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SENSITIVITY ANALYSIS OF PARAMETRIC ELLIPTIC OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper we investigate the sensitivity (variational stability) of parametric optimal control problems driven by nonlinear elliptic equations. We prove the continuity properties of the value function and of the multifunction of the optimal state-control pairs.

1. Introduction

One of the main problems in optimal control theory is the analysis of variations of the optimal solutions and of the value of the problem when we perturb the data, namely the governing equation and the cost (objective) functional. Such a sensitivity analysis (also known in the literature as "variational stability") is important because it provides information concerning the tolerances that are permitted in the specification of the mathematical models, it suggests ways to solve parametric problems, and also can lead to numerical methods to treat the problem.

In this paper, we conduct such a study for a class of nonlinear optimal control problems. We mention that Buttazzo-Dal Maso [5] provided a framework for the sensitivity analysis of optimal control problems using the formalism of multiple $\Gamma-$operators. They illustrated their method on optimal control problems driven by ordinary differential equations. Later, Migorski [11] considered systems driven by linear elliptic equations. A more detailed presentation of the subject can be found in the books of Buttazzo [4] and Dontchev-Zolezzi [7]. Finally, we also mention the relevant books of Ahmed [1] (identification of evolution systems) and Barbu [2] (optimal control of stationary and dynamic variational inequalities).

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$ and E a compact metric space (the parameter space). We deal with the following parametric nonlinear

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elliptic optimal control problem

$$(P_\lambda) \qquad \begin{array}{l} J(x,u,\lambda) := \int_\Omega L(z,x(z),Dx(z),u(z),\lambda)\,dz \to \inf =: m(\lambda), \\ -div\ a(Dx(z),\lambda) = f(z,x(z),\lambda)\,u(z) \text{ in } \Omega,\ x\mid_{\partial\Omega}=0, \\ |u(z)| \le \theta(z,\lambda) \text{ for a .a. } z\in\Omega \end{array}$$

where $L : \Omega\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times E\to\mathbb{R}$, $f : \Omega\times\mathbb{R}\times E\to\mathbb{R}$, $a : \mathbb{R}^N\times E\to\mathbb{R}^N$ and $\theta : \Omega\times E\to\mathbb{R}_+$.

In the next section, we recall the main mathematical notions which will be used in the analysis of problem (P_λ).

2. Mathematical background

Let Z, W be Hausdorff topological spaces. We say that $G : Z\to 2^W\setminus\{\varnothing\}$ is "upper semicontinuous" ("usc" for short), if for all open subsets $U\subseteq W$, the set $G^+(U) := \{z\in Z : G(z)\subseteq U\}$ is open.

Also, let (X,τ) be a Hausdorff topological space with τ denoting the topology of X, and let $\{C_n\}_{n\ge1}\subseteq 2^X\setminus\{\varnothing\}$. We define

$$K_\tau - \liminf_{n\to\infty} C_n = \left\{x\in X : x = \tau - \lim_{n\to\infty} x_n,\ x_n\in C_n \text{ for all } n\in\mathbb{N}\right\},$$

$$K_\tau - \limsup_{n\to\infty} C_n = \left\{x\in X : x = \tau - \lim_{k\to\infty} x_{n_k},\ x_{n_k}\in C_{n_k},\ n_k < n_{k+1}, \forall k\in\mathbb{N}\right\},$$

If $C = K_\tau - \liminf_{n\to\infty} C_n = K_\tau - \limsup_{n\to\infty} C_n$, then we write $C_n \overset{K_\tau}{\to} C$.

Now suppose that $(X,\|.\|)$ is a Banach space with dual X^* and s denotes the strong (norm) topology on X and w denotes the weak topology on X. Again let $\{C_n\}_{n\ge1}\subseteq 2^X\setminus\{\varnothing\}$. We say that the sequence $\{C_n\}_{n\ge1}$ converges to C in the sense of Mosco, denoted by $C_n \overset{M}{\to} C$, if and only if we have

$$C = K_w - \limsup_{n\to\infty} C_n = K_s - \liminf_{n\to\infty} C_n.$$

Let $(X,\|.\|)$ be a Banach space with dual X^*, and $\varphi : X\to\mathbb{R}$. The duality pairing between X^* and X is denoted by $\langle .,.\rangle$. We say that φ is "locally Lipschitz", if for every $x\in X$, we can find an open neighborhood $O(x)$ of x and a constant $k(x)>0$ such that

$$|\varphi(y)-\varphi(z)| \le k(x)\,\|y-z\| \text{ for all } y,\ z\in O(x).$$

If this inequality holds for all y, $z\in X$ and $k(x)=k>0$ is independent of x, then we have a Lipschitz continuous function. Clearly, if $\varphi : X\to\mathbb{R}$ is Lipschitz continuous on every bounded subset of X, then φ is locally Lipschitz. The converse is true provided X is finite dimensional. We know that if $\varphi : X\to\mathbb{R}$ is continuous and convex or if $\varphi\in C^1(X,\mathbb{R})$, then φ is locally Lipschitz.

Given a locally Lipschitz function $\varphi : X\to\mathbb{R}$, the generalized directional derivative of φ at $x\in X$ in the direction $h\in X$, denoted by $\varphi^0(x;h)$, is defined by

$$\varphi^0(x;h) = \limsup_{\substack{x'\to x\\ \lambda\downarrow 0}} \frac{\varphi(x'+\lambda h)-\varphi(x')}{\lambda}.$$

Using $\varphi^0(x;h)$ we can define the "generalized (or Clarke) subdifferential" of φ at $x \in X$, denoted by $\partial_c\varphi(x)$, as the set

$$\partial_c\varphi(x) = \left\{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^0(x;h) \text{ for all } h \in X\right\}.$$

This set is always nonempty, convex and w^*-compact. For a convex function $\varphi : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, we can define the "subdifferential in the sense of convex analysis" of φ at $x \in X$, to be the set

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq \varphi(y) - \varphi(x) \text{ for all } y \in X\}.$$

For continuous convex functions (hence locally Lipschitz, too), we have

$$\partial_c\varphi(x) = \partial\varphi(x) \text{ for all } x \in X.$$

Finally, let X_1 and X_2 be Hausdorff topological spaces and let $f_n : X_1 \times X_2 \to \mathbb{R}$ be a sequence of functions. By $Z(+)$ we denote the "sup" operator and by $Z(-)$ we denote the "inf" operator. For $h \in \{1,2\}$, let S_h denote the set of sequences $\left\{x_n^h\right\}_{n\geq 1}$ converging to some x_h in X_h. Also, let α_h be one of the signs $+$ or $-$. We set

$$\Gamma_{\text{seq}}(X_1^{\alpha_1}, X_2^{\alpha_2}) \lim_{n\to\infty} f_n(x_1,x_2) = \underset{\{x_n^1\}_{n\geq1}\in S_1}{Z(\alpha_1)} \; \underset{\{x_n^2\}_{n\geq1}\in S_2}{Z(\alpha_2)} \; \underset{k\in\mathbb{N}}{Z(-)} \; \underset{n\geq k}{Z(+)} f_n\left(x_n^1, x_n^2\right).$$

