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TRAVELLING WAVES FOR REACTION-DIFFUSION PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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ABSTRACT. The paper is devoted to a reaction-diffusion system in an infinite two-dimensional strip with nonlinear boundary conditions. The existence of travelling waves is proved in the bistable case by the Leray-Schauder method. It is based on a topological degree for elliptic problems in unbounded domains and on a priori estimates of solutions.

1. INTRODUCTION

We will study the reaction-diffusion system of equations

$$\frac{\partial u}{\partial t} = D\Delta u + F(u) \tag{1.1}$$

with the boundary conditions

$$y = 0 : \quad \frac{\partial u}{\partial y} = 0, \quad y = l : \quad \frac{\partial u}{\partial y} = G(u), \tag{1.2}$$

where $u = (u_1, \ldots, u_n)$, $F = (F_1, \ldots, F_n)$, $G = (G_1, \ldots, G_n)$, D is a diagonal matrix with positive diagonal elements. The scalar equation ($n = 1$) was studied in [1]. In the case of systems of equations ($n > 1$) we will assume that the following condition is satisfied:

$$\frac{\partial F_i}{\partial u_j} \geq 0, \quad \frac{\partial G_i}{\partial u_j} \geq 0, \quad i, j = 1, \ldots, n, \quad i \neq j. \tag{1.3}$$

It is a class of systems for which the maximum principle is applicable.

We will look for a travelling wave solution $u(x, t) = w(x - ct)$ of problem (1.1), (1.2). It satisfies the problem

$$D\Delta w + c\frac{\partial w}{\partial x} + F(w) = 0, \tag{1.4}$$

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$$y=0: \quad \frac{\partial w}{\partial y}=0, \quad y=l: \quad \frac{\partial w}{\partial y}=G(w). \tag{1.5}$$

Solution in the cross section. Consider the problem

$$Dw''+F(w)=0, \tag{1.6}$$

$$w'(0)=0, \quad w'(l)=G(w(l)) \tag{1.7}$$

in the interval $0<y<l$. Here the prime denotes the derivative with respect to y. We suppose the following two conditions to be satisfied.

Condition 1. There exist two solutions, $w_+(y)$ and $w_-(y)$ of problem (1.6), (1.7) such that

$$w_+(y) \le w_-(y), \quad 0<y<l$$

and the eigenvalue problems

$$Dv''+F'(w_\pm(y))v=\lambda v,$$

$$v'(0)=0, \quad v'(l)=G'(w_\pm(l))v(l)$$

have all eigenvalues in the left-half plane.

Condition 2. For any other solution $w(y)$ of problem (1.6), (1.7) such that

$$w_+(y) \le w(y) \le w_-(y), \quad 0<y<l,$$

the eigenvalue problems

$$Dv''+F'(w_\pm(y))v=\lambda v,$$

$$v'(0)=0, \quad v'(l)=G'(w_\pm(l))v(l)$$

have at least one eigenvalue in the right-half plane.

In the next section will introduce the corresponding operators and function spaces. Section 3 is devoted to a priori estimates of solutions and existence of waves is proved in Section 4. We apply the general results of the work to a model of atherosclerosis in Section 5.

2. Linear and nonlinear operators

2.1. **Fredholm property.** Consider the operator corresponding to problem (1.4), (1.5) and linearized about a solution $u(x, y)$:

$$Av = D\Delta v + c\frac{\partial v}{\partial x} + a(x, y)v, \quad (x, y) \in \Omega, \tag{2.1}$$

$$Bv = \begin{cases} \frac{\partial v}{\partial y} & , \quad y = 0 \\ \frac{\partial v}{\partial y} - b(x)v & , \quad y = l \end{cases}, \tag{2.2}$$

where $\Omega = \{-\infty < x < \infty, \ \ 0 < y < l\}$, and

$$a(x, y) = F'(u(x, y)), \quad b(x) = G'(u(x, l)).$$

Here $a(x, y)$ and $b(x)$ are square matrices. We suppose that their coefficients are Hölder continuous. Then the operator $L = (A, B)$ is a bounded linear operator from the space $E = C^{2+\alpha}(\bar{\Omega})$ into the space $F = C^{\alpha}(\bar{\Omega}) \times C^{1+\alpha}(\partial\Omega)$ for some $\alpha \in (0, 1)$. We will also consider the limiting operators

$$A^{\pm}v = D\Delta v + c\frac{\partial v}{\partial x} + a_{\pm}(y)v, \quad (x, y) \in \Omega, \tag{2.3}$$

$$B^{\pm}v = \begin{cases} \frac{\partial v}{\partial y} & , \quad y = 0 \\ \frac{\partial v}{\partial y} - b_{\pm}v & , \quad y = l \end{cases} \tag{2.4}$$

and the corresponding equations

$$A^{\pm}v = 0, \quad B^{\pm}v = 0. \tag{2.5}$$

Here

$$a_{\pm}(y) = \lim_{x \to \pm\infty} a(x, y), \quad b_{\pm} = \lim_{x \to \pm\infty} b(x).$$

