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STOCHASTIC QUASILINEAR SYMMETRIC HYPERBOLIC SYSTEM PERTURBED BY LÉVY NOISE

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Abstract. In this work we establish the local and global solvability of the Cauchy problem for a stochastic quasilinear symmetric hyperbolic system perturbed by Lévy noise. The local monotonicity property of the nonlinear terms and a stochastic generalization of the localized Minty-Browder technique are exploited in the proofs.

1. Introduction

Quasilinear symmetric and symmetrizable hyperbolic systems have a wide range of applications in engineering and physics including unsteady Euler and potential equations of gas dynamics, inviscid magnetohydrodynamic (MHD) equations, shallow water equations, compressible viscoelastic fluid flow equations, and Einstein's field equations of general relativity (see for example [12, 24, 32, 35, 39, 42]). In the past, the Cauchy problem of smooth solutions for these systems has been studied using the semigroup approach and fixed point arguments (see [16–18, 22, 30, 38]). In [17], Tosio Kato established the existence and uniqueness of local in time mild solutions of the Cauchy problem for various quasilinear equations of evolution, and in [18], he established the local solvability of quasilinear symmetrizable hyperbolic systems in uniformly local Sobolev spaces using a semigroup approach. In this paper, we utilize the local monotonicity method developed in [30] to prove local and global solvability theorems for the corresponding stochastic case perturbed by Lévy noise. For multidimensional scalar conservation laws with stochastic forcing we refer the readers to [6, 7, 10, 13], etc.

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Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a given complete filtered probability space. We describe the stochastic quasilinear symmetric hyperbolic system as

$$
\left\{
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t}(t,x,\omega) + \sum_{j=1}^{n} \mathrm{A}^j(t,x,\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_j}(t,x,\omega) &= \mathrm{f}(t,x,\omega),\\
\mathbf{u}(0,x,\omega) &= \mathbf{u}_0(x,\omega),
\end{aligned}
\right.
\tag{1.1}
$$

for $(t,x,\omega) \in (0,T)\times\mathbb{R}^n\times\Omega$, where $\mathbf{u}(t,x,\omega) = (u_1(t,x,\omega),\cdots,u_m(t,x,\omega))$, $\mathrm{A}^j(\cdot,\cdot,\cdot)$'s are $m\times m$ symmetric matrices, $\mathrm{f}(t,x,\omega)$ is the external random forcing and $\mathbf{u}_0(x,\omega)$ is the $\mathscr{F}_0-$adapted initial data with $\mathbf{u}_0 \in \mathbb{L}^4(\Omega;\mathbb{H}^s(\mathbb{R}^n))$ for $s > n/2+2$. Local existence and uniqueness of mild solution for stochastic quasilinear evolution equations, including symmetric hyperbolic systems, with additive Gaussian noise in Hilbert spaces and UMD Banach Spaces is obtained in [14] and [31], respectively.

The system (1.1) perturbed by additive and multiplicative Gaussian noise is considered in [21], where the author established the local solvability of the system using a vanishing viscosity method, and a global solvability for multiplicative Gaussian noise under a smallness assumption on the initial data. In this paper, we establish the local and global solvability of the system (1.1) perturbed by multiplicative Lévy noise. The novelties of this paper are:

(*i*) stochastic quasilinear symmetric hyperbolic system with Lévy noise is studied for the first time,
(*ii*) a local monotonicity method for the solvability of such systems handles the Gaussian and Lévy case easily,
(*iii*) global solvability under a smallness assumption on initial data and certain conditions on noise coefficients.

The construction of the paper is as follows. In section 2, we formulate the Itô stochastic differential form of (1.1), discuss the hypothesis satisfied by noise coefficients, and state certain properties satisfied by the linear operator $\mathscr{A}(t,\mathbf{u}) := \sum_{j=1}^n \mathrm{A}^j(t,x,\mathbf{u})\frac{\partial}{\partial x_j}$. By defining a suitable cutoff function, a local monotonicity property of the nonlinear term $\mathscr{A}(t,\mathbf{u})\mathbf{u}$ and energy estimates for the corresponding cutoff problem are obtained in section 3. A local in time existence and uniqueness of strong solutions of (1.1) up to a stopping time is obtained in section 4 (Theorem 4.3). The global solvability results for the system (1.1) are also discussed in section 4 (Theorem 4.5).

The main theorem of this paper is

Theorem 1.1. (I) *Let the $\mathscr{F}_0-$measurable initial data $\mathbf{u}_0 \in \mathbb{L}^4(\Omega;\mathbb{H}^s(\mathbb{R}^n))$, for $s > n/2+2$. Then under Property 2.5 and (2.6) (see section 2):*

(I) There exists a unique local strong solution $(\mathbf{u},\tau)$ *to the stochastic quasilinear symmetric hyperbolic system (2.1). Here $\tau > 0$, for almost all $\omega\in\Omega$, is a stopping time with respect to $\{\mathscr{F}_t\}_{t\geq 0}$ such that*

$$\tau(\omega) = \lim_{N\to\infty}\tau_N(\omega) \textit{ for alomost all } \omega,$$

where we define, for $N \in \mathbb{N}$,

$$\tau_N(\omega) := \inf_{t\geq 0}\Big\{t : \|\mathbf{u}(t)\|_{\mathbb{H}^s} \geq N\Big\},$$

and

$$\mathbf{u} \in \mathbb{L}^4(\Omega; \mathrm{D}(0,\tau(\omega); \mathbb{H}^s(\mathbb{R}^n))),$$

where $\mathrm{D}(0,\tau(\omega);\mathbb{H}^s(\mathbb{R}^n))$ *is the space of all càdlàg paths from* $[0,\tau)$ *to* $\mathbb{H}^s(\mathbb{R}^n)$, *and* $\mathbf{u}(\cdot)$ *satisfies*

$$\begin{aligned}\mathbf{u}(t\wedge\tau_N) = \mathbf{u}_0 - \int_0^{t\wedge\tau_N}\sum_{j=1}^n \mathrm{A}^j(s,x,\mathbf{u})\frac{\partial \mathbf{u}(s)}{\partial x_j}\mathrm{d}s + \int_0^{t\wedge\tau_N}\sigma(s,\mathbf{u}(s))\mathrm{dW}(s)\\ + \int_0^{t\wedge\tau_N}\int_{\mathrm{Z}}\gamma(s-,\mathbf{u}(s-),z)\widetilde{\mathcal{N}}(\mathrm{d}s\,dz),\end{aligned} \tag{1.2}$$

for all $t \in [0,T]$ *and all* $N \geq 1$ *and for almost all* $\omega \in \Omega$.

(II) *Choose any* $0 < \delta < 1$, *and let* $\beta \geq 1$. *Then, under the assumption (4.44) (see section 4), we have*

$$\mathbb{P}\Big\{\omega\in\Omega : \tau > \delta\Big\} \geq 1 - C\delta^{\frac{2}{\beta}}\Big\{1 + 2\mathbb{E}\left[\|\mathbf{u}_0\|^2_{\mathbb{H}^s}\right]\Big\}, \tag{1.3}$$

for some positive constant C *independent of* $\mathbf{u}_0$ *and* δ.

(III) *Let* $\varepsilon > 0$ *be given. Under the assumptions (4.52), (4.53) and (4.54) (see section 4), there exists a* $\kappa(\varepsilon)$ *such that if* $\mathbb{E}\left(\|\mathbf{u}_0\|^4_{\mathbb{H}^s}\right) < \kappa(\varepsilon)$, *then*

$$\mathbb{P}\Big\{\omega\in\Omega : \tau = +\infty\Big\} > 1-\varepsilon. \tag{1.4}$$

2. Stochastic Quasilinear Symmetric Hyperbolic System

The stochastic quasilinear symmetric hyperbolic system (1.1) perturbed by Lévy noise can be written in the Itô stochastic differential form as

$$\begin{cases}\mathrm{d}\mathbf{u}(t) = -\displaystyle\sum_{j=1}^n \mathrm{A}^j(t,x,\mathbf{u})\frac{\partial\mathbf{u}(t)}{\partial x_j}\mathrm{d}t + \sigma(t,\mathbf{u}(t))\mathrm{dW}(t)\\ \qquad\qquad + \displaystyle\int_{\mathrm{Z}}\gamma(t-,\mathbf{u}(t-),z)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z),\\ \mathbf{u}(0) = \mathbf{u}_0,\end{cases} \tag{2.1}$$

where $t \in (0,T)$, $\mathbf{u}_0 \in \mathbb{L}^4(\Omega;\mathbb{H}^s(\mathbb{R}^n))$, for $s > n/2+2$, and Z is a measurable subspace of some Hilbert space (for example measurable subspaces of $\mathbb{R}^n$, $\mathbb{L}^2(\mathbb{R}^n)$ etc). In (2.1), $\mathrm{W}(\cdot)$ is an $\mathbb{L}^2-$valued Wiener process with a nuclear covariance operator Q and $\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z)$ is a compensated Poisson random measure. The processes $\mathrm{W}(\cdot)$ and $\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z) = \mathcal{N}(\mathrm{d}t,\mathrm{d}z) - \lambda(\mathrm{d}z)\mathrm{d}t$ are mutually independent. The properties of the noise coefficient $\sigma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot,\cdot)$ are given in the next subsection.

2.1. **Hypothesis.** Let $\mathbb{H}$ and $\mathbb{U}$ be Hilbert spaces and let $Q : \mathbb{H} \to \mathbb{H}$ be a symmetric, positive, trace class operator such that $Qe_j = \lambda_j e_j$, where $\{\lambda_j\}_{j=1}^{\infty}$ are the eigenvalues of Q and $\{e_j\}_{j=1}^{\infty}$ are the corresponding eigenvectors in $\mathbb{H}$ with $\mathrm{Tr}(Q) = \sum_{j=1}^{\infty}\lambda_j < +\infty$.

Definition 2.1. Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete filtered probability space. A stochastic process $\{\mathrm{W}(t)\}_{0\leq t\leq T}$ is said to be an $\mathbb{H}-$*valued* $\mathscr{F}_t-$*adapted Wiener process* with covariance operator Q if

- (i) for each non-zero $h \in \mathbb{H}$, $|Q^{\frac{1}{2}}h|^{-1}(\mathrm{W}(t), h)_{\mathbb{H}}$ is a standard one dimensional Wiener process,
- (ii) for any $h \in \mathbb{H}, (\mathrm{W}(t), h)_{\mathbb{H}}$ is a martingale adapted to $\mathscr{F}_t$.

If $\mathrm{W}(\cdot)$ is an $\mathbb{H}-$valued Wiener process with covariance operator Q with $\mathrm{Tr}\, Q < +\infty$, then $\mathrm{W}(\cdot)$ is a Gaussian process on $\mathbb{H}$ and $\mathbb{E}[\mathrm{W}(t)] = 0$, $\mathrm{Cov}[\mathrm{W}(t)] = tQ$, $t \geq 0$. The space $\mathbb{H}_0 = Q^{\frac{1}{2}}\mathbb{H}$ is a Hilbert space equipped with the inner product $(\cdot, \cdot)_0$,

$$(u, v)_0 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k}(u, e_k)_{\mathbb{L}^2}(v, e_k)_{\mathbb{L}^2} = \left(Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v\right)_{\mathbb{L}^2}, \ \forall\, u, v \in \mathbb{H}_0,$$

where $Q^{-\frac{1}{2}}$ is the pseudo-inverse of $Q^{\frac{1}{2}}$. Since Q is a trace class operator, the imbedding of $\mathbb{H}_0$ in $\mathbb{H}$ is Hilbert-Schmidt.

Let $\mathcal{L}(\mathbb{H}, \mathbb{U})$, $\mathcal{L}_2(\mathbb{H}, \mathbb{U})$ and $\mathcal{L}_Q(\mathbb{H}, \mathbb{U})$ denote the space of all bounded linear operators from $\mathbb{H}$ to $\mathbb{U}$, Hilbert-Schmidt operators from $\mathbb{H}$ to $\mathbb{U}$, and Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathbb{H}$ to $\mathbb{U}$, respectively. Let $\{e_j\}_{j=1}^{\infty}$, $\{g_j\}_{j=1}^{\infty} = \{\lambda_j^{\frac{1}{2}} e_j\}_{j=1}^{\infty}$ and $\{f_j\}_{j=1}^{\infty}$ be orthonormal bases for $\mathbb{H}, \mathbb{H}_0$ and $\mathbb{U}$ respectively. Then the space $\mathcal{L}_Q(\mathbb{H}, \mathbb{U})$ is also a separable Hilbert space, equipped with the norm

$$\begin{aligned}
\|\Psi\|^2_{\mathcal{L}_Q(\mathbb{H},\mathbb{U})} &= \sum_{h=1}^{\infty} \|\Psi g_h\|^2_{\mathbb{U}} = \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} |(\Psi g_h, f_k)_{\mathbb{U}}|^2 \\
&= \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \left|\left(\Psi \lambda_h^{\frac{1}{2}} e_h, f_k\right)_{\mathbb{U}}\right|^2 = \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \left|\left(\Psi Q^{\frac{1}{2}} e_h, f_k\right)_{\mathbb{U}}\right|^2 \\
&= \left\|\Psi Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathbb{H},\mathbb{U})} = \mathrm{Tr}\left((\Psi Q^{\frac{1}{2}})^*\Psi Q^{\frac{1}{2}}\right) = \mathrm{Tr}\left(\Psi Q^{\frac{1}{2}}(\Psi Q^{\frac{1}{2}})^*\right) \\
&= \mathrm{Tr}(\Psi Q \Psi^*), \qquad (2.2)
\end{aligned}$$

where we used the fact that for a Hilbert-Schmidt operator S, $\mathrm{Tr}(\mathrm{S}^*\mathrm{S}) = \mathrm{Tr}(\mathrm{S}\mathrm{S}^*)$. In this paper, we take $\mathbb{H} = \mathbb{L}^2(\mathbb{R}^n)$ and $\mathbb{U} = \mathbb{H}^s(\mathbb{R}^n)$.

Let us assume that the noise coefficient $\sigma(\cdot, \cdot) : [0, T] \times \mathbb{H}^s \to \mathcal{L}(\mathbb{L}^2, \mathbb{H}^s)$, for any $T > 0$ and $s \geq 0$, satisfies the following conditions:

(C.1) For all $t \in [0, T]$ and $\mathbf{u} \in \mathbb{H}^s$, there exists a positive constant $\widetilde{K}$ such that

$$\|\sigma(t, \mathbf{u})\|_{\mathcal{L}(\mathbb{L}^2,\mathbb{H}^s)} \leq \widetilde{K}(1 + \|\mathbf{u}\|_{\mathbb{H}^s}).$$

(C.2) For all $t \in [0, T]$ and $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{H}^s$, there exists a positive constant $\widetilde{L}$ such that

$$\|\sigma(t, \mathbf{u}_1) - \sigma(t, \mathbf{u}_2)\|_{\mathcal{L}(\mathbb{L}^2,\mathbb{H}^s)} \leq \widetilde{L}\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbb{H}^s}.$$

Now we obtain the main hypotheses regarding the noise coefficient $\sigma(\cdot,\cdot):[0,T]\times\mathbb{H}^s\to\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)$ from the above two conditions.

Lemma 2.2. *We have*

(*i*) *For all $t\in[0,T]$ and $\mathbf{u}\in\mathbb{H}^s$, there exists a positive constant K such that*

$$\|\sigma(t,\mathbf{u})\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}\leq K\left(1+\|\mathbf{u}\|^2_{\mathbb{H}^s}\right).$$

(*ii*) *For all $t\in[0,T]$ and $\mathbf{u}_1,\mathbf{u}_2\in\mathbb{H}^s$, there exists a positive constant L such that*

$$\|\sigma(t,\mathbf{u}_1)-\sigma(t,\mathbf{u}_2)\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}\leq L\|\mathbf{u}_1-\mathbf{u}_2\|^2_{\mathbb{H}^s}.$$

Proof. For the sequence $\{e_j\}_{j=1}^\infty$ defined as above, we have

$$\begin{aligned}\|\sigma(t,\mathbf{u})\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}&=\sum_{j=1}^\infty\|\sigma(t,\mathbf{u})Q^{1/2}e_j\|^2_{\mathbb{H}^s}=\sum_{j=1}^\infty\lambda_j\|\sigma(t,\mathbf{u})e_j\|^2_{\mathbb{H}^s}\\&\leq\sum_{j=1}^\infty\lambda_j\|\sigma(t,\mathbf{u})\|^2_{\mathcal{L}(\mathbb{L}^2,\mathbb{H}^s)}\|e_j\|^2_{\mathbb{L}^2}=\mathrm{Tr}(Q)\|\sigma(t,\mathbf{u})\|^2_{\mathcal{L}(\mathbb{L}^2,\mathbb{H}^s)}\\&\leq\mathrm{Tr}(Q)\widetilde{K}^2(1+\|\mathbf{u}\|_{\mathbb{H}^s})^2\leq K(1+\|\mathbf{u}\|^2_{\mathbb{H}^s}),\end{aligned}\tag{2.3}$$

where $K=4\mathrm{Tr}(Q)\widetilde{K}^2$. Similarly, one can prove that

$$\|\sigma(t,\mathbf{u}_1)-\sigma(t,\mathbf{u}_2)\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}\leq L\|\mathbf{u}_1-\mathbf{u}_2\|^2_{\mathbb{H}^s},\tag{2.4}$$

for some positive constant $L=\widetilde{L}^2\mathrm{Tr}(Q)$. □

Definition 2.3. A càdlàg adapted process (paths are right continuous with left limits), $(\mathbf{L}_t)_{t\geq0}$, is called a *Lévy process* if it has stationary independent increments and is stochastically continuous.

Let $\mathbb{H}$ and $\mathbb{U}$ be Hilbert spaces and let $(\mathbf{L}_t)_{t\geq0}$ be an $\mathbb{H}-$valued Lévy process. Then, for every $\omega\in\Omega$, $\mathbf{L}_t(\omega)$ has countable number of jumps on $[0,t]$ with jump $\triangle\mathbf{L}_t(\omega)=\mathbf{L}_t(\omega)-\mathbf{L}_{t-}(\omega)$. Let us define

$$\mathcal{N}(t,\mathrm{Z})=\mathcal{N}(t,\mathrm{Z},\omega)=\#\left\{s\in(0,\infty):\triangle\mathbf{L}_s(\omega)\in\mathrm{Z}\right\},$$

for $t>0,\mathrm{Z}\in\mathscr{B}(\mathbb{H}\backslash\{0\})$, $\omega\in\Omega$, as the *Poisson random measure associated with the Lévy process* $(\mathbf{L}_t)_{t\geq0}$ (see page 100, [1]).

The differential form of the measure $\mathcal{N}(t,\mathrm{Z},\omega)$ is written as $\mathcal{N}(\mathrm{d}t,\mathrm{d}z)(\omega)$. We call $\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z)=\mathcal{N}(\mathrm{d}t,\mathrm{d}z)-\lambda(\mathrm{d}z)\mathrm{d}t$, a *compensated Poisson random measure (cPrm)*, i.e., $\mathbb{E}(\mathcal{N}(\mathrm{d}t,\mathrm{d}z))=\lambda(\mathrm{d}z)\mathrm{d}t$ (Theorem 35, section 4, [34]), where $\lambda(\mathrm{d}z)\mathrm{d}t$ is known as *compensator* of the Lévy process $(\mathbf{L}_t)_{t\geq0}$. Here $\mathrm{d}t$ denotes the Lebesgue measure on $\mathscr{B}(\mathbb{R}^+)$, and $\lambda(\mathrm{d}z)$ is a σ-finite Lévy measure on $(\mathrm{Z},\mathscr{B}(\mathrm{Z}))$.

Let us denote by $\mathrm{D}([0,T];\mathbb{H})$, the set of all $\mathbb{H}-$valued functions defined on $[0,T]$, which are right continuous and have left limits (càdlàg functions) for every $t\in[0,T]$. Also, let

$$\mathcal{M}^{2p}_T(\mathbb{H};\mathbb{U}):=\mathbb{L}^{2p}\left(\Omega\times(0,T]\times\mathrm{Z},\mathscr{B}((0,T]\times\mathscr{F}\times\mathrm{Z}),\mathrm{d}t\otimes\mathbb{P}\otimes\lambda;\mathbb{U}\right),\tag{2.5}$$

be the space of all $\mathscr{B}((0,T]\times\mathscr{F}\times \mathrm{Z})$ measurable functions $\gamma:[0,T]\times\Omega\times\mathrm{Z}\to\mathbb{U}$ such that

$$\mathbb{E}\left[\int_0^T\int_{\mathrm{Z}}\|\gamma(t,\cdot,z)\|_{\mathbb{U}}^{2p}\lambda(\mathrm{d}z)\mathrm{d}t\right]<+\infty.$$

Let us assume that the following conditions hold for the noise coefficient $\gamma(\cdot,\cdot,\cdot):[0,T]\times\mathbb{H}^s\times\mathrm{Z}\to\mathbb{H}^s$, for $s\geq 0$, corresponding to the jump processes ([41], [15]):

Assumption 2.4. The noise coefficient $\gamma(\cdot,\cdot,\cdot):[0,T]\times\mathbb{H}^s\times\mathrm{Z}\to\mathbb{H}^s$ satisfies:

(A.1) For all $t\in[0,T]$ and $\mathbf{u}\in\mathbb{H}^s$, there exists a positive constant K such that

$$\int_{\mathrm{Z}}\|\gamma(t,\mathbf{u},z)\|_{\mathbb{H}^s}^2\lambda(\mathrm{d}z)\leq K\left(1+\|\mathbf{u}\|_{\mathbb{H}^s}^2\right).$$

(A.2) For all $t\in[0,T]$ and $\mathbf{u}_1,\mathbf{u}_2\in\mathbb{H}^s$, there exists a positive constant L such that

$$\int_{\mathrm{Z}}\|\gamma(t,\mathbf{u}_1,z)-\gamma(t,\mathbf{u}_2,z)\|_{\mathbb{H}^s}^2\lambda(\mathrm{d}z)\leq L\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{H}^s}^2.$$

(A.3) We fix the measurable subset Z_m of Z with $\mathrm{Z}_m\uparrow\mathrm{Z}$ and $\lambda(\mathrm{Z}_m)<+\infty$ such that

$$\sup_{\|\mathbf{u}\|_{\mathbb{H}^s}\leq M}\int_{\mathrm{Z}_m^c}\|\gamma(t,\mathbf{u},z)\|_{\mathbb{H}^s}^2\lambda(\mathrm{d}z)\to 0,\ \text{ as } m\to\infty,\ \text{ for } M>0.$$

If Z is of finite measure, i.e., if $\lambda(\mathrm{Z})<+\infty$, then (A.3) is satisfied automatically. Let us combine Assumption 2.2 and Assumption 2.4 to get the properties of the noise coefficients $\sigma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot,\cdot)$, namely linear growth and a Lipschitz condition.

Property 2.5. For all $s\geq 0$, the noise coefficients $\sigma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot,\cdot)$ satisfy

(P.1) (Growth Condition) For all $\mathbf{u}\in\mathbb{H}^s(\mathbb{R}^n)$ and for all $t\in[0,T]$, there exists a positive constant K such that

$$\|\sigma(t,\mathbf{u})\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}^2+\int_{\mathrm{Z}}\|\gamma(t,\mathbf{u},z)\|_{\mathbb{H}^s}^2\lambda(\mathrm{d}z)\leq K\left(1+\|\mathbf{u}\|_{\mathbb{H}^s}^2\right).$$

(P.2) (Lipschitz Condition) For all $\mathbf{u}_1,\mathbf{u}_2\in\mathbb{H}^s(\mathbb{R}^n)$ and for all $t\in[0,T]$, there exists a positive constant L such that

$$\begin{aligned}\|\sigma(t,\mathbf{u}_1)-\sigma(t,\mathbf{u}_2)\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}^2+\int_{\mathrm{Z}}\|\gamma(t,\mathbf{u}_1,z)-\gamma(t,\mathbf{u}_2,z)\|_{\mathbb{H}^s}^2\lambda(\mathrm{d}z)\\ \leq L\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{H}^s}^2.\end{aligned}$$

(P.3) Assumption (A.3).

