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# NEUMANN BOUNDARY CONDITIONS IN SHAPE OPTIMIZATION

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Abstract. We discuss the case of shape optimization problems with Neumann boundary conditions via a fixed domain method involving functional variations. An important tool is the implicit parametrization theorem.

## 1. Introduction

Shape optimization problems have a similar structure with optimal control problems, the difference being that the minimization parameter is the domain $\Omega$ itself, where the state system is defined. This also shows the high degree of complexity of shape optimization problems: the relationship $\Omega \to y_\Omega$ (the state) is strongly nonlinear, the geometry is the main unknown of the problem and it is very difficult to be handled, both from the theoretical and computational points of view. Even to pass to the limit on a minimizing sequence of states $\{y_{\Omega_n}\}$ may be very tricky since each term is defined on a different domain.

For the existence theory involving compactness assumptions like the uniform segment property (domains of class $C$), the uniform cone property (Lipschitzian domains), the Sverak [28] condition in dimension two, etc., we quote the monographs Pironneau [23], Bucur and Buttazzo [2], Henrot and Pierre [9], Neittaanmäki, Sprekels and Tiba [17].

In order to effectively obtain the solution (i.e. to find at least a local minimum, in a certain sense) many types of variations and transformations have been considered during the time: boundary variations, Hadamard [7], interior variations, Garabedian and Spencer [5], speed method Zolesio [34], mapping method Simon [26], topological variations Schumacher [25], Sokolowski and Zochowski [27], Nazarov and Sokolowski [15].

In general, high regularity assumptions are required on the admissible domains $\Omega \in \mathcal{O}$, the variations have a prescribed geometric form and, consequently, the class of domains where the minimization is really performed may be quite restricted.

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Another drawback is the difficult numerical implementation of many such methods: the unknown domain changes and has to be remeshed in each iteration of the algorithm; the mass matrix has to be recomputed as well, etc.

An answer to such questions is searched by the so-called "fixed domain methods", which have received much attention in the recent mathematical literature. Even the classical mapping method is of this type: the unknown domains $\Omega$ are transported on a given domain $B$ (a ball, for instance) via a smooth bijective transformation $T : \Omega \to B$. Then, $T$ and its derivatives appear in the coefficients of the transformed state equation and the shape optimization problem becomes a control by the coefficients problem.

There are other numerous ways to reduce certain shape optimization problems to optimal control problems via geometric controllability properties or the use of characteristic functions, etc., Lions [12], Pironneau [23], Neittaanmäki, Sprekels and Tiba [17]. In Neittaanmaki and Tiba [19] the case of Neumann boundary conditions is discussed via the controllability approach.

This paper is devoted to a new fixed domain approach using *functional variations* of the geometry, introduced in Neittaanmäki, Pennanen and Tiba [16]. See as well the recent survey Neittaanmäki and Tiba [18]. In [31], the case of boundary cost functionals is discussed in this setting. These papers discuss mainly the case of Dirichlet boundary conditions and use essentially a penalization idea going back to Natori and Kawarada [14], that cannot be extended to other boundary conditions. It is the aim of the present article to extend this methodology to Neumann conditions.

Functional variations are based on implicit representations of domains, but are different from the level set method of Osher and Sethian [21] (for instance, no time variable and no "evolution" of domains is used, no Hamilton-Jacobi equation is needed). A certain comparison may be made with the second method discussed in the classical work of Santosa [24].

The paper is organized as follows. In the next section, we briefly recall some preliminaries: the recent implicit parametrization theorem [30], [20] which will play a key role in the sequel and the case with Dirichlet boundary conditions. For recent related approaches, see [11], [33]. The last section is devoted to the examination of the main objective of the paper: shape optimization problems with Neumann boundary conditions.

## 2. Preliminaries

We briefly discuss the implicit parametrization approach in dimension two, according to [30], [20]. A general treatment of the implicit parametrization theorem, including the critical case and applications in nonlinear programming can be found in the recent preprint [32]. Notice that in geometric optimization problems, the case of dimension two is a case of interest. Moreover, starting with dimension three, iterated Hamiltonian systems are necessary for implicit parametrizations and the complexity of numerical approaches in shape optimization becomes prohibitive.

We consider the implicit equation:

$$g(x, y) = 0, \quad (x, y) \in D \subset R^2. \tag{2.1}$$

and assume that $D$ is a bounded domain, $g \in C^1(D)$ and $g(x_0, y_0) = 0$, for some given (noncritical) $(x_0, y_0)$ in $D$.