Denote by $\tilde{v}(\xi, y)$ the partial Fourier transform of $v(x, y)$ with respect to x. Then from (2.5) we obtain

$$D\tilde{v}'' + (-D\xi^2 + ci\xi + a_{\pm}(y))\tilde{v} = 0, \quad 0 < y < l, \tag{2.6}$$

$$\tilde{v}'(\xi, 0) = 0, \ \ \tilde{v}'(\xi, l) = b_{\pm}\tilde{v}(\xi, l). \tag{2.7}$$

Since we consider the bistable case, then the eigenvalue problem

$$Dv'' + a_{\pm}(y)v = \lambda v, \ \ 0 < y < l, \quad v'(0) = 0, \ \ v'(l) = b_{\pm}v(l) \tag{2.8}$$

has all eigenvalues in the left-half plane. Therefore for each $\xi \in \mathbb{R}$, problem (2.6), (2.7) has only zero solution. Hence $v(x, y) \equiv 0$, and thus we have proved that limiting problems do not have nonzero bounded solutions. This is also true for the formally adjoint operator. Therefore the operator L satisfies the Fredholm property.

It remains also true if the operator acts from $W^{2,2}_\infty(\Omega)$ into $L^2_\infty(\Omega) \times W^{1/2,2}_\infty(\partial\Omega)$ ([10], page 163) where the ∞-spaces are defined as follows. Let E be a Banach space with the norm $\|\cdot\|$ and ϕ_i be a partition of unity. Then E_∞ is the space of functions for which the expression

$$\|u\|_\infty = \sup_i \|u\phi_i\|$$

is bounded. This is the norm in this space.

Theorem 2.1. *If both problems (2.8) have all eigenvalues in the left-half plane, then the operator $L = (A, B)$ acting from $C^{2+\alpha}(\bar\Omega)$ into $F = C^\alpha(\bar\Omega) \times C^{1+\alpha}(\partial\Omega)$ or from $W^{2,2}_\infty(\Omega)$ into $L^2_\infty(\Omega) \times W^{1/2,2}_\infty(\partial\Omega)$ satisfies the Fredholm property.*

2.2. **Properness and topological degree.** Consider the nonlinear operator in the domain Ω

$$T_0^\tau(w) = D\Delta w + c\frac{\partial w}{\partial x} + F_\tau(w), \quad (x,y) \in \Omega, \tag{2.9}$$

and the boundary operator

$$Q_0^\tau(w) = \begin{cases} \frac{\partial w}{\partial y} & , \quad y = 0 \\ \frac{\partial w}{\partial y} - G_\tau(w) & , \quad y = l \end{cases}, \tag{2.10}$$

where the functions F_τ and G_τ depend on the parameter $\tau \in [0,1]$. Everywhere below we will assume that the functions $F_\tau(w), G_\tau(w)$ are bounded and continuous together with their derivatives of the third order with respect to w and of the second order with respect to τ. These conditions allow the construction of the topological degree [10].

Let $w = u + \psi$, where $\psi(x,y)$ is an infinitely differentiable function such that $\psi(x,y) = u_+(y)$ for $x \geq 1$ and $\psi(x,y) = u_-(y)$ for $x \leq -1$. Set

$$T_\tau(u) = T_0^\tau(u+\psi) = D\Delta u + c\frac{\partial u}{\partial x} + F_\tau(u+\psi) + \Delta\psi + c\,\frac{\partial \psi}{\partial x}\,, \quad (x,y) \in \Omega, \tag{2.11}$$

$$Q_\tau(u) = Q_0^\tau(u+\psi) = \begin{cases} \frac{\partial u}{\partial y} & , \quad y = 0 \\ \frac{\partial u}{\partial y} - G_\tau(u+\psi) + \frac{\partial \psi}{\partial y} & , \quad y = l \end{cases}. \tag{2.12}$$

We consider the operator $P_\tau = (T_\tau, Q_\tau)$ acting in weighted spaces,

$$P_\tau = (T_\tau, Q_\tau) : W^{2,2}_{\infty,\mu}(\Omega) \to L^2_{\infty,\mu}(\Omega) \times W^{1/2,2}_{\infty,\mu}(\partial\Omega).$$

with the weight function $\mu(x) = \sqrt{1+x^2}$. The norm in the weighted space is defined as follows:

$$\|u\|_{\infty,\mu} = \|u\mu\|_\infty.$$

Consider the problem

$$Dw'' + F_\tau(w) = 0, \tag{2.13}$$

$$w'(0) = 0, \quad w'(l) = G_\tau(w(l)) \tag{2.14}$$

in the interval $0 < y < l$. We suppose the following two conditions to be satisfied.

