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EXTREMAL SOLUTIONS FOR QUASILINEAR PARABOLIC SYSTEMS IN TRAPPING REGIONS

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Abstract. We consider the initial-Dirichlet boundary value problem for quasilinear parabolic systems in a cylindrical domain $Q = \Omega \times (0, \tau)$ of the form $(i = 1, 2)$

$$\frac{\partial u_i}{\partial t} - \Delta_{p_i} u_i = f_i(x, t, u_1, u_2, \nabla u_1, \nabla u_2) \text{ in } Q,$$

with a diagonal (p_1, p_2)-Laplacian as leading elliptic operator, and with a lower order vector field $f = (f_1, f_2)$ that may depend also on the gradient of the solution $u = (u_1, u_2)$. We establish an enclosure and existence result for weak solutions in terms of trapping regions which stand for rectangles formed by pairs of appropriately defined sub-supersolutions, and prove the existence of extremal solutions within trapping regions without imposing any monotonicity conditions on the lower order vector field. Finally, we provide conditions that allow us to construct trapping regions. It should be noted that the results obtained in this paper may be extended to more general quasilinear systems, where the p_i-Laplacian is replaced by a general divergence form Leray-Lions operator div $A_i(x, t, u_i, \nabla u_i)$.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, $Q = \Omega \times (0, \tau)$, and $\Gamma = \partial\Omega \times (0, \tau)$, with $\tau > 0$, and let $W^{1,p_i}(\Omega)$ and $W_0^{1,p_i}(\Omega)$, $i = 1, 2$, denote the usual Sobolev spaces with dual spaces $(W^{1,p_i}(\Omega))^*$ and $W^{-1,q_i}(\Omega)$, respectively, where q_i is the Hölder conjugate satisfying $1/p_i + 1/q_i = 1$. For the sake of simplicity we assume throughout this paper $2 \le p_i < \infty$. Then $W^{1,p_i}(\Omega) \subset L^2(\Omega) \subset (W^{1,p_i}(\Omega))^*$ as well as $W_0^{1,p_i}(\Omega) \subset L^2(\Omega) \subset (W_0^{1,p_i}(\Omega))^*$ forms an evolution triple with all the embeddings being dense and compact, cf. [10].

Further we set $X_i = L^{p_i}(0, \tau; W^{1,p}(\Omega))$, $X_{0i} = L^{p_i}(0, \tau; W_0^{1,p}(\Omega))$, and introduce the Leray-Lions spaces W_i, and W_{0i} defined by

$$W_i = \{u \in X_i : u' \in X_i^*\}, \quad W_{0i} = \{u \in X_{0i} : u' \in X_{0i}^*\}$$

where the derivative $u' := \partial u/\partial t$ is understood in the sense of vector-valued distributions, and $X_i^* = L^q(0, \tau; (W^{1,p_i}(\Omega))^*)$ is the dual space of X_i, resp. $X_{0i}^* = L^{q_i}(0, \tau; (W_0^{1,p_i}(\Omega))^*)$ is the dual of X_{0i}. The spaces W_i and W_{0i} endowed with the

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graph norm of the operator $\partial/\partial t$

$$\|u\|_{W_i} = \|u\|_{X_i} + \|u'\|_{X_i^*}, \quad \|u\|_{W_{0i}} = \|u\|_{X_{0i}} + \|u'\|_{X_{0i}^*}$$

are Banach spaces which are separable and reflexive due to the separability and reflexivity of X_i and X_i^*, and X_{0i} and X_{0i}^*, respectively. It is well known that the embedding $W_i \hookrightarrow C([0,\tau], L^2(\Omega))$ (resp. $W_{0i} \hookrightarrow C([0,\tau], L^2(\Omega))$) is continuous, and by Aubin's lemma the embedding $W_i \hookrightarrow\hookrightarrow L^{p_i}(Q)$ (resp. $W_{0i} \hookrightarrow\hookrightarrow L^{p_i}(Q)$) is compact due to the compact embedding $W^{1,p_i}(\Omega) \hookrightarrow\hookrightarrow L^{p_i}(\Omega)$.

The notation $\langle\cdot,\cdot\rangle$ stands for any of the dual pairings between X_i and X_i^*, X_{0i} and X_{0i}^*, $W^{1,p_i}(\Omega)$ and $(W^{1,p_i}(\Omega))^*$, and $W_0^{1,p_i}(\Omega)$ and $W^{-1,q_i}(\Omega)$, such as for example, if $f \in X_i^*$ and $u \in X_i$, then

$$\langle f, u\rangle = \int_0^\tau \langle f(t), u(t)\rangle \, dt.$$

In what follows we denote by $L_i u := u' = \partial u/\partial t$ the time derivative operator with its domain of definition, $D(L_i)$, given by

$$D(L_i) = \left\{ u \in X_{0i} : u' \in X_{0i}^* \text{ and } u(\cdot,0) = 0 \right\}.$$

It is known that the linear operator $L_i : D(L_i) \subset X_{0i} \to X_{0i}^*$ is closed, densely defined and maximal monotone, e.g., cf. [10, Chap. 32]. Finally, for any number $r \in \mathbb{R}$ we set $r^\pm := \max\{\pm r, 0\}$, so $r = r^+ - r^-$.

In this paper we consider the initial-Dirichlet boundary value problem for the following quasilinear parabolic system of the form $(i = 1,2)$

$$u_i' - \Delta_{p_i} u_i = f_i(x,t,u_1,u_2,\nabla u_1,\nabla u_2) \text{ in } Q, \; u_i|_\Gamma = 0, \;\; u_i(\cdot,0)|_\Omega = 0, \tag{1.1}$$

where $\Delta_{p_i} u = \operatorname{div}(|\nabla u|^{p_i-2}\nabla u)$ is the p_i-Laplacian operator, and the right-hand side vector field $(f_1,f_2) : Q\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}^2$ is a Carathéodory map, i.e., $(x,t) \mapsto f_i(x,t,s_1,s_2,\xi_1,\xi_2)$ is measurable in Q for all (s_1,s_2,ξ_1,ξ_2) in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$, and $(s_1,s_2,\xi_1,\xi_2) \mapsto f_i(x,t,s_1,s_2,\xi_1,\xi_2)$ is continuous in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$ for a.a. $(x,t) \in Q$.

Definition 1.1. A weak solution of problem (1.1) is a pair $(u_1,u_2) \in W_{01}\times W_{02}$ such that $u_i(\cdot,0) = 0$ on Ω, $f_i(\cdot,\cdot,u_1,u_2,\nabla u_1,\nabla u_2) \in X_{0i}^*$ for $i = 1,2$, and

$$\langle u_1', v_1\rangle + \int_Q |\nabla u_1|^{p_1-2}\nabla u_1 \nabla v_1 = \langle f_1(\cdot,\cdot,u_1,u_2,\nabla u_1,\nabla u_2), v_1\rangle,$$
$$\langle u_2', v_2\rangle + \int_Q |\nabla u_2|^{p_2-2}\nabla u_2 \nabla v_2 = \langle f_2(\cdot,\cdot,u_1,u_2,\nabla u_1,\nabla u_2), v_2\rangle$$

for all $(v_1,v_2) \in X_{01}\times X_{02}$.

Remark 1.2. Note, here and throughout this paper we use the notation $\int_Q \cdot = \int_Q \cdot \, dxdt$. Further we remark that homogeneous initial- and boundary conditions in (1.1) have been assumed without loss of generality. Inhomogeneous initial-and boundary values of W_i functions can be considered without any difficulties.