We associate to (2.1) the Hamiltonian system:

$$x'(t) = -\frac{\partial g}{\partial y}(x(t), y(t)), \tag{2.2}$$

$$y'(t) = \frac{\partial g}{\partial x}(x(t), y(t)), \tag{2.3}$$

$$x(0) = x_0,\ y(0) = y_0. \tag{2.4}$$

By the Peano theorem, the system (2.2)–(2.4) has at least one local solution around $(x_0, y_0)$, for $t \in I_{\max} = (T_-, T_+)$, the maximal existence interval and we have:

$$g(x(t), y(t)) = 0, \quad \forall\, t \in I_{\max}.$$

This is in fact the conservation property of Hamiltonian systems. The geometric remark behind the system (2.2)–(2.4) is that $\nabla g(x, y)$ is the normal vector to the level curve (2.1), if nonzero. Then $\left(-\frac{\partial g}{\partial y}, \frac{\partial g}{\partial x}\right)$ is the tangent vector. This is valid just in dimension two and is well known [29], p.61. In higher dimension, the construction is more involved, [32].

**Remark 2.1.** We have a local solution for (2.2)–(2.4) on $t \in I_{\max}$, in the usual sense for differential equations. We underline that this sense is different from that of local solution as appearing in the implicit function theorem. Namely, in the classical implicit function theorem, the requirement to obtain locally a function is essentially influenced by the choice of the axes, which is completely arbitrary. Consequently, the parametrization $[x(t), y(t)]$ provides a better description of the manifold.

We consider functional variations in (2.1):

$$g(x, y) + \lambda h(x, y) = 0, \quad \text{in } D, \tag{2.5}$$

$\lambda \in R$ and $h \in C^1(D)$ with $g(x_0, y_0) = h(x_0, y_0) = 0$.

We associate to (2.5) the perturbed Hamiltonian system

$$x'_\lambda = -\frac{\partial g}{\partial y}(x_\lambda, y_\lambda) - \lambda \frac{\partial h}{\partial y}(x_\lambda, y_\lambda), \tag{2.6}$$

$$y'_\lambda = \frac{\partial g}{\partial x}(x_\lambda, y_\lambda) + \lambda \frac{\partial h}{\partial y}(x_\lambda, y_\lambda), \tag{2.7}$$

$$x_\lambda(0) = x_0,\ y_\lambda(0) = y_0. \tag{2.8}$$

By well known properties of ordinary differential systems, [8], the solutions $(x_\lambda(t), y_\lambda(t))$ exist for $|\lambda|$ small on a common compact interval $I$, with $0 \in \text{int } I$. We denote by:

$$z_\lambda = \frac{x_\lambda - x}{\lambda},\ w_\lambda = \frac{y_\lambda - y}{\lambda},\ t \in I, \lambda \neq 0. \tag{2.9}$$

It is known that

**Proposition 2.2.** *Assume that $g \in C^2(D)$, $h \in C^1(D)$, with locally Lipschitzian derivatives of the highest order. We have $z_\lambda \to z$, $w_\lambda \to w$ in $C^1(I)$ and $z, w$ satisfy the system in variations:*

$$z' = -\nabla \left[\frac{\partial g}{\partial y}(x,y)\right] \cdot (z,w) - \frac{\partial h}{\partial y}(x,y), \text{ in } I, \tag{2.10}$$

$$w' = \nabla \left[\frac{\partial g}{\partial x}(x,y)\right] \cdot (z,w) + \frac{\partial h}{\partial x}(x,y), \text{ in } I, \tag{2.11}$$

$$z(0) = w(0) = 0. \tag{2.12}$$

**Remark 2.3.** The existence interval for (2.10)–(2.12) is the same as for (2.2)–(2.4), via some usual extension procedure.

We briefly comment now on the Dirichlet case in shape optimization problems, following [16], [18]:

$$\operatorname*{Min}_{\Omega \in \mathcal{O}} \int_\Lambda j(x, y_\Omega(x))dx, \tag{2.13}$$

$$-\Delta y_\Omega = f \quad \text{in } \Omega, \tag{2.14}$$

$$y_\Omega = 0 \quad \text{on } \partial\Omega. \tag{2.15}$$

Here, $\mathcal{O}$ is a family of admissible domains in $R^2$, satisfying certain regularity hypotheses and conditions like

$$E \subset \Omega \subset D, \ \forall\, \Omega \in \mathcal{O}, \tag{2.16}$$

where $E \subset D$ are given (bounded) domains, $E$ may be even void, etc.