Condition 1'. There exist two solutions, $w_+^\tau(y)$ and $w_-^\tau(y)$ of problem (2.13), (2.14) such that

$$w_+^\tau(y) \le w_-^\tau(y), \quad 0 < y < l$$

and the eigenvalue problems

$$Dv'' + F'_\tau(w_\pm(y))v = \lambda v,$$
$$v'(0) = 0, \quad v'(l) = G'_\tau(w_\pm(l))v(l)$$

have all eigenvalues in the left-half plane.

Condition 2'. For any other solution $w^\tau(y)$ of problem (2.13), (2.14) such that

$$w_+^\tau(y) \le w(y) \le w_-^\tau(y), \quad 0 < y < l,$$

the eigenvalue problems

$$Dv'' + F'_\tau(w_\pm(y))v = \lambda v,$$
$$v'(0) = 0, \quad v'(l) = G'_\tau(w_\pm(l))v(l)$$

have at least one eigenvalue in the right-half plane.

For simplicity of presentation we will suppose in what follows that the functions $w_\pm^\tau(y)$ do not depend on τ.

3. A priori estimates

3.1. **Auxiliary results.** We begin with some auxiliary results. Consider the problem

$$D\Delta u + c\frac{\partial u}{\partial x} + F(u) = 0, \tag{3.1}$$

$$y = 0 : \frac{\partial u}{\partial y} = 0, \quad y = l : \frac{\partial u}{\partial y} = G(u). \tag{3.2}$$

The subscript τ is omitted where it is not necessary. We look for the solutions with the limits

$$\lim_{x\to\pm\infty} u(x, y) = u_\pm(y), \quad 0 < y < l \tag{3.3}$$

at infinity, $u_-(y) > u_+(y)$. The proofs of the following lemmas are similar to those in [1].

Lemma 3.1. Let $U_0(x, y)$ be a solution of problem (3.1), (3.2) such that $\frac{\partial U_0}{\partial x} \le 0$ for all $(x, y) \in \bar{\Omega}$. Then the last inequality is strict.

Lemma 3.2. Let $u_n(x, y)$ be a sequence of solutions of problem (3.1), (3.2) such that $u_n \to U_0$ in $C^1(\bar{\Omega})$, where $U_0(x, y)$ is a solution monotonically decreasing with respect to x. Then for all n sufficiently large $\frac{\partial u_n}{\partial x} < 0$, $(x, y) \in \bar{\Omega}$.

We will now determine the sign of the speed of the wave connecting a stable and an unstable solutions. This result will be used below for estimates of solutions.

Lemma 3.3. Suppose $u_0(y)$ is a solution of problem (1.6), (1.7) in the cross section of the domain, and $u_+(y) < u_0(y) < u_-(y)$. Assume, next, that the corresponding eigenvalue problem

$$v'' + F'(u_0)v = \lambda v, \quad v'(0) = 0, \quad v'(l) = G'(u_0(l))v(l) \tag{3.4}$$

has some eigenvalues in the right-half plane. If a monotone with respect to x function $w(x, y)$ satisfies the problem

$$D\Delta w + c\frac{\partial w}{\partial x} + F(w) = 0, \tag{3.5}$$

$$y = 0 : \frac{\partial w}{\partial y} = 0, \quad y = l : \frac{\partial w}{\partial y} = G(w), \tag{3.6}$$

$$\lim_{x\to-\infty} w(x, y) = u_-(y), \quad \lim_{x\to\infty} w(x, y) = u_0(y), \tag{3.7}$$

then $c > 0$. If

$$\lim_{x\to-\infty} w(x, y) = u_0(y), \quad \lim_{x\to\infty} w(x, y) = u_+(y),$$

instead of (3.7), then $c < 0$.

Lemma 3.4. *If problem (3.1)-(3.3) has a solution w, then the value of the speed admits the estimate $|c| \leq M$, where the constant M depends only on* $\max_{u\in[u_+,u_-]} |F'(u)|, |G'(u)|$.

3.2. **Functionalization of the parameter.** Let $w_0(x, y)$ be a solution of problem (3.1)-(3.3). Then the functions

$$w_h(x, y) = w_0(x + h, y), \quad h \in \mathbb{R}$$

are also solutions of this problem. The existence of the family of solutions does not allow one to use directly the topological degree because there is a zero eigenvalue of the linearized problem and a uniform a priori estimate of solutions in the weighted spaces does not occur.