In order to get the p^{th} moment estimate, for $p=2,3,\cdots$, we also assume, there exists a positive constant $\widehat{K}$ such that

$$\|\sigma(t,\mathbf{u})\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}^{2p}+\int_{\mathrm{Z}}\|\gamma(t,\mathbf{u},z)\|_{\mathbb{H}^s}^{2p}\lambda(\mathrm{d}z)\leq\widehat{K}\left(1+\|\mathbf{u}\|_{\mathbb{H}^s}^{2p}\right),\tag{2.6}$$

for all $\mathbf{u}\in\mathbb{H}^s(\mathbb{R}^n)$ and for all $t\in[0,T]$.

2.2. **Commutator Estimates and Moser Estimates.** We have $\mathrm{J}^s := (\mathrm{I}-\Delta)^{s/2}$ so that $\|\mathrm{f}\|_{\mathbb{H}^s} = \|\mathrm{J}^s\mathrm{f}\|_{\mathbb{L}^2}$, for $\mathrm{f} \in \mathbb{H}^s(\mathbb{R}^n)$. Let us recall the commutator estimates ([19]) and Moser estimates ([38]) used in this paper.

Lemma 2.6. *If $s \geq 0$ and $1 < p < \infty$, then*

$$\|\mathrm{J}^s(fg) - f(\mathrm{J}^s g)\|_{\mathbb{L}^p} \leq C_p \left(\|\nabla f\|_{\mathbb{L}^\infty}\|\mathrm{J}^{s-1}g\|_{\mathbb{L}^p} + \|\mathrm{J}^s f\|_{\mathbb{L}^p}\|g\|_{\mathbb{L}^\infty}\right). \tag{2.7}$$

Proof. See Lemma XI, [19]. □

Lemma 2.7. *Let $\mathrm{F}(\cdot)$ be a smooth function of $\mathbf{u} \in \mathbb{L}^\infty(\mathbb{R}^n) \cap \mathbb{H}^{s,p}(\mathbb{R}^n)$ and assume $\mathrm{F}(0) = 0$, then for $s > 0$ and $p \in (1,\infty)$, we have*

$$\|\mathrm{F}(\mathbf{u})\|_{\mathbb{H}^{s,p}} \leq C_{s,p}\left(\|\mathbf{u}\|_{\mathbb{L}^\infty}\right)\left(1 + \|\mathbf{u}\|_{\mathbb{H}^{s,p}}\right). \tag{2.8}$$

Proof. See Proposition 3.1.A., [38], Chapter 2, page 102, [40]. □

Remark 2.8. Note that for $s > \frac{n}{2} + k$, $\mathbb{H}^s(\mathbb{R}^n) \subset \mathrm{C}^k(\mathbb{R}^n)$, and

$$\|\mathrm{f}\|_{\mathrm{C}^k} \leq C\|\mathrm{f}\|_{\mathbb{H}^s},$$

where $\|\mathrm{f}\|_{\mathrm{C}^k} := \sup_{x\in\mathbb{R}^n}\{|\mathrm{f}|, |\nabla\mathrm{f}|, \cdots, |\nabla^k\mathrm{f}|\}$. In particular $\|\mathbf{f}\|_{\mathbb{L}^\infty} \leq C\|\mathbf{f}\|_{\mathbb{H}^s}$ for $s > n/2$ and $\|\nabla\mathbf{f}\|_{\mathbb{L}^\infty} \leq C\|\mathbf{f}\|_{\mathbb{H}^{s-1}}$ for $s > n/2+1$. Also, $\mathbb{H}^s$ is an algebra for $s > n/2$, i.e.,

$$\|\mathbf{fg}\|_{\mathbb{H}^s} \leq \|\mathbf{f}\|_{\mathbb{H}^s}\|\mathbf{g}\|_{\mathbb{H}^s},$$

for all $\mathbf{f}, \mathbf{g} \in \mathbb{H}^s$, $s > n/2$.

2.3. **Properties of the Operators $\mathscr{A}$ and $\mathscr{B}$.** For the stochastic quasilinear symmetric hyperbolic system (2.1), we obtain the following properties satisfied by the nonlinear operator $\mathscr{A}(\cdot,\cdot)$ under which we establish the local solvability of (2.1) (see [30]).

(F.1) The linear operator $\mathscr{A}(t,\mathbf{u}) := \sum_{j=1}^n \mathrm{A}^j(t,x,\mathbf{u})\frac{\partial}{\partial x_j}$, where $\mathrm{A}^j(\cdot,\cdot,\cdot)$'s are $m \times m$ symmetric matrices for $j = 1,\cdots,n$, satisfies

$$(\mathscr{A}(t,\mathbf{u})\mathbf{v},\mathbf{v})_{\mathbb{L}^2} \geq -\frac{1}{2}\|\nabla\mathrm{A}\|_{\mathbb{L}^\infty}\|\mathbf{v}\|_{\mathbb{L}^2}^2, \tag{2.9}$$

for $\mathbf{u},\mathbf{v} \in \mathbb{H}^s(\mathbb{R}^n)$, $s > n/2+2$, where

$$\|\nabla\mathrm{A}\|_{\mathbb{L}^\infty} = \sum_{j=1}^n \max_{1\leq i\leq m}\sum_{k=1}^m \sup_{x\in\mathbb{R}^n}\left|\frac{\partial}{\partial x_j}\mathrm{a}_{ik}^j(t,x,\mathbf{u})\right|$$

and $\mathrm{a}_{ik}^j(\cdot,\cdot,\cdot)$ is an entry of $\mathrm{A}^j(\cdot,\cdot,\cdot)$. For $\frac{1}{2}\|\nabla\mathrm{A}\|_{\mathbb{L}^\infty} \leq \mu$, the operator $\mathscr{A}(t,\mathbf{u}) + \mu\mathrm{I}$ is *monotone*.

(F.2) There exists an operator

$$\mathscr{B}(t,\mathbf{u}) := \mathrm{J}^s\mathscr{A}(t,\mathbf{u})\mathrm{J}^{-s} - \mathscr{A}(t,\mathbf{u}), \tag{2.10}$$

where $\mathrm{J}^s = (\mathrm{I}-\Delta)^{s/2}$, such that $\mathscr{B}(t,\mathbf{u}) \in \mathcal{L}(\mathbb{L}^2,\mathbb{L}^2)$ with

$$\begin{aligned}&\|\mathscr{B}(t,\mathbf{u})\mathbf{v}\|_{\mathbb{L}^2}\\ &\leq C\|\nabla\mathrm{A}\|_{\mathbb{L}^\infty}\|\nabla\mathrm{J}^{-s}\mathbf{v}\|_{\mathbb{H}^{s-1}} + C(\|\mathbf{u}\|_{\mathbb{L}^\infty})(1+\|\mathbf{u}\|_{\mathbb{H}^s})\|\nabla\mathrm{J}^{-s}\mathbf{v}\|_{\mathbb{L}^\infty},\end{aligned} \tag{2.11}$$

for $\mathbf{u},\mathbf{v} \in \mathbb{H}^s(\mathbb{R}^n)$, $s \geq n/2+2$.

(F.3) We have $\mathbb{H}^s(\mathbb{R}^n) \subset \mathrm{Dom}(\mathscr{A}(t,\mathbf{u}))$, so that $\mathscr{A}(t,\mathbf{u}) \in \mathcal{L}(\mathbb{H}^s, \mathbb{L}^2)$ with

$$\|\left(\mathscr{A}(t,\mathbf{u}) - \mathscr{A}(t,\mathbf{v})\right)\mathbf{w}\|_{\mathbb{L}^2} \leq \|\nabla_{\mathbf{u}}\mathrm{A}\|_{\mathbb{L}^\infty}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{L}^2}\|\nabla\mathbf{w}\|_{\mathbb{L}^\infty}, \tag{2.12}$$

and

$$\|\mathscr{A}(t,\mathbf{u}) - \mathscr{A}(t,\mathbf{v})\|_{\mathcal{L}(\mathbb{H}^s,\mathbb{L}^2)} \leq C\|\nabla_{\mathbf{u}}\mathrm{A}\|_{\mathbb{L}^\infty}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{L}^2}, \tag{2.13}$$

for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{H}^s(\mathbb{R}^n)$, $s > n/2 + 2$, where

$$\|\nabla_{\mathbf{u}}\mathrm{A}\|_{\mathbb{L}^\infty} = \left(\sum_{j=1}^n \max_{1\leq i\leq m}\sum_{k=1}^m \sup_{(x,\tau)\in\mathbb{R}^n\times[0,1]}\left|\nabla_{\mathbf{u}}a_{ik}^j(t,x,\tau\mathbf{u}+(1-\tau)\mathbf{v})\right|^2\right)^{1/2}.$$

(F.4) We also have

$$\|(\mathscr{A}(t,\mathbf{u}) - \mathscr{A}(t,\mathbf{v}))\mathbf{v}\|_{\mathbb{H}^{s-1}} \leq C\|\nabla_{\mathbf{u}}\mathrm{A}\|_{\mathbb{H}^{s-1}}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}^{s-1}}\|\mathbf{v}\|_{\mathbb{H}^s}, \tag{2.14}$$

where $\|\nabla_{\mathbf{u}}\mathrm{A}\|_{\mathbb{H}^{s-1}} = \left(\sum_{j=1}^n \left\|\nabla_{\mathbf{u}}\mathrm{A}^j\right\|_{\mathbb{H}^{s-1}}^2\right)^{1/2}$ with

$$\left\|\nabla_{\mathbf{u}}\mathrm{A}^j\right\|_{\mathbb{H}^{s-1}}^2 = \sup_{\tau\in[0,1]}\sum_{i,k=1}^m\left\|\nabla_{\mathbf{u}}a_{ik}^j(t,x,\tau\mathbf{u}+(1-\tau)\mathbf{v})\right\|_{\mathbb{H}^{s-1}}^2.$$

Remark 2.9. Condition (F.2) (see (2.11)) is obtained by making use of the commutator estimates (Lemma 2.6) and Moser estimates (Lemma 2.7) (see [30] for more details).

2.4. **Local and Global Strong Solution.** Let us now define the notion of local and global strong solutions of stochastic quasilinear symmetric hyperbolic system perturbed by Lévy noise.

Definition 2.10 (Local Strong Solution)**.** We say that the pair $(\mathbf{u}, \tau)$ is a *local strong (pathwise) solution* for the stochastic quasilinear symmetric hyperbolic system (2.1) with $\mathbf{u}_0 \in \mathbb{L}^4(\Omega; \mathbb{H}^s(\mathbb{R}^n))$, for $s > n/2+2$, if

(i) the symbol τ is a strictly positive stopping time, i.e.,

$$\mathbb{P}\Big\{\omega\in\Omega : \tau(\omega) > 0\Big\} = 1,$$

and

$$\tau(\omega) = \lim_{N\to\infty}\tau_N(\omega), \text{ for almost all } \omega\in\Omega,$$

where we define, for $N \in \mathbb{N}$,

$$\tau_N(\omega) = \inf_{t\geq 0}\Big\{t : \|\mathbf{u}(t)\|_{\mathbb{H}^s} \geq N\Big\},$$

(ii) for all $t \in [0, T)$, the symbol $\mathbf{u}$ denotes a right continuous progressively measurable stochastic process such that

(a) the process $\mathbf{u}(\cdot) \in \mathbb{L}^4(\Omega; \mathrm{D}(0, \tau(\omega); \mathbb{H}^s(\mathbb{R}^n)))$,

(b) $\mathbf{u}(\cdot)$ satisfies

$$\mathbf{u}(t\wedge\tau_N) = \mathbf{u}_0 - \sum_{j=1}^n\int_0^{t\wedge\tau_N}\mathrm{A}^j(s,x,\mathbf{u})\frac{\partial\mathbf{u}(s)}{\partial x_j}\mathrm{d}s + \int_0^{t\wedge\tau_N}\sigma(s,\mathbf{u}(s))\mathrm{dW}(s)$$

$$+\int_0^{t\wedge\tau_N}\int_{\mathrm{Z}}\gamma(s-,\mathbf{u}(s-),z)\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z),$$

for all $t\in[0,T)$ and $N\geq 1$.

Definition 2.11. A local strong solution $(\mathbf{u},\tau)$ to (2.1) is called a *unique local strong solution* if $(\widetilde{\mathbf{u}},\widetilde{\tau})$ is an another local strong solution, then

$$\tau=\widetilde{\tau},\ \text{for almost all } \omega\in\Omega,$$

and

$$\mathbf{u}(\cdot)=\widetilde{\mathbf{u}}(\cdot)\ \text{on } [0,\tau),$$

for almost all $\omega\in\Omega$.

Definition 2.12. The pair $(\mathbf{u},\tau)$ is a *global (pathwise) strong solution* for the stochastic quasilinear symmetric hyperbolic system (2.1) if

$$\mathbb{P}\Big\{\omega\in\Omega:\tau(\omega)=+\infty\Big\}=1.$$

3. Local Monotonicity and Energy Estimates

In this section, we prove the local monotonicity property of the nonlinear operator and energy estimates.

3.1. **Local Monotonicity.** We establish that the nonlinear term $\mathscr{A}(t,\mathbf{u})\mathbf{u}$ is locally monotone (in $\mathbb{L}^2$−norm), i.e., $\mathscr{A}(t,\cdot)\cdot+C(M)\mathrm{I}(\cdot)$ is a monotone operator in a closed ball $\mathbb{B}_M\subset\mathbb{H}^{s'}(\mathbb{R}^n)$ of radius M, for $s'\leq s$, $s'>n/2+2$.

Theorem 3.1. *For any given $M>0$, we consider the following (closed) ball:*

$$\mathbb{B}_M:=\Big\{\mathbf{z}\in\mathbb{H}^{s'}(\mathbb{R}^n):\|\mathbf{z}\|_{\mathbb{H}^{s'}}\leq M\Big\},\tag{3.1}$$

then for any $\mathbf{u},\mathbf{v}\in\mathbb{B}_M$ and each $t\in(0,T)$, we have

$$(\mathscr{A}(t,\mathbf{u})\mathbf{u}-\mathscr{A}(t,\mathbf{v})\mathbf{v},\mathbf{u}-\mathbf{v})_{\mathbb{L}^2}+C(M)\|\mathbf{u}-\mathbf{v}\|_{\mathbb{L}^2}^2\geq 0.\tag{3.2}$$

Proof. See Theorem 3.1, [30]. □

Next, we prove the local monotonicity of the nonlinear operator $\mathscr{A}(t,\mathbf{u})\mathbf{u}$ in the $\mathbb{H}^{s'-1}$−norm.

Theorem 3.2. *For any $\mathbf{u},\mathbf{v}\in\mathbb{B}_M$ and each $t\in(0,T)$, we have*

$$(\mathscr{A}(t,\mathbf{u})\mathbf{u}-\mathscr{A}(t,\mathbf{v})\mathbf{v},\mathbf{u}-\mathbf{v})_{\mathbb{H}^{s'-1}}+C(M)\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}^{s'-1}}^2\geq 0.\tag{3.3}$$

Proof. Let us consider

$$\begin{aligned}
&(\mathscr{A}(t,\mathbf{u})\mathbf{u}-\mathscr{A}(t,\mathbf{v})\mathbf{v},\mathbf{u}-\mathbf{v})_{\mathbb{H}^{s'-1}}\\
&=\Big(\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{u})\mathbf{u}\right]-\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{v})\mathbf{v}\right],\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}\\
&=\Big(\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{u})(\mathbf{u}-\mathbf{v})\right],\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}\\
&\quad+\Big(\mathrm{J}^{s'-1}\left[(\mathscr{A}(t,\mathbf{u})-\mathscr{A}(t,\mathbf{v}))\mathbf{v}\right],\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}\\
&=\Big(\mathscr{A}(t,\mathbf{u})\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v}),\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}
\end{aligned}$$

$$+\Big(\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{u})(\mathbf{u}-\mathbf{v})\right]-\mathscr{A}(t,\mathbf{u})\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v}),\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}$$

$$+\Big(\mathrm{J}^{s'-1}\left[(\mathscr{A}(t,\mathbf{u})-\mathscr{A}(t,\mathbf{v}))\mathbf{v}\right],\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}. \tag{3.4}$$

Now we take the first term from the right hand side of the equality (3.4) and use (2.9) to obtain

$$\begin{aligned}&\Big(\mathscr{A}(t,\mathbf{u})\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v}),\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}\\ &\geq -\frac{1}{2}\|\nabla\mathrm{A}\|_{\mathbb{L}^\infty}\|\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\|_{\mathbb{L}^2}^2=-\frac{1}{2}\|\nabla\mathrm{A}\|_{\mathbb{L}^\infty}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}^{s'-1}}^2.\end{aligned} \tag{3.5}$$

The second term from the right hand side of the equality (3.4) can be estimated using the Cauchy-Schwarz inequality and (2.11) as

$$\begin{aligned}&\left|\Big(\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{u})(\mathbf{u}-\mathbf{v})\right]-\mathscr{A}(t,\mathbf{u})\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v}),\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}\right|\\ &\leq\left\|\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{u})(\mathbf{u}-\mathbf{v})\right]-\mathscr{A}(t,\mathbf{u})\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{L}^2}\left\|\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{L}^2}\\ &\leq\Big[C\|\nabla\mathrm{A}\|_{\mathbb{L}^\infty}\|\nabla(\mathbf{u}-\mathbf{v})\|_{\mathbb{H}^{s'-2}}+C(\|\mathbf{u}\|_{\mathbb{L}^\infty})\left(1+\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\right)\|\nabla(\mathbf{u}-\mathbf{v})\|_{\mathbb{L}^\infty}\Big]\\ &\quad\times\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}^{s'-1}}\\ &\leq\Big(C\|\nabla\mathrm{A}\|_{\mathbb{L}^\infty}+C(\|\mathbf{u}\|_{\mathbb{L}^\infty})\left(1+\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\right)\Big)\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}^{s'-1}}^2\end{aligned} \tag{3.6}$$

The final term from the right hand side of the equality (3.4) can be estimated using the Cauchy-Schwarz inequality and (2.14) as

$$\begin{aligned}&\left|\Big(\mathrm{J}^{s'-1}\left[(\mathscr{A}(t,\mathbf{u})-\mathscr{A}(t,\mathbf{v}))\mathbf{v}\right],\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}\right|\\ &\leq\left\|\mathrm{J}^{s'-1}\left[(\mathscr{A}(t,\mathbf{u})-\mathscr{A}(t,\mathbf{v}))\mathbf{v}\right]\right\|_{\mathbb{L}^2}\left\|\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{L}^2}\\ &\leq C\|\nabla_{\mathbf{u}}\mathrm{A}\|_{\mathbb{H}^{s'-1}}\|\mathbf{v}\|_{\mathbb{H}^{s'}}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}^{s'-1}}^2.\end{aligned} \tag{3.7}$$

Combining (3.5), (3.6) and (3.7), and substituting it in (3.4) to get

$$\begin{aligned}&(\mathscr{A}(t,\mathbf{u})\mathbf{u}-\mathscr{A}(t,\mathbf{v})\mathbf{v},\mathbf{u}-\mathbf{v})_{\mathbb{H}^{s'-1}}\\ &\quad+\Big(C\|\nabla\mathrm{A}\|_{\mathbb{L}^\infty}+C(\|\mathbf{u}\|_{\mathbb{L}^\infty})\left(1+\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\right)\\ &\qquad+C\|\nabla_{\mathbf{u}}\mathrm{A}\|_{\mathbb{H}^{s'-1}}\|\mathbf{v}\|_{\mathbb{H}^{s'}}\Big)\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}^{s'-1}}^2\geq 0.\end{aligned} \tag{3.8}$$

Thus for any $\mathbf{u},\mathbf{v}\in\mathbb{B}_M$, we have

$$(\mathscr{A}(t,\mathbf{u})\mathbf{u}-\mathscr{A}(t,\mathbf{v})\mathbf{v},\mathbf{u}-\mathbf{v})_{\mathbb{H}^{s'-1}}+C(M)\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}^{s'-1}}^2\geq 0. \tag{3.9}$$

□

Remark 3.3. Let us denote $\mathcal{F}(\mathbf{u})=\mathscr{A}(t,\mathbf{u})\mathbf{u}$. From the local monotonicity theorem (Theorem 3.1 and Theorem 3.2), it can be easily seen that $\mathcal{F}(\cdot)+C(M)\mathrm{I}$ is a monotone operator in $\mathbb{B}_M\subset\mathbb{H}^{s'}(\mathbb{R}^n)$ (see (3.9)) and in fact one can prove that $\mathcal{F}(\cdot)+C(M)\mathrm{I}$ is a maximal monotone operator in $\mathbb{B}_M$ (see Remark 1, [30]).

Let us define a function $\psi : [0,\infty) \to [0,1]$ by

$$\psi_N(y) = \begin{cases} 1, & \text{for } 0 \le y \le N, \\ N+1-y, & \text{for } N < y \le N+1, \\ 0, & \text{for } y > N+1, \end{cases} \tag{3.10}$$

where N is a positive integer. Note that $\psi_N(\cdot)$ is a continuous function. We now consider the operator $\widetilde{\mathscr{A}}(t,\mathbf{u}) := \psi_N(\|\mathbf{u}\|_{\mathbb{H}^{s'-1}})\mathscr{A}(t,\mathbf{u})$ and prove the local monotonicity in $\mathbb{H}^{s'-1}-$norm.

Theorem 3.4. *For any $\mathbf{u},\mathbf{v} \in \mathbb{B}_M$ and each $t \in (0,T)$, we have*

$$\begin{aligned} &\left(\widetilde{\mathscr{A}}(t,\mathbf{u})\mathbf{u} - \widetilde{\mathscr{A}}(t,\mathbf{v})\mathbf{v}, \mathbf{u}-\mathbf{v}\right)_{\mathbb{H}^{s'-1}} + \left(C_N M + \frac{L}{2}\right)\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{H}^{s'-1}} \\ &\ge \frac{1}{2}\left[\|\sigma(t,\mathbf{u}) - \sigma(t,\mathbf{v})\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s'-1})} + \int_Z \|\gamma(t,\mathbf{u},z) - \gamma(t,\mathbf{v},z)\|^2_{\mathbb{H}^{s'-1}}\lambda(\mathrm{d}z)\right] \end{aligned} \tag{3.11}$$

where L is the Lipschitz constant appearing in (P.2) of Property 2.5.