Function $f \in L^2(D)$ and $\Lambda$ may be either $E$ (if nonvoid) or $\Omega$, $\partial\Omega$. The integrand $j(\cdot,\cdot) : D \times R \to R$ is of Carathéodory type.

A more general setting concerning the operators, the integrand functionals, the restrictions (for instance state constraints on the state $y_\Omega$), etc., may be considered as well.

A first property, specific to Dirichlet boundary conditions, ensures the approximate extension of the boundary value problem (2.14), (2.15) from the unknown and variable domain $\Omega$ to the fixed given domain $D$. We associate the approximating problem ($\varepsilon > 0$):

$$-\Delta \widehat{y} + \frac{1}{\varepsilon}(1 - H_\Omega)\widehat{y} = f \quad \text{in } D, \tag{2.17}$$

$$\widehat{y} = 0 \quad \text{on } \partial D. \tag{2.18}$$

Here, $H_\Omega : D \to R$ is the characteristic function of $\Omega$ in $D$.

In order to use functional variations, we define the class $\mathcal{O}$ of admissible domains by:

$$\Omega = \Omega_g = \operatorname{int}\{x \in D; g(x) \geq 0\}, \tag{2.19}$$

where $g \in X(D) \subset C(\overline{D})$ and $X(D)$ is some functional space on $D$. Examples of such spaces $X(D)$ may be finite element spaces. In numerical applications, it is enough to take $g$ piecewise continuous and bounded, Philip and Tiba [22].

Under representation formula (2.19), if we impose

$$g \geq 0 \quad \text{in } E, \tag{2.20}$$

then constraint (2.16) is fulfilled. If $H : R \to R$ is the Heaviside function, then $H(g) : D \to R$ is the characteristic function of $\overline{\Omega}_g$. Notice that in the critical case for $g$, the set $\{x \in D; g(x) = 0\}$ may have even positive measure, but $\Omega_g$ is always a Carathéodory set, i.e. $\Omega_g = \operatorname{int} \overline{\Omega}_g$.

Denote by $H^\varepsilon : R \to R$ a smoothing of the Yosida approximation of the maximal monotone extension of $H$ to $R \times R$. Then $H^\varepsilon(g) : D \to R$ is an approximation of the characteristic function $H_\Omega$. Such regularization procedures have been introduced in Makinen, Neittaanmäki and Tiba [13]. They are frequently used in image reconstruction problems and are sometimes called the Chan-Vese regularization, [3].

We further approximate (2.17) by

$$-\Delta y_\varepsilon + \frac{1}{\varepsilon}(1 - H^\varepsilon(g))y_\varepsilon = f \quad \text{in } D, \tag{2.21}$$

$$y_\varepsilon = 0 \quad \text{on } \partial D. \tag{2.22}$$

**Proposition 2.4.** *If $\Omega = \Omega_g$ is of class $C$, then $y_\varepsilon|_{\Omega_g} \to y$ (the solution of (2.14), (2.15)) weakly in $H^1(\Omega_g)$.*

This is a consequence of the Hedberg - Keldys stability property for domains of class $C$ (i.e. with continuous boundary), [17], [16]. Notice the very weak regularity assumptions on the admissible domains $\Omega \in \mathcal{O}$.

Based on Proposition 2.4, we approximate the shape optimization problem (2.13)–(2.16) by (2.13), (2.21), (2.22), (2.16). We underline that this is again a control by coefficients problem with the control $g$ entering just the lower order term of the differential operator and the state equation (2.21) being a simple penalization of (2.14). This is a very direct approach compared, for instance, with the mapping method. Its numerical implementation is very efficient and some examples are reported in [16], [22].

The "functional variations" [16], [18] $g + \lambda h$, $g, h \in X(D)$, $\lambda \in R$ may generate very complex geometric variations of $\Omega = \Omega_g$ combining boundary, interior and topological variations in (2.13)–(2.16). By combining Propositions 2.2, 2.4 one obtains the form of the gradient in the approximating shape optimization problem (2.13), (2.21), (2.22), (2.16). Theoretical results and numerical examples of this type can be found in [18], [31] as well.

It is the aim of this paper to provide at least a partial answer to the question how the above techniques can be extended to other boundary conditions. The extension and approximation provided by Proposition 2.4 are no more valid for Neumann or mixed boundary conditions, etc. Clearly, more regularity hypotheses have to be imposed on the admissible domains $\Omega \in \mathcal{O}$ in this case.