When the Γ_{seq} -limit is independent on the sign $+$ or $-$ associated to one of the spaces, then the sign is omitted. For example, if

$$\Gamma_{\text{seq}}\left(X_1^-, X_2^+\right) \lim_{n\to\infty} f_n(x_1,x_2) = \Gamma_{\text{seq}}\left(X_1^+, X_2^+\right) \lim_{n\to\infty} f_n(x_1,x_2)$$

then we indicate this common value by

$$\Gamma_{\text{seq}}\left(X_1, X_2^+\right) \lim_{n\to\infty} f_n(x_1,x_2).$$

The following notation will be in effect throughout the paper. We will use $|.|$ to indicate both the absolute value on $\mathbb{R}$ and the norm on $\mathbb{R}^N$. The inner product in $\mathbb{R}^N$ will be denoted denoted by $(.,.)_{\mathbb{R}^N}$. The norm in $L^p(\Omega)$ or $L^p\left(\Omega, \mathbb{R}^N\right)$, $1 < p < \infty$, will be designated by $\|.\|_p$, while $(.,.)_2$ will be used for the inner product in $L^2\left(\Omega, \mathbb{R}^N\right)$. We will use the symbol $\xrightarrow{w}$ to denote the weak convergence. Finally, we recall that

$$C_0^1\left(\overline{\Omega}\right) = \left\{u \in C^1\left(\overline{\Omega}\right) : u\,|_{\partial\Omega} = 0\right\}.$$

3. Sensitivity analysis

The hypotheses on the data of (P_λ) are the following:

$\mathbf{H}(a)$: $a(y,\lambda) = \partial_y\psi(y,\lambda)$ where $\psi : \mathbb{R}^N \times E \to \mathbb{R}$ is a function such that:

(i) for all $\lambda \in E$, $y \to \psi(y,\lambda)$ is convex, differentiable and $\psi(0,\lambda) = 0$;

(ii) there exist $0 < C_1 < C_2$ such that

$$C_1|y|^2 \leq \psi(y,\lambda) \leq C_2\left(1 + |y|^2\right) \text{ for all } (y,\lambda) \in \mathbb{R}\times E;$$

(iii) there exists $C_3 > 0$ such that for all y, $y' \in \mathbb{R}^N$ and all $\lambda \in E$, we have

$$C_3\left|y - y'\right|^2 \leq \left(a(y,\lambda) - a\left(y',\lambda\right), y - y'\right)_{\mathbb{R}^N};$$

(iv) if $\lambda_n \to \lambda$ in E, then $\psi(y,\lambda_n) \to \psi(y,\lambda)$ for all $y \in \mathbb{R}^N$.

Remark 3.1. Simple examples of functions $\psi(y,\lambda)$ which satisfy hypotheses $\mathbf{H}(a)$ are the following

$$\psi_1(y,\lambda) = \frac{a(\lambda)}{2}|y|^2 \text{ and } \psi_2(y,\lambda) = \frac{1}{2}|y|^2 + a(\lambda)|y|\ln(1+|y|),$$

with $\lambda \to a(\lambda)$ continuous from E into $(0,\infty)$.

$\mathbf{H}(f)$: $f : \Omega \times \mathbb{R} \times E \to \mathbb{R}$ is a function such that:

(i) for all $(x,\lambda) \in \mathbb{R} \times E$, $z \to f(z,x,\lambda)$ is measurable and $f(.,0,\lambda) \in L^2(\Omega)$ for all $\lambda \in E$;

(ii) for a. a. $z \in \Omega$, $(x,\lambda) \to f(z,x,\lambda)$ is continuous;

(iii) for a. a. $z \in \Omega$, all $\lambda \in E$ and all x, $x' \in \mathbb{R}$, we have

$$\left|f(z,x,\lambda) - f\left(z,x',\lambda\right)\right| \le k(z)\left|x - x'\right|$$

with $k \in L^\infty(\Omega)_+$ such that

$$k(z) \le C_3\widehat{\lambda}_1 \text{ for a. a. } z \in \Omega,$$

the inequality is strict on a set of positive measure, and $\widehat{\lambda}_1$ is the principal eigenvalue of $\left(-\triangle, H_0^1(\Omega)\right)$.

Remark 3.2. Whenever necessary, we will replace C_1 and C_3 in $\mathbf{H}(a)(ii)$, (iii) and $\mathbf{H}(f)(iii)$ by $\min\{C_1,C_3\}$.

Remark 3.3. We know that

$$\widehat{\lambda}_1 = \inf\left\{\frac{\|Dx\|_2^2}{\|x\|_2^2} : x \in H_0^1(\Omega)\right\}$$

and $\widehat{\lambda}_1 > 0$ (see Gasinski-Papageorgiou [8]). Also, hypothesis $\mathbf{H}(f)(iii)$ implies that

$$|f(z,x,\lambda)| \le k(z)|x| + |f(z,0,\lambda)| \text{ for a. a. } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and all } \lambda \in E. \tag{3.1}$$

$\mathbf{H}(\theta)$: $\theta : \Omega \times E \to \mathbb{R}_+$ is a Carathéodory function (that is, for all $\lambda \in E$, $z \to \theta(z,\lambda)$ is measurable and for a. a. $z \in \Omega$, $\lambda \to \theta(z,\lambda)$ is continuous) and $|\theta(z,\lambda)| \le 1$ for a. a. $z \in \Omega$, all $\lambda \in E$.

$\mathbf{H}(L)$: $L : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times E \to \mathbb{R}$ is an integrand such that

(i) for a. a. $z \in \Omega$, all $M > 0$, all $|x|, |x'|, |y|, |y'| \leq M$, all $|u| \leq 1$, all $\lambda \in E$:

$$L(.,0,0,u,\lambda) \in L^1(\Omega)$$

and

$$\left|L(z,x,y,u,\lambda) - L(z,x',y',u,\lambda)\right| \leq \eta_M(z)\left(|x-x'| + |y-y'|\right), \text{ with } \eta_M \in L^1(\Omega);$$

(ii) for a. a. $z \in \Omega$, all $(x,y,\lambda) \in \mathbb{R} \times \mathbb{R}^N \times E$, the function $u \to L(z,x,y,u,\lambda)$ is convex;

(iii) for a. a. $z \in \Omega$, all $(x,y) \in \mathbb{R} \times \mathbb{R}^N$, all $|u| \leq 1$, the function $\lambda \to L(z,x,y,u,\lambda)$ is continuous;

(iv) for a. a. $z \in \Omega$, all $(x,y) \in \mathbb{R} \times \mathbb{R}^N$, all $|u| \leq 1$, all $\lambda \in E$ we have

$$\beta(z) - C_4(|x| + |y|) \leq L(z,x,y,u,\lambda)$$

with $\beta \in L^1(\Omega)$, $C_4 > 0$.