Proof. Let us assume that $\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}, \|\mathbf{v}\|_{\mathbb{H}^{s'-1}} \le N$ and in this case the operator $\widetilde{\mathscr{A}}(\cdot,\cdot)$ becomes $\mathscr{A}(\cdot,\cdot)$ and from (3.9), for any $\mathbf{u},\mathbf{v} \in \mathbb{B}_M$, we have

$$\left(\widetilde{\mathscr{A}}(t,\mathbf{u})\mathbf{u} - \widetilde{\mathscr{A}}(t,\mathbf{v})\mathbf{v}, \mathbf{u}-\mathbf{v}\right)_{\mathbb{H}^{s'-1}} + C_N M\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{H}^{s'-1}} \ge 0. \tag{3.12}$$

Let us now assume that $N < \|\mathbf{u}\|_{\mathbb{H}^{s'-1}}, \|\mathbf{v}\|_{\mathbb{H}^{s'-1}} \le N+1$ and consider

$$\begin{aligned} &\left(\widetilde{\mathscr{A}}(t,\mathbf{u})\mathbf{u} - \widetilde{\mathscr{A}}(t,\mathbf{v})\mathbf{v}, \mathbf{u}-\mathbf{v}\right)_{\mathbb{H}^{s'-1}} \\ &= \Big(\mathrm{J}^{s'-1}\left[\psi_N(\|\mathbf{u}\|_{\mathbb{H}^{s'-1}})\mathscr{A}(t,\mathbf{u})\mathbf{u}\right] \\ &\quad -\mathrm{J}^{s'-1}\left[\psi_N(\|\mathbf{v}\|_{\mathbb{H}^{s'-1}})\mathscr{A}(t,\mathbf{v})\mathbf{v}\right], \mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2} \\ &= \Big(\psi_N(\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}) - \psi_N(\|\mathbf{v}\|_{\mathbb{H}^{s'-1}})\Big)\Big(\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{u})\mathbf{u}\right], \mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2} \\ &\quad + \psi_N(\|\mathbf{v}\|_{\mathbb{H}^{s'-1}})\Big(\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{u})\mathbf{u}\right] - \mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{v})\mathbf{v}\right], \mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}. \end{aligned} \tag{3.13}$$

The first term from the equality (3.13) can be estimated using the Cauchy-Schwarz inequality, algebra property of the $\mathbb{H}^{s'-1}$ norm, Moser estimate and reverse triangle inequality as

$$\begin{aligned} &\left|\Big(\psi_N(\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}) - \psi_N(\|\mathbf{v}\|_{\mathbb{H}^{s'-1}})\Big)\Big(\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{u})\mathbf{u}\right], \mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}\right| \\ &\le \left|\psi_N(\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}) - \psi_N(\|\mathbf{v}\|_{\mathbb{H}^{s'-1}})\right|\left\|\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{u})\mathbf{u}\right]\right\|_{\mathbb{L}^2}\left\|\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{L}^2} \\ &\le \left|\|\mathbf{v}\|_{\mathbb{H}^{s'-1}} - \|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\right|\|\mathrm{A}(\mathbf{u})\|_{\mathbb{H}^{s'-1}}\|\nabla\mathbf{u}\|_{\mathbb{H}^{s'-1}}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}^{s'-1}} \\ &\le C(\|\mathbf{u}\|_{\mathbb{L}^\infty})\left(1+\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\right)\|\mathbf{u}\|_{\mathbb{H}^{s'}}\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{H}^{s'-1}} \end{aligned} \tag{3.14}$$

Since $0 \le \psi_N(\cdot) \le 1$, the second term from the right hand side of the equality (3.13) can be estimated similarly as of (3.9). Hence, from (3.13), we have

$$\left(\widetilde{\mathscr{A}}(t,\mathbf{u})\mathbf{u} - \widetilde{\mathscr{A}}(t,\mathbf{v})\mathbf{v}, \mathbf{u}-\mathbf{v}\right)_{\mathbb{H}^{s'-1}}$$

$$+\Big(C\|\nabla \mathrm{A}\|_{\mathbb{L}^\infty}+C(\|\mathbf{u}\|_{\mathbb{L}^\infty})\left(1+\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\right)\left(1+\|\mathbf{u}\|_{\mathbb{H}^{s'}}\right)$$
$$+C\|\nabla_{\mathbf{u}}\mathrm{A}\|_{\mathbb{H}^{s'-1}}\|\mathbf{v}\|_{\mathbb{H}^{s'}}\Big)\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{H}^{s'-1}}\geq 0. \tag{3.15}$$

Hence, by choosing $\mathbf{u},\mathbf{v}\in\mathbb{B}_M$, we obtain (3.12).

Let us now consider the case $\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\leq N$ and $N<\|\mathbf{v}\|_{\mathbb{H}^{s'-1}}\leq N+1$. Then, we have

$$\Big(\widetilde{\mathscr{A}}(t,\mathbf{u})\mathbf{u}-\widetilde{\mathscr{A}}(t,\mathbf{v})\mathbf{v},\mathbf{u}-\mathbf{v}\Big)_{\mathbb{H}^{s'-1}}$$
$$=\Big(\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{u})\mathbf{u}\right]-\psi_N(\|\mathbf{v}\|_{\mathbb{H}^{s'-1}})\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{v})\mathbf{v}\right],\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}$$
$$=\Big(\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{u})\mathbf{u}\right]-\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{v})\mathbf{v}\right],\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}$$
$$+\left(\|\mathbf{v}\|_{\mathbb{H}^{s'-1}}-N\right)\Big(\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{v})\mathbf{v}\right],\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2} \tag{3.16}$$

The first term from the right hand side of the equality (3.16) can be estimated similarly as of (3.9). Note that by using reverse triangle inequality, we get

$$\|\mathbf{v}\|_{\mathbb{H}^{s'-1}}-N\leq\|\mathbf{v}\|_{\mathbb{H}^{s'-1}}-\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\leq\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}^{s'-1}}. \tag{3.17}$$

A calculation similar to (3.14) yields

$$\left|\left(\|\mathbf{v}\|_{\mathbb{H}^{s'-1}}-N\right)\Big(\mathrm{J}^{s'-1}\left[\mathscr{A}(t,\mathbf{v})\mathbf{v}\right],\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}\right|$$
$$\leq C(\|\mathbf{v}\|_{\mathbb{L}^\infty})\left(1+\|\mathbf{v}\|_{\mathbb{H}^{s'-1}}\right)\|\mathbf{v}\|_{\mathbb{H}^{s'}}\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{H}^{s'-1}}. \tag{3.18}$$

Thus from (3.16), we have

$$\Big(\widetilde{\mathscr{A}}(t,\mathbf{u})\mathbf{u}-\widetilde{\mathscr{A}}(t,\mathbf{v})\mathbf{v},\mathbf{u}-\mathbf{v}\Big)_{\mathbb{H}^{s'-1}}$$
$$+\Big(C\|\nabla\mathrm{A}\|_{\mathbb{L}^\infty}+C(\|\mathbf{u}\|_{\mathbb{L}^\infty})\left(1+\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\right)$$
$$+\left(C(\|\mathbf{v}\|_{\mathbb{L}^\infty})\left(1+\|\mathbf{v}\|_{\mathbb{H}^{s'-1}}\right)+C\|\nabla_{\mathbf{u}}\mathrm{A}\|_{\mathbb{H}^{s'-1}}\right)\|\mathbf{v}\|_{\mathbb{H}^{s'}}\Big)\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{H}^{s'-1}}\geq 0. \tag{3.19}$$

The case of $N<\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\leq N+1$ and $\|\mathbf{v}\|_{\mathbb{H}^{s'-1}}\leq N$ can also be handled similarly.

Now if $\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\leq N$ and $\|\mathbf{v}\|_{\mathbb{H}^{s'-1}}>N+1$, then

$$\Big(\widetilde{\mathscr{A}}(t,\mathbf{u})\mathbf{u}-\widetilde{\mathscr{A}}(t,\mathbf{v})\mathbf{v},\mathbf{u}-\mathbf{v}\Big)_{\mathbb{H}^{s'-1}}=\Big(\mathrm{J}^{s'-1}\mathscr{A}(t,\mathbf{u})\mathbf{u},\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}, \tag{3.20}$$

and note that

$$\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}^{s'-1}}\geq\|\mathbf{v}\|_{\mathbb{H}^{s'-1}}-\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\geq 1. \tag{3.21}$$

By using the Cauchy-Schwarz inequality, algebra property of $\mathbb{H}^{s'-1}$ norm, and Moser estimate, we have

$$\left|\Big(\mathrm{J}^{s'-1}\mathscr{A}(t,\mathbf{u})\mathbf{u},\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\Big)_{\mathbb{L}^2}\right|$$
$$\leq\|\mathrm{A}(\mathbf{u})\|_{\mathbb{H}^{s'-1}}\|\nabla\mathbf{u}\|_{\mathbb{H}^{s'-1}}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}^{s'-1}}$$
$$\leq C(\|\mathbf{u}\|_{\mathbb{L}^\infty})\left(1+\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\right)\|\mathbf{u}\|_{\mathbb{H}^{s'}}\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{H}^{s'-1}}. \tag{3.22}$$

Thus from (3.20), we obtain

$$\begin{aligned}&\left(\widetilde{\mathscr{A}}(t,\mathbf{u})\mathbf{u}-\widetilde{\mathscr{A}}(t,\mathbf{v})\mathbf{v},\mathbf{u}-\mathbf{v}\right)_{\mathbb{H}^{s'-1}}\\ &\quad+C(\|\mathbf{u}\|_{\mathbb{L}^\infty})\left(1+\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\right)\|\mathbf{u}\|_{\mathbb{H}^{s'}}\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{H}^{s'-1}}\geq 0.\end{aligned}\tag{3.23}$$

The case of $\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}>N+1$ and $\|\mathbf{v}\|_{\mathbb{H}^{s'-1}}\leq N$ can be handled in a similar fashion.

We now take $N<\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\leq N+1$ and $\|\mathbf{v}\|_{\mathbb{H}^{s'-1}}>N+1$. Then, we have

$$\begin{aligned}&\left(\widetilde{\mathscr{A}}(t,\mathbf{u})\mathbf{u}-\widetilde{\mathscr{A}}(t,\mathbf{v})\mathbf{v},\mathbf{u}-\mathbf{v}\right)_{\mathbb{H}^{s'-1}}\\ &\quad=\left(N+1-\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\right)\left(\mathrm{J}^{s'-1}\mathscr{A}(t,\mathbf{u})\mathbf{u},\mathrm{J}^{s'-1}(\mathbf{u}-\mathbf{v})\right)_{\mathbb{L}^2}.\end{aligned}\tag{3.24}$$

But we know that

$$\left(N+1-\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\right)\leq\|\mathbf{v}\|_{\mathbb{H}^{s'-1}}-\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}\leq\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}^{s'-1}},\tag{3.25}$$

and a calculation similar to (3.22) yields (3.23). The case of $\|\mathbf{u}\|_{\mathbb{H}^{s'-1}}>N+1$ and $N<\|\mathbf{v}\|_{\mathbb{H}^{s'-1}}\leq N+1$ follows similarly. The inequality (3.12) can be regained from the estimates (3.19) and (3.23) by choosing $\mathbf{u},\mathbf{v}\in\mathbb{B}_M$.

For $\|\mathbf{u}\|_{\mathbb{H}^{s'-1}},\|\mathbf{v}\|_{\mathbb{H}^{s'-1}}>N+1$, we have $\widetilde{\mathscr{A}}(t,\mathbf{u})\mathbf{u}=\widetilde{\mathscr{A}}(t,\mathbf{v})\mathbf{v}=0$ and the property (3.12) is trivially satisfied. Adding $\frac{L}{2}\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{H}^{s'-1}}$ on both sides of (3.12) and using property (P.2) from Property 2.5, we finally obtain (3.11). □

Similar results for Navier-Stokes equations is obtained in [4, 5]. For more details about monotone operators, we refer the readers to [2, 3].

3.2. **Energy Estimates.**

We next establish the $\mathbb{L}^2$ and $\mathbb{H}^s$ energy estimates for the stochastic hyperbolic system (2.1). We consider the truncated system corresponding to (2.1) in the Itô stochastic differential form in $(0,T)$ with the cutoff $\psi_N(\cdot)$ as

$$\begin{cases}\mathrm{d}\mathbf{u}(t)=-\psi_N(\|\mathbf{u}(t)\|_{\mathbb{H}^{s-1}})\mathscr{A}(t,\mathbf{u})\mathbf{u}(t)\mathrm{d}t+\sigma(t,\mathbf{u})\mathrm{dW}(t)\\ \qquad\qquad+\displaystyle\int_{\mathrm{Z}}\gamma(t-,\mathbf{u}(t-),z)\widetilde{\mathscr{N}}(\mathrm{d}t,\mathrm{d}z),\\ \mathbf{u}(0)=\mathbf{u}_0,\end{cases}\tag{3.26}$$

for $\mathbf{u}_0\in\mathbb{H}^s(\mathbb{R}^n)$ for $s>n/2+2$.

Let us consider a finite−dimensional Galerkin approximation of the system (2.1) perturbed by Lévy noise. Let $\{e_1,e_2,\cdots\}$ be a complete orthonormal system in $\mathbb{L}^2(\mathbb{R}^n)$ belonging to $\mathbb{H}^s(\mathbb{R}^n)$ and let $\mathbb{L}^2_n(\mathbb{R}^n)$ be the $n-$dimensional subspace of $\mathbb{L}^2(\mathbb{R}^n)$. Let P_n denote the orthogonal projection of $\mathbb{L}^2$ to $\mathbb{L}^2_n$. Note that $\mathrm{W}(\cdot)$ is an $\mathbb{L}^2-$valued $Q-$Wiener process such that $Qe_j=\lambda_je_j$ with $\mathrm{Tr}(Q)<+\infty$ and thus we define $\mathrm{W}_n(\cdot)=\mathrm{P}_n\mathrm{W}(\cdot)$, $\sigma^n(\cdot,\mathbf{u}^n)=\mathrm{P}_n\sigma(\cdot,\mathbf{u}^n)$ and $\gamma^n(\cdot,\mathbf{u}^n,\cdot)=\mathrm{P}_n\gamma(\cdot,\mathbf{u}^n,\cdot)$. Let us now consider the following system of finite−dimensional stochastic ODE satisfied by $\mathbf{u}^n(\cdot)$ in the variational form in $(0,T)$:

$$\begin{cases}\mathrm{d}(\mathbf{u}^n(t),\mathbf{v})_{\mathbb{L}^2}=-\left(\psi_N(\|\mathbf{u}^n(t)\|_{\mathbb{H}^{s-1}})\mathscr{A}(t,\mathbf{u}^n)\mathbf{u}^n(t),\mathbf{v}(t)\right)_{\mathbb{L}^2}\mathrm{d}t\\ \qquad\qquad+\left(\sigma^n(t,\mathbf{u}^n)\mathrm{dW}_n(t),\mathbf{v}(t)\right)_{\mathbb{L}^2}\\ \qquad\qquad+\displaystyle\int_{\mathrm{Z}_n}\left(\gamma^n(t-,\mathbf{u}^n(t-),z),\mathbf{v}(t)\right)_{\mathbb{L}^2}\widetilde{\mathscr{N}}(\mathrm{d}t,\mathrm{d}z),\\ \qquad\mathbf{u}^n(0)=\mathbf{u}^n_0,\end{cases}\tag{3.27}$$

with $\mathbf{u}_0^n = \mathrm{P}_n\mathbf{u}_0$ for each $\mathbf{v} \in \mathbb{L}_n^2$. Since the system (3.27) is finite-dimensional with bounded drift and locally Lipschitz coefficients (see (2.13) and (2.14)), it has a unique solution in $\mathbb{L}^2(\Omega; \mathrm{D}(0,T;\mathbb{L}_n^2))$ (see [27]).

Proposition 3.5 ($\mathbb{L}^2-$energy estimate)**.** *Let $\mathbf{u}^n(\cdot)$ be the unique solution of the system of stochastic ODE's (3.26) with $\mathbf{u}_0 \in \mathbb{L}^2(\Omega;\mathbb{L}^2(\mathbb{R}^n))$. Then, we have the following a-priori energy estimate:*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\|\mathbf{u}^n(t)\|_{\mathbb{L}^2}^2\right] \leq \left(1+2\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{L}^2}^2\right]\right)e^{2(C_N+9K)T}. \tag{3.28}$$

Proof. We first define the sequence of stopping times τ_M^n to be

$$\tau_M^n := \inf_{t\geq 0}\Big\{t : \|\mathbf{u}^n(t)\|_{\mathbb{L}^2} \geq M\Big\}. \tag{3.29}$$

Let us apply the Itô formula (see Theorem 3.7.2, [25], section 4.4, [1], section 2.3, [27]) to $\|\mathbf{u}^n(\cdot)\|_{\mathbb{L}^2}^2$ to obtain

$$\begin{aligned}
&\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{L}^2}^2\\
&= \|\mathbf{u}^n(0)\|_{\mathbb{L}^2}^2 - 2\int_0^{t\wedge\tau_M^n}(\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\mathscr{A}(s,\mathbf{u}^n)\mathbf{u}^n(s),\mathbf{u}^n(s))_{\mathbb{L}^2}\,\mathrm{d}s\\
&\quad+\int_0^{t\wedge\tau_M^n}\|\sigma^n(s,\mathbf{u}^n(s))\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{L}^2)}^2\mathrm{d}s\\
&\quad+2\int_0^{t\wedge\tau_M^n}(\sigma^n(s,\mathbf{u}^n(s))\mathrm{dW}_n(s),\mathbf{u}^n(s))_{\mathbb{L}^2}\\
&\quad+\int_0^{t\wedge\tau_M^n}\int_{\mathrm{Z}_n}\|\gamma^n(s,\mathbf{u}^n(s),z)\|_{\mathbb{L}^2}^2\mathcal{N}(\mathrm{d}s,\mathrm{d}z)\\
&\quad+2\int_0^{t\wedge\tau_M^n}\int_{\mathrm{Z}_n}(\gamma^n(s-,\mathbf{u}^n(s-),z),\mathbf{u}^n(s-))_{\mathbb{L}^2}\,\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z).
\end{aligned} \tag{3.30}$$

The term $-2\,(\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\mathscr{A}(s,\mathbf{u}^n)\mathbf{u}^n(s),\mathbf{u}^n(s))_{\mathbb{L}^2}$ from (3.30) can be estimated using (2.9) as

$$\begin{aligned}
&-2\,(\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\mathscr{A}(s,\mathbf{u}^n)\mathbf{u}^n(s),\mathbf{u}^n(s))_{\mathbb{L}^2}\\
&\leq \psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\|\nabla \mathrm{A}\|_{\mathbb{L}^\infty}\|\mathbf{u}^n(s)\|_{\mathbb{L}^2}^2.
\end{aligned} \tag{3.31}$$

By using (3.31) in (3.31), we get

$$\begin{aligned}
\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{L}^2}^2 &\leq \|\mathbf{u}^n(0)\|_{\mathbb{L}^2}^2 + \int_0^{t\wedge\tau_M^n}\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\|\nabla\mathrm{A}\|_{\mathbb{L}^\infty}\|\mathbf{u}^n(s)\|_{\mathbb{L}^2}^2\mathrm{d}s\\
&\quad+\int_0^{t\wedge\tau_M^n}\|\sigma^n(s,\mathbf{u}^n(s))\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{L}^2)}^2\mathrm{d}s\\
&\quad+2\int_0^{t\wedge\tau_M^n}(\sigma^n(s,\mathbf{u}^n(s))\mathrm{dW}_n(s),\mathbf{u}^n(s))_{\mathbb{L}^2}\\
&\quad+\int_0^{t\wedge\tau_M^n}\int_{\mathrm{Z}_n}\|\gamma^n(s,\mathbf{u}^n(s),z)\|_{\mathbb{L}^2}^2\mathcal{N}(\mathrm{d}s,\mathrm{d}z)
\end{aligned}$$

$$+2\int_0^{t\wedge\tau_M^n}\int_{Z_n}(\gamma^n(s-,\mathbf{u}^n(s-),z),\mathbf{u}^n(s-))_{\mathbb{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z). \tag{3.32}$$

Note that the fourth and final terms on the right hand side of the inequality (3.32) are local martingales having zero expectation. Let us take the expectation of (3.32) and use this fact to obtain

$$\begin{aligned}
&\mathbb{E}\left[\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{L}^2}^2\right]\\
&\leq \mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{L}^2}^2\right]+\mathbb{E}\left[\int_0^{t\wedge\tau_M^n}\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\|\nabla \mathrm{A}\|_{\mathbb{L}^\infty}\|\mathbf{u}^n(s)\|_{\mathbb{L}^2}^2\mathrm{d}s\right]\\
&\quad+\mathbb{E}\left[\int_0^{t\wedge\tau_M^n}\left(\|\sigma^n(s,\mathbf{u}^n(s))\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{L}^2)}^2+\int_{Z_n}\|\gamma^n(s,\mathbf{u}^n(s),z)\|_{\mathbb{L}^2}^2\lambda(\mathrm{d}z)\right)\mathrm{d}s\right],
\end{aligned}\tag{3.33}$$

where we also used the fact that the expectation of the Quadratic variation process and Meyer process of $\mathbf{u}^n(\cdot)$ are equal, i.e.,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{t\wedge\tau_M^n}\|\sigma^n(s,\mathbf{u}^n)\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{L}^2)}^2\mathrm{d}s+\int_0^{t\wedge\tau_M^n}\int_{Z_n}\|\gamma^n(s,\mathbf{u}^n,z)\|_{\mathbb{L}^2}^2\mathcal{N}(\mathrm{d}s,\mathrm{d}z)\right]\\
&=\mathbb{E}\left[\int_0^{t\wedge\tau_M^n}\|\sigma^n(s,\mathbf{u}^n)\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{L}^2)}^2\mathrm{d}s+\int_0^{t\wedge\tau_M^n}\int_{Z_n}\|\gamma^n(s,\mathbf{u}^n,z)\|_{\mathbb{L}^2}^2\lambda(\mathrm{d}z)\mathrm{d}s\right].
\end{aligned}\tag{3.34}$$

Let us use the property of the cutoff function $\psi_N(\cdot)$ and Property 2.5 in (3.34) to find

$$\begin{aligned}
\mathbb{E}\left[\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{L}^2}^2\right]\leq \mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{L}^2}^2\right]&+C_N\mathbb{E}\left[\int_0^{t\wedge\tau_M^n}\|\mathbf{u}^n(s)\|_{\mathbb{L}^2}^2\mathrm{d}s\right]\\
&+K\mathbb{E}\left[\int_0^{t\wedge\tau_M^n}\left(1+\|\mathbf{u}^n(s)\|_{\mathbb{L}^2}^2\right)\mathrm{d}s\right].
\end{aligned}\tag{3.35}$$

Thus from (3.35), we get

$$\begin{aligned}
&\mathbb{E}\left[\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{L}^2}^2\right]\\
&\leq \mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{L}^2}^2\right]+(C_N+K)\int_0^t\mathbb{E}\left[1+\|\mathbf{u}^n(s\wedge\tau_M^n)\|_{\mathbb{L}^2}^2\right]\mathrm{d}s.
\end{aligned}\tag{3.36}$$

An application of the Gronwall's inequality on (3.36) yields

$$\mathbb{E}\left[\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{L}^2}^2\right]\leq\left(1+\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{L}^2}^2\right]\right)e^{(C_N+K)t}, \tag{3.37}$$

for all $t\in[0,T]$. On the other hand, we have

$$\begin{aligned}
&\mathbb{E}\left[\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{L}^2}^2\right]\\
&=\mathbb{E}\left[\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{L}^2}^2\chi_{\{\tau_M^n<t\}}\right]+\mathbb{E}\left[\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{L}^2}^2\chi_{\{\tau_M^n\geq t\}}\right]\\
&=\mathbb{E}\left[\|\mathbf{u}^n(\tau_M^n)\|_{\mathbb{L}^2}^2\chi_{\{\tau_M^n<t\}}\right]+\mathbb{E}\left[\|\mathbf{u}^n(t)\|_{\mathbb{L}^2}^2\chi_{\{\tau_M^n\geq t\}}\right],
\end{aligned}\tag{3.38}$$

where χ is the indicator function. From the right continuity of the process $\mathbf{u}^n(\cdot)$, we know that $\|\mathbf{u}^n(\tau_M^n)\|_{\mathbb{L}^2} \geq M$ (see (3.29)), and note that

$$\mathbb{E}\left[\chi_{\{\tau_M^n<t\}}\right] = \mathbb{P}\Big\{\omega \in \Omega : \tau_N < t\Big\}.$$

Equation (3.38) gives

$$\begin{aligned}\mathbb{E}\left[\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{L}^2}^2\right] &= \mathbb{E}\left[\|\mathbf{u}^n(\tau_M^n)\|_{\mathbb{L}^2}^2\chi_{\{\tau_M^n<t\}}\right] + \mathbb{E}\left[\|\mathbf{u}^n(t)\|_{\mathbb{L}^2}^2\chi_{\{\tau_M^n\geq t\}}\right]\\ &\geq \mathbb{E}\left[\|\mathbf{u}^n(\tau_M^n)\|_{\mathbb{L}^2}^2\chi_{\{\tau_M^n<t\}}\right]\\ &\geq M^2\mathbb{P}\Big\{\omega\in\Omega : \tau_M^n < t\Big\}. \end{aligned}\tag{3.39}$$

Thus by using (3.37), we finally obtain

$$\begin{aligned}\mathbb{P}\Big\{\omega\in\Omega:\tau_M^n<t\Big\} &\leq \frac{1}{M^2}\mathbb{E}\left[\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{L}^2}^2\right]\\ &\leq \frac{1}{M^2}\left(1+\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{L}^2}^2\right]\right)e^{(C_N+K)t}.\end{aligned}\tag{3.40}$$

Hence, we have

$$\lim_{M\to\infty}\mathbb{P}\Big\{\omega\in\Omega:\tau_M^n<t\Big\} = 0 \text{ for all } t\in[0,T],\tag{3.41}$$

and $t\wedge\tau_M^n \to t$ as $M\to\infty$. Then on taking limit $M\to\infty$ in (3.37) and using the dominated convergence theorem, we get

$$\mathbb{E}\left[\|\mathbf{u}^n(t)\|_{\mathbb{L}^2}^2\right] \leq \left(1+\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{L}^2}^2\right]\right)e^{(C_N+K)t},\tag{3.42}$$

for $0\leq t\leq T$.