In order to overcome this difficulty, we replace the unknown parameter c, the wave speed, by a functional $c(w_h)$. This approach was suggested in [6] for periodic solutions of ordinary differential systems of equations, and then used for travelling waves in [8]. This functional determines a function of h, $s(h) = c(w_h)$. We will construct this functional in such a way that $s'(h) < 0$ and $s(h) \to \pm\infty$ as $h \to \mp\infty$. Then instead of the family of solutions we obtain a single solution for the value of h for which $c = s(h)$.

Let

$$\rho(w_h) = \int_\Omega (w_0(x + h, y) - u_+(y))r(x)dxdy,$$

where $r(x)$ is an increasing function satisfying the conditions:

$$r(-\infty) = 0, \quad r(+\infty) = 1, \quad \int_{-\infty}^{0} r(x)dx < \infty.$$

Since $w_0(x,y)$ is a decreasing function of x, then $\rho(w_h)$ is a decreasing function of h, and

$$\rho(w_h) \to \begin{cases} 0 & , \quad h \to +\infty \\ +\infty & , \quad h \to -\infty \end{cases} .$$

Hence the function $s(h) = c(w_h) = \ln \rho(w_h)$ possesses the required properties.

3.3. **Estimates of solutions.** We consider next the problem

$$D\Delta w + c\frac{\partial w}{\partial x} + F_\tau(w) = 0, \tag{3.8}$$

$$y = 0 : \frac{\partial w}{\partial y} = 0, \quad y = l : \frac{\partial w}{\partial y} = G_\tau(w(l)), \tag{3.9}$$

$$w(\pm\infty, y) = u_\pm(y). \tag{3.10}$$

The proof of the following lemma is similar to the proof for the single equation [1], [2].

Lemma 3.5. *Suppose that solution $w(x,y)$ of problem (3.8)-(3.10) satisfies the estimate $|w| \le M$ with some positive constant M, and*

$$|F_\tau^{(i)}(w)|, \;\; |G_\tau^{(i)}(w)| \le K \;\text{ for }\; |w| \le M, \quad i = 0,1,2,3,$$

where K is a positive constant. Then the Hölder norm $C^{2+\alpha}(\bar\Omega)$, $0 < \alpha < 1$ of the solution is bounded by a constant which depends only on K, M and c.

Denote by w_τ a solution of problem (3.8)-(3.10). We need to obtain a uniform estimate of the solution $u_\tau = w_\tau - \psi$ in the norm of the space $W^{2,2}_{\infty,\mu}(\Omega)$. Here $\psi(x,y)$ is an infinitely differentiable function such that $\psi(x,y) = u_+(y)$ for $x \ge 1$ and $\psi(x,y) = u_-(y)$ for $x \le -1$. Since $u \in C^{2+\alpha}(\bar\Omega)$, then the norm $W^{2,2}_\infty(\Omega)$ of the solution is also uniformly bounded. However, the boundedness of the norm in the weighted space does not follow from this and should be proved. In order to obtain the estimate, it is sufficient to prove that the solution is bounded in the weighted space, that is

$$\sup_{(x,y)\in\Omega} |(w_\tau(x,y) - \psi(x,y))\mu(x)| \le M \tag{3.11}$$

with some constant M independent of τ. If this estimate is satisfied, then the derivatives of the solution up to the order two are also bounded. Indeed, the function $u_\tau = w_\tau - \psi$ satisfies the problem

$$D\Delta u + c\frac{\partial u}{\partial x} + F_\tau(u+\psi) + \gamma(x,y) = 0,$$

$$y = 0 : \frac{\partial u}{\partial y} = 0, \quad y = l : \frac{\partial u}{\partial y} = G_\tau(u+\psi),$$

where $\gamma(x,y) = \Delta\psi + c\frac{\partial \psi}{\partial x}$. Then the function $v_\tau = u_\tau\mu$ satisfies the problem

$$D\Delta v + (c - 2\mu_1)\frac{\partial v}{\partial x} + (-c\mu_1 + 2\mu_1^2 - \mu_2)v + \tag{3.12}$$
$$(F_\tau(u+\psi) - F_\tau(\psi))\mu + (\gamma + F_\tau(\psi))\mu = 0,$$

$$y = 0 : \frac{\partial v}{\partial y} = 0, \quad y = l : \frac{\partial v}{\partial y} = (G_\tau(u+\psi) - G_\tau(\psi))\mu + G_\tau(\psi)\mu, \tag{3.13}$$

where

$$\mu_1 = \frac{\mu'}{\mu}, \quad \mu_2 = \frac{\mu''}{\mu}$$

are bounded infinitely differentiable functions converging to zero at infinity. Since

$$|(F_\tau(u+\psi) - F_\tau(\psi))\mu| \le \sup_s |F'_\tau(s)||u\mu|,$$

$$|(G_\tau(u+\psi) - G_\tau(\psi))\mu| \le \sup_s |G'_\tau(s)||u\mu|,$$

then, by virtue of (3.11), the functions

$$\Phi(u,x) = (F_\tau(u+\psi) - F_\tau(\psi))\mu + (\gamma + F_\tau(\psi))\mu,$$

$$\Psi(u,x) = (G_\tau(u+\psi) - G_\tau(\psi))\mu + G_\tau(\psi)\mu$$

are bounded together with their second derivatives. Therefore solutions of problem (3.12), (3.13) are uniformly bounded in the space $C^{2+\alpha}(\Omega)$. Then the norm $W^{2,2}_\infty(\Omega)$ is also bounded.