For $\lambda \in E$ let

$$\mathcal{U}(\lambda) = \left\{u \in L^1(\Omega) : |u(z)| \leq \theta(z,\lambda) \text{ for a. a. } z \in \Omega\right\}.$$

This is the set of admissible control functions. A pair $(x,u) \in H_0^1(\Omega) \times L^1(\Omega)$ such that $u \in \mathcal{U}(\lambda)$ and x is a solution of the Dirichlet elliptic equation

$$-div\ a(Dx(z),\lambda) = f(z,x(z),\lambda)u(z) \text{ in } \Omega,\ x|_{\partial\Omega} = 0 \tag{3.2}$$

governing the system is said to be an "admissible state-control pair". We denote by $\mathcal{S}(\lambda)$ the set of all admissible state-control pairs. If for a pair $(x,u) \in \mathcal{S}(\lambda)$ we have

$$J(x,u,\lambda) = m(\lambda)$$

then we say that $(x,u) \in \mathcal{S}(\lambda)$ is an *optimal pair.* By $\mathcal{Q}(\lambda)$ we denote the set of all optimal pairs. Obviously $\mathcal{Q}(\lambda) \subseteq \mathcal{S}(\lambda)$.

We start by examining the state equation of (P_λ). So, for $\lambda \in E$, we consider equation (3.2).

Proposition 3.4. *If hypotheses* $\mathbf{H}(a)$, $\mathbf{H}(f)$ *hold,* $\lambda \in E$ *and* $u \in \mathcal{U}(\lambda)$, *then problem* (3.2) *has a unique solution* $x = x(u) \in C_0^1(\overline{\Omega})$.

Proof. Let $\varphi_\lambda : H_0^1(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (3.1), defined by

$$\varphi_\lambda(x) = \int_\Omega \psi(Dx,\lambda)\,dz - \int_\Omega F(z,x,\lambda)\,u\ dz \text{ for all } x \in H_0^1(\Omega),$$

with $F(z,x,\lambda) = \int_0^x f(z,s,\lambda)\,ds$. Hypotheses $\mathbf{H}(a)$ imply that

$$x \to \psi_\lambda(x) = \int_\Omega \psi(Dx,\lambda)\,dz$$

is continuous and convex, thus locally Lipschitz. Also, let $\sigma_\lambda : H_0^1(\Omega) \to \mathbb{R}$ be defined by

$$\sigma_\lambda(x) = \int_\Omega F(z,x,\lambda)\,u\ dz \text{ for all } x \in H_0^1(\Omega).$$

Evidently, $\sigma_\lambda \in C^1\left(H_0^1(\Omega), \mathbb{R}\right)$, hence it is also locally Lipschitz. Then

$$x \to \varphi_\lambda(x) = \psi_\lambda(x) - \sigma_\lambda(x)$$

is locally Lipschitz as well. Moreover, the convex functional $\psi_\lambda(.)$ is sequentially weakly lower semicontinuous.

Let $x_n \xrightarrow{w} x$ in $H_0^1(\Omega)$. By passing to a subsequence if necessary, we have

$$\begin{aligned} & x_n \to x \text{ in } L^2(\Omega),\ x_n(z) \to x(z) \text{ for a. a. } z \in \Omega, \\ & |x_n(z)| \le \eta(z) \text{ for a. a. } z \in \Omega, \text{ all } n \in \mathbb{N}, \text{ with } \eta \in L^2(\Omega). \end{aligned} \tag{3.3}$$

Using (3.3) and Fatou's lemma, we have

$$\begin{aligned} \limsup_{n\to\infty} \int_\Omega F(z, x_n(z), \lambda) u(z) dz &\le \int_\Omega \limsup_{n\to\infty} F(z, x_n(z), \lambda) u(z) dz \\ &= \int_\Omega F(z, x(z), \lambda) u(z) dz, \end{aligned}$$

hence $x \to \sigma_\lambda(x)$ is sequentially weakly upper semicontinuous on $H_0^1(\Omega)$. Therefore we infer that

$$x \to \varphi_\lambda(x) = \psi_\lambda(x) - \sigma_\lambda(x)$$

is sequentially weakly lower semicontinuous on $H_0^1(\Omega)$. Also, using hypotheses $\mathbf{H}(a)(ii)$, $\mathbf{H}(f)(i), (iii)$, (3.1) and Remark 3.2, for every $x \in H_0^1(\Omega)$ we have

$$\begin{aligned} \varphi_\lambda(x) &\ge C_1 \|Dx\|_2^2 - \frac{1}{2}\int_\Omega k(z) x^2 dz - C_5 \\ &\ge C_6\left(\|Dx\|_2^2 - 1\right) \text{ for some } C_5,\ C_6 > 0, \end{aligned}$$

(see Papageorgiou-Kyritsi [12], Lemma 5.1.3, p.356), hence φ_λ is coercive on $H_0^1(\Omega)$. So, by the Weierstrass-Tonelli theorem, we can find $x \in H_0^1(\Omega)$ such that

$$\varphi_\lambda(x) = \inf\left\{\varphi_\lambda(v) : v \in H_0^1(\Omega)\right\},$$

hence

$$\begin{aligned} 0 &\in \partial_C(\psi_\lambda - \sigma_\lambda)(x) \text{ (Fermat's rule)} \\ &\subseteq \partial_C\psi_\lambda(x) - \partial_C\sigma_\lambda(x) \text{ (see Clarke [6], pp. 38-39).} \end{aligned}$$

We know that

$$\partial_C\sigma_\lambda(x) = \sigma_\lambda'(x) = N_{f_\lambda}(x) u$$

with $f_\lambda(z, x) = f(z, x, \lambda)$ and $N_{f_\lambda}(y)(.) = f_\lambda(., y(.))$ for all $y \in H_0^1(\Omega)$ (see Papageorgiou-Kyritsi [12], Proposition 1.1.28, p.12).