In order to prove (3.28), let us take the supremum from 0 to $T\wedge\tau_M^n$ before taking the expectation in (3.32) and use the cutoff property of the function $\psi_N(\cdot)$ to obtain

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{L}^2}^2\right]\\ &\leq \mathbb{E}\left[\|\mathbf{u}^n(0)\|_{\mathbb{L}^2}^2\right] + C_N\mathbb{E}\left[\int_0^{T\wedge\tau_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{L}^2}^2\mathrm{d}t\right]\\ &\quad+\mathbb{E}\left[\int_0^{T\wedge\tau_M^n}\left\{\|\sigma^n(t,\mathbf{u}^n(t))\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{L}^2)}^2 + \int_{\mathrm{Z}_n}\|\gamma^n(\mathbf{u}^n(t),z)\|_{\mathbb{L}^2}^2\lambda(\mathrm{d}z)\right\}\right]\mathrm{d}t\\ &\quad+2\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_M^n}\left|\int_0^t(\sigma^n(s,\mathbf{u}^n(s))\mathrm{dW}_n(s),\mathbf{u}^n(s))_{\mathbb{L}^2}\right|\right]\\ &\quad+2\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_M^n}\left|\int_0^t\int_{\mathrm{Z}_n}(\gamma^n(s-,\mathbf{u}^n(s-),z),\mathbf{u}^n(s-))_{\mathbb{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z)\right|\right].\end{aligned}\tag{3.43}$$

Let us take the fourth term from the right hand side of the inequality (3.43) and use the Burkholder-Davis-Gundy inequality (see Theorem 73, Chapter 4, [34], Theorem 3.50, [33]), Hölder inequality and Young's inequality to get

$$\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_M^n}\left|\int_0^t\big(\sigma^n(s,\mathbf{u}^n(s))\mathrm{dW}_n(s),\mathbf{u}^n(s)\big)_{\mathbb{L}^2}\right|\right]$$

$$\leq \sqrt{2}\mathbb{E}\left[\int_0^{T\wedge\tau_M^n} \|\sigma^n(t,\mathbf{u}^n(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{L}^2)}\|\mathbf{u}^n(t)\|^2_{\mathbb{L}^2}\mathrm{d}t\right]^{1/2}$$

$$\leq \frac{1}{8}\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_M^n}\|\mathbf{u}^n(t)\|^2_{\mathbb{L}^2}\right] + 4\mathbb{E}\left[\int_0^{T\wedge\tau_M^n}\|\sigma^n(t,\mathbf{u}^n(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{L}^2)}\mathrm{d}t\right]. \tag{3.44}$$

Using the Burkholder-Davis-Gundy inequality, Hölder inequality and Young's inequality on the final term from the right hand side of the inequality (3.43), we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_M^n}\left|\int_0^t\int_{\mathrm{Z}_n}(\gamma^n(s-,\mathbf{u}^n(s-),z),\mathbf{u}^n(s-))_{\mathbb{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z)\right|\right]$$

$$\leq \sqrt{2}\mathbb{E}\left[\int_0^{T\wedge\tau_M^n}\int_{\mathrm{Z}_n}\|\gamma^n(\mathbf{u}^n(t),z)\|^2_{\mathbb{L}^2}\|\mathbf{u}^n(t)\|^2_{\mathbb{L}^2}\lambda(\mathrm{d}z)\mathrm{d}t\right]^{1/2}$$

$$\leq \frac{1}{8}\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_M^n}\|\mathbf{u}^n(t)\|^2_{\mathbb{L}^2}\right] + 4\mathbb{E}\left[\int_0^{T\wedge\tau_M^n}\int_{\mathrm{Z}_n}\|\gamma^n(\mathbf{u}^n(t),z)\|^2_{\mathbb{L}^2}\lambda(\mathrm{d}z)\mathrm{d}t\right]. \tag{3.45}$$

Applying (3.44) and (3.45) in (3.43), we get

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_M^n}\|\mathbf{u}^n(t)\|^2_{\mathbb{L}^2}\right] &\leq 2\mathbb{E}\left[\|\mathbf{u}_0\|^2_{\mathbb{L}^2}\right] + 2C_N\mathbb{E}\left[\int_0^{T\wedge\tau_M^n}\|\mathbf{u}^n(t)\|^2_{\mathbb{L}^2}\mathrm{d}t\right]\\ &\quad + 18\mathbb{E}\left[\int_0^{T\wedge\tau_M^n}\Big(\|\sigma^n(t,\mathbf{u}^n(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{L}^2)}\right.\\ &\quad\left. + \int_{\mathrm{Z}_n}\|\gamma^n(t,\mathbf{u}^n(t),z)\|^2_{\mathbb{L}^2}\lambda(\mathrm{d}z)\Big)\right]\mathrm{d}t.\end{aligned} \tag{3.46}$$

Using Property 2.5 in (3.46), we find

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_M^n}\|\mathbf{u}^n(t)\|^2_{\mathbb{L}^2}\right]\\ &\leq 2\mathbb{E}\left[\|\mathbf{u}_0\|^2_{\mathbb{L}^2}\right] + 2(C_N+9K)\int_0^T\mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau_M^n}\left(1+\|\mathbf{u}^n(s)\|^2_{\mathbb{L}^2}\right)\right]\mathrm{d}t.\end{aligned} \tag{3.47}$$

An application of Gronwall's inequality on (3.47) yields

$$\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_M^n}\|\mathbf{u}^n(t)\|^2_{\mathbb{L}^2}\right] \leq \left(1+2\mathbb{E}\left[\|\mathbf{u}_0\|^2_{\mathbb{L}^2}\right]\right)e^{2(C_N+9K)T}. \tag{3.48}$$

A calculation similar to (3.41) yields $\lim_{M\to\infty}\mathbb{P}\Big\{\omega\in\Omega : \tau_M^n < T\Big\} = 0$ and thus as $M\to\infty$, $T\wedge\tau_M^n\to T$. Let us take the limit $M\to\infty$ in (3.48) and use the dominated convergence theorem to obtain the estimate (3.28). □

The Itô stochastic differential equation satisfied by $\mathrm{J}^s\mathbf{u}(\cdot)$ with the cutoff function $\psi_N(\cdot)$ in $(0,T)$ can be written as

$$\mathrm{dJ}^s\mathbf{u}(t) = -\psi_N(\|\mathbf{u}(t)\|_{\mathbb{H}^{s-1}})\mathrm{J}^s\left[\mathscr{A}(t,\mathbf{u})\mathbf{u}(t)\right]\mathrm{d}t + \mathrm{J}^s\sigma(t,\mathbf{u}(t))\mathrm{dW}(t)$$

$$+\int_{\mathrm{Z}}\mathrm{J}^s\gamma(t-,\mathbf{u}(t-),z)\widetilde{\mathscr{N}}(\mathrm{d}t,\mathrm{d}z),$$

with $\mathbf{u}(0)=\mathbf{u}_0$.

Proposition 3.6. *Let $\mathbf{u}^n(\cdot)$ be the unique solution of the system of stochastic ODE's (3.26) with $\mathbf{u}_0\in\mathbb{L}^{2p}(\Omega;\mathbb{H}^s(\mathbb{R}^n))$, for $s>n/2+2$ and $p=1,2,\cdots$. Then, under Property 2.5 and (2.6), we have the following a-priori energy estimate:*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2p}\right]\leq\left(1+2\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{H}^s}^{2p}\right]\right)e^{C(N,K,\widehat{K},p,T)T}. \tag{3.49}$$

Proof. Let us define the sequence of stopping times τ_M^n to be

$$\tau_M^n:=\inf_{t\geq 0}\Big\{t:\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}\geq M\Big\}. \tag{3.50}$$

Now we apply Itô's formula to $\|\mathrm{J}^s\mathbf{u}^n(\cdot)\|_{\mathbb{L}^2}^{2p}$ to get

$$\begin{aligned}
&\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{H}^s}^{2p}=\|\mathbf{u}^n(0)\|_{\mathbb{H}^s}^{2p}\\
&\quad-2p\int_0^{t\wedge\tau_M^n}\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p-2}\left(\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\mathrm{J}^s\left[\mathscr{A}(s,\mathbf{u}^n)\mathbf{u}^n\right],\mathrm{J}^s\mathbf{u}^n(s)\right)_{\mathbb{L}^2}\mathrm{d}s\\
&\quad+2p\int_0^{t\wedge\tau_M^n}\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p-2}\left(\mathrm{J}^s\sigma^n(s,\mathbf{u}^n(s))\mathrm{dW}_n(s),\mathrm{J}^s\mathbf{u}^n(s)\right)_{\mathbb{L}^2}\\
&\quad+p(2p-1)\int_0^{t\wedge\tau_M^n}\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p-2}\|\sigma^n(s,\mathbf{u}^n(s))\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}^2\mathrm{d}s\\
&\quad+\int_0^{t\wedge\tau_M^n}\int_{\mathrm{Z}_n}\Big(\|\mathbf{u}^n(s)+\gamma^n(s,\mathbf{u}^n(s),z)\|_{\mathbb{H}^s}^{2p}-\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p}\\
&\qquad-2p\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p-2}\left(\mathrm{J}^s\mathbf{u}^n(s),\mathrm{J}^s\gamma^n(s,\mathbf{u}^n(s),z)\right)_{\mathbb{L}^2}\Big)\mathscr{N}(\mathrm{d}s,\mathrm{d}z)\\
&\quad+2p\int_0^{t\wedge\tau_M^n}\int_{\mathrm{Z}_n}\|\mathbf{u}^n\|_{\mathbb{H}^s}^{2p-2}\left(\mathrm{J}^s\gamma^n(s-,\mathbf{u}^n(s-),z),\mathrm{J}^s\mathbf{u}^n(s-)\right)_{\mathbb{L}^2}\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z).
\end{aligned} \tag{3.51}$$

We write the term $\left(\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\mathrm{J}^s\left[\mathscr{A}(s,\mathbf{u}^n)\mathbf{u}^n\right],\mathrm{J}^s\mathbf{u}^n(s)\right)_{\mathbb{L}^2}$ as

$$\begin{aligned}
&\left(\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\mathrm{J}^s\left[\mathscr{A}(s,\mathbf{u}^n)\mathbf{u}^n\right],\mathrm{J}^s\mathbf{u}^n(s)\right)_{\mathbb{L}^2}\\
&=\left(\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\mathscr{A}(s,\mathbf{u}^n)\mathrm{J}^s\mathbf{u}^n(s),\mathrm{J}^s\mathbf{u}^n(s)\right)_{\mathbb{L}^2}\\
&\quad+\left(\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\left(\mathrm{J}^s\left[\mathscr{A}(s,\mathbf{u}^n)\mathbf{u}^n(s)\right]-\mathscr{A}(s,\mathbf{u}^n)\mathrm{J}^s\mathbf{u}^n(s)\right),\mathrm{J}^s\mathbf{u}^n(s)\right)_{\mathbb{L}^2}.
\end{aligned} \tag{3.52}$$

The term $\left(\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\mathscr{A}(s,\mathbf{u}^n)\mathrm{J}^s\mathbf{u}^n(s),\mathrm{J}^s\mathbf{u}^n(s)\right)_{\mathbb{L}^2}$ from (3.52) can be estimated using (2.9) as

$$\begin{aligned}
&\left(\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\mathscr{A}(s,\mathbf{u}^n)\mathrm{J}^s\mathbf{u}^n(s),\mathrm{J}^s\mathbf{u}^n(s)\right)_{\mathbb{L}^2}\\
&\geq-\frac{1}{2}\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\|\nabla\mathrm{A}\|_{\mathbb{L}^\infty}\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^2.
\end{aligned} \tag{3.53}$$

For estimating the second term from the right hand side of the equality (3.52), we use the Cauchy-Schwarz inequality and (2.11) to obtain

$$\left|\left(\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\left(\mathrm{J}^s\left[\mathscr{A}(s,\mathbf{u}^n)\mathbf{u}^n(s)\right]-\mathscr{A}(s,\mathbf{u}^n)\mathrm{J}^s\mathbf{u}^n(s)\right),\mathrm{J}^s\mathbf{u}^n(s)\right)_{\mathbb{L}^2}\right|$$

$$
\begin{aligned}
&\leq \|\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\mathscr{B}(s,\mathbf{u}^n)\mathrm{J}^s\mathbf{u}^n(s)\|_{\mathbb{L}^2}\,\|\mathrm{J}^s\mathbf{u}^n(s)\|_{\mathbb{L}^2}\\
&\leq \psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\Big(C\|\nabla \mathrm{A}\|_{\mathbb{L}^\infty}+C(\|\mathbf{u}^n\|_{\mathbb{L}^\infty})\,(1+\|\nabla\mathbf{u}^n\|_{\mathbb{L}^\infty})\Big)\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^2, \qquad (3.54)
\end{aligned}
$$

for $s>n/2+2$. Next we use (3.53) and (3.54) in (3.51) to get

$$
\begin{aligned}
&\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{H}^s}^{2p}\\
&\leq \|\mathbf{u}^n(0)\|_{\mathbb{H}^s}^{2p}+\int_0^{t\wedge\tau_M^n}\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\Big((2C+1)\|\nabla\mathrm{A}\|_{\mathbb{L}^\infty}\\
&\quad+C(\|\mathbf{u}^n\|_{\mathbb{L}^\infty})\,(1+\|\nabla\mathbf{u}^n\|_{\mathbb{L}^\infty})\Big)\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p}\mathrm{d}s\\
&\quad+2p\int_0^{t\wedge\tau_M^n}\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p-2}\left(\mathrm{J}^s\sigma^n(s,\mathbf{u}^n(s))\mathrm{dW}_n(s),\mathrm{J}^s\mathbf{u}^n(s)\right)_{\mathbb{L}^2}\\
&\quad+p(2p-1)\int_0^{t\wedge\tau_M^n}\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p-2}\|\sigma^n(s,\mathbf{u}^n(s))\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}^2\mathrm{d}s\\
&\quad+\int_0^{t\wedge\tau_M^n}\int_{\mathrm{Z}_n}\Big(\|\mathbf{u}^n(s)+\gamma^n(s,\mathbf{u}^n(s),z)\|_{\mathbb{H}^s}^{2p}-\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p}\\
&\quad-2p\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p-2}\left(\mathrm{J}^s\mathbf{u}^n(s),\mathrm{J}^s\gamma^n(s,\mathbf{u}^n(s),z)\right)_{\mathbb{L}^2}\Big)\mathcal{N}(\mathrm{d}s,\mathrm{d}z)\\
&\quad+2p\int_0^{t\wedge\tau_M^n}\int_{\mathrm{Z}_n}\|\mathbf{u}^n\|_{\mathbb{H}^s}^{2p-2}\left(\mathrm{J}^s\gamma^n(s-,\mathbf{u}^n(s-),z),\mathrm{J}^s\mathbf{u}^n(s-)\right)_{\mathbb{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z). \qquad (3.55)
\end{aligned}
$$

Now we take expectation in (3.55), use the property of the cutoff function, and use the fact that the third and final terms from the right hand side of the inequality (3.55) are local martingales having zero expectation to get

$$
\begin{aligned}
&\mathbb{E}\left[\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{H}^s}^{2p}\right]\\
&\leq \mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{H}^s}^{2p}\right]+C_N\mathbb{E}\left[\int_0^{t\wedge\tau_M^n}\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p}\mathrm{d}s\right]\\
&\quad+p(2p-1)\mathbb{E}\left[\int_0^{t\wedge\tau_M^n}\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p-2}\|\sigma^n(s,\mathbf{u}^n(s))\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}^2\mathrm{d}s\right]\\
&\quad+\mathbb{E}\Big[\int_0^{t\wedge\tau_M^n}\int_{\mathrm{Z}_n}\Big(\|\mathbf{u}^n(s)+\gamma^n(s,\mathbf{u}^n(s),z)\|_{\mathbb{H}^s}^{2p}-\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p}.\\
&\qquad-2p\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p-2}\left(\mathrm{J}^s\mathbf{u}^n(s),\mathrm{J}^s\gamma^n(s,\mathbf{u}^n(s),z)\right)_{\mathbb{L}^2}\Big)\lambda(\mathrm{d}z)\mathrm{d}s\Big]. \qquad (3.56)
\end{aligned}
$$

By virtue of Taylor's formula, we have

$$
\begin{aligned}
&\left|\|\mathbf{u}^n+\gamma^n(\cdot,\mathbf{u}^n,z)\|_{\mathbb{H}^s}^{2p}-\|\mathbf{u}^n\|_{\mathbb{H}^s}^{2p}-2p\|\mathbf{u}^n\|_{\mathbb{H}^s}^{2p-2}\left(\mathbf{u}^n,\gamma^n(\cdot,\mathbf{u}^n,z)\right)_{\mathbb{H}^s}\right|\\
&\leq C_p\left(\|\mathbf{u}^n\|_{\mathbb{H}^s}^{2p-2}\|\gamma^n(\cdot,\mathbf{u}^n,z)\|_{\mathbb{H}^s}^2+\|\gamma^n(\cdot,\mathbf{u}^n,z)\|_{\mathbb{H}^s}^{2p}\right). \qquad (3.57)
\end{aligned}
$$

Note that for $p=1$, we have

$$
\begin{aligned}
&\|\mathbf{u}^n+\gamma^n(\cdot,\mathbf{u}^n,z)\|_{\mathbb{H}^s}^{2p}-\|\mathbf{u}^n\|_{\mathbb{H}^s}^{2p}-2p\|\mathbf{u}^n\|_{\mathbb{H}^s}^{2p-2}\left(\mathbf{u}^n,\gamma^n(\cdot,\mathbf{u}^n,z)\right)_{\mathbb{H}^s}\\
&=\|\gamma^n(\cdot,\mathbf{u}^n,z)\|_{\mathbb{H}^s}^2. \qquad (3.58)
\end{aligned}
$$

By using (3.57) in (3.56), we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{H}^s}^{2p}\right]\\
&\leq \mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{H}^s}^{2p}\right]+C_N\mathbb{E}\left[\int_0^{t\wedge\tau_M^n}\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p}\mathrm{d}s\right]\\
&\quad+C_p\mathbb{E}\bigg[\int_0^{t\wedge\tau_M^n}\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p-2}\bigg(\|\sigma^n(s,\mathbf{u}^n(s))\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}^2\\
&\qquad+\int_{\mathrm{Z}_n}\|\gamma^n(\cdot,\mathbf{u}^n,z)\|_{\mathbb{H}^s}^2\lambda(\mathrm{d}z)\bigg)\mathrm{d}s\bigg]\\
&\quad+C_p\mathbb{E}\bigg[\int_0^{t\wedge\tau_M^n}\int_{\mathrm{Z}_n}\|\gamma^n(\cdot,\mathbf{u}^n,z)\|_{\mathbb{H}^s}^{2p}\lambda(\mathrm{d}z)\mathrm{d}s\bigg].
\end{aligned}\tag{3.59}
$$

Let us use Property 2.5, (2.6) and the fact that $|x|^{2p-2}\leq 1+|x|^{2p}$, for all $p\geq 1$, in (3.59) to get

$$
\begin{aligned}
&\mathbb{E}\left[\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{H}^s}^{2p}\right]\\
&\leq \mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{H}^s}^{2p}\right]+(C_N+C_p(K+\widehat{K}))\left[\int_0^t\mathbb{E}\left(1+\|\mathbf{u}^n(s\wedge\tau_M^n)\|_{\mathbb{H}^s}^{2p}\right)\mathrm{d}s\right].
\end{aligned}\tag{3.60}
$$

An application of Gronwall's inequality on (3.60) yields

$$
\mathbb{E}\left[\|\mathbf{u}^n(t\wedge\tau_M^n)\|_{\mathbb{H}^s}^{2p}\right]\leq\left(1+\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{H}^s}^{2p}\right]\right)e^{(C_N+C_p(K+\widehat{K}))t},\tag{3.61}
$$

for all $t\in[0,T]$. A calculation similar to (3.41) shows that $\lim_{M\to\infty}\mathbb{P}\Big\{\omega\in\Omega:\tau_M^n<t\Big\}=0$ and this implies $t\wedge\tau_M^n\to t$ as $M\to\infty$. Taking the limit as $M\to\infty$ in (3.61) and using the dominated convergence theorem, we get

$$
\mathbb{E}\left[\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2}\right]\leq\left(1+\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{H}^s}^{2}\right]\right)e^{(C_N+C_p(K+\widehat{K}))t},\tag{3.62}
$$

for $0\leq t\leq T$.