Let us introduce operators as follows:

$$\begin{aligned} &L : D(L_1)\times D(L_2) \to X_{01}^*\times X_{02}^* : \langle Lu, v\rangle := \langle L_1u_1, v_1\rangle + \langle L_2u_2, v_2\rangle, \\ &u = (u_1,u_2) \in D(L_1)\times D(L_2) \subset X_{01}\times X_{02}, \; v = (v_1,v_2) \in X_{01}\times X_{02}, \end{aligned} \tag{1.2}$$

(1.3) $$Au = (-\Delta_{p_1} u_1, -\Delta_{p_2} u_2) : X_{01} \times X_{02} \to X_{01}^* \times X_{02}^*, \quad \text{defined by}$$

$$\langle Au, v\rangle = \sum_{k=1}^{2} \langle -\Delta_{p_k} u_k, v_k\rangle = \sum_{k=1}^{2} \int_Q |\nabla u_k|^{p_k-2} \nabla u_k \nabla v_k,$$

and the Nemytskij operators N_{f_i} generated by the right-hand side f_i through

$$N_{f_i}(u_1, u_2)(x, t) = f_i(x, t, u_1, u_2, \nabla u_1, \nabla u_2),$$

which under certain growth conditions specified later give rise to the operator N_f : $X_{01} \times X_{02} \to X_{01}^* \times X_{02}^*$ defined by

(1.4) $$\langle N_f(u), v\rangle = \sum_{k=1}^{2} \int_Q f_k(x, t, u_1, u_2, \nabla u_1, \nabla u_2) v_k.$$

With the operators introduced above, Definition 1.1 is equivalent to the following operator equation: Find $u = (u_1, u_2) \in D(L_1) \times D(L_2)$ such that

(1.5) $$\langle Lu + Au, v\rangle = \langle N_f(u), v\rangle \quad \text{for all} \quad v \in X_0 = X_{01} \times X_{02}.$$

We next introduce our basic notion of trapping region formed by a pair of sub-supersolution.

Definition 1.3. We say that $\underline{u} = (\underline{u}_1, \underline{u}_2)$, $\overline{u} = (\overline{u}_1, \overline{u}_2) \in W_1 \times W_2$ form a pair of sub-supersolution for problem (1.1) if the following holds true:

(i) $\underline{u}_i \le \overline{u}_i$ a.e. in Q, $\underline{u}_i \le 0 \le \overline{u}_i$ a.e. on $\Omega \times 0$, $\underline{u}_i \le 0 \le \overline{u}_i$ on Γ for $i = 1, 2$.

(ii) $f_1(\cdot, \cdot, \underline{u}_1, w_2, \nabla \underline{u}_1, \nabla w_2), f_1(\cdot, \cdot, \overline{u}_1, w_2, \nabla \overline{u}_1, \nabla w_2) \in X_{01}^*$,
$f_2(\cdot, \cdot, w_1, \underline{u}_2, \nabla w_1, \nabla \underline{u}_2), f_2(\cdot, \cdot, w_1, \overline{u}_2, \nabla w_1, \nabla \overline{u}_2) \in X_{02}^*$.

(iii)

$$\langle \underline{u}_1', v_1\rangle + \int_Q |\nabla \underline{u}_1|^{p_1-2} \nabla \underline{u}_1 \nabla v_1 - \langle f_1(\cdot, \cdot, \underline{u}_1, w_2, \nabla \underline{u}_1, \nabla w_2), v_1\rangle$$
$$+\langle \underline{u}_2', v_2\rangle + \int_Q |\nabla \underline{u}_2|^{p_2-2} \nabla \underline{u}_2 \nabla v_2 - \langle f_2(\cdot, \cdot, w_1, \underline{u}_2, \nabla w_1, \nabla \underline{u}_2), v_2\rangle \le 0,$$

and

$$\langle \overline{u}_1', v_1\rangle + \int_Q |\nabla \overline{u}_1|^{p_1-2} \nabla \overline{u}_1 \nabla v_1 - \langle f_1(\cdot, \cdot, \overline{u}_1, w_2, \nabla \overline{u}_1, \nabla w_2), v_1\rangle$$
$$+\langle \overline{u}_2', v_2\rangle + \int_Q |\nabla \overline{u}_2|^{p_2-2} \nabla \overline{u}_2 \nabla v_2 - \langle f_2(\cdot, \cdot, w_1, \overline{u}_2, \nabla w_1, \nabla \overline{u}_2), v_2\rangle \ge 0$$

for all $(v_1, v_2) \in X_{01} \times X_{02}$ with $v_i \ge 0$, and all $(w_1, w_2) \in W_1 \times W_2$ with $\underline{u}_i \le w_i \le \overline{u}_i$ for $i = 1, 2$.

Definition 1.4. If $\underline{u} = (\underline{u}_1, \underline{u}_2), \overline{u} = (\overline{u}_1, \overline{u}_2)$ is a pair of sub-supersolution, then the rectangle $[\underline{u}, \overline{u}] = [\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2]$ is called a **trapping region**. Here we have denoted $[\underline{u}_i, \overline{u}_i] = \{u \in X_i : \underline{u}_i \le u \le \overline{u}_i \text{ a.e. in } Q\}$.

Remark 1.5. The two inequalities of Definition 1.3 are equivalent to the following four inequalities (in their respective corresponding weak form):

$$\underline{u}_1' - \Delta_{p_1} \underline{u}_1 - f_1(x, t, \underline{u}_1, w_2, \nabla \underline{u}_1, \nabla w_2) \le 0, \quad \text{for all } w_2 \in [\underline{u}_2, \overline{u}_2],$$
$$\underline{u}_2' - \Delta_{p_2} \underline{u}_2 - f_2(x, t, w_1, \underline{u}_2, \nabla w_1, \nabla \underline{u}_2) \le 0, \quad \text{for all } w_1 \in [\underline{u}_1, \overline{u}_1],$$

$$\overline{u}_1' - \Delta_{p_1}\overline{u}_1 - f_1(x,t,\overline{u}_1,w_2,\nabla\overline{u}_1,\nabla w_2) \geq 0, \quad \text{for all } w_2 \in [\underline{u}_2,\overline{u}_2],$$
$$\overline{u}_2' - \Delta_{p_2}\overline{u}_2 - f_2(x,t,w_1,\overline{u}_2,\nabla w_1,\nabla\overline{u}_2) \geq 0, \quad \text{for all } w_1 \in [\underline{u}_1,\overline{u}_1].$$

In the present paper we establish an enclosure and existence result for solutions of the parabolic system (1.1) in terms of trapping regions which stand for rectangles formed by pairs of sub-supersolutions. This provides not only the existence of solutions, but also their location in trapping regions in the sense of Definition 1.3. First, we prove an abstract result when the trapping region is prescribed by a given pair of sub-supersolution.

Second, by applying the abstract result, we prove the existence of extremal solutions within trapping regions without imposing any monotonicity conditions on the lower order vector field. More precisely, we prove the existence of minimal and maximal solutions within a trapping region, where the notion *maximal* and *minimal* refer to the partial ordering of vector-valued functions introduced by the order cone $L_+^{p_1}(Q) \times L_+^{p_2}(Q)$. Finally, we establish the existence of positive and negative solutions of (1.1) under verifiable conditions on the vector field (f_1, f_2)..