Let $K_\lambda : L^2\left(\Omega, \mathbb{R}^N\right) \to \mathbb{R}$ be the integral functional defined by

$$K_\lambda(y) = \int_\Omega \psi(y, \lambda) dz \text{ for all } y \in L^2\left(\Omega, \mathbb{R}^N\right).$$

Then K_λ is continuous and convex, and we have

$$\psi_\lambda = K_\lambda \circ D$$

where $D \in \mathcal{L}\left(H_0^1(\Omega), L^2\left(\Omega, \mathbb{R}^N\right)\right)$ is the gradient operator. Using the nonlinear Green's identity (see, for example, Gasinski-Papageorgiou [8], p. 210), we have

$$D^* = -div \in \mathcal{L}\left(L^2\left(\Omega, \mathbb{R}^N\right), H^{-1}(\Omega)\right).$$

From the nonsmooth chain rule of Clarke ([6], Theorem 2.3.10, p. 45), we have

$$\partial_C \psi_\lambda (x) \subseteq -div\left(\partial_C K_\lambda (Dx)\right).$$

But since $K_\lambda (.)$ is continuous and convex, we have

$$\partial_C K_\lambda (y) = \partial K_\lambda (y) \text{ for all } y \in L^2 \left(\Omega, \mathbb{R}^N\right)$$

and from Theorem 4.5.16 of Gasinski-Papageorgiou ([8], p.570) it follows that

$$\partial K_\lambda (y) = \left\{h \in L^2 \left(\Omega, \mathbb{R}^N\right) : h(z) = \partial \psi (y(z), \lambda) = a(y(z), \lambda) \text{ for a. a. } z \in \Omega\right\}.$$

So, finally we have

$$-div\ h^* = N_{f_\lambda} (x)\, u \text{ with } h^*(z) = a(Dx(z), \lambda) \text{ for a. a. } z \in \Omega,$$

therefore

$$-div\ a(Dx(z), \lambda) = f(z, x(z), \lambda)\, u(z) \quad \text{for a. a. } z \in \Omega,\ x\mid_{\partial\Omega} = 0.$$

From Ladyzhenskaia-Uraltseva [9] (Theorem 7.1, p. 286) we know that

$$x \in L^\infty (\Omega).$$

So, we can apply Theorem 1 of Lieberman [10], and conclude that

$$x \in C_0^1 \left(\overline{\Omega}\right).$$

Now we show the uniqueness of this solution. So, suppose that $x_1,\ x_2 \in H_0^1(\Omega)$ are two solutions of (3.2). From the first part of the proof we have $x_1,\ x_2 \in C_0^1\left(\overline{\Omega}\right)$. We can find $h_1^*,\ h_2^* \in L^2\left(\Omega, \mathbb{R}^N\right)$ such that

$$h_1^*(z) = a(Dx_1(z), \lambda),\ h_2^*(z) = a(Dx_2(z), \lambda) \text{ for a. a. } z \in \Omega,$$
$$-div\ h_1^* = N_{f_\lambda}(x_1)\, u\ ,\ -div\ h_2^* = N_{f_\lambda}(x_2)\, u\ .$$

So, we have

$$\int_\Omega (h_1^* - h_2^*, Dx_1 - Dx_2)_{\mathbb{R}^N}\, dz = \int_\Omega (f(z, x_1(z), \lambda) - f(z, x_2(z), \lambda))\, u(x_1 - x_2)\, dz,$$

hence

$$C_3 \|Dx_1 - Dx_2\|_2^2 \leq \int_\Omega k(z)\, |x_1(z) - x_2(z)|^2\, dz$$

(see hypotheses $\mathbf{H}(a)(iii), \mathbf{H}(f)(iii)$), therefore

$$C_7 \|Dx_1 - Dx_2\|_2^2 \leq 0 \text{ for some } C_7 > 0$$

(see hypothesis $\mathbf{H}(f)(iii)$ and [12], p.356), and we conclude that $x_1 = x_2$. This proves the uniqueness of the solution $x \in C_0^1\left(\overline{\Omega}\right)$ of (3.2). □

In the next proposition, we determine the behavior of the set $\mathcal{S}(\lambda)$ as λ changes.

Proposition 3.5. *If hypotheses* $\mathbf{H}(a), \mathbf{H}(f), \mathbf{H}(\theta)$ *hold and* $\lambda_n \to \lambda$ *in* E, *then*

$$\mathcal{S}(\lambda_n) \overset{K\times M}{\longrightarrow} \mathcal{S}(\lambda) \ \text{ in } C_0^1\left(\overline{\Omega}\right) \times L^1(\Omega).$$

Proof. Let $(x,u) \in \mathcal{S}(\lambda)$. Hypothesis $\mathbf{H}(\theta)$ implies that

$$\mathcal{U}(\lambda_n) \xrightarrow{M} \mathcal{U}(\lambda) \text{ in } L^1(\Omega).$$

So, we can find $u_n \in \mathcal{U}(\lambda_n)$ such that

$$u_n \to u \text{ in } L^1(\Omega).$$

Let $x_n \in C_0^1(\overline{\Omega})$ be the unique state generated by the admissible control function u_n (see Proposition 3.4). We have

$$-div\ h_n^* = N_{f_{\lambda_n}}(x_n)\, u_n \text{ for all } n \in \mathbb{N}, \tag{3.4}$$

with $h_n^* \in L^2(\Omega, \mathbb{R}^N)$ satisfying

$$h_n^* = a(Dx_n(z), \lambda_n) \text{ for a. a. } z \in \Omega, \text{all } n \in \mathbb{N}. \tag{3.5}$$

On (3.4) we act with $x_n \in C_0^1(\overline{\Omega})$ and obtain

$$\langle -div\ h_n^*, x_n \rangle = \langle N_{f_{\lambda_n}}(u_n), x_n \rangle$$

where $\langle .,. \rangle$ denotes the duality brackets for $\left(H^{-1}(\Omega), H_0^1(\Omega)\right)$. Then, by the nonlinear Green's identity (see [8], p. 210), it follows that

$$\int_\Omega (h_n^*, Dx_n)_{\mathbb{R}^N}\, dz = \int_\Omega f(z, x_n, \lambda_n)\, u_n x_n dz. \tag{3.6}$$

By (3.5) and since $a(y,\lambda) = \partial\psi(y,\lambda)$, we have

(3.7)

$$(h_n^*, Dx_n)_{\mathbb{R}^N} \geq \psi(Dx_n(z), \lambda_n) \geq C_1 |Dx_n(z)|^2 \text{ for for a. a. } z \in \Omega, \text{ all } n \in \mathbb{N}$$