It remains to prove estimate (3.11). Consider first of all the behavior of solutions at the vicinity of infinity. By virtue of the Fredholm property, $|w_\tau(x,y) - u_\pm(y)|$ decay exponentially as $x \to \pm\infty$. The decay rate is determined by the principal eigenvalue of the corresponding operators in the cross-section of the cylinder. They can be estimated independently of τ.

Let $\epsilon > 0$ be small enough, $N_-(\tau)$ and $N_+(\tau)$ be such that $|w_\tau(x,y) - u_+(y)| \le \epsilon$ for $x \ge N_+(\tau)$ and $|w_\tau(x,y) - u_-(y)| \le \epsilon$ for $x \le N_-(\tau)$. For a polynomial weight function $\mu(x)$ there exists a constant K independent of $\tau \in [0,1]$ such that

$$|w_\tau(x,y) - u_\pm(y)|\mu(x) \le K, \quad x \gtrless N_\pm(\tau), \quad \tau \in [0,1].$$

Since the functions $w_\tau(x,y)$ are uniformly bounded, then (3.11) will follow from the uniform boundedness of the values $N_\pm(\tau)$.

First, let us note that the difference between them is uniformly bounded. Indeed, if this is not the case and $N_+(\tau) - N_-(\tau) \to \infty$ as $\tau \to \tau_0$ for some τ_0, then there are two solutions of problem (3.8), (3.9) for $\tau = \tau_0$, w_1 and w_2 with the limits

$$w_1(x,y) \to \begin{cases} u_-(y) & , \quad x \to -\infty \\ u_0(y) & , \quad x \to +\infty \end{cases}, \quad w_2(x,y) \to \begin{cases} u_0(y) & , \quad x \to -\infty \\ u_+(y) & , \quad x \to +\infty \end{cases}.$$

These solutions are obtained as limits of the solution w_τ as $\tau \to \tau_0$. In order to obtain them, consider a sequence of functions $w_{\tau_k}(x,y)$, $\tau_k \to \tau_0$ and two sequences of shifted functions: $w_{\tau_k}(x + N_-(\tau_k), y)$ and $w_{\tau_k}(x + N_+(\tau_k), y)$. The first sequence gives in the limit the first solution, the second limit gives the second solution.

The existence of such solutions contradicts Lemma 3.3 since the first one affirms that the speed is positive while the second one that it is negative.

Next, if one of the values $|N_\pm(\tau)|$ tends to infinity as $\tau \to \tau_0$, then the modulus $|c(w_h)|$ of the functional introduced in Section 3.2 also tends to infinity as $\tau \to \tau_0$. This contradicts a priori estimates of the wave speed. Thus, we have proved the following theorem.

Theorem 3.6. *Let the functions $F_\tau(w), G_\tau(w)$ be bounded and continuous together with their derivatives of the third order with respect to w and of the second order with respect to τ. Suppose further that Conditions 1' and 2' are satisfied. If there exists a solution w_τ of problem (3.8)-(3.10) such that $u_\tau = w_\tau - \psi \in W^{2,2}_{\infty,\mu}(\Omega)$, then the norm $\|u_\tau\|_{W^{2,2}_{\infty,\mu}(\Omega)}$ is bounded independently of τ and of the solution w_τ.*

4. Wave existence

4.1. **Model problem.** Consider the problem

$$D\Delta w + c\frac{\partial w}{\partial x} + F_0(w) = 0, \tag{4.1}$$

$$y = 0 : \frac{\partial w}{\partial y} = 0, \quad y = l : \frac{\partial w}{\partial y} = 0, \tag{4.2}$$

$$w(\pm\infty, y) = u_\pm, \tag{4.3}$$

where we put 0 instead of $G(w)$ in the boundary condition, u_+ and u_- are some vectors such that $F_0(u_\pm) = 0$, and the matrices $F_0'(u_\pm)$ have all eigenvalues in the left-half plane. Suppose, next, that for any other zero u_0 such that $u_+ < u_0 < u_-$ (inequalities are component-wise), the matrix $F_0'(u_0)$ has an eigenvalue in the right-half plane. In this case the problem

$$Dw'' + cw' + F_0(w) = 0, \quad w(\pm\infty) = u_\pm$$

has a solution $w_0(x)$ for a unique value of c (see, e.g., [8]). This function is also a solution of problem (4.1)-(4.3). The uniqueness of this solution as a solution of the two-dimensional problem is proved in the following lemma.