The main difficulty in our study is represented by the fact that the right-hand side (f_1, f_2) in the system depends not only on the solution (u_1, u_2) but on its gradient $(\nabla u_1, \nabla u_2)$, too. It is for the first time that this is considered for an evolutionary system of quasilinear equations. It should be noted that the results obtained in this paper may be extended to more general quasilinear systems, where the p_i-Laplacian is replaced by a general divergence form Leray-Lions operator div $A_i(x,t,u_i,\nabla u_i)$. Only for the sake of simplifying the presentation and in order to emphasize the main idea, we have restricted to the quasiliear parabolic system (1.1) as the model case. The elliptic counterpart of system (1.1) was treated in [3]. However, the quasilinear parabolic system considered here is by no means a straightforward extension of the elliptic case, and requires new tools for its treatment. Existence and enclosure results for parabolic systems with linear elliptic diagonal operators and right-hand side vector fields not depending on the gradient have been obtained earlier in [5, 6, 7]. A parabolic equation whose leading differential operator is $\partial_t u - \Delta_p u$, so containing the p-Laplacian and actually corresponding to the equation case of our system, but without gradient dependence in the right-hand side, is studied in [2]. In [1], under strong regularity assumptions on the data, existence of classical solutions of parabolic systems have been studied with linear elliptic diagonal operators having Hölder continuous coefficients, and right-hand side vector fields that are allowed to depend on the gradient but are required to satisfy Hölder conditions with respect to the space-time variables and local Lipschitz conditions with respect to the dependent variables.

2. Existence of solutions in trapping regions

Given a pair of sub-supersolution $\underline{u} = (\underline{u}_1, \underline{u}_2)$, $\overline{u} = (\overline{u}_1, \overline{u}_2) \in W_1 \times W_2$ for problem (1.1), the following hypothesis on the right-hand side vector field (f_1, f_2) is supposed:

(H1) For $i = 1, 2$, the functions $f_i : Q \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory and satisfy the growth conditions

$$|f_1(x, t, s_1, s_2, \xi_1, \xi_2)| \le k_1(x, t) + c_1\Big(|\xi_1|^{\frac{p_1}{q_1}} + |\xi_2|^{\frac{p_2}{q_1}}\Big), \tag{2.1}$$

$$|f_2(x, t, s_1, s_2, \xi_1, \xi_2)| \le k_2(x, t) + c_2\Big(|\xi_1|^{\frac{p_1}{q_2}} + |\xi_2|^{\frac{p_2}{q_2}}\Big), \tag{2.2}$$

for a.a. $(x, t) \in Q$, for all $(s_1, s_2) \in [\underline{u}(x, t), \overline{u}(x, t)]$, and for all $\xi_i \in \mathbb{R}^N$, with constants $c_i \ge 0$, and functions $k_i \in L^{q_i}_+(Q)$.

Corresponding to the trapping region $[\underline{u}, \overline{u}]$ we consider the truncation operators $T_i : X_i \to X_i$ $(i = 1, 2)$ given by

$$(T_i u)(x, t) = \begin{cases} \underline{u}_i(x, t) & \text{if} \quad u(x, t) < \underline{u}_i(x, t), \\ u(x, t) & \text{if} \quad \underline{u}_i(x, t) \le u(x, t) \le \overline{u}_i(x, t), \\ \overline{u}_i(x, t) & \text{if} \quad u(x, t) > \overline{u}_i(x, t). \end{cases} \tag{2.3}$$

The operators T_i are known to be continuous and bounded.

In addition, we consider the cut-off functions $b_i : Q \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ defined by

$$b_i(x, t, s) = \begin{cases} -(\underline{u}_i(x, t) - s)^{p_i - 1} & \text{if} \quad s < \underline{u}_i(x, t), \\ 0 & \text{if} \quad \underline{u}_i(x, t) \le s \le \overline{u}_i(x, t), \\ (s - \overline{u}_i(x, t))^{p_i - 1} & \text{if} \quad s > \overline{u}_i(x, t). \end{cases} \tag{2.4}$$

The functions b_i are Carathéodory with the growth

$$|b_i(x, t, s)| \le \tilde{k}_i(x, t) + \tilde{c}_i |s|^{p_i - 1} \tag{2.5}$$

for a.a. $(x, t) \in Q$, for all $s \in \mathbb{R}$, with constants $\tilde{c}_i \ge 0$, and functions $\tilde{k}_i \in L^{q_i}_+(\Omega)$. Furthermore, there are positive constants $a_1^{(i)}$ and $a_2^{(i)}$ such that

$$\int_Q b_i(x, t, u) u \ge a_1^{(i)} \, \|u\|^{p_i}_{L^{p_i}(Q)} - a_2^{(i)}, \quad \forall \; u \in L^{p_i}(Q). \tag{2.6}$$

By (2.5) it turns out that the associated Nemytskij operators $B_i : L^{p_i}(Q) \to L^{q_i}(Q)$ defined by $B_i(u_i)(x, t) = b_i(x, t, u_i(x, t))$ are well defined, continuous and bounded.

From (2.1) and (2.2 it follows that the Nemytskij operators $N_{f_i} : [\underline{u}, \overline{u}] \to L^{q_i}(Q) \subset X_i^*$ $(i = 1, 2)$ are well defined, continuous and bounded.

Let $Tu := (T_1 u_1, T_2 u_2)$, then $T : X_1 \times X_2 \to X_1 \times X_2$ is continuous and bounded, and thus the compositions $N_{f_i} \circ T$ are well defined and

$$N_{f_i} \circ T : X_1 \times X_2 \to X_i^* \quad \text{is bounded and continuous,} \tag{2.7}$$

which implies that the operator $N_f \circ T : X_1 \times X_2 \to X_1^* \times X_2^*$ given by

$$\langle N_f \circ T(u), v \rangle = \sum_{k=1}^{2} \int_Q f_k(x, t, T_1 u_1, T_2 u_2, \nabla T_1 u_1, \nabla T_2 u_2) v_k \tag{2.8}$$

is bounded and continuous as well. In order to handle system (1.1) (or equivalently (1.5)), we consider next the following truncated, auxiliary problem: Find $(u_1, u_2) \in D(L_1) \times D(L_2)$ such that

$$\langle Lu + Au + \lambda \, B(u), v \rangle = \langle N_f \circ T(u), v \rangle \quad \text{for all} \;\; v \in X_0 = X_{01} \times X_{02}, \tag{2.9}$$

where the operators L, A, $N_f \circ T$ are given by (1.2), (1.3) and (2.8), respectively, and $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 > 0$ and $\lambda_2 > 0$ to be chosen appropriately, and $\lambda B(u) := (\lambda_1 B_1(u_1), \lambda_2 B_2(u_2))$. Our next goal is to show the existence of solutions for the truncated, auxiliary problem (2.9) using an abstract surjectivity result for evolution equations, see [4, Theorem 2.152]), which adapted to the situation of problem (2.9) reads as follows.

Theorem 2.1. *Let $L : D(L) \subset X_0 \to X_0^*$ be as given by (1.2) with $X_0 := X_{01} \times X_{02}$, and let $\mathcal{A} : X_0 \to X_0^*$ be bounded, demicontinuous, and pseudomonotone with respect to $D(L)$. If $\mathcal{A}$ is coercive, then $L + \mathcal{A} : D(L) \to X_0^*$ is surjective, i.e., $(L + \mathcal{A})(D(L)) = X_0^*$.*

Problem (2.9) can equivalently be reformulated as

$$u \in D(L) : Lu + \mathcal{A}u = 0, \quad \text{with } \mathcal{A} = A + \lambda\, B - N_f \circ T. \tag{2.10}$$

Lemma 2.2. *The operator $\mathcal{A} = A + \lambda\, B - N_f \circ T : X_0 \to X_0^*$ is pseudomonotone with respect to the domain $D(L) = D(L_1) \times D(L_2)$.*