(see hypothesis $\mathbf{H}(a)(iii)$). Returning to (3.6) and using (3.7) , (3.1) and $\mathbf{H}(\theta)$, we obtain

$$C_1 \|Dx_n\|_2^2 - \int_\Omega k(z)\, |x_n(z)|^2\, dz \leq \int_\Omega f(z, 0, \lambda_n)\, |x_n(z)|\, dz,$$

hence (see Remark 3.2 and [12], p. 356)

$$\|Dx_n\|_2^2 \leq C_8 \|Dx_n\|_2 \text{ for some } C_8 > 0, \text{ all } n \in \mathbb{N},$$

therefore

$$\{x_n\}_{n \geq 1} \subseteq H_0^1(\Omega) \text{ is bounded.}$$

Then as before, from Ladyzhenskaya-Uraltseva ([9], p. 286), it follows that there exists $C_9 > 0$ such that

$$\|x_n\|_\infty \leq C_9 \text{ for all } n \in \mathbb{N}.$$

So, from the regularity theory of Lieberman [10], we know that there exist $\alpha \in (0,1)$ and $C_{10} > 0$ such that

$$x_n \in C_0^{1,\alpha}(\overline{\Omega}), \ \|x_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq C_{10} \text{ for all } n \in \mathbb{N}.$$

Exploiting the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and passing to a subsequence if necessary, we have

$$x_n \to \widehat{x} \text{ in } C_0^1(\overline{\Omega}). \tag{3.8}$$

Using the notation from the proof of Proposition 3.4, we conclude that

$$h_n^* = \partial K_{\lambda_n}(Dx_n). \tag{3.9}$$

From (3.5) and (3.8) it follows that

$$\{h_n^*\}_{n\geq 1} \subseteq L^2\left(\Omega, \mathbb{R}^N\right) \text{ is bounded.}$$

So, we may assume that

$$h_n^* \xrightarrow{w} \widehat{h}^* \text{ in } L^2\left(\Omega, \mathbb{R}^N\right). \tag{3.10}$$

From (3.9) we have

$$\begin{aligned} \left(h_n^*, v - x_n\right)_2 &\leq K_{\lambda_n}(v) - K_{\lambda_n}(Dx_n) \\ &= \int_\Omega \left[\psi\left(v(z), \lambda_n\right) - \psi\left(Dx_n, \lambda_n\right)\right] dz, \text{ for all } v \in L^2\left(\Omega, \mathbb{R}^N\right). \end{aligned} \tag{3.11}$$

Hypothesis $\mathbf{H}(a)(iv)$, (3.8) and Theorem 10.6, p. 88 of Rockafellar [13] imply that

$$\psi\left(v(z), \lambda_n\right) \to \psi\left(v(z), \lambda\right) \text{ and } \psi\left(Dx_n(z), \lambda_n\right) \to \psi\left(D\widehat{x}(z), \lambda\right) \text{ for all } z \in \overline{\Omega},$$

hence (cf. $\mathbf{H}(a)(iii)$)

$$\int_\Omega \left[\psi\left(v(z), \lambda_n\right) - \psi\left(Dx_n, \lambda_n\right)\right] dz \to \int_\Omega \left[\psi\left(v(z), \lambda\right) - \psi\left(D\widehat{x}(z), \lambda\right)\right] dz. \tag{3.12}$$

So, if in (3.11) we pass to the limit as $n \to \infty$ and use (3.8), (3.10), (3.12), we obtain

$$\left(\widehat{h}^*, v - \widehat{x}\right)_2 \leq K_\lambda(v) - K_\lambda(D\widehat{x}) \text{ for all } v \in L^2\left(\Omega, \mathbb{R}^N\right),$$

hence

$$\widehat{h}^* = \partial K_\lambda(D\widehat{x}),$$

therefore

$$-div\ \widehat{h}^* \in \partial\psi_\lambda(\widehat{x}). \tag{3.13}$$

Also, from (3.8), (3.1) and the dominated convergence theorem, we have

$$N_{f_{\lambda_n}}(x_n)\, u_n \to N_{f_\lambda}(\widehat{x})\, u \text{ in } L^2(\Omega). \tag{3.14}$$

Passing to the limit as $n \to \infty$ in (3.4), (3.5), and using (3.10), (3.13) and (3.14), we obtain

$$-div\ \widehat{h}^* = N_{f_\lambda}(\widehat{x})\, u \text{ with } \widehat{h}^* = a\left(D\widehat{x}(z), \lambda\right) \text{ for a. a. } z \in \Omega,$$

hence

$$(\widehat{x}, u) \in S(\lambda),$$

and we have

$$\widehat{x} = x \text{ (see Proposition 3.4).}$$

Therefore we have produced a sequence $\{(x_n, u_n)\}_{n\geq 1}$ (denoted by the same index) such that

$$x_n \to x \text{ in } C_0^1\left(\overline{\Omega}\right) \text{ and } u_n \to u \text{ in } L^1(\Omega). \tag{3.15}$$

The uniqueness of x and Urysohn's criterion for convergence of sequences imply that (3.15) holds for the original sequence and so

$$\mathcal{S}(\lambda) \subseteq K_{ss} - \liminf_{n\to\infty} \mathcal{S}(\lambda_n) = K - \liminf_{n\to\infty} \mathcal{S}(\lambda_n) \text{ in } C_0^1\left(\overline{\Omega}\right) \times L^1(\Omega). \tag{3.16}$$

Next consider $(x,u) \in K_{sw} - \limsup_{n\to\infty} \mathcal{S}(\lambda_n)$. Denoting subsequences with the same index as the initial sequence, we can find $(x_n,u_n) \in \mathcal{S}(\lambda_n)$ for all $n \in \mathbb{N}$ such that

$$x_n \to x \text{ in } C_0^1(\overline{\Omega}) \text{ and } u_n \xrightarrow{w} u \text{ in } L^1(\Omega). \tag{3.17}$$

Evidently $u \in \mathcal{U}(\lambda)$ (note that on account of hypothesis $\mathbf{H}(\theta)$, $\mathcal{U}(\lambda_n) \xrightarrow{M} \mathcal{U}(\lambda)$).