Let us take the supremum from 0 to $\widetilde{\tau}_M^n:=T\wedge\tau_M^n$ before taking the expectation in (3.55) to get

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq\widetilde{\tau}_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2p}\right]\\
&\leq\mathbb{E}\left[\|\mathbf{u}^n(0)\|_{\mathbb{H}^s}^{2p}\right]+C_N\mathbb{E}\left[\int_0^{\widetilde{\tau}_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2p}\mathrm{d}t\right]\\
&\quad+2p\mathbb{E}\left[\sup_{0\leq t\leq\widetilde{\tau}_M^n}\left|\int_0^t\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p-2}\left(\mathrm{J}^s\sigma^n(s,\mathbf{u}^n(s))\mathrm{dW}_n(s),\mathrm{J}^s\mathbf{u}^n(s)\right)_{\mathbb{L}^2}\right|\right]\\
&\quad+p(2p-1)\mathbb{E}\left[\int_0^{\widetilde{\tau}_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2p-2}\|\sigma^n(t,\mathbf{u}^n(t))\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}^2\mathrm{d}t\right]\\
&\quad+\mathbb{E}\bigg[\sup_{0\leq t\leq\widetilde{\tau}_M^n}\bigg|\int_0^t\int_{\mathrm{Z}_n}\bigg(\|\mathbf{u}^n(s)+\gamma^n(s,\mathbf{u}^n(s),z)\|_{\mathbb{H}^s}^{2p}-\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p}
\end{aligned}
$$

$$- 2p\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p-2}\left(\mathrm{J}^s\mathbf{u}^n(s),\mathrm{J}^s\gamma^n(s,\mathbf{u}^n(s),z)\right)_{\mathbb{L}^2}\Big)\mathcal{N}(\mathrm{d}s,\mathrm{d}z)\Big|\Bigg]$$

$$+ 2p\mathbb{E}\Bigg[\sup_{0\le t\le\widetilde{\tau}_M^n}\Big|\int_0^t\int_{\mathrm{Z}_n}\|\mathbf{u}^n\|_{\mathbb{H}^s}^{2p-2}\left(\mathrm{J}^s\gamma^n(s-,\mathbf{u}^n,z),\mathrm{J}^s\mathbf{u}^n\right)_{\mathbb{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z)\Big|\Bigg]. \tag{3.63}$$

Now we take the third term from the right hand side of the inequality (3.63) and use Burkholder-Davis-Gundy inequality and Young's inequality to obtain

$$\begin{aligned}
&2p\mathbb{E}\left[\sup_{0\le t\le\widetilde{\tau}_M^n}\left|\int_0^t\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p-2}\left(\mathrm{J}^s\sigma^n(s,\mathbf{u}^n(s))\mathrm{dW}_n(s),\mathrm{J}^s\mathbf{u}^n(s)\right)_{\mathbb{L}^2}\right|\right]\\
&\le C_p\mathbb{E}\left[\int_0^{\widetilde{\tau}_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{4p-2}\|\sigma^n(t,\mathbf{u}^n(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}\mathrm{d}t\right]^{1/2}\\
&\le C_p\mathbb{E}\left[\sup_{0\le t\le\widetilde{\tau}_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2p-1}\left(\int_0^{\widetilde{\tau}_M^n}\|\sigma^n(t,\mathbf{u}^n(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}\mathrm{d}t\right)^{1/2}\right]\\
&\le \frac{1}{4}\mathbb{E}\left[\sup_{0\le t\le\widetilde{\tau}_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2p}\right]+C_p\mathbb{E}\left[\int_0^{\widetilde{\tau}_M^n}\|\sigma^n(t,\mathbf{u}^n(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}\mathrm{d}t\right]^p\\
&\le \frac{1}{4}\mathbb{E}\left[\sup_{0\le t\le\widetilde{\tau}_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2p}\right]+C_pT^{p-1}\mathbb{E}\left[\int_0^{\widetilde{\tau}_M^n}\|\sigma^n(t,\mathbf{u}^n(t))\|^{2p}_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}\mathrm{d}t\right].
\end{aligned}\tag{3.64}$$

For the fifth term from the right hand side of the inequality can be estimated using (3.57) as

$$\begin{aligned}
&\mathbb{E}\Bigg[\sup_{0\le t\le\widetilde{\tau}_M^n}\Big|\int_0^t\int_{\mathrm{Z}_n}\Big(\|\mathbf{u}^n(s)+\gamma^n(s,\mathbf{u}^n(s),z)\|_{\mathbb{H}^s}^{2p}-\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p}\\
&\qquad-2p\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p-2}\left(\mathrm{J}^s\mathbf{u}^n(s),\mathrm{J}^s\gamma^n(s,\mathbf{u}^n(s),z)\right)_{\mathbb{L}^2}\Big)\mathcal{N}(\mathrm{d}t,\mathrm{d}z)\Big|\Bigg]\\
&\le C_p\mathbb{E}\left[\int_0^{\widetilde{\tau}_M^n}\int_{\mathrm{Z}_n}\Big(\|\mathbf{u}^n\|_{\mathbb{H}^s}^{2p-2}\|\gamma^n(t,\mathbf{u}^n,z)\|_{\mathbb{H}^s}^2+\|\gamma^n(t,\mathbf{u}^n,z)\|_{\mathbb{H}^s}^{2p}\Big)\mathcal{N}(\mathrm{d}t,\mathrm{d}z)\right]\\
&= C_p\mathbb{E}\left[\int_0^{\widetilde{\tau}_M^n}\int_{\mathrm{Z}_n}\Big(\|\mathbf{u}^n\|_{\mathbb{H}^s}^{2p-2}\|\gamma^n(t,\mathbf{u}^n,z)\|_{\mathbb{H}^s}^2+\|\gamma^n(t,\mathbf{u}^n,z)\|_{\mathbb{H}^s}^{2p}\Big)\lambda(\mathrm{d}z)\mathrm{d}t\right].
\end{aligned}\tag{3.65}$$

The final term from the right hand side of the inequality (3.63) can be estimated by using Burkholder-Davis-Gundy inequality and Young's inequality as

$$\begin{aligned}
&2p\mathbb{E}\Bigg[\sup_{0\le t\le\widetilde{\tau}_M^n}\Big|\int_0^t\int_{\mathrm{Z}_n}\|\mathbf{u}^n\|_{\mathbb{H}^s}^{2p-2}\left(\mathrm{J}^s\gamma^n(s-,\mathbf{u}^n(s-),z),\mathrm{J}^s\mathbf{u}^n\right)_{\mathbb{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z)\Big|\Bigg]\\
&\le C_p\mathbb{E}\left[\int_0^{\widetilde{\tau}_M^n}\int_{\mathrm{Z}_n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{4p-2}\|\gamma^n(t,\mathbf{u}^n(t),z)\|_{\mathbb{H}^s}^2\lambda(\mathrm{d}z)\mathrm{d}t\right]^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq C_p\mathbb{E}\left[\sup_{0\leq t\leq\widetilde{\tau}_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2p-1}\left(\int_0^{\widetilde{\tau}_M^n}\int_{\mathrm{Z}_n}\|\gamma^n(t,\mathbf{u}^n(t),z)\|_{\mathbb{H}^s}^2\lambda(\mathrm{d}z)\mathrm{d}t\right)^{1/2}\right]\\
&\leq \frac{1}{4}\mathbb{E}\left[\sup_{0\leq t\leq\widetilde{\tau}_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2p}\right]+C_p\mathbb{E}\left[\int_0^{\widetilde{\tau}_M^n}\int_{\mathrm{Z}_n}\|\gamma^n(t,\mathbf{u}^n(t),z)\|_{\mathbb{H}^s}^2\lambda(\mathrm{d}z)\mathrm{d}t\right]^p\\
&\leq \frac{1}{4}\mathbb{E}\left[\sup_{0\leq t\leq\widetilde{\tau}_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2p}\right]\\
&\quad+C_pT^{p-1}\mathbb{E}\left[\int_0^{\widetilde{\tau}_M^n}\int_{\mathrm{Z}_n}\|\gamma^n(t,\mathbf{u}^n(t),z)\|_{\mathbb{H}^s}^{2p}\lambda(\mathrm{d}z)\mathrm{d}t\right].
\end{aligned}\tag{3.66}
$$

Let us combine (3.64), (3.65) and (3.66), and substitute it in (3.63) to get

$$
\begin{aligned}
\frac{1}{2}\mathbb{E}\left[\sup_{0\leq t\leq\widetilde{\tau}_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2p}\right]&\leq \mathbb{E}\left[\|\mathbf{u}^n(0)\|_{\mathbb{H}^s}^{2p}\right]+C_N\mathbb{E}\left[\int_0^{\widetilde{\tau}_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2p}\mathrm{d}t\right]\\
&\quad+C_p\mathbb{E}\Bigg[\int_0^{\widetilde{\tau}_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2p-2}\Bigg(\|\sigma^n(t,\mathbf{u}^n)\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}^2\\
&\quad\quad+\int_{\mathrm{Z}_n}\|\gamma^n(t,\mathbf{u}^n(t),z)\|_{\mathbb{H}^s}^2\lambda(\mathrm{d}z)\Bigg)\mathrm{d}s\Bigg]\\
&\quad+C_pT^{p-1}\mathbb{E}\Bigg[\int_0^{\widetilde{\tau}_M^n}\Bigg(\|\sigma^n(t,\mathbf{u}^n(t))\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}^{2p}\\
&\quad\quad+\int_{\mathrm{Z}_n}\|\gamma^n(t,\mathbf{u}^n(t),z)\|_{\mathbb{H}^s}^{2p}\lambda(\mathrm{d}z)\Bigg)\mathrm{d}t\Bigg].
\end{aligned}\tag{3.67}
$$

By using Property 2.5, (2.6) and the fact that $|x|^{2p-2}\leq 1+|x|^{2p}$, for all $p\geq 1$, in (3.67), we obtain

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq\widetilde{\tau}_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2p}\right]&\leq 2\left[\|\mathbf{u}^n(0)\|_{\mathbb{H}^s}^{2p}\right]\\
&\quad+2\left(C_N+C_p(K+T^{p-1}\widehat{K})\right)\int_0^T\mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau_M^n}\left(1+\|\mathbf{u}^n(s)\|_{\mathbb{H}^s}^{2p}\right)\right]\mathrm{d}t.
\end{aligned}\tag{3.68}
$$

An application of Gronwall's inequality in (3.68) yields

$$
\mathbb{E}\left[\sup_{0\leq t\leq\widetilde{\tau}_M^n}\|\mathbf{u}^n(t)\|_{\mathbb{H}^s}^{2p}\right]\leq\left(1+2\left[\|\mathbf{u}^n(0)\|_{\mathbb{H}^s}^{2p}\right]\right)e^{2\left(C_N+C_p(K+T^{p-1}\widehat{K})\right)T}.\tag{3.69}
$$

A calculation similar to (3.41) yields $\lim_{M\to\infty}\mathbb{P}\Big\{\omega\in\Omega:\tau_M^n<T\Big\}=0$ and thus as $M\to\infty$, $T\wedge\tau_M^n\to T$. Let us take the limit $M\to\infty$ in (3.69) and use the dominated convergence theorem to get the estimate (3.49). □

Corollary 3.7 ($\mathbb{H}^s-$energy estimate)**.** *Let* $\mathbf{u}^n(\cdot)$ *be the unique solution of the system of stochastic ODE's (3.26) with* $\mathbf{u}_0\in\mathbb{L}^2(\Omega;\mathbb{H}^s(\mathbb{R}^n))$*, for* $s>n/2+2$*. Then, we* *have the following a-priori energy estimate:*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\|\mathbf{u}^n(t)\|^2_{\mathbb{H}^s}\right]\leq\left(1+2\mathbb{E}\left[\|\mathbf{u}_0\|^2_{\mathbb{H}^s}\right]\right)e^{2(C_N+9K)T}. \tag{3.70}$$

4. Existence and Uniqueness of Strong Solutions

In this section, we discuss the local, and global solvability (under the smallness assumptions on initial data and some extra assumptions on the noise coefficients) of the symmetric hyperbolic system (2.1). In order to do this, we first prove the existence and uniqueness of the system (3.26).

4.1. **Local Strong Solution.**

Let us now prove that the system (3.26) has a unique solution by exploiting the local monotonicity property (Theorem 3.4) of the nonlinear operator with cutoff function and a stochastic generalization of the Minty-Browder technique. Similar existence results for deterministic quasilinear symmetric hyperbolic system can be found in [30] and $2-D$ stochastic Navier-Stokes equations can be found in [28, 36].

Theorem 4.1 (Local Existence and Uniqueness)**.** *Let* $\mathbf{u}_0\in\mathbb{L}^4(\Omega;\mathbb{H}^s(\mathbb{R}^n))$ *be* $\mathscr{F}_0-$*measurable with* $s>n/2+2$ *be given. Then there exists a strong solution* $\mathbf{u}(\cdot)$ *to the problem (3.26) such that*

(*i*) $\mathbf{u}\in\mathbb{L}^4(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H}^s(\mathbb{R}^n)))$,

(*ii*) *the* $\mathscr{F}_t-$*adapted paths of* $\mathbf{u}(\cdot)$ *are càdlàg.*

Proof. Let us prove Theorem 4.1 by using a stochastic generalization of the Minty-Browder technique of local monotonicity in the following steps:

Step (1). Finite-dimensional Galerkin approximation of (2.1) and energy equality:

Let $\{e_1,e_2,\cdots\}$ be a fixed complete orthonormal system in $\mathbb{L}^2(\mathbb{R}^n)$ belonging to $\mathbb{H}^s(\mathbb{R}^n)$. Let $\mathbb{L}^2_n(\mathbb{R}^n):=\text{span}\{e_1,e_2,\cdots,e_n\}$ be the $n-$dimensional subspace of $\mathbb{L}^2(\mathbb{R}^n)$. Let us now consider the following Itô stochastic differential equation satisfied by $\{\mathbf{u}^n(\cdot)\}$:

$$\begin{cases}\mathrm{d}\mathbf{u}^n(t)=-\mathrm{F}(\mathbf{u}^n(t))\mathrm{d}t+\sigma^n(t,\mathbf{u}^n(t))\mathrm{dW}_n(t)\\ \qquad\qquad+\displaystyle\int_{\mathrm{Z}_n}\gamma^n(t-,\mathbf{u}^n(t-),z)\widetilde{\mathscr{N}}(\mathrm{d}t,\mathrm{d}z),\\ \mathbf{u}^n(0)=\mathbf{u}_0^n,\end{cases} \tag{4.1}$$

where $\mathrm{F}(\mathbf{u}^n(t))=\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\mathscr{A}(t,\mathbf{u}^n)\mathbf{u}^n(t)$ and the energy equality

$$\begin{aligned}\|\mathbf{u}^n(t)\|^2_{\mathbb{H}^{s-1}}&=\|\mathbf{u}^n(0)\|^2_{\mathbb{H}^{s-1}}-2\int_0^t(\mathrm{F}(\mathbf{u}^n(s)),\mathbf{u}^n(s))_{\mathbb{H}^{s-1}}\,\mathrm{d}s\\&\quad+2\int_0^t(\sigma^n(s,\mathbf{u}^n(s))\mathrm{dW}_n(s),\mathbf{u}^n(s))_{\mathbb{H}^{s-1}}\\&\quad+\int_0^t\|\sigma^n(s,\mathbf{u}^n(s))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\mathrm{d}s\\&\quad+\int_0^t\int_{\mathrm{Z}_n}\|\gamma^n(s-,\mathbf{u}^n(s-),z)\|^2_{\mathbb{H}^{s-1}}\mathscr{N}(\mathrm{d}s,\mathrm{d}z)\end{aligned}$$

$$+2\int_0^t\int_{\mathrm{Z}_n}(\gamma^n(s-,\mathbf{u}^n(s-),z),\mathbf{u}^n(s-))_{\mathbb{H}^{s-1}}\widetilde{\mathscr{N}}(\mathrm{d}t,\mathrm{d}z), \tag{4.2}$$

for all $t\in[0,T]$ and $s>n/2+2$. Let us now apply Itô's formula to $e^{-r(t)}\|\mathbf{u}^n(\cdot)\|^2_{\mathbb{H}^{s-1}}$ to get

$$\begin{aligned}
&\mathrm{d}\left(e^{-r(t)}\|\mathbf{u}^n(t)\|^2_{\mathbb{H}^{s-1}}\right)\\
&=-e^{-r(t)}\left(2\mathrm{F}(\mathbf{u}^n(t))+\dot{r}(t)\mathbf{u}^n(t),\mathbf{u}^n(t)\right)_{\mathbb{H}^{s-1}}\mathrm{d}t\\
&\quad+2e^{-r(t)}\left(\sigma^n(t,\mathbf{u}^n(t))\mathrm{dW}_n(t),\mathbf{u}^n(t)\right)_{\mathbb{H}^{s-1}}\\
&\quad+e^{-r(t)}\|\sigma^n(t,\mathbf{u}^n(t))\|^2_{\mathcal{L}(\mathbb{L}^2,\mathbb{H}^{s-1})}\mathrm{d}t\\
&\quad+e^{-r(t)}\int_{\mathrm{Z}_n}\|\gamma^n(t,\mathbf{u}^n(t),z)\|^2_{\mathbb{H}^{s-1}}\mathscr{N}(\mathrm{d}t,\mathrm{d}z)\\
&\quad+2e^{-r(t)}\int_{\mathrm{Z}_n}(\gamma^n(t-,\mathbf{u}^n(t-),z),\mathbf{u}^n(t-))_{\mathbb{H}^{s-1}}\widetilde{\mathscr{N}}(\mathrm{d}t,\mathrm{d}z).
\end{aligned} \tag{4.3}$$

Note that the second and final terms from the right hand side of the equality (4.3) are martingales having zero expectation. Let us now integrate the equality (4.3) from 0 to t and then take the expectation to obtain

$$\begin{aligned}
&\mathbb{E}\left[e^{-r(t)}\|\mathbf{u}^n(t)\|^2_{\mathbb{H}^{s-1}}\right]\\
&=\mathbb{E}\left[e^{-r(0)}\|\mathbf{u}^n(0)\|^2_{\mathbb{H}^{s-1}}\right]\\
&\quad-\mathbb{E}\left[\int_0^t e^{-r(s)}\left(2\mathrm{F}(\mathbf{u}^n(s))+\dot{r}(s)\mathbf{u}^n(s),\mathbf{u}^n(s)\right)_{\mathbb{H}^{s-1}}\mathrm{d}s\right]\\
&\quad+\mathbb{E}\left[\int_0^t e^{-r(s)}\left(\|\sigma^n(s,\mathbf{u}^n(s))\|^2_{\mathcal{L}(\mathbb{L}^2,\mathbb{H}^{s-1})}\mathrm{d}s\right.\right.\\
&\qquad\left.\left.+\int_{\mathrm{Z}_n}\|\gamma^n(t,\mathbf{u}^n(t),z)\|^2_{\mathbb{H}^{s-1}}\lambda(\mathrm{d}z)\right)\mathrm{d}s\right],
\end{aligned} \tag{4.4}$$

for all $t\in[0,T]$.

Step (2). Weak convergence of the sequences $\mathbf{u}^n(\cdot)$, $\mathrm{F}(\mathbf{u}^n(\cdot))$, $\sigma^n(\cdot,\cdot)$ and $\gamma^n(\cdot,\cdot,\cdot)$:

Using the energy estimates in Proposition 3.6 and Corollary 3.7, and the fact that $\mathbb{L}^4\left(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H}^s(\mathbb{R}^n))\right)\cong\left(\mathbb{L}^{4/3}\left(\Omega;\mathbb{L}^1(0,T;\mathbb{H}^{-s}(\mathbb{R}^n))\right)\right)^*$, where X^* denotes the dual of X, along with the Banach-Alaoglu theorem, we can extract a subsequence $\{\mathbf{u}^{n_k}\}$ of $\{\mathbf{u}^n\}$ which converges to the following limits (for notational simplicity, we denote the index n_k by n):

$$\begin{cases}
\mathbf{u}^n(\cdot)\xrightarrow{w^*}\mathbf{u}(\cdot)\text{ in }\mathbb{L}^4(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H}^s(\mathbb{R}^n))),\\
\mathbf{u}^n(T)\xrightarrow{w}\eta\in\mathbb{L}^2(\Omega;\mathbb{H}^s(\mathbb{R}^n)),\\
\mathrm{F}(\mathbf{u}^n(\cdot))\xrightarrow{w^*}\mathrm{F}_0(\cdot)\text{ in }\mathbb{L}^2(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H}^{s-1}(\mathbb{R}^n))).
\end{cases} \tag{4.5}$$

The final convergence (4.5) is obtained by using the Moser estimates (2.8) and the algebra property of $\mathbb{H}^{s-1}$ as

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}\|\psi_N(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\mathscr{A}(t,\mathbf{u}^n)\mathbf{u}^n(t)\|^2_{\mathbb{H}^{s-1}}\right]\\
&\leq \mathbb{E}\left[\sup_{0\leq t\leq T}\left(\psi_N^2(\|\mathbf{u}^n\|_{\mathbb{H}^{s-1}})\sum_{j=1}^n\left\|\mathrm{A}^j(t,x,\mathbf{u}^n)\right\|^2_{\mathbb{H}^{s-1}}\left\|\frac{\partial\mathbf{u}^n(t)}{\partial x_j}\right\|^2_{\mathbb{H}^{s-1}}\right)\right]\\
&\leq C_N\mathbb{E}\left[\sup_{0\leq t\leq T}\|\mathbf{u}^n(t)\|^2_{\mathbb{H}^s}\right]\\
&\leq C_N\left(1+2\mathbb{E}\left[\|\mathbf{u}_0\|^2_{\mathbb{H}^s}\right]\right)e^{2(C_N+9K)T},
\end{aligned}\tag{4.6}$$

and the right hand side of (4.6) is finite, since $\mathbf{u}^n\in\mathbb{L}^2(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H}^s(\mathbb{R}^n)))$, and independent of n. From the linear growth property (Property 2.5) and energy estimates given Proposition 3.6 and Corollary 3.7, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\|\sigma^n(t,\mathbf{u}^n(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}\mathrm{d}t+\int_0^T\int_{\mathrm{Z}_n}\|\gamma^n(t,\mathbf{u}^n(t),z)\|^2_{\mathbb{H}^s}\lambda(\mathrm{d}z)\mathrm{d}t\right]\\
&\leq K\mathbb{E}\left[\int_0^T\left(1+\|\mathbf{u}^n(t)\|^2_{\mathbb{H}^s}\right)\mathrm{d}t\right]\\
&\leq KT\Big(1+\left(1+2\mathbb{E}\left[\|\mathbf{u}_0\|^2_{\mathbb{H}^s}\right]\right)e^{2(C_N+9K)T}\Big)<+\infty.
\end{aligned}\tag{4.7}$$

Thus, we can extract subsequences $\{\sigma^{n_k}(\cdot,\mathbf{u}^{n_k})\}$ and $\{\gamma^{n_k}(\cdot,\mathbf{u}^{n_k},\cdot)\}$ which converge to the following limits (denoting the index n_k by n):

$$\begin{cases}\sigma^n(\cdot,\mathbf{u}^n)\mathrm{P}_n\xrightarrow{w}\Phi(\cdot)\text{ in }\mathbb{L}^2(\Omega;\mathbb{L}^2(0,T;\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s))),\\ \gamma^n(\cdot,\mathbf{u}^n,\cdot)\xrightarrow{w}\Psi(\cdot,\cdot)\text{ in }\mathcal{M}^2_T(\mathbb{L}^2;\mathbb{H}^s).\end{cases}\tag{4.8}$$

As discussed in Chapter 6, Theorem 7.5, [11] (see Theorem 2.6, [36], Theorem 3.2, [37] also), we extend the time interval from $[0,T]$ to an open interval $(-\mu,T+\mu)$ with $\mu>0$, and set the terms in the equation (4.1) equal to zero outside the interval $[0,T]$. Let $\phi(t)$ be a function in $\mathrm{H}^1(-\mu,T+\mu)$ with $\phi(0)=1$. Let us define $e_j(t)=\phi(t)e_j$ for all $j\geq 1$, where $\{e_j\}$ is the fixed orthonormal basis in $\mathbb{L}^2(\mathbb{R}^n)$ belonging to $\mathbb{H}^s(\mathbb{R}^n)$. Applying the Itô formula to the process $(\mathbf{u}^n(t),e_j(t))_{\mathbb{L}^2}$, one obtains

$$\begin{aligned}
&(\mathbf{u}^n(T),e_j(T))_{\mathbb{L}^2}\\
&=(\mathbf{u}^n(0),e_j)_{\mathbb{L}^2}+\int_0^T\left(\mathbf{u}^n(t),\frac{\mathrm{d}e_j(t)}{\mathrm{d}t}\right)_{\mathbb{L}^2}\mathrm{d}t\\
&\quad-\int_0^T(\mathrm{F}(\mathbf{u}^n(t)),e_j(t))_{\mathbb{L}^2}\,\mathrm{d}t+\int_0^T(\sigma^n(t,\mathbf{u}^n(t))\mathrm{d}\mathrm{W}_n(t),e_j(t))_{\mathbb{L}^2}\\
&\quad+\int_0^T\int_{\mathrm{Z}}(\gamma^n(t-,\mathbf{u}^n(t-),z),e_j(t-))_{\mathbb{L}^2}\,\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z).
\end{aligned}\tag{4.9}$$

We can take the term by term limit $n \to \infty$ in (4.9) by using the weak convergence given in (4.5) and (4.8). For instance, let us consider the stochastic integral present in the fourth term from the right hand side of the equality (4.9) with j fixed. Let $\mathcal{P}_T$ denote the class of predictable processes with values in $\mathbb{L}^2(\Omega;\mathbb{L}^2(0,T;\mathcal{L}_Q(\mathbb{L}^2,\mathbb{L}^2)))$ with the inner product defined by

$$(\sigma,\zeta)_{\mathcal{P}_T}=\mathbb{E}\left[\int_0^T \mathrm{Tr}(\sigma(t)Q\zeta^*(t))\mathrm{d}t\right] \text{ for all } \sigma,\zeta\in\mathcal{P}_T.$$

Also, let us define the map $\Upsilon:\mathcal{P}_T\to\mathbb{L}^2(\Omega;\mathbb{L}^2(0,T))$ by

$$\Upsilon(G)=\int_0^t (G(s)\mathrm{d}\mathrm{W}(s),e_j(s))_{\mathbb{L}^2}$$

for all $t\in[0,T]$. Clearly the map Υ is linear and continuous. Note that the weak convergence of $\sigma^n(\cdot,\mathbf{u}^n)\mathrm{P}_n \xrightarrow{w} \Phi(\cdot)$ in $\mathbb{L}^2(\Omega;\mathbb{L}^2(0,T;\mathcal{L}_Q(\mathbb{L}^2,\mathbb{L}^2)))$ (see (4.8)) implies that $(\sigma^n(t,\mathbf{u}^n(t))\mathrm{P}_n,\zeta)_{\mathcal{P}_T}\to(\Phi(t)\mathrm{d}\mathrm{W}(t),\zeta)_{\mathcal{P}_T}$ for all $\zeta\in\mathcal{P}_T$ as $n\to\infty$. From this, as $n\to\infty$, we conclude that

$$\begin{aligned}\Upsilon(\sigma^n(t,\mathbf{u}^n(t))\mathrm{P}_n)&=\int_0^t(\sigma^n(t,\mathbf{u}^n(t))\mathrm{P}_n\mathrm{d}\mathrm{W}(s),e_j(s))_{\mathbb{L}^2}\\&\to\int_0^t(\Phi(t)\mathrm{d}\mathrm{W}(s),e_j(s))_{\mathbb{L}^2},\end{aligned}$$

for all $t\in[0,T]$ and for each j.