## 3. Extension of the Neumann problem and shape optimization

We fix now $\Omega = \Omega_g$ as in (2.19), under the assumption that $g \in C^1(\overline{D})$ and $|\nabla g(x,y)| > 0$ for $g(x,y) = 0$. In this non critical case, we have $\Omega_g = \{g(x,y) > 0\}$ and regularity conditions may be imposed on $\partial\Omega_g$ by increasing the regularity of $g$. We denote by $z_g \in C^1(I_g)^2$ a parametrization of $\partial\Omega_g$, obtained as in the previous section, with $I_g$ being its maximal existence interval (see (2.2)–(2.4)). If we also impose $g(x) < 0$ on $\partial D$, then $\partial\Omega_g$ does not meet $\partial D$.

Notice that under the above hypothesis and the implicit functions theorem, the curve $\{z_g(t), t \in I_g\}$ has no selfintersections. It may be periodic, Clarke [4], p.279, or intersect $\partial D$ or it may have a spiral type structure as in the Bendixson theorem, [8]. It is unclear how to handle this last case and we assume that $\{z_g(t), t \in I_g\}$, $I_g$ bounded, provides a complete description of $\partial\Omega_g$. In the case $\Omega_g$ is not simply connected, such parametrizations are supposed for each of the (assumed finitely many) components of the boundary $\partial\Omega_g$ and, in particular, the integral in relation (3.3) below has to be replaced by a corresponding sum of integrals. The subsequent arguments remain valid under straightforward modifications.

We discuss the Neumann boundary value problem:

$$-\Delta y_g + y_g = f \quad \text{in } \Omega_g, \tag{3.1}$$

$$\frac{\partial y_g}{\partial n} = 0 \quad \text{on } \partial\Omega_g. \tag{3.2}$$

More complex elliptic operators may be handled in a similar way.

We associate to (3.1), (3.2) a distributed optimal control problem in $D$ ($s > 2$):

$$\underset{u \in L^s(D)}{\operatorname{Min}} \int_{I_g} |\nabla g(z_g) \cdot \nabla y_u(z_g)|^2 dt, \tag{3.3}$$

$$-\Delta y_u + y_u = f + (1 - H(g))u \quad \text{in } D, \tag{3.4}$$

$$\frac{\partial y_u}{\partial n} = 0 \quad \text{on } \partial D, \tag{3.5}$$

where $\nabla g(z_g) \neq 0$, under our assumptions.

We assume that $u, f \in L^s(D)$ and, consequently, $y_u \in W^{2,s}(D)$ with $\nabla y_u \in C(\overline{D})^2$ by the Sobolev theorem (due to $s > 2$) if $\partial D$ is $C^{1,1}$. Then, the cost functional (3.3) makes sense.

Here, as in §2, $H : R \to R$ is the Heaviside function and (3.4), (3.5) represents the alternative for the Neumann problem to the "extension system" (2.17), (2.18) used for Dirichlet boundary conditions.

**Proposition 3.1.** *The optimal control problem (3.3)–(3.5) has at least one optimal pair $[u^*, y^*] \in L^s(D) \times W^{2,s}(D)$. The optimal value is 0.*

*Proof.* We have smoothness for $\partial\Omega_g$ by the implicit functions theorem and $\partial D$ is assumed smooth. By the trace theorem, there is an extension $\widetilde{y}_g \in W^{2,s}(D)$, not necessarily unique, with $\widetilde{y}_g|_{\Omega_g} = y_g$, $\frac{\partial \widetilde{y}_g}{\partial n} = 0$ on $\partial D$.

We denote by $u_g = -\Delta\widetilde{y}_g + \widetilde{y}_g - f \in L^s(D)$ and the pair $[u_g, \widetilde{y}_g]$ is admissible for (3.4), (3.5).

Notice that $\nabla g(z_g) \cdot \nabla\widetilde{y}_g(z_g) = 0$ due to the condition $\frac{\partial \widetilde{y}_g}{\partial n} = 0$ on $\partial\Omega_g$, i.e. due to (3.2), since $\nabla g(z_g)$ is normal to $\partial\Omega_g$.