We have

$$-div\ h_n^* = N_{f_{\lambda_n}}(x_n)\,u_n \text{ for all } n \in \mathbb{N}, \tag{3.18}$$

with $h_n^* \in L^2(\Omega,\mathbb{R}^N)$ satisfying

$$h_n^*(z) = a(Dx_n(z),\lambda_n) \text{ for a. a. } z \in \Omega, \text{all } n \in \mathbb{N}.$$

As before, we may assume that

$$h_n^* \xrightarrow{w} h^* \text{ in } L^2(\Omega,\mathbb{R}^N). \tag{3.19}$$

Again we have

$$h^*(z) = a(Dx(z),\lambda) \text{ for a. a. } z \in \Omega. \tag{3.20}$$

Also, we have

$$N_{f_{\lambda_n}}(x_n)\,u_n \xrightarrow{w} N_{f_\lambda}(x)\,u \text{ in } L^2(\Omega) \tag{3.21}$$

(see (3.14), (3.17) and hypothesis $\mathbf{H}(\theta)$). Passing to the limit as $n \to \infty$ in (3.18) and using (3.19) and (3.21), we obtain

$$-div\ h^* = N_{f_\lambda}(x)\,u$$

hence

$$(x,u) \in \mathcal{S}(\lambda) \text{ (see (3.20), and recall that } u \in \mathcal{U}(\lambda)\text{)},$$

therefore

$$K_{sw} - \limsup_{n\to\infty} \mathcal{S}(\lambda_n) \subseteq \mathcal{S}(\lambda). \tag{3.22}$$

From (3.16) and (3.22), we conclude that

$$\mathcal{S}(\lambda_n) \xrightarrow{K\times M} \mathcal{S}(\lambda) \text{ in } C_0^1(\overline{\Omega}) \times L^1(\Omega).$$

□

So far, we have examined only the constraints of problem (P_λ). Now we bring the cost functional into the picture. First we show that for each $\lambda \in E$, problem (P_λ) admits an optimal pair (that is, for all $\lambda \in E$, $\mathcal{Q}(\lambda) \neq \varnothing$).

Proposition 3.6. *If hypotheses* $\mathbf{H}(a)$, $\mathbf{H}(f)$, $\mathbf{H}(\theta)$, $\mathbf{H}(L)$ *hold, then for every* $\lambda \in E$, $\mathcal{Q}(\lambda) \neq \varnothing$.

Proof. Let $\{(x_n,u_n)\}_{n\geq 1} \subseteq \mathcal{S}(\lambda)$ be a minimizing sequence for problem (P_λ), that is

$$J(x_n,u_n,\lambda) \downarrow m(\lambda) \text{ as } n \to \infty.$$

We know that

$$\{(x_n,u_n)\}_{n\geq 1} \subseteq H_0^1(\Omega) \times L^\infty(\Omega) \text{ is bounded}$$

(see the proof of Proposition 3.5 and hypothesis $\mathbf{H}(\theta)$). By passing to a suitable subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \text{ in } H_0^1(\Omega) \text{ and } u_n \xrightarrow{w^*} u \text{ in } L^\infty(\Omega). \tag{3.23}$$

From Theorem 2.1.28 of Papageorgiou-Kyritsi ([12], p. 72), we know that $J(.,.,\lambda)$ is sequentially lower semicontinuous on $L^2(\Omega) \times L^\infty(\Omega)_{w^*}$. So, from (3.23) and the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, it follows

$$J(x,u,\lambda) \le \liminf_{n\to\infty} J(x_n,u_n,\lambda) = m(\lambda). \tag{3.24}$$

On the other hand, from (3.23) and the regularity theory of Lieberman [10], we have

$$x_n \to x \text{ in } C_0^1\left(\overline{\Omega}\right),$$

hence

$$(x,u) \in \mathcal{S}(\lambda) \text{ (see Proposition 3.5),}$$

therefore

$$J(x,u,\lambda) = m(\lambda)$$

and we conclude that

$$(x,u) \in \mathcal{Q}(\lambda) \neq \varnothing.$$

□

Now we are in a position to prove the theorem concerning the variational stability of problem (P_λ). We show that (P_λ) is Hadamard well-posed.

The result reads as follows:

Theorem 3.7. *If hypotheses $\mathbf{H}(a), \mathbf{H}(f), \mathbf{H}(\theta), \mathbf{H}(L)$ hold, then the value function $\lambda \to m(\lambda)$ is continuous from E into $\mathbb{R}$, and the solution multifunction $\mathcal{Q}$: $E \to 2^{C_0^1(\overline{\Omega})\times L^1(\Omega)_w} \setminus \{\varnothing\}$ is upper semicontinuous.*

Proof. Let $\lambda_n \to \lambda$ in E. For every $n \in \mathbb{N}$, let $(x_n,u_n) \in \mathcal{Q}(\lambda_n)$ (see Proposition 3.6). Then

$$m(\lambda_n) = J(x_n,u_n,\lambda_n) \text{ for all } n \in \mathbb{N}. \tag{3.25}$$

From the proof of Proposition 3.5, we know that at least for a subsequence, we have

$$x_n \to x \text{ in } C_0^1\left(\overline{\Omega}\right) \text{ and } u_n \xrightarrow{w} u \text{ in } L^1(\Omega). \tag{3.26}$$

Then Proposition 3.5 implies that

$$(x,u) \in \mathcal{S}(\lambda). \tag{3.27}$$

From the lower semicontinuity result of Berkovitz [3] we have

$$\int_\Omega L(z,x(z),Dx(z),u(z),\lambda)\,dz \le \liminf_{n\to\infty} \int_\Omega L(z,x_n(z),Dx_n(z),u_n(z),\lambda_n)\,dz,$$

hence

$$m(\lambda) \le \liminf_{n\to\infty} m(\lambda_n) \text{ (see (3.25), (3.27)).} \tag{3.28}$$

Next, let $(x,u) \in \mathcal{Q}(\lambda)$. Then

$$m(\lambda) = J(x,u,\lambda). \tag{3.29}$$

Proposition 3.5 implies that we can find $(x_n, u_n) \in \mathcal{S}(\lambda_n)$ for all $n \in \mathbb{N}$ such that

$$x_n \to x \text{ in } C_0^1(\overline{\Omega}) \text{ and } u_n \to u \text{ in } L^1(\Omega). \tag{3.30}$$

We have

$$\begin{aligned} m(\lambda_n) &\leq J(x_n, u_n, \lambda_n) \\ &= \textstyle\int_\Omega L(z, x_n, Dx_n, u_n, \lambda_n)\,dz \\ &= \textstyle\int_\Omega [L(z, x_n, Dx_n, u_n, \lambda_n) - L(z, x, Dx, u_n, \lambda_n) \\ &\qquad\qquad + L(z, x, Dx, u_n, \lambda_n)]\,dz \\ &\leq \textstyle\int_\Omega \eta_M(z)\,[|x_n - x| + |Dx_n - Dx|]\,dz \\ &\quad + \textstyle\int_\Omega L(z, x, Dx, u_n, \lambda_n)\,dz \end{aligned} \tag{3.31}$$

with $M = \sup\left\{\|x_n\|_{C^1(\overline{\Omega})} : n \geq 1\right\}$ (see (3.30)). Note that

$$\int_\Omega \eta_M(z)\,[|x_n - x| + |Dx_n - Dx|]\,dz \to 0 \text{ (see (3.30))} \tag{3.32}$$

and

$$\int_\Omega L(z, x(z), Dx(z), u_n(z), \lambda_n)\,dz \to \int_\Omega L(z, x(z), Dx(z), u(z), \lambda)\,dz \tag{3.33}$$

(see hypotheses $\mathbf{H}(L)\,(ii)$, (iii) and Theorem 10.6, p. 88 of [13]).