Now we consider the stochastic integral present in the final term from the right hand side of the equality (4.9) with j fixed. Let $\mathscr{P}_T$ denote the class of predictable processes with values in $\mathcal{M}_T^2$ (see (2.5) for definition and Chapter 3, [25]) associated with the inner product

$$(\gamma,\xi)_{\mathscr{P}_T}=\mathbb{E}\left[\int_0^T\int_{\mathrm{Z}}(\gamma(t),\xi(t))_{\mathbb{L}^2}\lambda(\mathrm{d}z)\mathrm{d}t\right] \text{ for all } \gamma,\xi\in\mathscr{P}_T.$$

Let us now define the map $\Sigma:\mathscr{P}_T\to\mathbb{L}^2(\Omega;\mathbb{L}^2(0,T))$ by

$$\Sigma(K)=\int_0^t\int_{\mathrm{Z}_n}(K(s-,\omega,z),e_j(s-))_{\mathbb{L}^2}\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z),$$

for all $t\in[0,T]$. It can be easily seen that the map Σ is linear and continuous. Also, the weak convergence of $\gamma^n(\cdot,\mathbf{u}^n,\cdot)\xrightarrow{w}\Psi(\cdot,\cdot)$ in $\mathcal{M}_T^2(\mathbb{L}^2;\mathbb{L}^2)$ implies that $(\gamma^n(t,\mathbf{u}^n(t),z),\xi)_{\mathscr{P}_T}\to(\Psi(t,z),\xi)_{\mathscr{P}_T}$, for all $\xi\in\mathscr{P}_T$ and $t\in[0,T]$, as $n\to\infty$. Thus, as $n\to\infty$, we have

$$\begin{aligned}\Sigma(\gamma^n(t,\mathbf{u}^n(t),z))&=\int_0^t\int_{\mathrm{Z}_n}(\gamma^n(s-,\mathbf{u}^n(s-),z),e_j(s-))_{\mathbb{L}^2}\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z)\\&\to\int_0^t\int_{\mathrm{Z}}(\Psi(s-,z),e_j(s-))_{\mathbb{L}^2}\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z) \text{ as } n\to\infty,\end{aligned}$$

for all $t\in[0,T]$ and for each j. Here, we used Property 2.5 (see (P.3)) and that $\mathrm{Z}_n\uparrow\mathrm{Z}$ with $\lambda(\mathrm{Z}_n)<+\infty$.

Let us pass to limits termwise in the equation (4.9) to get

$$
\begin{aligned}
(\eta, e_j)_{\mathbb{L}^2}\phi(T) = (\mathbf{u}_0, e_j)_{\mathbb{L}^2} &+ \int_0^T \left(\mathbf{u}^n(t), \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} e_j\right)_{\mathbb{L}^2} \mathrm{d}t \\
&- \int_0^T \phi(t)\left(\mathrm{F}_0(t), e_j\right)_{\mathbb{L}^2} \mathrm{d}t + \int_0^T \phi(t)\left(\Phi(t)\mathrm{dW}(t), e_j\right)_{\mathbb{L}^2} \\
&+ \int_0^T \int_{\mathrm{Z}} \phi(t-)\left(\Psi(t-,z), e_j\right)_{\mathbb{L}^2} \widetilde{\mathscr{N}}(\mathrm{d}t, \mathrm{d}z). \qquad (4.10)
\end{aligned}
$$

Now we choose a subsequence $\{\phi_k\} \in \mathrm{H}^1(-\mu, T+\mu)$ with $\phi_k(0) = 1$, for $k \in \mathbb{N}$, such that $\phi_k \to \chi_t$ and the time derivative of ϕ_k converges to δ_t, where $\chi_t(s) = 1$, for $s \leq t$ and 0 otherwise, and $\delta_t(s) = \delta(t-s)$ is the Dirac $\delta-$distribution. Using ϕ_k in place of ϕ in (4.10) and then letting $k \to \infty$, we obtain

$$
\begin{aligned}
(\mathbf{u}(t), e_j)_{\mathbb{L}^2} = (\mathbf{u}_0, e_j)_{\mathbb{L}^2} &- \int_0^t \left(\mathrm{F}_0(s), e_j\right)_{\mathbb{L}^2} \mathrm{d}s \int_0^t \left(\Phi(s)\mathrm{dW}(s), e_j\right)_{\mathbb{L}^2} \\
&+ \int_0^t \int_{\mathrm{Z}} \left(\Psi(s-,z), e_j\right)_{\mathbb{L}^2} \widetilde{\mathscr{N}}(\mathrm{d}s, \mathrm{d}z), \qquad (4.11)
\end{aligned}
$$

for all $t < T$ with $(\mathbf{u}(T), e_j) = (\eta, e_j)$ for all j. Thus, we have

$$
\mathbf{u}(t) = \mathbf{u}_0 - \int_0^t \mathrm{F}_0(s)\mathrm{d}s + \int_0^t \Phi(s)\mathrm{dW}(s) + \int_0^t \int_{\mathrm{Z}} \Psi(s-,z)\widetilde{\mathscr{N}}(\mathrm{d}s, \mathrm{d}z), \qquad (4.12)
$$

with $(\mathbf{u}(T), e_j) = (\eta, e_j)$. Also $\mathbf{u}(\cdot)$ satisfies the Itô stochastic differential

$$
\begin{cases}
\mathrm{d}\mathbf{u}(t) = -\mathrm{F}_0(t)\mathrm{d}t + \Phi(t)\mathrm{dW}(t) + \displaystyle\int_{\mathrm{Z}} \Psi(t-,z)\widetilde{\mathscr{N}}(\mathrm{d}t, \mathrm{d}z), \\
\mathbf{u}(0) = \mathbf{u}_0,
\end{cases} \qquad (4.13)
$$

and the energy equality

$$
\begin{aligned}
\|\mathbf{u}(t)\|_{\mathbb{H}^{s-1}}^2 = \|\mathbf{u}_0\|_{\mathbb{H}^{s-1}}^2 &- 2\int_0^t \left(\mathrm{F}_0(s), \mathbf{u}(s)\right)_{\mathbb{H}^{s-1}} \mathrm{d}s \\
&+ 2\int_0^t \left(\Phi(s)\mathrm{dW}(s), \mathbf{u}(s)\right)_{\mathbb{H}^{s-1}} + \int_0^t \|\Phi(s)\|_{\mathcal{L}_Q(\mathbb{L}^2, \mathbb{H}^{s-1})}^2 \mathrm{d}s \\
&+ \int_0^t \int_{\mathrm{Z}} \|\Psi(s,z)\|_{\mathbb{H}^{s-1}}^2 \mathscr{N}(\mathrm{d}s, \mathrm{d}z) \\
&+ 2\int_0^t \int_{\mathrm{Z}} \left(\Psi(s-,z), \mathbf{u}(s-)\right)_{\mathbb{H}^{s-1}} \widetilde{\mathscr{N}}(\mathrm{d}s, \mathrm{d}z), \qquad (4.14)
\end{aligned}
$$

for all $t \in [0, T]$. A calculation similar to (4.4) yields

$$
\begin{aligned}
&\mathbb{E}\left[e^{-r(t)}\|\mathbf{u}(t)\|_{\mathbb{H}^{s-1}}^2\right] \\
&= \mathbb{E}\left[e^{-r(0)}\|\mathbf{u}_0\|_{\mathbb{H}^{s-1}}^2\right] - \mathbb{E}\left[\int_0^t e^{-r(s)}\left(2\mathrm{F}_0(s) + \dot{r}(s)\mathbf{u}(s), \mathbf{u}(s)\right)_{\mathbb{H}^{s-1}} \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\int_0^t e^{-r(s)}\left(\|\Phi(s)\|_{\mathcal{L}(\mathbb{L}^2, \mathbb{H}^{s-1})}^2 \mathrm{d}s + \int_{\mathrm{Z}} \|\Psi(s,z)\|_{\mathbb{H}^{s-1}}^2 \lambda(\mathrm{d}z)\right) \mathrm{d}s\right], \qquad (4.15)
\end{aligned}
$$

for all $t \in [0,T]$. Also, it should be noted that the initial value $\mathbf{u}^n(0)$ converges to $\mathbf{u}(0)$ strongly, i.e.,

$$\lim_{n\to\infty} \mathbb{E}\left[\|\mathbf{u}^n(0)-\mathbf{u}_0\|_{\mathbb{H}^s}^2\right]=0. \tag{4.16}$$

Step (3). Local Minty-Browder Technique and Local Strong Solution:

Let us now prove that $\mathrm{F}(\mathbf{u}(\cdot))=\mathrm{F}_0(\cdot)$, $\sigma(\cdot,\mathbf{u}(\cdot))=\Phi(\cdot)$ and $\gamma(\cdot,\mathbf{u}(\cdot),\cdot)=\Psi(\cdot,\cdot)$. For $\mathbf{v}\in\mathbb{L}^4(\Omega;\mathbb{L}^\infty(0,T;\mathbb{L}^2_m(\mathbb{R}^n)))$ with $m<n$, let us define

$$r(t)=2\int_0^t \left(C_N\|\mathbf{v}(s)\|_{\mathbb{H}^s}+\frac{L}{2}\right)\mathrm{d}s, \tag{4.17}$$

so that $\dot{r}(t)=2\left(C_N\|\mathbf{v}(s)\|_{\mathbb{H}^s}+\frac{L}{2}\right)$, a.e. For $\mathbf{u}^n,\mathbf{v}\in\mathbb{B}_M$, from the local monotonicity theorem (Remark 3.4), by using (3.11), we have

$$\begin{aligned}
&\mathbb{E}\Bigg[\int_0^T e^{-r(t)}\Big(2\left(\mathrm{F}(\mathbf{v}(t))-\mathrm{F}(\mathbf{u}^n(t)),\mathbf{v}(t)-\mathbf{u}^n(t)\right)_{\mathbb{H}^{s-1}}\\
&\quad+\dot{r}(t)\left(\mathbf{v}(t)-\mathbf{u}^n(t),\mathbf{v}(t)-\mathbf{u}^n(t)\right)_{\mathbb{H}^{s-1}}\Big)\mathrm{d}t\Bigg]\\
&\geq \mathbb{E}\left[\int_0^T e^{-r(t)}\|\sigma^n(t,\mathbf{v}(t))-\sigma^n(t,\mathbf{u}^n(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\mathrm{d}t\right]\\
&\quad+\mathbb{E}\left[\int_0^T e^{-r(t)}\int_{\mathrm{Z}_n}\|\gamma^n(t,\mathbf{v}(t),z)-\gamma^n(t,\mathbf{u}^n(t),z)\|^2_{\mathbb{H}^{s-1}}\lambda(\mathrm{d}z)\mathrm{d}t\right].
\end{aligned} \tag{4.18}$$

In (4.18), rearranging the terms and using the energy equality (4.4) to get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{-r(t)}\left(2\mathrm{F}(\mathbf{v}(t))+\dot{r}(t)\mathbf{v}(t),\mathbf{v}(t)-\mathbf{u}^n(t)\right)_{\mathbb{H}^{s-1}}\mathrm{d}t\right]\\
&\quad-\mathbb{E}\Bigg[\int_0^T e^{-r(t)}\Big(\|\sigma^n(t,\mathbf{v}(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\\
&\qquad+\int_{\mathrm{Z}_n}\|\gamma^n(t,\mathbf{v}(t),z)\|^2_{\mathbb{H}^{s-1}}\lambda(\mathrm{d}z)\Big)\mathrm{d}t\Bigg]\\
&\quad+2\mathbb{E}\left[\int_0^T e^{-r(t)}\left(\sigma^n(t,\mathbf{v}(t)),\sigma^n(t,\mathbf{u}^n(t))\right)_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\mathrm{d}t\right]\\
&\quad+2\mathbb{E}\left[\int_0^T e^{-r(t)}\int_{\mathrm{Z}_n}\left(\gamma^n(t,\mathbf{v}(t),z),\gamma^n(t,\mathbf{u}^n(t),z)\right)_{\mathbb{H}^{s-1}}\lambda(\mathrm{d}z)\mathrm{d}t\right]\\
&\geq \mathbb{E}\left[\int_0^T e^{-r(t)}\left(2\mathrm{F}(\mathbf{u}^n(t))+\dot{r}(t)\mathbf{u}^n(t),\mathbf{v}(t)\right)_{\mathbb{H}^{s-1}}\mathrm{d}t\right]\\
&\quad-\mathbb{E}\left[\int_0^T e^{-r(t)}\left(2\mathrm{F}(\mathbf{u}^n(t))+\dot{r}(t)\mathbf{u}^n(t),\mathbf{u}^n(t)\right)_{\mathbb{H}^{s-1}}\mathrm{d}t\right]\\
&\quad+\mathbb{E}\Bigg[\int_0^T e^{-r(t)}\Big(\|\sigma^n(t,\mathbf{u}^n(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\\
&\qquad+\int_{\mathrm{Z}_n}\|\gamma^n(t,\mathbf{u}^n(t),z)\|^2_{\mathbb{H}^{s-1}}\lambda(\mathrm{d}z)\Big)\mathrm{d}t\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\int_0^T e^{-r(t)}\left(2\mathrm{F}(\mathbf{u}^n(t)) + \dot{r}(t)\mathbf{u}^n(t), \mathbf{v}(t)\right)_{\mathbb{H}^{s-1}} \mathrm{d}t\right] \\
&\quad + \mathbb{E}\left[e^{-r(T)}\|\mathbf{u}^n(T)\|^2_{\mathbb{H}^{s-1}} - \|\mathbf{u}^n(0)\|^2_{\mathbb{H}^{s-1}}\right]. \qquad (4.19)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\mathbb{E}\Bigg[\int_0^T e^{-r(t)}\Big(2\left(\sigma^n(t,\mathbf{v}(t)), \sigma^n(t,\mathbf{u}^n(t))\right)_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})} \\
&\qquad -\|\sigma^n(t,\mathbf{v}(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\Big)\,\mathrm{d}t\Bigg] \\
&= \mathbb{E}\left[\int_0^T e^{-r(t)}2\left(\sigma(t,\mathbf{v}(t)), \sigma^n(t,\mathbf{u}^n(t))\right)_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\,\mathrm{d}t\right] \\
&\quad + \mathbb{E}\left[\int_0^T e^{-r(t)}2\left(\sigma^n(t,\mathbf{v}(t)) - \sigma(t,\mathbf{v}(t)), \sigma^n(t,\mathbf{u}^n(t))\right)_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\,\mathrm{d}t\right] \\
&\quad - \mathbb{E}\left[\int_0^T e^{-r(t)}\|\sigma^n(t,\mathbf{v}(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\mathrm{d}t\right] \\
&\leq \mathbb{E}\left[\int_0^T e^{-r(t)}2\left(\sigma(t,\mathbf{v}(t)), \sigma^n(t,\mathbf{u}^n(t))\right)_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\,\mathrm{d}t\right] \\
&\quad + 2C\left(\mathbb{E}\left[\int_0^T e^{-2r(t)}\,\|\sigma^n(t,\mathbf{v}(t)) - \sigma(t,\mathbf{v}(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\,\mathrm{d}t\right]\right)^{1/2} \\
&\quad + \mathbb{E}\left[\int_0^T -e^{-r(t)}\|\sigma^n(t,\mathbf{v}(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\mathrm{d}t\right], \qquad (4.20)
\end{aligned}
$$

where $C = \left(\mathbb{E}\left[\int_0^T e^{-2r(t)}\,\|\sigma^n(t,\mathbf{u}^n(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\,\mathrm{d}t\right]\right)^{1/2}$. Then applying the weak convergence (4.8) of $\{\sigma^n(\cdot,\mathbf{u}^n(\cdot)) : n \in \mathbb{N}\}$ to the first term and the Lebesgue Dominated Convergence Theorem to the second and third terms on the right hand side of the inequality (4.20), we deduce that (see Proposition 4.6, [8])

$$
\begin{aligned}
&\mathbb{E}\Bigg[\int_0^T e^{-r(t)}\Big(2\left(\sigma^n(t,\mathbf{v}(t)), \sigma^n(t,\mathbf{u}^n(t))\right)_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})} \\
&\qquad -\|\sigma^n(t,\mathbf{v}(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\Big)\,\mathrm{d}t\Bigg] \\
&\to \mathbb{E}\left[\int_0^T e^{-r(t)}\left(2\left(\sigma(t,\mathbf{v}), \Phi(t)\right)_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})} - \|\sigma(t,\mathbf{v}(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\right)\mathrm{d}t\right], \qquad (4.21)
\end{aligned}
$$

as $n \to \infty$. Similarly one can prove that

$$
\begin{aligned}
&\mathbb{E}\Bigg[\int_0^T e^{-r(t)}\int_{\mathrm{Z}_n}\Big(2\left(\gamma^n(t,\mathbf{v}(t),z), \gamma^n(t,\mathbf{u}^n(t),z)\right)_{\mathbb{H}^{s-1}} \\
&\qquad -\|\gamma^n(t,\mathbf{v}(t),z)\|^2_{\mathbb{H}^{s-1}}\Big)\,\lambda(\mathrm{d}z)\mathrm{d}t\Bigg]
\end{aligned}
$$

$$\to \mathbb{E}\Bigg[\int_0^T e^{-r(t)}\int_{\mathrm{Z}_n}\left(2\left(\gamma(t,\mathbf{v}(t),z),\Psi(t,z)\right)_{\mathbb{H}^{s-1}}\right.$$

$$\left.-\|\gamma(t,\mathbf{v}(t),z)\|^2_{\mathbb{H}^{s-1}}\right)\lambda(\mathrm{d}z)\mathrm{d}t\Bigg], \tag{4.22}$$

as $n\to\infty$. On taking liminf on both sides of (4.19), and using (4.21) and (4.22), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{-r(t)}\left(2\mathrm{F}(\mathbf{v}(t))+\dot{r}(t)\mathbf{v}(t),\mathbf{v}(t)-\mathbf{u}(t)\right)_{\mathbb{H}^{s-1}}\mathrm{d}t\right]\\
&-\mathbb{E}\left[\int_0^T e^{-r(t)}\left(\|\sigma(t,\mathbf{v}(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}+\int_{\mathrm{Z}}\|\gamma(\mathbf{v}(t),z)\|^2_{\mathbb{H}^{s-1}}\lambda(\mathrm{d}z)\right)\mathrm{d}t\right]\\
&+2\mathbb{E}\left[\int_0^T e^{-r(t)}\left(\sigma(t,\mathbf{v}(t)),\Phi(t)\right)_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\mathrm{d}t\right]\\
&+2\mathbb{E}\left[\int_0^T e^{-r(t)}\int_{\mathrm{Z}}\left(\gamma(t,\mathbf{v}(t),z),\Psi(t,z)\right)_{\mathbb{H}^{s-1}}\lambda(\mathrm{d}z)\mathrm{d}t\right]\\
&\geq \mathbb{E}\left[\int_0^T e^{-r(t)}\left(2\mathrm{F}_0(t)+\dot{r}(t)\mathbf{u}(t),\mathbf{v}(t)\right)_{\mathbb{H}^{s-1}}\mathrm{d}t\right]\\
&+\liminf_{n\to\infty}\mathbb{E}\left[e^{-r(T)}\|\mathbf{u}^n(T)\|^2_{\mathbb{H}^{s-1}}-\|\mathbf{u}^n(0)\|^2_{\mathbb{H}^{s-1}}\right].
\end{aligned}\tag{4.23}$$

By using the lower semicontinuity property of the $\mathbb{L}^2-$norm and the strong convergence of the initial data $\mathbf{u}^n(0)$ (see (4.16)), the second term on the right hand side of the inequality satisfies the following inequality:

$$\begin{aligned}
&\liminf_{n\to\infty}\mathbb{E}\left[e^{-r(T)}\|\mathbf{u}^n(T)\|^2_{\mathbb{H}^{s-1}}-\|\mathbf{u}^n(0)\|^2_{\mathbb{H}^{s-1}}\right]\\
&\geq \mathbb{E}\left[e^{-r(T)}\|\mathbf{u}(T)\|^2_{\mathbb{H}^{s-1}}-\|\mathbf{u}_0\|^2_{\mathbb{H}^{s-1}}\right].
\end{aligned}\tag{4.24}$$

Hence by using the energy equality (4.15) and (4.24) in (4.23), we find

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{-r(t)}\left(2\mathrm{F}(\mathbf{v}(t))+\dot{r}(t)\mathbf{v}(t),\mathbf{v}(t)-\mathbf{u}(t)\right)_{\mathbb{H}^{s-1}}\mathrm{d}t\right]\\
&\geq \mathbb{E}\left[\int_0^T e^{-r(t)}\left(\|\sigma(t,\mathbf{v}(t))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}+\int_{\mathrm{Z}}\|\gamma(t,\mathbf{v}(t),z)\|^2_{\mathbb{H}^{s-1}}\lambda(\mathrm{d}z)\right)\mathrm{d}t\right]\\
&-2\mathbb{E}\left[\int_0^T e^{-r(t)}\left(\sigma(t,\mathbf{v}(t)),\Phi(t)\right)_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\mathrm{d}t\right]\\
&-2\mathbb{E}\left[\int_0^T e^{-r(t)}\int_{\mathrm{Z}}\left(\gamma(t,\mathbf{v}(t),z),\Psi(t,z)\right)_{\mathbb{H}^{s-1}}\lambda(\mathrm{d}z)\mathrm{d}t\right]\\
&+\mathbb{E}\left[\int_0^T e^{-r(t)}\left(\|\Phi(t)\|^2_{\mathcal{L}(\mathbb{L}^2,\mathbb{H}^{s-1})}\mathrm{d}t+\int_{\mathrm{Z}}\|\Psi(t,z)\|^2_{\mathbb{H}^{s-1}}\lambda(\mathrm{d}z)\right)\mathrm{d}t\right]\\
&+\mathbb{E}\left[\int_0^T e^{-r(t)}\left(2\mathrm{F}_0(t)+\dot{r}(t)\mathbf{u}(t),\mathbf{v}(t)-\mathbf{u}(t)\right)_{\mathbb{H}^{s-1}}\mathrm{d}t\right].
\end{aligned}\tag{4.25}$$

Thus, by rearranging the terms in (4.25), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{-r(t)}\left(2\mathrm{F}(\mathbf{v}(t))-2\mathrm{F}_0(t)+\dot{r}(t)(\mathbf{v}(t)-\mathbf{u}(t)),\mathbf{v}(t)-\mathbf{u}(t)\right)_{\mathbb{H}^{s-1}}\mathrm{d}t\right]\\
&\geq \mathbb{E}\Bigg[\int_0^T e^{-r(t)}\Big(\|\sigma(t,\mathbf{v}(t))-\Phi(t)\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^{s-1})}\\
&\quad+\int_{\mathrm{Z}}\|\gamma(t,\mathbf{v}(t),z)-\Psi(t,z)\|^2_{\mathbb{H}^{s-1}}\lambda(\mathrm{d}z)\Big)\mathrm{d}t\Bigg]\geq 0. \qquad (4.26)
\end{aligned}$$

The estimate (4.26) holds for any $\mathbf{v}\in\mathbb{L}^4(\Omega;\mathbb{L}^\infty(0,T;\mathbb{L}^2_m(\mathbb{R}^n)))$ for any $m\in\mathbb{N}$, since the estimate (4.26) is independent of m and n. It can be easily seen by a density argument that the inequality (4.26) remains true for any $\mathbf{v}\in\mathbb{L}^4(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H}^s(\mathbb{R}^n)))$ for $s>n/2+2$. Indeed, for any $\mathbf{v}\in\mathbb{L}^4(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H}^s(\mathbb{R}^n)))$, there exists a strongly convergent subsequence $\mathbf{v}_m\in\mathbb{L}^4(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H}^s(\mathbb{R}^n)))$ that satisfies the inequality (4.26).