Then, the cost (3.3) is null for $[u_g, \widetilde{y}_g]$ and $[u_g, \widetilde{y}_g]$ is an optimal pair denoted by $[u^*, y^*]$. $\square$

**Proposition 3.2.** *If $[u^*, y^*] \in L^s(D) \times W^{2,s}(D)$ is an optimal pair of (3.3)–(3.5), then $y^*|_{\Omega_g}$ is the solution of (3.1), (3.2).*

*Proof.* As the coefficient $1 - H(g)$ is null in $\Omega_g$, we see that $y^*|_{\Omega_g}$ satisfies (3.1) in the strong sense. We also have

$$\int_{I_g} |\nabla g(z_g) \cdot \nabla y^*(z_g)|^2 = 0$$

and this gives $\frac{\partial y^*}{\partial n}|_{\partial\Omega_g} = 0$ since $\nabla g(z_g) \neq 0$ and $z_g$ is a parametrization of the whole $\partial\Omega_g$, according to our hypotheses. $\square$

**Remark 3.3.** In the work of Joly and Rhaouty [10], the same problem is handled via a Lagrange multipliers approach, under $C^{1,1}$ regularity assumptions on the boundary $\partial\Omega$. The approach that we develop is purely analytic, not involving the geometry, in order to allow an efficient use in shape optimization problems. In the case of hyperbolic equations, C. Tsogka, J. Rodriguez, E. Bécache [1], apply an approach similar to [10].

We introduce now the approximation/regularization of (3.3)–(3.5), which will play an essential role later:

$$\text{Min}\{\frac{1}{2}\int_{I_g} |\nabla g(z_g) \cdot \nabla y_\varepsilon(z_g)|^2 + \frac{c(\varepsilon)}{2}|(1 - H^\varepsilon(g))u|^2_{L^s(D)}\}, \tag{3.6}$$

where $c(\varepsilon) \to 0$ for $\varepsilon \to 0$ and is defined in (3.15),

$$-\Delta y_\varepsilon + y_\varepsilon = f + (1 - H^\varepsilon(g))u \quad \text{in } D, \tag{3.7}$$

$$\frac{\partial y_\varepsilon}{\partial n} = 0 \quad \text{on } \partial D. \tag{3.8}$$

Without loss of generality, we may require $H^\varepsilon(r) = 1$, $r \geq 0$, $H^\varepsilon(r) = 0$, $r \leq -\varepsilon$ and $0 < H^\varepsilon(r) < 1$ for $-\varepsilon < r < 0$. See [13], [16].

**Theorem 3.4.** *The problem (3.6)–(3.8) has an optimal pair $[u^*_\varepsilon, y^*_\varepsilon]$ such that $[(1 - H^\varepsilon(g))u^*_\varepsilon, y^*_\varepsilon] \in L^s(D) \times W^{2,s}(D)$ and the optimal value $J(u^*_\varepsilon, y^*_\varepsilon) \to 0$ for $\varepsilon \to 0$.*

*Moreover, on a subsequence, we have*

$$y^*_\varepsilon \to y^* \quad \textit{weakly in } W^{2,s}(D), \tag{3.9}$$

$$(1 - H^\varepsilon(g))u^*_\varepsilon \to u^* \quad \textit{weakly in } L^s(D), \tag{3.10}$$

*where $[u^*, y^*]$ is an optimal pair of (3.3)–(3.5).*

*Proof.* Take in (3.7), $u = u_g$ as defined in the proof of Proposition 3.1. We compute

$$v_g^\varepsilon = f + (1 - H^\varepsilon(g))u_g + \Delta\widetilde{y}_g - \widetilde{y}_g = -H^\varepsilon(g)u_g, \tag{3.11}$$

again as in the proof of Proposition 3.1.

We have that $\{v_g^\varepsilon\}$ is bounded in $L^s(D)$, $s > 2$, by $|H^\varepsilon(g)| \leq 1$ a.e. in $D$ and $u_g \in L^s(D)$.

Moreover, by the properties of $H^\varepsilon$ and since $u_g = 0$ in $\Omega_g$, we have $v_g^\varepsilon \neq 0$ just in the set $\{x \in D; -\varepsilon < g < 0\} = G_\varepsilon$ and $\mu(G_\varepsilon) \to 0$ for $\varepsilon \to 0$ due to the regularity assumptions on $g$ and Weyl tube formula [6]. Namely, on a subsequence, $\overline{G}_\varepsilon \to \widetilde{G}$ in the Hausdorff-Pompeiu metric and their distance gives the radius of the tube. As $g$ is continuous, we get immediately that $\widetilde{G} = \{(x,y) \in D; g(x,y) = 0\}$.

As the limit is unique, we have the convergence on the whole sequence. Then $G_\varepsilon$ is contained in a "shrinking" tube around the curve $\widetilde{G}$ (and $\widetilde{G}$ has zero measure by the Stampacchia property [30]).

Then, on a subsequence, $v_g^\varepsilon \to 0$ a.e. in $D$.