Proof. As for the definition of pseudomonotone with respect to the domain $D(L_1) \times D(L_2)$ we refer to [4, Definition 2.151]. In order to prove this, let $(u_n) = (u_{1n}, u_{2n}) \subset D(L) = D(L_1) \times D(L_2)$ be a sequence such that $u_{in} \rightharpoonup u_i$ in X_{0i}, $u'_{in} \rightharpoonup u'_i$ in X_{0i}^*, for $i = 1, 2$, i.e., $u_n \rightharpoonup u = (u_1, u_2)$ in $W_0 = W_{01} \times W_{02}$ and

$$\limsup_{n\to\infty} \langle \mathcal{A}(u_n), u_n - u\rangle \le 0. \tag{2.11}$$

We have to show that

$$\mathcal{A}(u_n) \rightharpoonup \mathcal{A}(u) \text{ in } X_0^* = X_{01}^* \times X_{02}^* \text{ and } \langle \mathcal{A}(u_n), u_n\rangle \to \langle \mathcal{A}(u), u\rangle. \tag{2.12}$$

It is known that $W_0 = W_{01} \times W_{02}$ is compactly embedded in $L^{p_1}(Q) \times L^{p_2}(Q)$ (see [4, Theorem 2.141]) which yields

$$u_{in} \to u_i \quad \text{in } L^{p_i}(Q) \text{ for } i = 1, 2. \tag{2.13}$$

Since $B : L^{p_1}(Q) \times L^{p_2}(Q) \to L^{q_1}(Q) \times L^{q_2}(Q)$ is continuous and bounded, we get from (2.13)

$$\lim_{n\to\infty} \langle B(u_n), u_n - u\rangle = 0. \tag{2.14}$$

From $u_n \rightharpoonup u = (u_1, u_2)$ in $W_0 = W_{01} \times W_{02}$, we infer that (u_n) is, in particular, bounded in $X_0 = X_{01} \times X_{02}$, and thus $N_f \circ T(u_n)$ is bounded in $L^{q_1}(Q) \times L^{q_1}(Q)$ due to (H1), which by taking into account (2.8) and (2.13) yields

$$\lim_{n\to\infty} \langle N_f \circ T(u_n), u_n - u\rangle = 0. \tag{2.15}$$

Then by (2.14) and (2.15) we infer that

$$\lim_{n\to\infty} \langle \lambda\, B(u_n) - N_f \circ T(u_n), u_n - u\rangle = 0. \tag{2.16}$$

With (2.11) and (2.16) and $\mathcal{A} = A + \lambda\, B - N_f \circ T$ we finally obtain

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \le 0, \tag{2.17}$$

which due to the definition of A given by (1.3) means

$$\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle = \limsup_{n\to\infty}\big[\langle -\Delta_{p_1}u_{1n}, u_{1n}-u_1\rangle + \langle -\Delta_{p_2}u_{2n}, u_{2n}-u_2\rangle\big] \le 0. \tag{2.18}$$

From the weak convergence $u_{in} \rightharpoonup u_i$ in X_{0i}, and $-\Delta_{p_i} : X_{0i} \to X_{0i}^*$ being continuous, bounded and strictly monotone, we get from (2.18) by using

$$\lim_{n\to\infty}\langle -\Delta_{p_i}u_i, u_{in}-u_i\rangle = 0,$$

the following equality

$$\begin{aligned}\limsup_{n\to\infty}\big[&\langle -\Delta_{p_1}u_{1n} - (-\Delta_{p_1}u_1), u_{1n}-u_1\rangle \\ &+\langle -\Delta_{p_2}u_{2n} - (-\Delta_{p_2}u_2), u_{2n}-u_2\rangle\big] \\ = \limsup_{n\to\infty}\big[&\langle -\Delta_{p_1}u_{1n}, u_{1n}-u_1\rangle + \langle -\Delta_{p_2}u_{2n}, u_{2n}-u_2\rangle\big] \le 0.\end{aligned}$$

From this we infer by taking into account the strict monotonicity of the p_i-Laplacian that

$$0 \le \limsup_{n\to\infty}\big[\langle -\Delta_{p_i}u_{in} - (-\Delta_{p_i}u_i), u_{in}-u_i\rangle \le 0,$$

and thus

$$0 = \lim_{n\to\infty}\big[\langle -\Delta_{p_i}u_{in} - (-\Delta_{p_i}u_i), u_{in}-u_i\rangle = \lim_{n\to\infty}\langle -\Delta_{p_i}u_{in}, u_{in}-u_i\rangle. \tag{2.19}$$

Since the negative p_i-Laplacian $-\Delta_{p_i} : W_0^{1,p_i}(\Omega) \to (W_0^{1,p_i}(\Omega))^*$ has the (S_+)-property, it follows that its time-extension $-\Delta_{p_i} : X_{0i} \to X_{0i}^*$ has the (S_+)-property with respect to $D(L_i)$ for $i = 1, 2$ (see [4, Theorem 2.153]), which along with the weak convergence $u_{in} \rightharpoonup u_i$ in X_{0i} results in

$$u_{in} \to u_i \quad \text{(strongly) in } X_{0i},\ i = 1, 2. \tag{2.20}$$

Now it is straightforward to obtain from (2.20) that (2.12) holds true. The claim that the operator $\mathcal{A}$ is pseudomonotone with respect to $D(L_1) \times D(L_2)$ is verified. □

Lemma 2.3. *Assume that $\underline{u} = (\underline{u}_1, \underline{u}_2)$, $\overline{u} = (\overline{u}_1, \overline{u}_2) \in W_1 \times W_2$ is a pair of sub-supersolution for problem (1.1) such that hypothesis (H1) is satisfied. Then problem (2.9) has a solution provided $\lambda_1 > 0$ and $\lambda_2 > 0$ are sufficiently large.*

Proof. The operators A, B, $N_f \circ T : X_0 \to X_0^*$ are bounded and continuous, and due to Lemma 2.2, the operator $\mathcal{A} = A + \lambda B - N_f \circ T : X_0 \to X_0^*$ is pseudomonotone with respect to the domain $D(L) = D(L_1) \times D(L_2)$. Thus we may apply Theorem 2.1 to ensure the existence of solutions of the truncated auxiliary problem (2.9) provided that $\mathcal{A}$ is also coercive, which reads as (note: $X_0 = X_{01} \times X_{02}$, $u = (u_1, u_2)$)

$$\frac{\langle \mathcal{A}(u), u\rangle}{\|u\|_{X_0}} \to 0 \quad \text{as} \quad \|u\|_{X_0} \to \infty, \tag{2.21}$$

where $\|u\|_{X_0} = \|u_1\|_{X_{01}} + \|u_2\|_{X_{02}} = \|\nabla u_1\|_{L^{p_1}(Q)} + \|\nabla u_2\|_{L^{p_2}(Q)}$. Towards this we have

$$\langle Au, u\rangle = \|\nabla u_1\|^{p_1}_{L^{p_1}(Q)} + \|\nabla u_2\|^{p_2}_{L^{p_2}(Q)}. \tag{2.22}$$