Taking $\mathbf{v}(\cdot)=\mathbf{u}(\cdot)$ in (4.26) immediately gives $\sigma(\cdot,\mathbf{v}(\cdot))=\Phi(\cdot)$ and $\gamma(\cdot,\mathbf{v}(\cdot),\cdot)=\Psi(\cdot,\cdot)$. Let us now take $\mathbf{v}(\cdot)=\mathbf{u}(\cdot)+\lambda\mathbf{w}(\cdot)$, $\lambda>0$, where $\mathbf{w}\in\mathbb{L}^4(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H}^s(\mathbb{R}^n)))$, and substitute for $\mathbf{v}$ in (4.26) to get

$$\mathbb{E}\left[\int_0^T e^{-r(t)}\left(2\mathrm{F}(\mathbf{u}(t)+\lambda\mathbf{w}(t))-2\mathrm{F}_0(t)+\dot{r}(t)\lambda\mathbf{w}(t),\lambda\mathbf{w}(t)\right)_{\mathbb{H}^{s-1}}\mathrm{d}t\right]\geq 0. \qquad (4.27)$$

Let us divide the inequality (4.27) by λ, use the continuity of $\psi_N(\cdot)$, the hemicontinuity property of $\mathscr{A}(\cdot,\cdot)\cdot$, and let $\lambda\to 0$ to obtain (see [30])

$$\mathbb{E}\left[\int_0^T e^{-r(t)}\left(\mathrm{F}(\mathbf{u}(t))-\mathrm{F}_0(t),\mathbf{w}(t)\right)_{\mathbb{H}^{s-1}}\mathrm{d}t\right]\geq 0. \qquad (4.28)$$

The final term from (4.27) tends to 0 as $\lambda\to 0$, since

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{-r(t)}\dot{r}(t)(\mathbf{w}(t),\mathbf{w}(t))_{\mathbb{H}^{s-1}}\mathrm{d}t\right]\\
&=2\mathbb{E}\left[\int_0^T e^{-r(t)}\left(C_N\|\mathbf{v}(t)\|_{\mathbb{H}^s}+\frac{L}{2}\right)\|\mathbf{w}(t)\|^2_{\mathbb{H}^{s-1}}\mathrm{d}t\right]\\
&\leq 2C_NT\mathbb{E}\left\{\sup_{0\leq t\leq T}\|\mathbf{v}(t)\|^2_{\mathbb{H}^s}\right\}^{1/2}\mathbb{E}\left[\sup_{0\leq t\leq T}\|\mathbf{w}(t)\|^4_{\mathbb{H}^{s-1}}\mathrm{d}t\right]^{1/2}\\
&\quad+LT\mathbb{E}\left[\sup_{0\leq t\leq T}\|\mathbf{w}(t)\|^2_{\mathbb{H}^{s-1}}\mathrm{d}t\right]<+\infty. \qquad (4.29)
\end{aligned}$$

Thus from (4.28), we have $\mathrm{F}(\mathbf{u}(t))=\mathrm{F}_0(t)$ in $\mathbb{L}^2(0,T;\mathbb{H}^{s-1})$ and hence $\mathbf{u}(\cdot)$ is a solution of the system (3.26) and $\mathbf{u}\in\mathbb{L}^4(\Omega,\mathbb{L}^\infty(0,T;\mathbb{H}^s(\mathbb{R}^n)))$ for $s>n/2+2$. From the energy estimates (see Proposition 3.5, Proposition 3.6 and Corollary 3.7), $\mathbf{u}^n(\cdot)$ is almost surely uniformly convergent on finite intervals $[0,T]$ to $\mathbf{u}(\cdot)$, from which it follows that $\mathbf{u}(\cdot)$ is adapted and càdlàg (Theorem 6.2.3, [1]). □

Now, we prove that the strong solution $\mathbf{u}(\cdot)$ of the system (3.26) is pathwise unique.

Theorem 4.2. *Let* $\mathbf{u}_0 \in \mathbb{L}^4(\Omega;\mathbb{H}^s(\mathbb{R}^n))$ *be* $\mathscr{F}_0-$*measurable for* $s > n/2+2$. *Let* $\mathbf{u}_j \in \mathbb{L}^4(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H}^s(\mathbb{R}^n)))$, $j=1,2$, *be two* $\mathscr{F}_t-$*adapted processes with càdlàg paths that are strong solutions of (3.26) having same initial value* $\mathbf{u}_j(0)=\mathbf{u}_0$. *Then*

$$\mathbf{u}_1(t)=\mathbf{u}_2(t), \ \textit{for all } t\in[0,T], \ \textit{a. s.}$$

Proof. For $i=1,2$, we define sequences of stopping times as

$$\zeta_{i,M} = \begin{cases} \inf_{0\leq t\leq T}\Big\{t : \|\mathbf{u}_i(t)\|_{\mathbb{H}^s}\geq M\Big\}, \\ T, \text{ if the set } \{\cdots\} \text{ is empty,}\end{cases} \tag{4.30}$$

and

$$\zeta_M = \zeta_{1,M}\wedge\zeta_{2,M}.$$

Let us apply Itô's formula to $\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{H}^{s-1}}^2$ to obtain

$$\begin{aligned}
&\|\mathbf{u}_1(t\wedge\zeta_M)-\mathbf{u}_2(t\wedge\zeta_M)\|_{\mathbb{H}^{s-1}}^2\\
&= -2\int_0^{t\wedge\zeta_M}\Big(\psi_N(\|\mathbf{u}_1\|_{\mathbb{H}^{s-1}})\mathscr{A}(s,\mathbf{u}_1)\mathbf{u}_1\\
&\quad -\psi_N(\|\mathbf{u}_2\|_{\mathbb{H}^{s-1}})\mathscr{A}(s,\mathbf{u}_2)\mathbf{u}_2, \mathbf{u}_1-\mathbf{u}_2\Big)_{\mathbb{H}^{s-1}}\mathrm{d}s\\
&\quad +2\int_0^{t\wedge\zeta_M}((\sigma(s,\mathbf{u}_1)-\sigma(s,\mathbf{u}_2))\mathrm{dW}(s),\mathbf{u}_1-\mathbf{u}_2)_{\mathbb{H}^{s-1}}\\
&\quad +\int_0^{t\wedge\zeta_M}\|\sigma(s,\mathbf{u}_1)-\sigma(s,\mathbf{u}_2)\|_{\mathcal{L}(\mathbb{L}^2,\mathbb{H}^{s-1})}^2\mathrm{d}s\\
&\quad +\int_0^{t\wedge\zeta_M}\int_{\mathrm{Z}}\|\gamma(s,\mathbf{u}_1,z)-\gamma(s,\mathbf{u}_2,z)\|_{\mathbb{H}^{s-1}}^2\mathscr{N}(\mathrm{d}s,\mathrm{d}z)\\
&\quad +2\int_0^{t\wedge\zeta_M}\int_{\mathrm{Z}}(\gamma(s-,\mathbf{u}_1,z)-\gamma(s-,\mathbf{u}_2,z),\mathbf{u}_1-\mathbf{u}_2)_{\mathbb{H}^{s-1}}\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z).
\end{aligned} \tag{4.31}$$

By using (2.9), (2.11), (2.14) and properties of the cutoff function, we estimate the first term from the right hand side of the equality (4.31) as (see (3.11), Theorem 3.4)

$$\begin{aligned}
&|(\psi_N(\|\mathbf{u}_1\|_{\mathbb{H}^{s-1}})\mathscr{A}(s,\mathbf{u}_1)\mathbf{u}_1-\psi_N(\|\mathbf{u}_2\|_{\mathbb{H}^{s-1}})\mathscr{A}(s,\mathbf{u}_2)\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)_{\mathbb{H}^{s-1}}|\\
&\leq C_N\left(\|\mathbf{u}_1\|_{\mathbb{H}^s}+\|\mathbf{u}_2\|_{\mathbb{H}^s}\right)\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{H}^{s-1}}^2
\end{aligned} \tag{4.32}$$

Let us take supremum over 0 to T and then take expectation in (4.31) to obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}\|\mathbf{u}_1(t\wedge\zeta_M)-\mathbf{u}_2(t\wedge\zeta_M)\|_{\mathbb{H}^{s-1}}^2\right]\\
&\leq C_{N,M}\mathbb{E}\left[\int_0^{T\wedge\zeta_M}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{H}^{s-1}}^2\mathrm{d}s\right]\\
&\quad +\mathbb{E}\Bigg[\int_0^{T\wedge\zeta_M}\Big(\|\sigma(s,\mathbf{u}_1)-\sigma(s,\mathbf{u}_2)\|_{\mathcal{L}(\mathbb{L}^2,\mathbb{H}^{s-1})}^2\\
&\quad +\int_{\mathrm{Z}}\|\gamma(s,\mathbf{u}_1,z)-\gamma(s,\mathbf{u}_2,z)\|_{\mathbb{H}^{s-1}}^2\lambda(\mathrm{d}z)\Big)\mathrm{d}s\Bigg]
\end{aligned}$$

$$+ 2\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^{t\wedge\zeta_M}((\sigma(s,\mathbf{u}_1)-\sigma(s,\mathbf{u}_2))\mathrm{dW}(s),\mathbf{u}_1-\mathbf{u}_2)_{\mathbb{H}^{s-1}}\right|\right]$$

$$+ 2\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^{t\wedge\zeta_M}\int_{\mathrm{Z}}(\gamma(\mathbf{u}_1,z)-\gamma(\mathbf{u}_2,z),\mathbf{u}_1-\mathbf{u}_2)_{\mathbb{H}^{s-1}}\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z)\right|\right]. \tag{4.33}$$

By using Burkholder-Davis-Gundy inequality and (P.3) (see Property 2.5), we get

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\|\mathbf{u}_1(t\wedge\zeta_M)-\mathbf{u}_2(t\wedge\zeta_M)\|_{\mathbb{H}^{s-1}}^2\right]$$

$$\leq 2(C_{N,M}+9L)\mathbb{E}\left[\int_0^T\sup_{0\leq s\leq t\wedge\zeta_M}\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_{\mathbb{H}^{s-1}}^2\mathrm{d}s\right]. \tag{4.34}$$

An application of Gronwall's inequality in (4.34) yields

$$\mathbf{u}_1(t\wedge\zeta_M)=\mathbf{u}_2(t\wedge\zeta_M),\ \text{for all } t\in[0,T],\ \text{a. s.} \tag{4.35}$$

Let us now pass $M\to\infty$ in (4.35) to find

$$\mathbf{u}_1(t)=\mathbf{u}_2(t),\ \text{for all } t\in[0,T],\ \text{a. s.}, \tag{4.36}$$

and hence the uniqueness of strong solution to the system (3.26) follows. □

Let us now prove the unique local solvability of the system (2.1).

Theorem 4.3. *Let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq0},\mathbb{P})$ be a given probability space and $\mathbf{u}_0\in\mathbb{L}^4(\Omega;\mathbb{H}^s(\mathbb{R}^n))$ be $\mathscr{F}_0-$measurable with $s>n/2+2$ be given. Then there exists a unique strong solution $(\mathbf{u},\tau)$ to the problem (2.1) such that*

(i) $\mathbf{u}\in\mathbb{L}^4(\Omega;\mathbb{L}^\infty(0,\tau(\omega);\mathbb{H}^s(\mathbb{R}^n)))$,
(ii) the $\mathscr{F}_t-$adapted paths of $\mathbf{u}(\cdot)$ are càdlàg.

Proof. From the unique global solvability of the system (3.26), for each fixed integer $N\geq3$ and $0<T<\infty$, there exists progressively measurable càdlàg process $\mathbf{u}_N(\cdot)$ such that

$$\mathbf{u}_N\in\mathbb{L}^4(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H}^s(\mathbb{R}^n))) \tag{4.37}$$

such that

$$\mathbf{u}_N(t)=\mathbf{u}_0-\int_0^t\psi_N(\|\mathbf{u}_N\|_{\mathbb{H}^{s-1}})\mathscr{A}(t,\mathbf{u}_N)\mathbf{u}_N\mathrm{d}s+\int_0^t\sigma(s,\mathbf{u}_N(s))\mathrm{dW}(s)$$

$$+\int_0^t\int_{\mathrm{Z}}\gamma(s-,\mathbf{u}_N(s-),z)\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z), \tag{4.38}$$

for all $0\leq t\leq T$ and almost all $\omega\in\Omega$. Let us now define

$$\tau_N=\inf_{t\geq0}\Big\{t:\|\mathbf{u}_N(t)\|_{\mathbb{H}^s}\geq N\Big\}. \tag{4.39}$$

For $3\leq N_1<N_2$, by an estimate similar to (4.34), we have

$$\mathbf{u}_{N_1}(t)=\mathbf{u}_{N_2}(t)\ \text{for all } t\in[0,\tau_{N_1}\wedge\tau_{N_2}],\ \text{a. s.}, \tag{4.40}$$

since $T > 0$ is arbitrary. From the definition of the stopping time (4.39), we have $\tau_{N_1} \leq \tau_{N_2}$, a. s. We can now define

$$\tau(\omega) = \lim_{N\uparrow\infty} \tau_N(\omega) \text{ a. s.,} \tag{4.41}$$

and

$$\mathbf{u}(t) = \lim_{N\to\infty} \mathbf{u}_N(t), \text{ for } 0 \leq t < \tau, \text{ a. s.}$$

Hence, $(\mathbf{u}, \tau)$ is a local strong solution to the problem (2.1).

Let us now prove that the local strong solution obtained above is unique. Let us assume that the pair $(\widetilde{\mathbf{u}}, \widetilde{\tau})$ is another local strong solution. Thus there exists an increasing sequence of stopping times $\{\widetilde{\tau}_N, N \geq 1\}$ converging to $\widetilde{\tau}$ such that $(\widetilde{\mathbf{u}}_N, \widetilde{\tau}_N)$ is a strong solution to (3.26) and

$$\widetilde{\tau}_N = \inf_{t\geq 0} \Big\{ t : \|\widetilde{\mathbf{u}}_N(t)\|_{\mathbb{H}^s} \geq N \Big\}.$$

But from the uniqueness theorem (Theorem 4.2), we have $\mathbf{u}_N(t) = \widetilde{\mathbf{u}}_N(t)$, for all $t \in [0, \tau_N \wedge \widetilde{\tau}_N]$, a. s., for $N \geq 1$. Let us take $N \uparrow \infty$ so that we get

$$\mathbf{u}(t) = \widetilde{\mathbf{u}}(t), \text{ for all } t \in [0, \tau \wedge \widetilde{\tau}], \text{ a. s.} \tag{4.42}$$

From (4.42), we can easily conclude that $\tau = \widetilde{\tau}$, a. s. If $\tau \neq \widetilde{\tau}$, then either $\tau > \widetilde{\tau}$ or $\tau < \widetilde{\tau}$, a. s. If $\widetilde{\tau} < \tau$, then by using uniqueness (4.42), we have

$$\begin{aligned}\lim_{t\uparrow\tau}\left[\sup_{0\leq s\leq t}\left\|\chi_{\{\widetilde{\tau}<\tau\}}\mathbf{u}(s)\right\|_{\mathbb{H}^s}\right] &= \lim_{N\uparrow\infty}\left[\sup_{0\leq s\leq \tau_N}\left\|\chi_{\{\widetilde{\tau}<\tau\}}\mathbf{u}(s)\right\|_{\mathbb{H}^s}\right] \\ &= \lim_{N\uparrow\infty}\left[\sup_{0\leq s\leq \widetilde{\tau}_N}\left\|\chi_{\{\widetilde{\tau}<\tau\}}\widetilde{\mathbf{u}}(s)\right\|_{\mathbb{H}^s}\right] = \infty,\end{aligned} \tag{4.43}$$

contradicts the fact that $\mathbf{u}$ does not explode before the stopping time τ. Similarly $\widetilde{\tau} > \tau$ is also not possible and hence $\tau = \widetilde{\tau}$, a. s. ☐

In order to prove the probabilistic estimate of τ, we assume that

$$\Big(C\|\nabla \mathrm{A}\|_{\mathbb{L}^\infty} + C(\|\mathbf{u}\|_{\mathbb{L}^\infty})\left(1 + \|\nabla\mathbf{u}\|_{\mathbb{L}^\infty}\right)\Big) \leq C\left(1 + \|\mathbf{u}\|_{\mathbb{H}^{s-1}}^{\beta}\right), \tag{4.44}$$

for $\beta \geq 1$ and $s > n/2 + 2$.

Theorem 4.4. *For a given $0 < \delta < 1$, we have*

$$\mathbb{P}\Big\{\tau > \delta\Big\} \geq 1 - C\delta^{\frac{2}{\beta}}\Big\{1 + 2\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{H}^s}^2\right]\Big\}, \tag{4.45}$$

for some positive constant C independent of $\mathbf{u}_0$ and δ.

Proof. Let $\mathbf{u}(\cdot)$ be the solution of (3.26) constructed in Theorem 4.3, and define the stopping times

$$\tau := \lim_{N\to\infty} \tau_N, \text{ a. s., where } \tau_N := \inf_{t\geq 0}\Big\{t : \|\mathbf{u}(t)\|_{\mathbb{H}^s} \geq N\Big\}. \tag{4.46}$$

For each $\delta > 0$, by using (4.44), a calculation similar to (3.63) yields

$$\mathbb{E}\left[\sup_{0\leq t\leq \delta\wedge\tau_N}\|\mathbf{u}(t)\|_{\mathbb{H}^s}^2\right]$$

$$
\begin{aligned}
&\leq \mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{H}^s}^2\right] + C(1+N^\beta)\mathbb{E}\left[\int_0^{\delta\wedge\tau_N} \|\mathbf{u}(t)\|_{\mathbb{H}^s}^2 \mathrm{d}t\right] \\
&\quad + 2\mathbb{E}\left[\sup_{0\leq t\leq\delta\wedge\tau_N}\left|\int_0^t (\sigma(s,\mathbf{u}(s))\mathrm{d}\mathrm{W}_n(s), \mathbf{u}(s))_{\mathbb{H}^s}\right|\right] \\
&\quad + \mathbb{E}\left[\int_0^{\delta\wedge\tau_N} \|\sigma(t,\mathbf{u}(t))\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}^2 \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\int_0^{\delta\wedge\tau_N}\int_{\mathrm{Z}_n} \|\gamma(t,\mathbf{u}(t),z)\|_{\mathbb{H}^s}^2 \lambda(\mathrm{d}z)\mathrm{d}t\right] \\
&\quad + 2\mathbb{E}\left[\sup_{0\leq t\leq\delta\wedge\tau_N}\left|\int_0^t\int_{\mathrm{Z}_n} (\gamma(s-,\mathbf{u}(s-),z),\mathbf{u}(s-))_{\mathbb{H}^s}\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z)\right|\right]. \qquad (4.47)
\end{aligned}
$$

Let us use Burkholder-Davis-Gundy inequality and Property 2.5 to obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq\delta\wedge\tau_N} \|\mathbf{u}(t)\|_{\mathbb{H}^s}^2\right] \\
&\leq 2\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{H}^s}^2\right] + 2C(1+N^\beta+9K)\mathbb{E}\left[\int_0^{\delta\wedge\tau_N} \|\mathbf{u}(t)\|_{\mathbb{H}^s}^2 \mathrm{d}t\right]. \qquad (4.48)
\end{aligned}
$$

An application of Gronwall's inequality in (4.48) yields

$$
\mathbb{E}\left[\sup_{0\leq t\leq\delta} \|\mathbf{u}(t\wedge\tau_N)\|_{\mathbb{H}^s}^2\right] \leq \left(1+2\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{H}^s}^2\right]\right) e^{2C(1+N^\beta+9K)\delta}, \qquad (4.49)
$$

where C is a positive constant independent of $\mathbf{u}$, N and δ.

For the given $0<\delta<1$, there exists a positive integer N such that

$$
\frac{1}{N+1} \leq \delta^{\frac{1}{\beta}} < \frac{1}{N}.
$$

From the definition of τ_N and τ(see (4.46)), we have

$$
\begin{aligned}
P &:= \left\{\omega\in\Omega : \sup_{0\leq t\leq\delta\wedge\tau_N} \|\mathbf{u}(t)\|_{\mathbb{H}^s} < N\right\} \\
&\subseteq Q := \left\{\omega\in\Omega : \tau_N > \delta\right\} \subseteq R := \left\{\omega\in\Omega : \tau > \delta\right\}. \qquad (4.50)
\end{aligned}
$$

In order to prove the first inclusion $P\subseteq Q$, let us take an $\omega\in P$. Then, there are two possibilities, either $\tau_N(\omega)>\delta$ or $\tau_N(\omega)\leq\delta$. If $\tau_N(\omega)>\delta$, then clearly $\omega\in Q$. Now, if $\tau_N(\omega)\leq\delta$, then $\tau_N\wedge\delta=\tau_N$ and the supremum norm inside P exceeds N for all the trajectories and the set P is empty and hence $P\subseteq Q$. For the second inclusion, we take $\omega\in Q$ and thus $\tau_N(\omega)>\delta$. Let us assume that $\omega\notin R$, so that we get $\tau(\omega)\leq\delta$. But from the definition of τ, we know that $\tau_N(\omega)\leq\tau(\omega)\leq\delta$, a contradiction. Hence $\omega\in R$ and $Q\subseteq R$. Thus it follows that

$$
\mathbb{P}\left\{\tau>\delta\right\} \geq \mathbb{P}\left\{\tau_N>\delta\right\} \geq \mathbb{P}\left\{\sup_{0\leq t\leq\delta} \|\mathbf{u}(t\wedge\tau_N)\|_{\mathbb{H}^s} < N\right\}
$$

$$
\begin{aligned}
&= \mathbb{P}\left\{\sup_{0\leq t\leq\delta}\|\mathbf{u}(t\wedge\tau_N)\|^2_{\mathbb{H}^s} < N^2\right\}\\
&\geq 1-\frac{1}{N^2}\mathbb{E}\left(\sup_{0\leq t\leq\delta}\|\mathbf{u}(t\wedge\tau_N)\|^2_{\mathbb{H}^s}\right)\\
&\geq 1-\frac{1}{N^2}\left(1+2\mathbb{E}\left[\|\mathbf{u}_0\|^2_{\mathbb{H}^s}\right]\right)e^{2C(1+N^\beta+9K)\delta}\\
&\geq 1-C\delta^{\frac{2}{\beta}}\left(1+2\mathbb{E}\left[\|\mathbf{u}_0\|^2_{\mathbb{H}^s}\right]\right), \qquad (4.51)
\end{aligned}
$$

where we used the Markov's inequality, (4.49) and

$$
\frac{1}{N^2}\leq\frac{1}{N^2}(N+1)^2\delta^{\frac{2}{\beta}}=\left(1+\frac{1}{N}\right)^2\delta^{\frac{2}{\beta}}\leq 4\delta^{\frac{2}{\beta}}.
$$

Note that in (4.51), C is a positive constant independent of δ and $\mathbf{u}_0$. □

Similar ideas for proving the positivity of the stopping time for stochastic quasilinear hyperbolic systems can be found in Theorem 1.3, [21], stochastic Euler equations can be found in Theorem 2.14, [29], and stochastic non-resistive MHD equations can be found in Theorem 3.17, [26].