Lemma 4.1. *There exists a unique monotone in x solution of problem (4.1)-(4.3) up to translation in space.*

Proof. Suppose that there exist two different monotone solutions of problem (4.1)-(4.3), (w_1, c_1) and (w_2, c_2). We recall that the corresponding values of the speed c can be different. Consider the equation

$$\frac{\partial v}{\partial t} = D\Delta v + c_1\frac{\partial v}{\partial x} + F_0(v) \tag{4.4}$$

with the boundary condition (4.2). The function $w_1(x, y)$ is a stationary solution of this problem. It is proved in [9] that it is globally stable with respect to all initial conditions $v(x, y, 0)$, which are monotone with respect to x and such that the norm $\|v(x, y, 0) - w_1(x, y)\|_{L^2(\Omega)}$ is bounded.

Consider the initial condition $v(x, y, 0) = w_2(x, y)$. It is monotone and the L^2 norm of the difference $w_2 - w_1$ is bounded since these functions approach exponentially their limits at infinity. According to the stability result, the solution converges to $w_1(x + h, y)$ with some h. On the other hand, the solution writes $u(x, y, t) = w_2(x - (c_2 - c_1)t, y)$, and it cannot converge to w_1. This contradiction proves the lemma. □

4.2. **Existence.** We consider next the problem (3.8)-(3.10) and the corresponding operators

$$T_\tau(u) = D\Delta(u+\psi) + c(u+\psi)\frac{\partial(u+\psi)}{\partial x} + F_\tau(u+\psi), \quad (x,y)\in\Omega, \tag{4.5}$$

$$Q_\tau(u) = \begin{cases} \frac{\partial u}{\partial y} & , \quad y=0 \\ \frac{\partial u}{\partial y} - G_\tau(u+\psi) & , \quad y=l \end{cases}, \tag{4.6}$$

$$P_\tau = (T_\tau, Q_\tau) : W^{2,2}_{\infty,\mu}(\Omega) \to L^2_{\infty,\mu}(\Omega) \times W^{1/2,2}_{\infty,\mu}(\partial\Omega).$$

Suppose that $g_\tau(u) \equiv 0$ for $\tau = 0$. Then the equation

$$P_\tau(u) = 0 \tag{4.7}$$

has a unique solution $u_0 = w_0 - \psi$ for $\tau = 0$. The index of this solution, that is the topological degree of this operator with respect to a small neighborhood of the solution, equal 1. Indeed, the index equals $(-1)^\nu$, where the ν is the number of positive eigenvalues of the linearized operator [8], [10]. In the case under consideration, the linearized operator has all eigenvalues in the left-half plane [9].

Theorem 4.2. *Suppose that the functions $F_\tau(w), G_\tau(w)$ are bounded and continuous together with their derivatives of the third order with respect to w and of the second order with respect to τ. Let $G_0(w) = 0$. Assume, further, that Conditions 1' and 2' are satisfied. Then equation (4.7) has a solution for all $\tau \in [0,1]$.*

Proof. The proof of the theorem is based on the Leray-Schauder method. We consider equation (4.7). The topological degree for the operator $P_\tau(u)$ is defined. Denote by Γ_m the ensemble of solutions of equation (4.7) for all $\tau \in [0,1]$ such that for any $u \in \Gamma_m$ the function $w = u+\psi$ is monotone with respect to x. Let Γ_n be the set of all solutions for which the function $w = u+\psi$ is not monotone with respect to x. Then the distance d between these two sets in the space $E = W^{2,2}_{\infty,\mu}(\Omega)$ is positive. Indeed, suppose that this is not true. Then there exist two sequences $u_k \in \Gamma_m$ and $v_k \in \Gamma_n$ such that $\|u_k - v_k\|_E \to 0$ as $k \to \infty$. From Lemma 3.2 it follows that the functions $w_k = v_k + \psi$ are monotone with respect to x for k sufficiently large. This contradiction shows that the convergence cannot occur.

From Theorem 3.6, applicable for solutions from Γ_m, it follows that the set Γ_m is bounded in E. Moreover, by virtue of properness of the operator P_τ it is compact. Hence there exists a bounded domain $G \subset E$ such that $\Gamma_m \subset G$ and $\Gamma_n \cap \bar{G} = \oslash$.