Using (2.1), (2.2) and Young's inequality, we find for any positive ε and with some positive constants d_i $(i = 1, 2)$ that

$$\begin{aligned}|\langle N_{f_i} \circ T(u), u_i\rangle| &\leq \int_Q |f_i(x, t, T_1u_1, T_2u_2, \nabla T_1u_1, \nabla T_2u_2)u_i| \\ &\leq \int_Q (k_i + c_i(|\nabla T_1u_1|^{p_1-1} + |\nabla T_2u_2|^{\frac{p_2}{q_1}})|u_i| \\ &\leq (\|k_i\|_{L^{q_i}(Q)} + d_i)\|u_1\|_{L^{p_1}(Q)} \\ &\quad +\varepsilon(\|\nabla u_1\|^{p_1}_{L^{p_1}(Q)} + \|\nabla u_2\|^{p_2}_{L^{p_2}(Q)}) \\ &\quad +C(\varepsilon)(\|u_1\|^{p_1}_{L^{p_1}(Q)} + \|u_2\|^{p_2}_{L^{p_2}(Q)}),\end{aligned} \tag{2.23}$$

where $C(\varepsilon)$ is a positive constant depending only on ε. By means of (2.6), (2.22), and (2.23), we then arrive at

$$\begin{aligned}\langle \mathcal{A}u, u\rangle \;\geq\; & (1 - \varepsilon)(\|\nabla u_1\|^{p_1}_{L^{p_1}(Q)} + \|\nabla u_2\|^{p_2}_{L^{p_2}(Q)}) \\ & +(\lambda_1 a_1^{(1)} - C(\varepsilon))\,\|u_1\|^{p_1}_{L^{p_1}(Q)} \\ & +(\lambda_2 a_1^{(2)} - C(\varepsilon))\,\|u_2\|^{p_2}_{L^{p_2}(Q)} \\ & -(\|k_1\|_{L^{q_1}(Q)} + d_1)\|u_1\|_{L^{p_1}(Q)} - (\|k_2\|_{L^{q_2}(Q)} + d_2)\|u_2\|_{L^{p_2}(Q)} \\ & -(\lambda_1 a_2^{(1)} + \lambda_2 a_2^{(2)}).\end{aligned} \tag{2.24}$$

Choosing $\varepsilon < 1$ and λ_i large enough such that $\lambda_i a_1^{(i)} - C(\varepsilon) > 0$, with $i = 1, 2,$, from (2.24) we infer that property (2.21) is valid, whence $\mathcal{A}$ is coercive. Applying Theorem 2.1, we obtain the existence of solutions of problem (2.9), which completes the proof. □

Now we are in the position to prove our main existence and enclosure result.

Theorem 2.4. *Assume that $\underline{u} = (\underline{u}_1, \underline{u}_2)$, $\overline{u} = (\overline{u}_1, \overline{u}_2) \in W_1 \times W_2$ is a pair of sub-supersolution for problem (1.1) such that hypothesis (H1) is satisfied. Then problem (1.1) has at least one solution $u = (u_1, u_2)$ satisfying the location property $\underline{u} \leq u \leq \overline{u}$.*

Proof. By Lemma 2.3 we know that the auxiliary truncated system (2.9) possesses a solution $u = (u_1, u_2) \in D(L_1) \times D(L_2)$ provided $\lambda_1 > 0$ and $\lambda_2 > 0$ are sufficiently large. We develop a comparison procedure aiming to prove the enclosure $\underline{u} \leq u \leq \overline{u}$ for any solution u of the auxiliary problem (2.9), which completes the proof of Theorem 2.4, since then $B(u) = 0$, and $Tu = u$, and thus u is a solution of (1.1) within the trapping region $[\underline{u}, \overline{u}]$. Let us verify the inequality $u \leq \overline{u}$ only, since the inequality $\underline{u} \leq u$ can be shown in a similar way.

Definition 1.3 with the test function $(v_1, v_2) = ((u_1 - \overline{u}_1)^+, (u_2 - \overline{u}_2)^+) \in X_{01} \times X_{02}$ gives

$$\begin{aligned}&\langle \overline{u}_1', (u_1 - \overline{u}_1)^+\rangle \\ &+ \int_Q \left(|\nabla \overline{u}_1|^{p_1-2}\nabla \overline{u}_1 \nabla (u_1 - \overline{u}_1)^+ - f_1(\cdot, \cdot, \overline{u}_2, w_2, \nabla \overline{u}_1, \nabla w_2)(u_1 - \overline{u}_1)^+\right) \\ &+\langle \overline{u}_2', (u_2 - \overline{u}_2)^+\rangle\end{aligned}$$

$$
\begin{aligned}
&+\int_Q \left(|\nabla \overline{u}_2|^{p_2-2}\nabla \overline{u}_2 \nabla (u_2-\overline{u}_2)^+ - f_2(\cdot,\cdot,w_1,\overline{u}_2,\nabla w_1,\nabla \overline{u}_2)(u_2-\overline{u}_2)^+\right)\\
&\geq 0
\end{aligned}
\tag{2.25}
$$

for all $(w_1, w_2) \in W_1 \times W_2$ with $\underline{u}_i \leq w_i \leq \overline{u}_i$ for $i = 1, 2$. Then (2.25) and system (2.9) with the test functions $(v_1, v_2) = ((u_1 - \overline{u}_1)^+, (u_2 - \overline{u}_2)^+)$ enable us to find

$$
\begin{aligned}
&\langle (u_1-\overline{u}_1)', (u_1-\overline{u}_1)^+\rangle + \langle (u_2-\overline{u}_2', (u_2-\overline{u}_2)^+\rangle\\
&+\int_Q \Big(|\nabla u_1|^{p_1-2}\nabla u_1 - |\nabla \overline{u}_1|^{p_1-2}\nabla \overline{u}_1\Big)\nabla (u_1-\overline{u}_1)^+\\
&+\int_Q \Big(|\nabla u_2|^{p_2-2}\nabla u_2 - |\nabla \overline{u}_2|^{p_2-2}\nabla \overline{u}_2\Big)\nabla (u_2-\overline{u}_2)^+\\
&+\lambda_1 \int_Q b_1(\cdot,\cdot,u_1)(u_1-\overline{u}_1)^+ + \lambda_2 \int_Q b_2(\cdot,\cdot,u_2)(u_2-\overline{u}_2)^+\\
&-\int_Q \Big(N_{f_1}\circ T(u) - f_1(\cdot,\cdot,\overline{u}_1,w_2,\nabla \overline{u}_1,\nabla w_2)\Big)(u_1-\overline{u}_1)^+\\
&-\int_Q \Big(N_{f_2}\circ T(u) - f_2(\cdot,\cdot,w_1,\overline{u}_2,\nabla w_1,\nabla \overline{u}_2)\Big)(u_2-\overline{u}_2)^+\\
&\leq 0
\end{aligned}
\tag{2.26}
$$

for all $(w_1, w_2) \in W_1 \times W_2$ with $\underline{u}_i \leq w_i \leq \overline{u}_i$ for $i = 1, 2$. Therefore it is permitted to insert $w_1 = T_1 u_1$ and $w_2 = T_2 u_2$. Then (2.26) implies

$$
\begin{aligned}
&\langle (u_1-\overline{u}_1)', (u_1-\overline{u}_1)^+\rangle + \langle (u_2-\overline{u}_2', (u_2-\overline{u}_2)^+\rangle\\
&+\lambda_1 \int_Q b_1(\cdot,\cdot,u_1)(u_1-\overline{u}_1)^+ + \lambda_2 \int_\Omega b_2(\cdot,\cdot,u_2)(u_2-\overline{u}_2)^+ \leq 0.
\end{aligned}
\tag{2.27}
$$

We note that

$$
\langle (u_i-\overline{u}_i)', (u_i-\overline{u}_i)^+\rangle = \frac{1}{2}\|(u_i-\overline{u}_i)^+(\cdot,\tau)\|^2_{L^2(\Omega)} \quad (i = 1, 2).
$$

In view of this and by the definition of the cut-off functions b_i in (2.4), inequality (2.27) gives rise to

$$
\lambda_1 \int_Q [(u_1-\overline{u}_1)^+]^{p_1} + \lambda_2 \int_Q [(u_2-\overline{u}_2)^+]^{p_2} \leq 0.
$$

Thus we are led to $u_i \leq \overline{u}_i$ for $i = 1, 2$.