Returning to (3.31), passing to the limit as $n \to \infty$ and using (3.32) and (3.33), we obtain

$$\limsup_{n\to\infty} m(\lambda_n) \leq J(x, u, \lambda) = m(\lambda). \tag{3.34}$$

From (3.28) and (3.34) it follows that

$$m(\lambda_n) \to m(\lambda)$$

hence

$$\lambda \to m(\lambda) \text{ is continuous from } E \text{ into } \mathbb{R}.$$

Next we show the upper semicontinuity of the multifunction $\lambda \to \mathcal{Q}(\lambda)$. To this end, let $C \subseteq C_0^1(\overline{\Omega}) \times L^1(\Omega)_w$ be a closed set. We need to show that

$$\mathcal{Q}^-(C) := \{\lambda \in E : \mathcal{Q}(\lambda) \cap C \neq \varnothing\}$$

is closed. So, let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathcal{Q}^-(C)$ and assume that $\lambda_n \to \lambda$ in E. Let

$$(x_n, u_n) \in \mathcal{Q}(\lambda_n) \cap C, \text{ for all } n \in \mathbb{N}.$$

From the proof of Proposition 3.5 and hypothesis $\mathbf{H}(\theta)$, we know that we may assume that at least for a subsequence we have

$$x_n \to x \text{ in } C_0^1(\overline{\Omega}) \text{ and } u_n \xrightarrow{w} u \text{ in } L^1(\Omega). \tag{3.35}$$

We have

$$m(\lambda_n) = J(x_n, u_n, \lambda_n) \text{ for all } n \in \mathbb{N},$$

hence

$$m(\lambda) = \liminf_{n\to\infty} J(x_n, u_n, \lambda_n) \geq J(x, u, \lambda) \tag{3.36}$$

(see (3.35) and recall that $\lambda \to m(\lambda)$ is continuous). From Proposition 3.5 we have $(x, u) \in \mathcal{S}(\lambda)$. Hence (3.36) becomes

$$m(\lambda) = J(x, u, \lambda),$$

hence

$$(x, u) \in \mathcal{Q}(\lambda) \cap C$$

therefore $\lambda \in \mathcal{Q}^{-}(C)$. So, the set $\mathcal{Q}^{-1}(C)$ is closed and the proof is complete.. □

4. A minimax problem

In this section we consider a particular case of the control system in problem (P_λ), in which the function $\theta(.)$ in the control constraint is independent of the parameter. In other words, the control constraint set is fixed and does not depend on the parameter $\lambda \in E$.

So, we deal with the following nonlinear elliptic equation

$$(P'_\lambda) \qquad \begin{cases} -div\ a(Dx(z), \lambda) = f(z, x(z), \lambda)\, u(z) \text{ in } \Omega,\ x\mid_{\partial\Omega} = 0, \\ |u(z)| \le \theta(z) \text{ for a. a. } z \in \Omega.,\ \lambda \in E. \end{cases}$$

In this case the function $\theta(.)$ satisfies

$\mathbf{H}(\theta)'$: $\theta : \Omega \to \mathbb{R}_+$ is a measurable function such that $0 \le \theta(z) \le 1$ for a. a. $z \in \Omega$.

For every admissible control u, problem (P'_λ) has a unique solution $x_\lambda(u) \in C_0^1(\overline{\Omega})$ (see Proposition 3.4).

To system (P'_λ) we associate the integral cost functional defined by

$$\widehat{J}(u, \lambda) = \int_\Omega L(z, x_\lambda(u)(z), Dx_\lambda(u)(z), u(z), \lambda)\, dz.$$

In this context we consider the following minimax problem

$$\inf_{u \in S_c} \sup_{\lambda \in E} \widehat{J}(u, \lambda) = m, \tag{4.1}$$

with

$$S_c = \left\{ u \in L^1(\Omega) : |u(z)| \le \theta(z) \text{ for a. a. } z \in \Omega \right\}.$$

So, in this problem, given an admissible control function $u \in S_c$, the system analyst determines the maximum cost (risk) over all possible parameter values and then minimizes the maximum value over all admissible controls.

In what follows, given $u \in S_c$, we set

$$\varphi(u) = \sup\left\{ \widehat{J}(u, \lambda) : \lambda \in E \right\}. \tag{4.2}$$

By a *solution* of the minimax control problem (4.1), we mean a control function $u^* \in S_c$ such that

$$\varphi(u^*) = \inf\{\varphi(u) : u \in S_c\}. \tag{4.3}$$

Now the hypotheses on the cost integrand L are the following:

$\mathbf{H}(L)'$: $L : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times E \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a measurable integrand such that

(i) for a. a. $z \in \Omega$, $(x, y, u, \lambda) \to L(z, x, y, u, \lambda)$ is proper, lower semicontinuous;

(ii) for a. a. $z \in \Omega$ and all $(x, y, \lambda) \in \mathbb{R} \times \mathbb{R}^N \times E$, the function $u \to L(z, x, y, u, \lambda)$ is convex;

(iii) for every $M > 0$, there exist $\widehat{\eta}_M \in L^1(\Omega)$ and $\widehat{C}_M > 0$ such that

$$\widehat{\eta}_M - \widehat{C}_M (|x| + |y|) \leq L(z, x, y, u, \lambda) \text{ for a. a. } z \in \Omega, \\ \text{all } |x|, |y| \leq M, \text{ all } |u| \leq 1, \text{ all } \lambda \in E.$$

Let $\tau : L^1(\Omega) \times E \to H_0^1(\Omega)$ be the map which to each pair $(u, \lambda) \in S_c \times E$ assigns the unique solution (state) $x \in H_0^1(\Omega)$ of equation (P'_λ) (see Proposition 3.4). A byproduct of the proof of Proposition 3.5 is the following result concerning the map $\tau(.)$:

Proposition 4.1. *If hypotheses* $\mathbf{H}(a), \mathbf{H}(f), \mathbf{H}(\theta)'$ *hold, then the map* $\tau : L^1(\Omega)_w \times E \to H_0^1(\Omega)$ *is sequentially continuous.*

Using this proposition, we can find a solution for the minimax problem (4.1).