4.2. **Global Strong Solution.** In this subsection, we obtain the global solvability results under the smallness assumptions on initial data and certain conditions satisfied by the noise coefficient. A similar theorem for multiplicative Gaussian noise was obtained in Theorem 1.4, [21]. For the global existence, we assume that each A^j's is independent of (t,x) and $\mathrm{A}^j\in \mathrm{C}^n(\mathbb{R}^m)$. Hence, there is a nondecreasing function $\phi_j:[0,\infty)\to[0,\infty)$ such that

$$
\begin{aligned}
&\Big(C\|\nabla\mathrm{A}\|_{\mathbb{L}^\infty}+C(\|\mathbf{u}\|_{\mathbb{L}^\infty})\left(1+\|\nabla\mathbf{u}\|_{\mathbb{L}^\infty}\right)\Big)\\
&\leq C\sum_{j=1}^n\phi_j\left(\|\mathbf{v}\|_{\mathbb{H}^{s-1}}\right)\|\mathbf{v}\|_{\mathbb{H}^{s-1}}\|\mathbf{v}\|^2_{\mathbb{H}^s},\ \text{ for all } \mathbf{v}\in\mathbb{H}^s(\mathbb{R}^n), \qquad (4.52)
\end{aligned}
$$

for some constant $C>0$.

Theorem 4.5. *Let $(\mathbf{u}(\cdot),\tau)$ be a solution of the stochastic quasilinear symmetric hyperbolic system (2.1) under the assumption: (4.52) and there exists constants $K_i>0, i=1,2,3$ such that*

$$
\left\{
\begin{aligned}
&K_1\|\mathbf{u}(t)\|^4_{\mathbb{H}^s}\leq\sum_{j=1}^\infty\left|\Big(\mathrm{J}^s\sigma(t,\mathbf{u}(t))Q^{1/2}e_j,\mathrm{J}^s\mathbf{u}(t)\Big)_{\mathbb{L}^2}\right|^2,\\
&\|\sigma(t,\mathbf{u}(t))\|_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}\leq K_2\|\mathbf{u}(t)\|_{\mathbb{H}^s},\\
&\|\gamma(t,\mathbf{u}(t),z)\|_{\mathbb{H}^s}\leq K_3\|\mathbf{u}(t)\|_{\mathbb{H}^s} \text{ and } \lambda(\mathrm{Z})<+\infty,
\end{aligned}
\right. \qquad (4.53)
$$

with K_3 sufficiently small and

$$
0<K_2^2<2K_1, \qquad (4.54)
$$

along with (P.2) in Property 2.5. Let $0<\varepsilon<1$ *be given, then there exists a* $\kappa(\varepsilon)$ *such that if* $\mathbb{E}\left(\|\mathbf{u}_0\|^2_{\mathbb{H}^s}\right)<\kappa(\varepsilon)$*, then we have*

$$\mathbb{P}\Big\{\omega\in\Omega:\tau=+\infty\Big\}>1-\varepsilon. \tag{4.55}$$

Proof. Let us define the sequence of stopping times τ_δ to be

$$\tau_\delta(\omega):=\inf_{t\geq 0}\Big\{t:\|\mathbf{u}(t)\|_{\mathbb{H}^s}\geq\delta\Big\}, \tag{4.56}$$

for $0<\delta<1$. Let us define $\mathrm{Y}(t)=\|\mathbf{u}(t\wedge\tau_\delta)\|^2_{\mathbb{H}^s}$, so that $\mathrm{Y}(\cdot)$ satisfies the Itô stochastic differential equation:

$$\begin{aligned}
\mathrm{dY}(t)&=\chi_{[0,\tau_\delta]}\bigg(-2\left(\left(\mathscr{B}(t\wedge\tau_\delta,\mathbf{u})+\mathscr{A}(t\wedge\tau_\delta,\mathbf{u})\right)\mathrm{J}^s\mathbf{u},\mathrm{J}^s\mathbf{u}\right)_{\mathbb{L}^2}\\
&\qquad+\|\sigma(t,\mathbf{u}(t\wedge\tau_\delta))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}+\int_{\mathrm{Z}}\|\gamma(t,\mathbf{u}(t\wedge\tau_\delta),z)\|^2_{\mathbb{H}^s}\lambda(\mathrm{d}z)\bigg)\mathrm{d}t\\
&\quad+2\sum_{j=1}^{\infty}\lambda_j^{1/2}\chi_{[0,\tau_\delta]}\left(\mathrm{J}^s\sigma(t\wedge\tau_\delta,\mathbf{u})e_j,\mathrm{J}^s\mathbf{u}\right)_{\mathbb{L}^2}\mathrm{d}\beta_j(t)\\
&\quad+\chi_{[0,\tau_\delta]}\int_{\mathrm{Z}}\Big[2\left(\mathrm{J}^s\gamma((t\wedge\tau_\delta)-,\mathbf{u},z),\mathrm{J}^s\mathbf{u}\right)_{\mathbb{L}^2}\\
&\qquad+\|\gamma(t\wedge\tau_\delta,\mathbf{u},z)\|^2_{\mathbb{H}^s}\Big]\widetilde{\mathscr{N}}(\mathrm{d}t,\mathrm{d}z),
\end{aligned} \tag{4.57}$$

where $\left(\mathscr{B}(\cdot,\mathbf{u})+\mathscr{A}(\cdot,\mathbf{u})\right)\mathrm{J}^s\mathbf{u}=\mathrm{J}^s[\mathscr{A}(\cdot,\mathbf{u})\mathbf{u}]$. Let us denote

$$\vartheta(t,\mathbf{u},z)=\chi_{[0,\tau_\delta]}\left[2\left(\mathrm{J}^s\gamma((t\wedge\tau_\delta)-,\mathbf{u},z),\mathrm{J}^s\mathbf{u}\right)_{\mathbb{L}^2}+\|\gamma(t\wedge\tau_\delta,\mathbf{u},z)\|^2_{\mathbb{H}^s}\right].$$

Let us choose $0<\alpha<\frac{1}{2}$ and $\eta>0$ and apply Itô's formula to $(\eta+\mathrm{Y}(\cdot))^\alpha$ to find

$$\begin{aligned}
&(\eta+\mathrm{Y}(t))^\alpha=(\eta+\mathrm{Y}(0))^\alpha\\
&+\alpha\int_0^t(\eta+\mathrm{Y}(s))^{\alpha-1}\chi_{[0,\tau_\delta]}\bigg(-2\left(\left(\mathscr{B}(s\wedge\tau_\delta,\mathbf{u})+\mathscr{A}(s\wedge\tau_\delta,\mathbf{u})\right)\mathrm{J}^s\mathbf{u},\mathrm{J}^s\mathbf{u}\right)_{\mathbb{L}^2}\\
&\quad+\|\sigma(s,\mathbf{u}(s\wedge\tau_\delta))\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}+\int_{\mathrm{Z}}\|\gamma(s\wedge\tau_\delta,\mathbf{u},z)\|^2_{\mathbb{H}^s}\lambda(\mathrm{d}z)\bigg)\mathrm{d}s\\
&+2\alpha\int_0^t(\eta+\mathrm{Y}(s))^{\alpha-1}\sum_{j=1}^{\infty}\lambda_j^{1/2}\chi_{[0,\tau_\delta]}\left(\mathrm{J}^s\sigma(s\wedge\tau_\delta,\mathbf{u})e_j,\mathrm{J}^s\mathbf{u}\right)_{\mathbb{L}^2}\mathrm{d}\beta_j(s)\\
&+2\alpha(\alpha-1)\int_0^t(\eta+\mathrm{Y}(s))^{\alpha-2}\sum_{j=1}^{\infty}\lambda_j\chi_{[0,\tau_\delta]}\left|\left(\mathrm{J}^s\sigma(s\wedge\tau_\delta,\mathbf{u})e_j,\mathrm{J}^s\mathbf{u}\right)_{\mathbb{L}^2}\right|^2\mathrm{d}s\\
&+\int_0^t\int_{\mathrm{Z}}\left((\eta+\mathrm{Y}(s-)+\vartheta(s-,\mathbf{u},x))^\alpha-(\eta+\mathrm{Y}(s-))^\alpha\right)\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z)\\
&+\int_0^t\int_{\mathrm{Z}}\Big((\eta+\mathrm{Y}(s)+\vartheta(s,\mathbf{u},z))^\alpha-(\eta+\mathrm{Y}(s))^\alpha\\
&\quad-\alpha(\eta+\mathrm{Y}(s))^{\alpha-1}\vartheta(s,\mathbf{u},z)\Big)\lambda(\mathrm{d}z)\mathrm{d}s.
\end{aligned} \tag{4.58}$$

Now we take the expectation on both sides of the equality (4.58) and note that the third and fifth term from the right hand side of the equality are martingales having zero expectation to get

$$
\begin{aligned}
&\mathbb{E}(\eta+\mathrm{Y}(t))^{\alpha}=\mathbb{E}(\eta+\mathrm{Y}(0))^{\alpha}\\
&+\alpha\mathbb{E}\bigg[\int_0^t(\eta+\mathrm{Y}(s))^{\alpha-1}\chi_{[0,\tau_\delta]}\bigg(-2\left(\left(\mathscr{B}(s\wedge\tau_\delta,\mathbf{u})+\mathscr{A}(s\wedge\tau_\delta,\mathbf{u})\right)\mathrm{J}^s\mathbf{u},\mathrm{J}^s\mathbf{u}\right)_{\mathbb{L}^2}\\
&\quad+\|\sigma(s\wedge\tau_\delta,\mathbf{u})\|^2_{\mathcal{L}_Q(\mathbb{L}^2,\mathbb{H}^s)}+\int_{\mathrm{Z}}\|\gamma(s,\mathbf{u}(s\wedge\tau_\delta),z)\|^2_{\mathbb{H}^s}\lambda(\mathrm{d}z)\bigg)\mathrm{d}s\bigg]\\
&-2\alpha(1-\alpha)\mathbb{E}\left[\int_0^t(\eta+\mathrm{Y}(s))^{\alpha-2}\sum_{j=1}^{\infty}\lambda_j\chi_{[0,\tau_\delta]}\left|(\mathrm{J}^s\sigma(s\wedge\tau_\delta,\mathbf{u})e_j,\mathrm{J}^s\mathbf{u})_{\mathbb{L}^2}\right|^2\mathrm{d}s\right]\\
&+\mathbb{E}\bigg[\int_0^t\int_{\mathrm{Z}}\Big((\eta+\mathrm{Y}(s)+\vartheta(s,\mathbf{u},z))^{\alpha}-(\eta+\mathrm{Y}(s))^{\alpha}\\
&\quad-\alpha(\eta+\mathrm{Y}(s))^{\alpha-1}\vartheta(s,\mathbf{u},z)\Big)\lambda(\mathrm{d}z)\mathrm{d}s\bigg].
\end{aligned}\tag{4.59}
$$

By using (2.9), commutator estimates, Moser estimates and (4.44), we get

$$
\begin{aligned}
&|((\mathscr{B}(t,\mathbf{u})+\mathscr{A}(t,\mathbf{u}))\,\mathrm{J}^s\mathbf{u},\mathrm{J}^s\mathbf{u})_{\mathbb{L}^2}|\\
&\leq\left(C\|\nabla\mathrm{A}\|_{\mathbb{L}^\infty}+C(\|\mathbf{u}\|_{\mathbb{L}^\infty})\left(1+\|\nabla\mathbf{u}\|_{\mathbb{L}^\infty}\right)\right)\|\mathbf{u}\|^2_{\mathbb{H}^s}\\
&\leq C\sum_{j=1}^{n}\phi_j\left(\|\mathbf{u}\|_{\mathbb{H}^{s-1}}\right)\|\mathbf{u}\|_{\mathbb{H}^{s-1}}\|\mathbf{u}\|^2_{\mathbb{H}^s}\leq CC_1\|\mathbf{u}\|_{\mathbb{H}^{s-1}}\|\mathbf{u}\|^2_{\mathbb{H}^s},
\end{aligned}\tag{4.60}
$$

where $C_1=\sum_{j=1}^{n}\phi_j(1)$. Let us use (4.60) and (4.53) in (4.59) to obtain

$$
\begin{aligned}
&\mathbb{E}(\eta+\mathrm{Y}(t))^{\alpha}\\
&\leq\mathbb{E}(\eta+\mathrm{Y}(0))^{\alpha}+\alpha\mathbb{E}\left[\int_0^t(\eta+\mathrm{Y}(s\wedge\tau_\delta))^{\alpha-1}\chi_{[0,\tau_\delta]}(2CC_1\delta+K_2^2)\mathrm{Y}(s)\mathrm{d}s\right]\\
&\quad-2\alpha(1-\alpha)K_1\mathbb{E}\left[\int_0^t(\eta+\mathrm{Y}(s))^{\alpha-2}\chi_{[0,\tau_\delta]}\mathrm{Y}(s)^2\mathrm{d}s\right]\\
&\quad+\mathbb{E}\bigg[\int_0^t\int_{\mathrm{Z}}\Big((\eta+\mathrm{Y}(s)+\vartheta(s,\mathbf{u},z))^{\alpha}-(\eta+\mathrm{Y}(s))^{\alpha}\\
&\quad-2\alpha(\eta+\mathrm{Y}(s))^{\alpha-1}\chi_{[0,\tau_\delta]}\left(\mathrm{J}^s\gamma(s\wedge\tau_\delta,\mathbf{u},z),\mathrm{J}^s\mathbf{u}\right)_{\mathbb{L}^2}\Big)\lambda(\mathrm{d}z)\mathrm{d}s\bigg].
\end{aligned}\tag{4.61}
$$

By using the inequality

$$
b^2\geq(1-\alpha)(b+c)^2-\frac{(1-\alpha)}{\alpha}c^2,
$$

for all $b,c\geq 0$, and for all $0<\alpha<1$, we can estimate $-2(1-\alpha)K_1(\eta+\mathrm{Y})^{\alpha-2}\mathrm{Y}^2$ as

$$
-2(1-\alpha)K_1(\eta+\mathrm{Y})^{\alpha-2}\mathrm{Y}^2\leq-2(1-\alpha)^2K_1(\eta+\mathrm{Y})^{\alpha}+2\frac{(1-\alpha)^2}{\alpha}K_1\eta^{\alpha},\tag{4.62}
$$

since $0 < \alpha < \frac{1}{2}$ and $\mathrm{Y} > 0$. We now simplify the integrand in the final term from the right hand side of the inequality (4.61), by using the inequality

$$(a+b)^\alpha \le 2^\alpha(|a|^\alpha + |b|^\alpha) \text{ for } a+b \ge 0 \text{ and all } \alpha \ge 0,$$

Cauchy-Schwarz inequality, and (4.53) as

$$\begin{aligned}
&\Big((\eta + \mathrm{Y}(s) + \vartheta(s,\mathbf{u},z))^\alpha - (\eta + \mathrm{Y}(s))^\alpha \\
&\quad - 2\alpha(\eta + \mathrm{Y}(s))^{\alpha-1}\chi_{[0,\tau_\delta]}\left(\mathrm{J}^s\gamma(s\wedge\tau_\delta,\mathbf{u},z), \mathrm{J}^s\mathbf{u}\right)_{\mathbb{L}^2}\Big) \\
&\le (2^\alpha - 1)(\eta + \mathrm{Y}(s))^\alpha + 2^\alpha|\vartheta(s,\mathbf{u},z)|^\alpha \\
&\quad + 2\alpha(\eta + \mathrm{Y}(s))^{\alpha-1}\chi_{[0,\tau_\delta]}\|\gamma(s\wedge\tau_\delta,\mathbf{u},z)\|_{\mathbb{H}^s}\|\mathbf{u}\|_{\mathbb{H}^s} \\
&\le \left((2^\alpha - 1) + 2^\alpha\left(2K_3 + K_3^2\right)^\alpha + 2\alpha K_3\right)(\eta + \mathrm{Y}(s))^\alpha.
\end{aligned} \tag{4.63}$$

Let us substitute (4.62) and (4.63) in (4.61) to find

$$\begin{aligned}
&\mathbb{E}(\eta + \mathrm{Y}(t))^\alpha \\
&\le \mathbb{E}(\eta + \mathrm{Y}(0))^\alpha + 2(1-\alpha)^2K_1\eta^\alpha t \\
&\quad + \Big[\left(\alpha(2CC_1\delta + K_2^2) + \left((2^\alpha - 1) + 2^\alpha(2K_3 + K_3^2)^\alpha + 2\alpha K_3\right)\lambda(\mathrm{Z})\right) \\
&\qquad - 2\alpha(1-\alpha)^2K_1\Big] \times \mathbb{E}\left[\int_0^{t\wedge\tau_\delta}(\eta + \mathrm{Y}(s))^\alpha \mathrm{d}s\right].
\end{aligned} \tag{4.64}$$

Since K_3 is sufficiently small, we can now choose sufficiently small $0 < \delta < 1$ and $0 < \alpha < \frac{1}{2}$ so that

$$\begin{aligned}
&\left(\alpha(2CC_1\delta + K_2^2) + \left((2^\alpha - 1) + 2^\alpha(2K_3 + K_3^2)^\alpha + 2\alpha K_3\right)\lambda(\mathrm{Z})\right) \\
&< 2\alpha(1-\alpha)^2K_1.
\end{aligned} \tag{4.65}$$

Thus from (4.65), we obtain

$$\mathbb{E}(\eta + \mathrm{Y}(t))^\alpha \le \mathbb{E}(\eta + \mathrm{Y}(0))^\alpha + 2(1-\alpha)^2K_1\eta^\alpha t. \tag{4.66}$$

Let us now pass $\eta \to 0$ in (4.66) to get

$$\mathbb{E}\left(\|\mathbf{u}(t\wedge\tau_\delta)\|_{\mathbb{H}^s}^{2\alpha}\right) \le \mathbb{E}\left(\|\mathbf{u}_0\|_{\mathbb{H}^s}^{2\alpha}\right), \text{ for all } t \ge 0. \tag{4.67}$$

Let us define the set $\mathscr{G}$ to be

$$\mathscr{G} := \Big\{\omega \in \Omega : \tau_\delta(\omega) < +\infty\Big\}.$$

Now, by using the Markov's inequality, (4.66), Jensen's inequality and Hölder's inequality, we obtain

$$\begin{aligned}
\mathbb{P}(\mathscr{G}) &= \mathbb{P}\Big\{\omega \in \Omega : \tau_\delta(\omega) < +\infty\Big\} \\
&= \mathbb{P}\bigg\{\liminf_{t\to\infty}\Big[\chi_{\mathscr{G}}\|\mathbf{u}(t\wedge\tau_\delta\|_{\mathbb{H}^s}\Big] \ge \delta\bigg\} \\
&\le \frac{1}{\delta^{2\alpha}}\mathbb{E}\bigg(\liminf_{t\to\infty}\Big[\chi_{\mathscr{G}}\|\mathbf{u}(t\wedge\tau_\delta)\|_{\mathbb{H}^s}^{2\alpha}\Big]\bigg) \\
&\le \frac{1}{\delta^{2\alpha}}\liminf_{t\to\infty}\mathbb{E}\Big(\chi_{\mathscr{G}}\|\mathbf{u}(t\wedge\tau_\delta)\|_{\mathbb{H}^s}^{2\alpha}\Big) \le \frac{1}{\delta^{2\alpha}}\mathbb{E}\left(\|\mathbf{u}_0\|_{\mathbb{H}^s}^{2\alpha}\right)
\end{aligned}$$

$$\leq \frac{1}{\delta^{2\alpha}} \left\{\mathbb{E}\left(\|\mathbf{u}_0\|_{\mathbb{H}^s}^2\right)\right\}^{\alpha} \leq \left\{\mathbb{E}\left(\|\mathbf{u}_0\|_{\mathbb{H}^s}^4\right)\right\}^{\frac{\alpha}{2}} \leq \varepsilon, \tag{4.68}$$

for $\mathbb{E}\left(\|\mathbf{u}_0\|_{\mathbb{H}^s}^2\right) \leq \varepsilon^{\frac{2}{\alpha}}\delta^4$. Hence, we have (4.55). □

Remark 4.6. If the Wiener noise $\mathrm{W}(\cdot,\cdot)$ contains only finite number of nodes, i. e., $\lambda_j = 0$ for all $j \geq k$, $k \geq 2$ and if $\sigma(t,\mathbf{u}) = \mathbf{u}$, then Theorem 4.5 holds, if there exists positive constants $K_i, i = 1, 2$ such that

$$\begin{cases} K_1\|\mathbf{u}\|_{\mathbb{H}^s}^4 \leq \displaystyle\sum_{j=1}^{k} \lambda_j \left|(\mathrm{J}^s\mathbf{u}e_j, \mathrm{J}^s\mathbf{u})_{\mathbb{L}^2}\right|^2, \\ \displaystyle\sum_{j=1}^{k} \lambda_j \|\mathbf{u}e_j\|_{\mathbb{H}^s}^2 \leq K_2\|\mathbf{u}\|_{\mathbb{H}^s}^2, \\ K_2 \leq 2K_1, \end{cases} \tag{4.69}$$

along with other assumptions in (4.53). Also, if the continuous martingale part in the noise is $\sum_{j=1}^{k} \alpha_j \mathbf{u}(\cdot)\beta_j(\cdot)$, where $\alpha_j \in \mathbb{R}$ and $\beta_j(\cdot)$'s are one dimensional Brownian motions, then the global existence holds for any $(\alpha_1, \cdots, \alpha_k) \in \mathbb{R}^k$.

If we consider more general forms of A^j's, that is, $\mathrm{A}^j = \mathrm{A}^j(t, x, \mathbf{u})$, then the estimate (4.60) reduces to

$$|((\mathscr{B}(t,\mathbf{u}) + \mathscr{A}(t,\mathbf{u}))\, \mathrm{J}^s\mathbf{u}, \mathrm{J}^s\mathbf{u})_{\mathbb{L}^2}| \leq C_2\|\mathbf{u}\|_{\mathbb{H}^s}^2,$$

for all $\mathbf{u} \in \mathbb{H}^s(\mathbb{R}^n)$, $\|\mathbf{u}\|_{\mathbb{H}^{s-1}} \leq 1$. Hence the necessary condition (4.65) becomes

$$\begin{aligned} &\left(\alpha(2C_2 + K_2^2) + \left((2^{\alpha} - 1) + 2^{\alpha}(2K_3 + K_3^2)^{\alpha} + 2\alpha K_3\right)\lambda(\mathrm{Z})\right) \\ &< 2\alpha(1-\alpha)^2 K_1. \end{aligned} \tag{4.70}$$

Note that (4.70) may not satisfy if C_2 is not relatively small to K_1, for sufficiently small α and K_3.

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