Consequently, we know that the solution $u = (u_1, u_2)$ of auxiliary truncated problem (2.9) verifies $\underline{u} \leq u \leq \overline{u}$. This makes that $T_i u_i = u_i$ and $B_i(u_i) = 0$ for $i = 1, 2$ (see (2.3) and (2.4)). Clearly, (2.9) becomes (1.1), which completes the proof. □

3. Extremal solutions in trapping regions

Given a trapping region $[\underline{u}, \overline{u}]$ for problem (1.1) formed by a pair of sub-supersolution, in this section we are going to show the existence of extremal solutions of (1.1) in $[\underline{u}, \overline{u}]$. Denote by $\mathcal{S}$ the set of all solutions of (1.1) in $[\underline{u}, \overline{u}]$, by extremal solutions we understand maximal and minimal solutions in $[\underline{u}, \overline{u}]$ or equivalently, maximal and minimal elements of $\mathcal{S}$ with respect to the underlying partial ordering in $W_0 = W_{01} \times W_{02}$, which is defined by the positive order cone L_+ given by

$$L_+ = L_+^{p_1}(Q) \times L_+^{p_2}(Q),$$

where $L_+^{p_i}(Q)$ is the set of all nonnegative $L^{p_i}(Q)$-functions. An element $u^* \in \mathcal{S}$ is called a maximal element of $\mathcal{S}$ if $w \in \mathcal{S}$ and $u^* \le w$ implies $u^* = w$. Similarly, a minimal element u_* of $\mathcal{S}$ is defined.

Theorem 3.1. *Assume that* $\underline{u} = (\underline{u}_1, \underline{u}_2)$, $\overline{u} = (\overline{u}_1, \overline{u}_2) \in W_1 \times W_2$ *is a pair of sub-supersolution for problem (1.1) such that hypothesis (H1) is satisfied. Then problem (1.1) has extremal solutions within* $[\underline{u}, \overline{u}]$*, i.e.,* $\mathcal{S}$ *has maximal and minimal elements.*

Proof. By Theorem 2.4, $\mathcal{S} \neq \emptyset$. Let us focus on the assertion regarding the minimal element of $\mathcal{S}$, because the existence of a maximal element can be proved by analogous reasoning.

Our basic tool is Zorn's lemma. To this end we consider a chain $\mathcal{C}$ in $\mathcal{S}$. As $\mathcal{C}$ is order bounded, i.e., $\mathcal{C}$ is order bounded above by $\overline{u}$ and below by $\underline{u}$, and since $L_+^{p_i}(Q)$ is a regular order cone (see [9, p.28]), we can apply [9, Proposition 1.3.2] to ensure that there is a sequence $u_n = (u_{1n}, u_{2n}) \subset \mathcal{C}$, with $u_{n+1} \le u_n$ for all $n \ge 1$ such that

$$\inf \mathcal{C} = \lim_{n\to\infty} u_n.$$

Following similar estimates as in (2.23) and (2.24), and taking into account that (u_{1n}, u_{2n}) are solutions of (1.1) belonging to the trapping region $[\underline{u}, \overline{u}]$, we obtain the estimate

$$\begin{aligned} &\frac{1}{2}(\|u_{1n}(\cdot,\tau)\|^2_{L^2(\Omega)} + \|u_{2n}(\cdot,\tau)\|^2_{L^2(\Omega)}) \\ &+(1-\eta)(\|\nabla u_{1n}\|^{p_1}_{L^{p_1}(Q)} + \|\nabla u_{2n}\|^{p_2}_{L^{p_2}(Q)}) \\ &\le (\|k_1\|_{L^{q_1}(Q)} + d_1)\|u_{1n}\|_{L^{p_1}(Q)} + (\|k_2\|_{L^{q_2}(Q)} + d_2)\|u_{2n}\|_{L^{p_2}(Q)} \\ &+C(\eta)(\|u_{1n}\|^{p_1}_{L^{p_1}(Q)} + \|u_{2n}\|^{p_2}_{L^{p_2}(Q)}) \end{aligned} \tag{3.1}$$

for every $\eta > 0$, with constants $C(\eta) > 0$ and $d_i > 0$, $i = 1, 2$. By (3.1) it follows that the sequence $u_n = (u_{1n}, u_{2n})$ is bounded in $X_{01} \times X_{02}$. Directly from the system (1.1) with the solution (u_{1n}, u_{2n}) we then infer that $(u_n') = (u_{1n}', u_{2n}')$ is bounded in $X_{01}^* \times X_{02}^*$. Consequently, there is a subsequence (again denoted by (u_n)) such that $u_{in} \rightharpoonup u_i$ in W_{0i} as $n \to \infty$, with $u_i \in D(L_i)$ for $i = 1, 2$. Here we have used that L_i is a linear closed operator.

Notice again from equation (1.1) (resp. (1.5) with solution $(u_n) = (u_{1n}, u_{2n})$ and test function $u_n - u$ we have

$$\langle Au_n, u_n - u\rangle = -\langle Lu_n, u_n - u\rangle + \langle N_f(u_n), u_n - u\rangle,$$

which in view of

$$-\langle Lu_n, u_n - u\rangle = -\langle Lu_n - Lu, u_n - u\rangle - \langle Lu, u_n - u\rangle \le -\langle Lu, u_n - u\rangle,$$

yields the estimate

$$\langle Au_n, u_n - u\rangle \le -\langle Lu, u_n - u\rangle + \langle N_f(u_n), u_n - u\rangle, \tag{3.2}$$

As the right-hand side of (3.2) tends to zero as $n \to \infty$ we get by passing to the lim sup in (3.2)

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \le 0,$$

or equivalently

$$\limsup_{n\to\infty} [\langle -\Delta_{p_1} u_{1n}, u_{1n} - u_1\rangle + \langle -\Delta_{p_2} u_{2n}, u_{2n} - u_2\rangle] \le 0. \tag{3.3}$$

The weak convergence of $(u_n) = (u_{1n}, u_{2n})$ as seen above along with (3.3) and the (S_+)-property of $-\Delta_{p_i}$ with respect to $D(L_i)$ implies the strong convergence $u_{in} \to u_i$ in X_{0i} for $i = 1, 2$. This allows us to pass to the limit in

$$\langle Lu_n + Au_n, v\rangle = \langle N_f(u_n), v\rangle, \quad \forall\, v \in X_0 = X_{01} \times X_{02},$$

which proves that $\inf \mathcal{C} = \lim_{n\to\infty} u_n = u$ belongs to $\mathcal{S}$. Therefore Zorn's Lemma can be applied providing a minimal element of $\mathcal{S}$. The proof is thus complete. □

Remark 3.2. Assuming hypothesis (H1), the main results of Section 2 and Section 3, Theorem 2.4 and Theorem 3.1, respectively, remain true for the following more general initial-Dirichlet boundary value problem ($i = 1, 2$): Find $u_i \in W_{0i}$ with $u_i(\cdot, 0) = 0$ such that

$$u_i' + A_i u_i = f_i(x, t, u_1, u_2, \nabla u_1, \nabla u_2) + h_i \quad \text{in } Q, \tag{3.4}$$

where $h_i \in X_{0i}^*$, and $A_i : X_{01} \to X_{0i}^*$ are given by

$$A_i u_i(x, t) = -\sum_{k=1}^{N} \frac{\partial}{\partial x_k} a_k^{(i)}(x, t, \nabla u_i(x, t)),$$

with coefficients $a_k^{(i)} : Q \times \mathbb{R}^N \to \mathbb{R}$ satisfying the following hypotheses of Leray-Lions type:

(A1) $a_k^{(i)} : Q \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions, satisfying the growth condition

$$|a_k^{(i)}(x, t, \xi)| \le k_0^{(i)}(x, t) + c_{0i}|\xi|^{p_i - 1}, \quad \text{for a.e. } (x, t) \in Q, \ \forall\, \xi \in \mathbb{R}^N,$$

where $k_0^{(i)} \in L_+^{q_i}(Q)$, and c_{0i} are positive constants.

