1} - I)(F''(\varphi_\infty)y) + \frac{\gamma}{\tau}(A^{-1} - I)z - \frac{\gamma l}{\tau}(A^{-1} - I)y$$

$$= f_\omega v + \frac{1}{\tau}(A^{-1} - I)F_{r,\varepsilon}(y), \qquad \text{in } (0,\infty) \times \Omega, \tag{3.3}$$

$$z_t + \frac{1}{\tau}(A - I)z + \frac{l}{\tau}(I - A)y = f_\omega u, , \qquad \text{in } (0,\infty) \times \Omega, \tag{3.4}$$

$$y(0) = y_0, \; z(0) = z_0, \qquad \text{in } \Omega, \tag{3.5}$$

where $F_{r,\varepsilon}$ is the rest of second order in the Taylor expansion of $F_\varepsilon'(y+\varphi_\infty)$, written in the integral form

$$F_{r,\varepsilon}(y) = y^2 \int_0^1 (1-s)F_\varepsilon'''(\varphi_\infty + sy)ds. \tag{3.6}$$

Therefore, we get the same linearized system (2.2)-(2.4) with the corresponding operator $\mathcal{A}$ given by (2.7) and consequently, all results in Sections 2.1 and 2.2, that is Propositions 2.1-2.4 remain valid, with the constants depending on the problem parameters, Ω and possibly on C_F via $\|F''(\varphi_\infty)\|_\infty$.

The nonlinear system in the closed loop form is

$$(3.7)\qquad \begin{aligned} \frac{d}{dt}(y(t),z(t))+\mathcal{A}(y(t),z(t)) &= \mathcal{G}_\varepsilon(y(t))-BB^*R(y(t),z(t)),\ \text{a.e. } t>0,\\ (y(0),z(0)) &= (y_0,z_0), \end{aligned}$$

where $\mathcal{G}_\varepsilon(y)=(G_\varepsilon(y),0)$, and

$$(3.8)\qquad G_\varepsilon(y)=\frac{1}{\tau}(A^{-1}-I)F_{r,\varepsilon}(y).$$

The main results are enunciated in Theorem 3.1, for the function F_ε and Theorem 3.2, for the function F.

Theorem 3.1. *Let $(y_0,z_0)\in D(A^{1/2})\times D(A^{1/2})$. There exists ρ (depending on the problem parameters, Ω and C_F) such that if (2.78) takes place, the closed loop system (3.7), corresponding to F_ε, has a unique solution in the spaces (2.79). The solution is exponentially stable, and satisfies (2.80), for some positive constants k_∞ and C_∞ which depend on the problem parameters, Ω and C_F.*

Proof. The arguments are the same as in the proof of Theorem 2.5, but some modifications in the computations are necessary due to the current expression of the potential. We shall point out only the computations which are different.

We define S_T as in (2.81). To prove the existence of the solution to (3.7) we proceed again by the Schauder fixed point technique, set $(\overline{y},\overline{z})\in S_T$ and introduce problem (2.82) with $\mathcal{G}$ replaced by $\mathcal{G}_\varepsilon$. We show that the new $G_\varepsilon(\overline{y})\in L^2(0,T;H)$. We have by (3.8),

$$(3.9)\qquad \|G_\varepsilon(\overline{y}(t))\|_H\le C_G\|F_{r,\varepsilon}(\overline{y}(t))\|_H\le C_GC_F\|A^{1/2}\overline{y}(t)\|_H^2,$$

where C_G denotes various constants depending on Ω and on the problem parameters. Next, since $(\overline{y},\overline{z})\in S_T$ we obtain

$$(3.10)\qquad \int_0^T\|G_\varepsilon(\overline{y}(t))\|_H^2dt\le C_GC_F^2\int_0^T\|A^{1/2}\overline{y}(t)\|_H^4dt\le C_GC_F^2r^4.$$

Therefore, problem (2.82) has a solution (2.83). To prove that $\Psi_T(S_T)\subset S_T$ we recall (2.86) and impose the condition

$$\frac{c_2}{c_1}\rho^2+\frac{4C_R^2}{c_1}C_GC_F^2r^4\le r^2$$

where $\frac{4C_R^2}{c_1}C_G:=\overline{C_2}$ depends on the problem parameters, Ω and C_F (via $\|F(\varphi_\infty)\|_\infty$). Here we can choose again

$$(3.11)\qquad \rho=r\sqrt{\frac{c_1}{2c_2}}$$

and $\overline{C_2}C_F^2r^4\le\frac{r^2}{2}$. This yields

$$(3.12)\qquad r\le r_1:=\frac{1}{C_F}\sqrt{\frac{1}{\overline{C_2}}}.$$

Further, the proof of the solution existence follows as in Theorem 2.5.