The Lions lemma (see lemma 2.2.8, [17]) gives $v_g^\varepsilon \to 0$ strongly in $L^p(D)$, for any $p \in ]2, s[$ (and without taking subsequence since the limit is uniquely determined).

Denote by $y_g^\varepsilon$ the solution of (3.7), (3.8) corresponding to $u_g$. By (3.11), we have

$$-\Delta(y_g^\varepsilon - \widetilde{y}_g) + y_g^\varepsilon - \widetilde{y}_g = v_g^\varepsilon. \tag{3.12}$$

Consequently $y_g^\varepsilon - \widetilde{y}_g \to 0$ strongly in $W^{2,p}(D)$ and $\nabla y_g^\varepsilon - \nabla\widetilde{y}_g \to 0$ strongly in $C(\overline{D})^2$.

That is, by (3.12) we infer

$$\int_{I_g} |\nabla g(z_g) \cdot \nabla y_g^\varepsilon(z_g)|^2 dt \to \int_{I_g} |\nabla g(z_g) \cdot \nabla\widetilde{y}_g(z_g)|^2 dt = 0. \tag{3.13}$$

The optimal value of the problem (3.6)–(3.8) is estimated from above by the value corresponding to the admissible pair $[y_g^\varepsilon, u_g]$ and (3.13) proves that its limit for $\varepsilon \to 0$ is indeed 0.

Moreover, the existence of an optimal pair such that $[(1 - H^\varepsilon(g)u_\varepsilon^*, y_\varepsilon^*] \in L^s(D) \times W^{2,s}(D)$ is obvious since the optimal control problem is linear quadratic in these two terms. Then $J(u_\varepsilon^*, y_\varepsilon^*) \to 0$ for $\varepsilon \to 0$.

We need now a more precise computation:

$$\int_{I_g} |\nabla g(z_g) \cdot \nabla y_\varepsilon^*|^2 dt \leq \int_{I_g} |\nabla g(z_g) \cdot \nabla y_g^\varepsilon(z_g)|^2 dt + \tag{3.14}$$

$$+\frac{c(\varepsilon)}{2}|(1 - H^\varepsilon(g))u_g|^2_{L^s(D)} \leq kc(\varepsilon) + c|y_g^\varepsilon - \widetilde{y}_g|^2_{C^1(\overline{D})} \leq kc(\varepsilon) + C|v_g^\varepsilon|^2_{L^p(D)}$$

with some $p \in ]2, s[$ and some constants $c, C$ independent of $\varepsilon$.

By (3.11), we get

$$\int_D |v_g^\varepsilon|^p \leq \int_{-\varepsilon<g<0} |u_g|^p = c(\varepsilon)^p \to 0 \text{ for } \varepsilon \to 0, \tag{3.15}$$

by the Hölder inequality ($u_g \in L^s(D), s > p$) and Weyl tube formula.

Notice that (3.15) gives the definition of $c(\varepsilon)$ as well.

From (3.14), we get

$$\frac{c(\varepsilon)}{2}|(1-H^\varepsilon(g))u_\varepsilon^*|^2_{L^s(D)} \leq \int\limits_{I_g} |\nabla g(z_g)\cdot \nabla y_g^\varepsilon(z_g)|^2 dt +$$

$$+\frac{c(\varepsilon)}{2}|(1-H^\varepsilon(g))u_g|^2_{L^s(D)} \leq kc(\varepsilon)+Cc(\varepsilon)^2.$$

This yields $\{(1-H^\varepsilon(g))u_\varepsilon^*\}$ bounded in $L^s(D)$. And (4.7) gives $\{y_\varepsilon^*\}$ bounded in $W^{2,s}(D)$.

Denote by $[\widehat{u},\widehat{y}]$ the weak limits on a subsequence, in $L^s(D)\times W^{2,s}(D)$ of these functions.

Since $1-H^\varepsilon(g)\equiv 0$ for $g\geq 0$, then we have $(1-H(g))\widehat{u}=\widehat{u}$ a.e. $D$ as $\{g(x)=0\}$ has zero measure in our hypotheses. One can pass to the limit in (3.7), (3.8) and see that $[\widehat{u},\widehat{y}]$ satisfies (3.4), (3.5).