(A2) For a.e. $(x, t) \in Q$ and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \ne \xi'$ the following monotonicity condition is satisfied

$$\sum_{k=1}^{N} \big(a_k^{(i)}(x, t, \xi) - a_k^{(i)}(x, t, \xi')\big)(\xi_k - \xi_k') > 0.$$

(A2) There are positive constants μ_i and functions $k_{i1} \in L^1(Q)$ such that

$$\sum_{k=1}^{N} a_k^{(i)}(x,t,\xi)\xi_k \geq \mu_i|\xi|^{p_i} + k_{i1}(x,t), \quad \text{for a.e. } (x,t) \in Q, \ \forall\, \xi \in \mathbb{R}^N.$$

The proofs of the main results, Theorem 2.4 and Theorem 3.1, with the more general leading quasilinear elliptic operator $A = (A_1, A_2)$ given above, can be done in a straightforward manner.

4. Constuction of trapping regions

The aim of this section is to provide a construction of specific trapping regions that will allow us to show the existence of at least one positive solution of the following initial-Dirichlet boundary value problem $(i = 1, 2)$ with nonnegative initial values $\psi_i \in L^\infty_+(\Omega)$, $(i = 1, 2)$: Find $u_i \in W_{0i}$ with $u_i(\cdot, 0) = \psi_i$ such that

$$u_i' - \Delta_{p_i} u_i = f_i(x,t,u_1,u_2,\nabla u_1,\nabla u_2) \quad \text{in } Q, \tag{4.1}$$

where we assume hypothesis (H1) to hold true. To make sure that problem (4.1) with inhomogeneous initial values and homogeneous Dirichlet-boundary conditions can be handled by the results obtained in the preceding sections, we are going to transform (4.1) into a problem of the type discussed in Remark 3.2. To this end let $w_i \in W_{0i}$ denote the solutions of the following problem: Find $w_i \in W_{0i}$ such that $w_i(\cdot, 0) = \psi_i$ and

$$w_i' - \Delta_{p_i} w_i = 0 \quad \text{in } Q. \tag{4.2}$$

The existence of a unique solution w_i of (4.2) (even for more general initial values $\psi_i \in L^2(\Omega)$) follows by standard existence results, see e.g. [10, Theorem 30.A]. Now we introduce the new function $\hat{u}_i = u_i - w_i$. Then problem (4.1) is transformed into the following equivalent problem for $\hat{u}_i$ with homogeneous initial and boundary values: Find $\hat{u}_i \in W_{0i}$ with $\hat{u}_i(\cdot, 0) = 0$ such that

$$\hat{u}_i' - \Delta_{p_i}(\hat{u}_i + w_i) = f_i(x,t,\hat{u}_1+w_1,\hat{u}_2+w_2,\nabla(\hat{u}_1+w_1),\nabla(\hat{u}_2+w_2)) - w_i' \quad \text{in } Q. \tag{4.3}$$

Setting

$$\begin{aligned}
&A_i\hat{u}_i = -\Delta_{p_i}(\hat{u}_i + w_i),\\
&\hat{f}_i(x,t,\hat{u}_1,\hat{u}_2,\nabla\hat{u}_1,\nabla\hat{u}_2)\\
&= f_i(x,t,\hat{u}_1 + w_1,\hat{u}_2 + w_2,\nabla(\hat{u}_1 + w_1),\nabla(\hat{u}_2 + w_2)),\\
&h_i = -w_i',
\end{aligned}$$

then (4.3) results in: Find $\hat{u}_i \in W_{0i}$ with $\hat{u}_i(\cdot, 0) = 0$

$$\hat{u}_i' + A_i\hat{u}_i = \hat{f}_i(x,t,\hat{u}_1,\hat{u}_2,\nabla\hat{u}_1,\nabla\hat{u}_2) + h_i \quad \text{in } Q. \tag{4.4}$$

which can easily be verified to be of the structure of the homogeneous initial-Dirichlet boundary value problem (3.4) discussed in Remark 3.2. It should be noted that hypotheses (H1) for the right-hand sides f_i hold true for the right-hand sides $\hat{f}_i$ of (4.4) as well. Moreover, a pair $(\underline{u}, \overline{u})$ of sub-supersolution for the problem (4.1) with the inhomogeneous initial values ψ_i yields a corresponding transformed pair of sub-supersolutions $(\underline{\hat{u}}, \overline{\hat{u}})$ of (4.4), where $\underline{\hat{u}} = \underline{u} - w$ and $\overline{\hat{u}} = \overline{u} - w$. Therefore, according to Remark 3.2 we may apply our main results to the problem (4.1). In

a similar way one can deal with inhomogeneous boundary values (in the sense of traces) as well.
Our goal is to formulate conditions on the right-hand side vector field that will allow us to construct trapping regions which imply the existence of nonnegative solutions of (4.1). Assuming homogeneous boundary conditions, we consider two cases: Homogeneous initial data and inhomogeneous initial data.

Case I: Homogeneous Initial Values

To this end, we recall the eigenvalue problem for $-\Delta_p$ on $W_0^{1,p}(\Omega)$ with $1 < p < \infty$ which reads as follows

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

It has a first eigenvalue $\lambda_{1,p} > 0$, which is characterized by

$$\lambda_{1,p} = \inf \left[\frac{\|\nabla u\|_{L^p(\Omega)}^p}{\|u\|_{L^p(\Omega)}^p} : \ u \in W_0^{1,p}(\Omega), \ u \neq 0 \right].$$

Denote by $\widehat{u}_{1,p}$ the corresponding L^p-normalized positive eigenfunction, i.e. $\widehat{u}_{1,p}$ solves (4.5) with $\lambda = \lambda_{1,p}$, $\widehat{u}_{1,p} > 0$ in Ω, and $\|\widehat{u}_{1,p}\|_{L^p(\Omega)} = 1$. In addition to hypothesis (H1) let us suppose the following hypotheses:

(H2-0) There exist continuously differentiable functions $\theta_i : [0, \tau] \to \mathbb{R}$, $i = 1, 2$, such that $\delta_i := \min_{t\in[0,\tau]} \theta_i(t) > 0$, and for a.a. $(x, t) \in Q$

$$f_1(x, t, \theta_1(t), s_2, 0, \xi_2) \le \theta_1'(t), \quad \forall\ s_2 \in [0, \theta_2(t)],\ \xi_2 \in \mathbb{R}^N,$$

$$f_2(x, t, s_1, \theta_2(t), \xi_1, 0) \le \theta_2'(t), \quad \forall\ s_1 \in [0, \theta_1(t)], \quad \xi_1 \in \mathbb{R}^N,$$

(H3-0) There exist positive constants δ, η_1, η_2 such that for a.a. $(x, t) \in Q$

$$f_1(x, t, s_1, s_2, \xi_1, \xi_2) \ge \eta_1, \quad \forall\ s_1 \in [0, \delta],\ |\xi_1| \le \delta,\ s_2 \in [0, \theta_2(t)],\ \xi_2 \in \mathbb{R}^N,$$

$$f_2(x, t, s_1, s_2, \xi_1, \xi_2) \ge \eta_2, \quad \forall\ s_1 \in [0, \theta_1(t)],\ \xi_1 \in \mathbb{R}^N,\ s_2 \in [0, \delta],\ |\xi_2| \le \delta.$$

Theorem 4.1. *Assume hypothesis (H1), (H2-0), and (H3-0), where the growth conditions (2.1), (2.1) of (H1) are supposed to hold for all* (s_1, s_2) *within* $[0, \theta_1(t)] \times [0, \theta_2(t)]$. *Then for* $\varepsilon > 0$ *sufficiently small, system (1.1) (resp. (4.1)), admits trapping regions of the form:*

$$[\varepsilon\, t\widehat{u}_{1,p_1}, \theta_1(t)] \times [\varepsilon\, t\widehat{u}_{1,p_2}, \theta_2(t)],$$

which give rise to the existence of a positive solution $(u_1, u_2) \in [\varepsilon\, t\widehat{u}_{1,p_1}, \theta_1(t)] \times [\varepsilon\, t\widehat{u}_{1,p_2}, \theta_2(t)]$.