In the part concerning the uniqueness there is only one change for the term involving $F'_\varepsilon(y+\varphi_\infty)$, namely

$$\begin{aligned}&\big((I-A^{-1})(F'_\varepsilon(y_1+\varphi_\infty)-F'_\varepsilon(y_2+\varphi_\infty)),y\big)_H\\&\quad=\big(F'_\varepsilon(Y_1)-F'_\varepsilon(Y_2),(I-A^{-1})Y\big)_H\\&\quad\le\|F'_\varepsilon(Y_1)-F'_\varepsilon(Y_2)\|_H\|(I-A^{-1})Y\|_H\le CC''_F\|Y\|^2_H.\end{aligned}$$

Here, we used the fact that F'_ε is Lipschitz with the constant C''_F.

The last part concerning the stabilization result is led in the same way as before, recalling (3.9). We resume the right-hand side in (2.90) and using (2.48) we have

$$\begin{aligned}&C_R\|G_\varepsilon(y(t))\|_H\left(\|Ay(t)\|_H+\|Az(t)\|_H\right)\\&\quad\le C_1^*C_F\|A^{1/2}y(t)\|^2_H\left(\|Ay(t)\|_H+\|Az(t)\|_H\right)\\&\quad\le C_1^*C_F\|A^{1/2}y(t)\|_H\left(\|Ay(t)\|^2_H+\|A^{1/2}y(t)\|_H\|Az(t)\|_H\right)\\&\quad\le C_1^*C_Fr\left(\|Ay(t)\|^2_H+\|A^{1/2}y(t)\|^2_H+\|Az(t)\|^2_H\right).\end{aligned}$$

Finally, we obtain

$$C_R\|G_\varepsilon(y(t))\|_H\left(\|Ay(t)\|_H+\|Az(t)\|_H\right)\le 2C_1^*C_F\left(\|Ay(t)\|^2_H+\|Az(t)\|^2_H\right)r.$$

Therefore, (2.91) becomes

$$\frac{d}{dt}(R(y(t),z(t)),(y(t),z(t)))_{H\times H}+\|Ay\|^2_H+\|Az\|^2_H\le 2C_1^*C_Fr\left(\|Ay\|^2_H+\|Az\|^2_H\right)$$

and we have a new condition $2C_1^*C_Fr<1$, which provides

$$r\le r_2:=\frac{1}{2C_1^*C_F}.$$

Thus, we can fix ρ by (3.11), with r set by

$$r\le r_0:=\max\{r_1,r_2\}\tag{3.13}$$

and the proof is continued as in Theorem 2.5. The constants in the relations established in this case depend on the problem parameters and on C_F given by (3.2). □

Theorem 3.1 provides a general result for a function F_ε which together with its derivatives up to the third order are continuous.

We present the consequence for the logarithmic function F.

Theorem 3.2. *Let $\varepsilon\in(0,1)$ be arbitrary but fixed. For all pairs $(y_0,z_0)\in D(A^{1/2})\times D(A^{1/2})$ with $\|y_0\|_{D(A^{1/2})}+\|z_0\|_{D(A^{1/2})}\le\rho$, the closed loop system (3.7) corresponding to the logarithmic potential F has, in the one-dimensional case, a unique solution belonging to the spaces (2.79). The solution is exponentially stable and satisfies (2.80).*