Consider the topological degree $\gamma(P_\tau, G)$. Since

$$P_\tau(u) \neq 0, \quad u \in \partial G,$$

then it is well defined. Since $\gamma(P_0, G) = 1$, then $\gamma(P_\tau, G) = 1$ for any $\tau \in [0,1]$. Hence equation (4.7) has a monotone solution for any $\tau \in [0,1]$.

It remains to verify its uniqueness. We recall that

$$\gamma(P_\tau, G) = \sum_i \text{ ind } u_i,$$

where ind u_i is the index of a solution u_i and the sum is taken with respect to all solutions $u_i \in G$. Since $\gamma(P_\tau, G) = 1$ and ind $u_i = 1$, then the solution is necessarily unique. □

5. Application to a model of atherosclerosis

In this section we apply the result obtained above to a model of atherosclerosis development. Let us recall that atherosclerosis is a chronic inflammation in blood vessel walls where inflammatory cytokines activate receptors at the surface of the endothelial cells separating intima (interior of the vessel wall) and blood flow. These receptors stop monocytes rolling along the vessel wall. They penetrate intima, differentiate into macrophages and eliminate inflammatory substances. However they remain trapped in the intima and transform into foam cells which produce even more inflammatory cytokines. This is a self-amplifying process which leads to the inflammation spreading in the tissue [3]-[5].

Consider the system of two equations

$$\Delta u + c\frac{\partial u}{\partial x} = 0, \tag{5.1}$$

$$\Delta v + c\frac{\partial v}{\partial x} + f(u) - v = 0, \tag{5.2}$$

in the two-dimensional domain Ω,

$$\Omega = \{(x, y), -\infty < x < \infty,\ 0 < y < l\}$$

with the boundary conditions

$$y = 0 : \frac{\partial u}{\partial y} = 0,\ \ \frac{\partial v}{\partial y} = 0, \quad y = l : \frac{\partial u}{\partial y} = g(v(x, l)) - u(x, l),\ \ \frac{\partial v}{\partial y} = 0. \tag{5.3}$$

Here the two-dimensional domain Ω corresponds to the longitudinal cross section of the blood vessel wall, u is the concentration of cells, v is the concentration of cytokines. The function $f(u)$ describes production of cytokines by cells, and the function $g(v)$ in the boundary condition shows how the cell flux through the boundary depends on the cytokine concentration. The term $-u$ in the same boundary condition describes the decrease of cell flux as a function of cell concentration due to the crowding effect.

5.1. Problem in the cross section.

Consider the problem in the cross-section

$$\begin{gathered} u'' = 0, \quad v'' + f(u) - v = 0, \\ u'(0) = v'(0) = 0,\ \ u'(l) = g(v(l)) - u(l),\ \ v'(l) = 0. \end{gathered} \tag{5.4}$$

We will study existence and stability of its solutions. The proof of the following lemmas is straightforward.

Lemma 5.1. *Problem (5.4) has only constant solutions that can be found as solutions of the equations*

$$f(u) - v = 0, \quad g(v) - u = 0.$$

Lemma 5.2. *A constant solution (u, v) of problem (5.4) is stable (the principle eigenvalue of the linearized problem is negative) if the eigenvalues of the matrix*

$$A = \begin{pmatrix} f'(u) & -1 \\ -1 & g'(v) \end{pmatrix}$$

are negative. It is unstable (the principle eigenvalue of the linearized problem is positive) if one of the eigenvalues of the matrix A is positive.

Next, we consider the following problem

$$\Delta u + c\frac{\partial u}{\partial x} + \delta(g(v) - u) = 0, \tag{5.5}$$

$$\Delta v + c\frac{\partial v}{\partial x} + f(u) - v = 0, \tag{5.6}$$

in the two-dimensional domain Ω,

$$\Omega = \{(x, y), -\infty < x < \infty,\ 0 < y < l\}$$

with the boundary conditions

$$y = 0 : \frac{\partial u}{\partial y} = 0,\ \ \frac{\partial v}{\partial y} = 0, \quad y = l : \frac{\partial u}{\partial y} = \epsilon(g(v(x, l)) - u(x, l)),\ \ \frac{\partial v}{\partial y} = 0. \tag{5.7}$$

For $\epsilon = 0$ this problem has a unique solution. It depends on x and does not depend on y. For $\epsilon = 1, \delta = 0$ we get problem (5.1)-(5.3).

Let us show that the problem in the cross section

$$u'' + \delta(g(v) - u) = 0, \quad v'' + f(u) - v = 0, \tag{5.8}$$

$$u'(0) = v'(0) = 0,\ \ u'(l) = \epsilon(g(v(l)) - u(l)),\ \ v'(l) = 0 \tag{5.9}$$

has only constant solution for sufficiently small ϵ and δ.