We have

$$0=\lim_{\varepsilon\to 0} J(u_\varepsilon^*,y_\varepsilon^*)=\frac{1}{2}\int\limits_{I_g}|\nabla g(z_g)\cdot\nabla\widehat{y}(z_g)|^2 dt.$$

This shows that the pair $[\widehat{u},\widehat{y}]$ is optimal for the problem (3.3)–(3.5) and we denote it by $[u^*,y^*]$. □

**Theorem 3.5.** *The first order optimality system for the control problem (3.6)–(3.8) is given by the state equation (3.7), (3.8), the adjoint state system ($p_\varepsilon \in L^{s'}(D)$, $\frac{1}{s}+\frac{1}{s'}=1$):*

$$\int\limits_D p_\varepsilon(-\Delta r+r)=\int\limits_{I_g}[\nabla g(z_g)\cdot\nabla y_\varepsilon^*(z_g)][\nabla g(z_g)\cdot\nabla r(z_g)]dt, \tag{3.16}$$

$$\forall\, r\in W^{2,s}(D),\ \frac{\partial r}{\partial n}=0 \text{ on } \partial D,$$

*and the maximum principle:*

$$(1-H^\varepsilon(g))p_\varepsilon + c(\varepsilon)F[(1-H^\varepsilon(g))u_\varepsilon^*]=0. \tag{3.17}$$

Above, $F: L^s(D)\to L^{s'}(D)$ is the duality mapping.

*Proof.* We take variations around $[u_\varepsilon^*,y_\varepsilon^*]$ of the form $[u_\varepsilon^*,y_\varepsilon^*]+\lambda[u,w]$, $\lambda\in R$, $v\in L^s(D)$ and $w$ defined by the equation in variations

$$-\Delta w+w=(1-H^\varepsilon(g))v \quad \text{in } D,$$

$$\frac{\partial w}{\partial n}=0 \quad \text{on } \partial D.$$

Comparing the cost associated to the above variation with the optimal one, dividing by $\lambda\neq 0$ and letting $\lambda\to 0$, we get

$$\int\limits_{I_g}[\nabla g(z_g)\cdot\nabla y_\varepsilon^*(z_g)][\nabla g(z_g)\cdot\nabla w(z_g)]+c(\varepsilon)F[(1-H^\varepsilon(g))u_\varepsilon^*]dx=0.$$

Taking into account (3.16) and the equation in variations, we infer

$$\int_D p^\varepsilon (1 - H^\varepsilon(g)) v dx + c(\varepsilon) \int_D v F[(1 - H^\varepsilon(g)) u_\varepsilon^*] dx = 0,$$

which yields (3.17) since $v$ is arbitrary. □

**Remark 3.6.** The unique solution of (3.16) is defined in the transposition sense, $p_\varepsilon \in L^{s'}(D)$. One can eliminate $u_\varepsilon$ from the state equation, via (3.17). The obtained result is a system of equations with unknowns $y_\varepsilon^*, p_\varepsilon$ that constitutes the approximate extension of the Neumann problem from $\Omega_g$ to $D$. We should note that the adjoint eqation (3.16) includes a source term on $\partial\Omega_g$, which is a similar situation with [1], [10]. Other related works are [16], [18]. The main novelty here is that everything is explicit, due to the tools developped in the previous section. The unknown geometry of the problem (i.e. $\Omega_g$ or $\partial\Omega_g$) is completely replaced by $g$ and $D$, that is our formulation is purely analytic.

**Proposition 3.7.** *We have $\{p_\varepsilon\}$ bounded in $L^{s'}(D)$, $p_\varepsilon \to p^*$ weakly in $L^{s'}(D)$ on a subsequence and $p^* = 0$ a.e. in $D \setminus \Omega_g$. Moreover:*

$$\int_D p^*(-\Delta r + r) = \int_{I_g} [\nabla g(z_g) \cdot \nabla y^*(z_g)][\nabla g(z_g) \cdot \nabla r(z_g)], \tag{3.18}$$

$$\forall\, r \in W^{2,s}(D),\ \frac{\partial r}{\partial n} = 0 \text{ on } \partial D.$$

*Proof.* The boundedness of $\{p_\varepsilon\}$ in $L^{s'}(D)$ is a consequence of (3.16) and (3.9), while $p^* = 0$ in $D \setminus \Omega_g$ is obtained by passing to the limit in (3.17). Relation (3.18) is again a consequence of (3.16), (3.9). □

**Remark 3.8.** The state system (3.4), (3.5), the equation (3.18) and the relation

$$(1 - H(g))p^* = 0 \quad \text{in } D \tag{3.19}$$

give the optimality conditions in the problem (3.3)–(3.5).