Proof. We recall the result of Theorem 3.1 for the system (3.3)–(3.5) corresponding to F_ε. To be more specific we write (3.3) in the form

$$\begin{aligned}&y_t+\frac{\nu}{\tau^2}(A+A^{-1}-2)y-\frac{1}{\tau}(A^{-1}-I)(F'_\varepsilon(y+\varphi_\infty)-F'(\varphi_\infty))\\&\quad+\frac{\gamma}{\tau}(A^{-1}-I)z-\frac{\gamma l}{\tau}(A^{-1}-I)y=f_\omega v,\ \text{in }(0,\infty)\times\Omega,\end{aligned}\tag{3.14}$$

which is exactly that before expanding $F'_\varepsilon(y+\varphi_\infty)$ in Taylor series. Obviously, here we can write $F'(\varphi_\infty)$ instead of $F'_\varepsilon(\varphi_\infty)$, because of (3.1). We know that there exists ρ given by (3.11) such that if the initial datum is in the ball with the radius ρ we have

$$\begin{aligned}\|y(t)\|_{D(A^{1/2})} + \|z(t)\|_{D(A^{1/2})} &\leq C_\infty e^{-k_\infty t}(\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/2})}) \\ &\leq C_\infty e^{-k_\infty t}\rho.\end{aligned} \tag{3.15}$$

The argument we use further is based on the compactness $H^1(\Omega)$ in $C(\overline{\Omega})$ which holds for $d=1$. Since

$$|y(t)| \leq \|y(t)\|_{C(\overline{\Omega})} \leq C_\Omega \|y(t)\|_{D(A^{1/2})} \leq C_\Omega C_\infty e^{-k_\infty t}\rho \tag{3.16}$$

relation (3.15) implies that $|y(t)| \to 0$, as $t \to \infty$ and so, for a sufficient large t, $t > \frac{1}{k_\infty} \ln \frac{\rho C_\infty C_\Omega}{1-\varepsilon}$, we can find the values of $|y(t)|$ in a ball with the radius less than $1-\varepsilon$.

Moreover, one can set a new ρ such that the solution remains less than $1-\varepsilon$ for all $t>0$. Because $|\varphi_\infty| < 1-\varepsilon$, we can write $|\varphi_\infty| \leq 1-\varepsilon-\delta$, with $\delta \in (0, 1-\varepsilon)$ and so we can impose in (3.16) that

$$|y(t)| \leq C_\Omega C_\infty e^{-k_\infty t}\rho \leq \delta \text{ for all } t \geq 0.$$

This happens if

$$\rho \leq \frac{\delta}{C_\Omega C_\infty}$$

and, recalling (3.11) we can set a new ρ as

$$\rho \leq \min\left\{\frac{\delta}{C_\Omega C_\infty}, r\sqrt{\frac{c_1}{2c_2}}\right\}$$

with $r < r_0$ by (3.13). Then, we have

$$|y(t)+\varphi_\infty| < \delta + 1 - \varepsilon - \delta = 1-\varepsilon \text{ for all } t \geq 0,$$

and consequently we can write $F'_\varepsilon(y+\varphi_\infty) = F'(y+\varphi_\infty)$ in (3.14). In conclusion, our solution $y(t)$ actually satisfies system (3.14), (3.4)-(3.5) corresponding to the function F and we have the stabilization result in the one-dimensional case. □

The following consequence for the system in θ and φ is immediate.

Theorem 3.3. *There exists ρ such that for all pairs $(\varphi_0, \theta_0) \in D(A^{1/2}) \times D(A^{1/2})$ with*

$$\|\varphi_0 - \varphi_\infty\|_{D(A^{1/2})} + \|(\theta_0 - \theta_\infty) + l(\varphi_0 - \varphi_\infty)\|_{D(A^{1/2})} \leq \rho,$$

the closed loop system (1.9)–(1.12), with $(1^*_\omega v, 1^*_\omega u)$ replaced by (2.74), corresponding to the double-well potential (1.6), has a unique solution belonging to the spaces (2.79). The solution is exponentially stable and satisfies

$$\begin{aligned}&\|\varphi(t) - \varphi_\infty\|_{D(A^{1/2})} + \|(\theta(t) - \theta_\infty) + l(\varphi(t) - \varphi_\infty)\|_{D(A^{1/2})} \\ &\leq C_\infty e^{-k_\infty t}(\|\varphi_0 - \varphi_\infty\|_{D(A^{1/2})} + \|(\theta_0 - \theta_\infty) + l(\varphi_0 - \varphi_\infty)\|_{D(A^{1/2})}), \\ &\qquad\qquad \textit{for all } t \geq 0.\end{aligned} \tag{3.17}$$

This result remains true for the logarithmic potential (1.7), but in the one-dimensional case.

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G. Marinoschi
Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Calea 13 Septembrie 13, 050711 Bucharest, Romania
E-mail address: gabriela.marinoschi@acad.ro