Theorem 4.2. *If hypotheses* $\mathbf{H}(a), \mathbf{H}(f), \mathbf{H}(\theta)', \mathbf{H}(L)'$ *hold, then problem* (4.1) *admits an optimal control* $u^* \in S_c$ *(see* $(4.2), (4.3)$*).*

Proof. From Proposition 4.1 and Theorem 2.1.28 of Papageorgiou-Kyritsi ([12], p.72), it follows that the function $(u, \lambda) \to \widehat{J}(u, \lambda)$ is sequentially lower semicontinuous on $L^1(\Omega)_w \times E$. Let φ be defined by (4.2). We claim that $\varphi : L^1(\Omega)_w \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is sequentially lower semicontinuous. To see this, let $\eta \in \mathbb{R}$ and consider the sublevel set

$$S_\eta := \left\{ u \in L^1(\Omega) : \varphi(u) \leq \eta \right\}.$$

We need to show that S_η is sequentially weakly closed.

So, let $\{u_n\}_{n \in \mathbb{N}} \subseteq S_\eta$ and assume that

$$u_n \xrightarrow{w} u \text{ in } L^1(\Omega).$$

Then

$$\widehat{J}(u, \lambda) \leq \liminf_{n \to \infty} \widehat{J}(u_n, \lambda) \leq \liminf_{n \to \infty} \varphi(u_n) \leq \eta,$$

hence

$$\widehat{J}(u, \lambda) \leq \eta \text{ for all } \lambda \in E,$$

therefore

$$\varphi(u) \leq \eta, \text{ that is } u \in S_\eta.$$

So, φ is sequentially lower semicontinuous on $L^1(\Omega)_w$. By the Eberlein-Smulian theorem, the set S_c is sequentially compact in $L^1(\Omega)_w$. Then, by the Weierstass-Tonelli theorem, we can find $u^* \in S_c$ satisfying (4.3). ☐

5. Another sensitivity result

In this section, using the method of multiple $\Gamma-$operators developed by Buttazzo-Dal Maso [5], we prove another sensitivity (variational stability) result.

Now the cost integrand L is independent of the gradient of the state. So, the conditions on L are the following:

$\mathbf{H}(L)''$: $L : \Omega \times \mathbb{R} \times \mathbb{R} \times E \to \mathbb{R}$ is a measurable integrand such that

(*i*) for a. a. $z \in \Omega$, and all $(x, \lambda) \in \mathbb{R} \times E$, $u \to L(z, x, u, \lambda)$ is convex;

(*ii*) for a. a. $z \in \Omega$, all $(x, u) \in \mathbb{R} \times \mathbb{R}$, and all $\lambda \in E$:

$$\widetilde{C}_1 |u|^2 \leq L(z, x, u, \lambda) \leq \widetilde{C}_2 \left(1 + x^2 + u^2\right), \text{ with } \widetilde{C}_1,\ \widetilde{C}_2 > 0;$$

(*iii*) for a. a. $z \in \Omega$, all $(u, \lambda) \in \mathbb{R} \times E$, and all $x, v \in \mathbb{R}$ with $|x - v| < 1$, we have

$$|L(z, x, u, \lambda) - L(z, v, u, \lambda)| \leq \rho(|x - v|) \left(1 + x^2 + u^2\right)$$

with $\rho : [0, 1] \to \mathbb{R}_+$ increasing, continuous and such that $\rho(0) = 0$;

(*iv*) if $\lambda_n \to \lambda$ in E, then $L^*(., x, u, \lambda_n) \xrightarrow{w} L^*(., x, u, \lambda)$ in $L^1(\Omega)_w$ for all $(x, u) \in \mathbb{R} \times \mathbb{R}$.

(Here $L^*(z, x, u, \lambda) = \sup_{\overline{u} \in \mathbb{R}} \{\overline{u}u - L(z, x, u, \lambda)\}$).

Now the cost functional is

$$J_0(x, u, \lambda) = \int_\Omega L(z, x(z), u(z), \lambda)\, dz.$$

Using Lemma 3.1 of Butazzo-Dal Maso [5] and the fact that $H_0^1(\Omega)$ is embedded compactly in $L^2(\Omega)$, we have

Proposition 5.1. *If hypotheses* $\mathbf{H}(L)''$ *hold and* $\lambda_n \to \lambda$ *in* E, *then*

$$J_0(x, u, \lambda) = \Gamma_{\text{seq}} \left(H_0^1(\Omega)_w, L^2(\Omega)_w^-\right) \lim_{n\to\infty} J_0(x, u, \lambda_n).$$

Also, combining our Proposition 3.5 with Example 2.1 of Butazzo-Dal Maso [5], we obtain a convergence result for the indicator functions

$$\delta_{S(\lambda_n)}(x, u) = \begin{cases} 0 & \text{if } (x, u) \in S(\lambda_n) \\ +\infty & \text{otherwise} \end{cases}, \quad n \in \mathbb{N}.$$

Proposition 5.2. *If hypotheses* $\mathbf{H}(a)$, $\mathbf{H}(f)$, $\mathbf{H}(\theta)$, $\mathbf{H}(L)''$ *hold and* $\lambda_n \to \lambda$ *in* E, *then*

$$\delta_{S(\lambda)} = \Gamma_{\text{seq}} \left(H_0^1(\Omega)_w, L^2(\Omega)_w^-\right) \lim_{n\to\infty} \delta_{S(\lambda_n)}.$$

Now Propositions 5.1 and 5.2 permit the use of Theorem 2.1 of [5], which leads to the following sensitivity result:

Theorem 5.3. *If hypotheses* $\mathbf{H}(a)$, $\mathbf{H}(f)$, $\mathbf{H}(\theta)$, $\mathbf{H}(L)''$ *hold and* $\lambda_n \to \lambda$ *in* E, *then for every* $n \in \mathbb{N}$, *problem* (P_{λ_n}) *(with* J *replaced by* J_0*), has an optimal pair* $(x_n^*, u_n^*) \in \mathcal{Q}(\lambda_n)$, *and*

$$(x_n^*, u_n^*) \to (x^*, u^*) \text{ in } C_0^1\left(\overline{\Omega}\right) \times L^1(\Omega)_w \text{ with } (x^*, u^*) \in \mathcal{Q}(\lambda).$$

Remark 5.4. In particular this theorem implies that:

(i) $\lambda \to m(\lambda)$ is lower semicontinuous;
(ii) $\lambda \to \mathcal{Q}(\lambda)$ is upper semicontinuous from E into $C_0^1\left(\overline{\Omega}\right) \times L^1(\Omega)_w$.

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