Lemma 5.3. *Problem (5.8), (5.9) has only constant solutions for all ϵ and δ sufficiently small.*

Proof. It can be easily verified that the assertion of the lemma is true for $\epsilon > 0$ and $\delta = 0$. From the implicit function theorem it follows that it remains true for sufficiently small $\delta > 0$ (depending on ϵ). Similarly, the lemma holds for $\delta > 0$ (sufficiently small) and $\epsilon > 0$ (depending on δ).

Let us consider the case where ϵ and δ are of the same order of magnitude and set $\delta = k\epsilon$ for a fixed positive k. For simplicity of presentation we restrict ourselves to the case $g(v) = v$ which is sufficient to the construction of homotopy. (We can also consider the case where $k \to 0$ or $\epsilon = k\delta$ and $k \to 0$.)

It can be easily verified that for $\epsilon = 0$ the problem has only constant solution. Since the corresponding operator is proper, then the set of solutions (u_ϵ, v_ϵ) is compact, and it converges to a constant solution (u_0, v_0) as $\epsilon \to 0$ such that $f(u_0) = v_0$. Hence we can write problem (5.8), (5.9) as follows:

$$u'' + k\epsilon(v - u) = 0, \quad v'' + f(u_0) + f'(u_0)(u - u_0) + \frac{1}{2}f''(\theta)(u - u_0)^2 - v = 0, \tag{5.10}$$

$$u'(0) = v'(0) = 0, \;\; u'(l) = \epsilon(v(l) - u(l)), \;\; v'(l) = 0, \tag{5.11}$$

where θ is some point, and search its solution in the form:

$$u = u_0 + \epsilon u_1, \;\; v = v_0 + \epsilon v_1.$$

We substitute these expressions into (5.10), (5.11) and keep the first-order terms with respect to ϵ:

$$u_1'' + k(v_0 - u_0) = 0, \quad v_1'' + f'(u_0)u_1 - v_1 = 0, \tag{5.12}$$

$$u_1'(0) = v_1'(0) = 0, \;\; u_1'(l) = v_0 - u_0, \;\; v_1'(l) = 0. \tag{5.13}$$

From the first equation in (5.12) we have $u_1(y) = a + by - \frac{k}{2}(v_0 - u_0)y^2$. From the left boundary condition we conclude that $b = 0$ and from the right boundary condition that $v_0 = u_0$ since $k > 0$. Hence $u_1 \equiv$ const. Then from the second equation and the boundary conditions we obtain that $v_1 \equiv$ const. If the solution $u = u_0$ of the equation $f(u) = u$ is isolated, then $u_1 = v_1 = 0$. □

Let us note that for $\epsilon = \delta = 0$ problem (5.8), (5.9) linearized about the constant solution has a zero eigenvalue. Therefore we cannot directly use the implicit function theorem to prove the uniqueness of solution. We should consider a subspace orthogonal to the eigenfunction corresponding to the zero eigenvalue (constant). The proof of the lemma represents another realization of this approach: in the subspace orthogonal to a constant (nonzero) function, problem (5.12), (5.13) has only zero solution. Therefore in this subspace the zero solution is unique for sufficiently small ϵ.

5.2. **Existence of solutions.** We can now prove the existence theorem.

Theorem 5.4. *Suppose that the functions $f(u)$ and $g(v)$ are bounded and continuous together with their third derivatives, and the system of equations $f(u) = v, g(v) = u$ has three solutions: $(u_\pm, v_\pm)$ and (u_0, v_0) such that $u_+ < u_0 < u_-, v_+ < v_0 < v_-$. If $f'(u_\pm)g'(v_\pm) < 1$ and $f'(u_0)g'(v_0) > 1$, then problem (5.5)-(5.7) has a solution for some value of c with the limits $u \to u_\pm, v \to v_\pm$ as $x \to \pm\infty$.*

Proof. Consider problem (5.5)-(5.7). For $\epsilon = 1, \delta = 0$ it coincides with problem (5.1)-(5.3). Let ϵ_0, δ_0 be such that problem (5.8), (5.9) has only constant solutions for $0 \le \epsilon \le \epsilon_0, 0 \le \delta \le \delta_0$ (Lemma 5.3).

We construct a homotopy of problem (5.1)-(5.3). It consists of the following steps:

1. the value of ϵ in (5.5)-(5.7) changes from 1 to ϵ_0, $\delta = 0$,
2. the value of δ changes from 0 to δ_0, $\epsilon = \epsilon_0$,
3. the value of ϵ changes from ϵ_0 to 0, $\delta = \delta_0$.

During this homotopy the problem in the cross section (5.8), (5.9) has only constant solutions $(u_\pm, v_\pm)$ and (u_0, v_0). Moreover the solutions $(u_\pm, v_\pm)$ are stable and (u_0, v_0) is unstable. We can now apply Theorem 4.2.

□

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