In particular, (3.19) expresses the fact that the gradient of the cost is null at the optimal pair. In applying iterative gradient algorithms to (3.3)–(3.5), in iteration $n$ one has $u_n$, computes $y_n$ by (3.4), (3.5), then $p_n$ by (3.18). The gradient is $(1 - H(g))p_n$ and gives the descent direction. In this case, it is not possible to eliminate the control mapping. The justification of the approximation/regularization approach of this section is given by the smooth character of the mapping $g \to H^\varepsilon(g)$. Then, the mapping $g \to y_\varepsilon$ as defined by (3.7), (3.8) is smooth as well.

We consider an example involving both boundary cost functionals and Neumann boundary conditions in the state system.

The family $\mathcal{O}$ of admissible domains in $R^2$ is defined by (2.19) with

$$g \in C^1(D),\ \nabla g(x,y) \neq 0 \text{ for } g(x,y) = 0,\ g(x,y) < 0 \text{ on } \partial D. \tag{3.20}$$

We also impose the Sverak [28] hypothesis that the number of the connected components of the complementary sets of the admissible domains is bounded from above (and, consequently, in (3.3) just finite sums may appear).

The shape optimization problem has the form

$$\underset{\Omega\in\mathcal{O}}{\operatorname{Min}}\{\int_{\partial\Omega} |y_\Omega - y_d|^2 d\sigma + \int_\Omega dx\}, \tag{3.21}$$

subject to

$$-\Delta y_\Omega + y_\Omega = f \quad \text{in } \Omega, \tag{3.22}$$

$$\frac{\partial y_\Omega}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{3.23}$$

where $f \in L^s(D)$, $s > 2$ is given.

By the well-known Stampacchia property [30], under conditions (3.20), the definition of $\Omega = \Omega_g$ can be written in the form

$$\Omega = \Omega_g = \{(x, y) \in D; g(x, y) > 0\}. \tag{3.24}$$

We have the description (2.2)–(2.4) for the boundary $\partial\Omega = \partial\Omega_g$ of $\Omega$ from (3.24) and, by the results in this section, the problem (3.21)–(3.23) is approximated by:

$$\underset{g\in G_{\mathrm{ad}}}{\operatorname{Min}}\{\int_{I_g} |y_\varepsilon(z_g) - y_d(z_g)|^2 \sqrt{(z_g')^2} dt + \int_D H^\varepsilon(g) dx\}, \tag{3.25}$$

subject to:

$$-\Delta y_\varepsilon + y_\varepsilon = f + c(\varepsilon)^{-1} F^{-1}[(H^\varepsilon(g) - 1)p_\varepsilon] \quad \text{in } D, \tag{3.26}$$

$$\frac{\partial y_\varepsilon}{\partial n} = 0 \quad \text{on } \partial D, \tag{3.27}$$

$$\int_D p_\varepsilon(-\Delta r + r) dx = \int_{I_g} [\nabla g(z_g) \cdot \nabla y_\varepsilon(z_g)][\nabla g(z_g) \cdot \nabla r(z_g)], \tag{3.28}$$

$$\forall\, r \in W^{2,s}(D),\ \frac{\partial r}{\partial n} = 0 \text{ on } \partial D.$$

This is a consequence of Theorem 3.5 and $G_{\mathrm{ad}}$ is the class of admissible shape functions $g$ from (3.20). In the numerical approximation of the optimization problem (3.21)–(3.23), one has to fix some $\varepsilon > 0$ and may fix $c(\varepsilon) = \varepsilon$ by adding in the definition (3.20), the condition $\int_{-\varepsilon<g<0} |u_g|^p = \varepsilon^p$. This set of controls is nonvoid since $g$ may be scaled by a positive function without affecting (3.24) or (3.20). Moreover, $u_g$ used in the proof of Proposition 3.1 and here, and $\Omega_g$ are not influenced by such a scaling. In fact, the family of admissible geometries $\Omega_g$ remains unchanged in case this supplementary restriction is added to (3.20).

We notice that the problem (3.25)–(3.28) is a control (mapping $g$) by the coefficients problem defined in the given domain $D$. The state system (3.26)–(3.28) is the optimality system for the optimal control problem (3.6)–(3.8), after the elimination of the control $u_\varepsilon$ (see Remark 3.6).

The geometry behind he formulation (3.25)–(3.28) is completely hidden. We look for a suboptimal solution $\widetilde{g}_\varepsilon$ of (3.25)–(3.28) and then find an approximation of the solution to the shape optimization problem (3.21)–(3.23) by using the definition (3.24). The approximation property is with respect to the value of the cost functional. A detailed analysis of the control problem (3.25)–(3.28) with the corresponding solution methods will be investigated in a future work.

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