Proof. From (H2-0) we immediately get for a.a. $(x, t) \in Q$

$$\begin{cases} \theta_1'(t) - \Delta_{p_1}(\theta_1(t)) - f_1(x, t, \theta_1(t), s_2, 0, \xi_2) \ge 0, \\ \ \forall\ s_2 \in [0, \theta_2(t)],\ \xi_2 \in \mathbb{R}^N, \\ \theta_2'(t) - \Delta_{p_2}(\theta_2(t)) - f_2(x, t, s_1, \theta_2(t), \xi_1, 0) \ge 0, \\ \ \forall\ s_1 \in [0, \theta_1(t)],\ \xi_1 \in \mathbb{R}^N. \end{cases} \tag{4.6}$$

Choose $\delta \in (0, \min\{\delta_1, \delta_2\})$, and take $\varepsilon > 0$ sufficiently small such that

$$\varepsilon\, t\widehat{u}_{1,p_i}(x) \le \delta, \ \text{and}\ \varepsilon\, t|\nabla \widehat{u}_{1,p_i}(x)| \le \delta,$$

for all $x \in \Omega$, $t \in [0,\tau]$, $i = 1,2$. For a possibly smaller δ we can make use of hypothesis (H3-0) obtaining the following inequalities for a.a. $(x,t) \in Q$

$$(4.7) \qquad \begin{cases} f_1(x,t,\varepsilon\, t\widehat{u}_{1,p_1}(x), s_2, \varepsilon\, t\nabla\widehat{u}_{1,p_1}(x), \xi_2) \geq \eta_1, \\ \ \forall\ s_2 \in [0,\theta_2(t)],\ \ \xi_2 \in \mathbb{R}^N, \\ f_2(x,t,s_1,\varepsilon\, t\widehat{u}_{1,p_2}(x), \xi_1, \varepsilon\, t\nabla\widehat{u}_{1,p_2}(x)) \geq \eta_2, \\ \ \forall\ s_1 \in [0,\theta_1(t)],\ \ \xi_1 \in \mathbb{R}^N. \end{cases}$$

Choosing $\varepsilon > 0$ even smaller (if necessary), enables us to suppose that

$$\varepsilon\,\widehat{u}_{1,p_i}(x) + \lambda_{1,p_i}\Big(\varepsilon\, t\widehat{u}_{1,p_i}(x)\Big)^{p_i-1} \leq \eta_i,$$

which in view of (4.7) yields

$$(4.8) \quad \begin{cases} (\varepsilon\, t\widehat{u}_{1,p_1})' - \Delta_{p_1}(\varepsilon\, t\widehat{u}_{1,p_1}) - f_1(x,t,\varepsilon\, t\widehat{u}_{1,p_1}(x), s_2, \varepsilon\, t\nabla\widehat{u}_{1,p_1}(x), \xi_2) \leq 0, \\ \ \forall\ s_2 \in [0,\theta_2(t)],\ \ \xi_2 \in \mathbb{R}^N, \\ (\varepsilon\, t\widehat{u}_{1,p_2})' - \Delta_{p_2}(\varepsilon\, t\widehat{u}_{1,p_2}) - f_2(x,t,s_1,\varepsilon\, t\widehat{u}_{1,p_2}(x), \xi_1, \varepsilon\, t\nabla\widehat{u}_{1,p_2}(x)) \leq 0, \\ \ \forall\ s_1 \in [0,\theta_1(t)],\ \ \xi_1 \in \mathbb{R}^N. \end{cases}$$

From (4.6) and (4.8) it follows that

$$\underline{u} = (\underline{u_1}, \underline{u_2}) := (\varepsilon\, t\widehat{u}_{1,p_1}, \varepsilon\, t\widehat{u}_{1,p_2}), \text{ and } \overline{u} = (\overline{u_1}, \overline{u_2}) = (\theta_1, \theta_2)$$

constitute a pair of sub-supersolution for (1.1) (resp. (4.1), which completes the proof. □

An explicit example for the right-hand side vector field (f_1, f_2) that satisfies hypotheses (H1), (H2-0), (H3-0) is as follows.

Example 4.2. Let (f_1, f_2) be given by

$$f_1(x,t,s_1,s_2,\xi_1,\xi_2) = \gamma_1 \sin(|s_1|^{p_1-2}s_1) + t\ \arctan(s_1)\,|\xi_1|^{\frac{1}{r}}|\xi_2|^{\frac{1}{r}} + \alpha_1,$$
$$f_2(x,t,s_1,s_2,\xi_1,\xi_2) = \gamma_2 \sin(|s_2|^{p_2-2}s_2) + t\ \arctan(s_2)\,|\xi_1|^{\frac{1}{r}}|\xi_2|^{\frac{1}{r}} + \alpha_2,$$

for all $(x,t,s_1,s_2,\xi_1,\xi_2) \in Q \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ with positive constants γ_i, α_i, r satisfying $\alpha_i > \gamma_i$ $(i = 1,2)$, and

$$\frac{2}{r} \leq \min\{p_1 - 1, p_2 - 1, \frac{p_2(p_1-1)}{p_1}, \frac{p_1(p_2-1)}{p_2}\}.$$

For δ_i, $\beta_i > 0$ with $\beta_i \geq \gamma_i + \alpha_i$, $i = 1,2$, we define

$$\theta_i(t) = \beta_i\, t + \delta_i, \quad t \in [0,\tau].$$

Let us check that hypotheses (H1), (H2-0), (H3-0) are fulfilled. Regarding (H1) we have

$$|f_1(x,t,s_1,s_2,\xi_1,\xi_2)| \leq \gamma_1 + \frac{\tau\pi}{4}\Big(|\xi_1|^{\frac{2}{r}} + |\xi_2|^{\frac{2}{r}}\Big) + \alpha_1 \leq a_1\Big(1 + |\xi_1|^{\frac{p_1}{q_1}} + |\xi_2|^{\frac{p_2}{q_1}}\Big)$$

with some positive constant a_1, and for a.a. $(x,t) \in Q$, for all $(s_1,s_2) \in [0,\theta_1(t)] \times [0,\theta_2(t)]$, and for all $(\xi_1,\xi_2) \in \mathbb{R}^N \times \mathbb{R}^N$. Similarly, for f_2.

Regarding (H2-0) we have

$$f_1(x,t,\theta_1(t),s_2,0,\xi_2) \leq \gamma_1 + \alpha_1 \leq \beta_1 = \theta_1'(t),$$

for a.a. $(x,t) \in Q$, for all $s_2 \in \mathbb{R}$, and for all $\xi_2 \in \mathbb{R}^N$, and similarly for f_2.

Finally, checking (H3-0), we have

$$f_1(x,t,s_1,s_2,\xi_1,\xi_2) \geq \alpha_1,$$

for a.a. $(x,t) \in Q$, for all $s_1 \in [0,\delta]$, for all $s_2 \in \mathbb{R}$, and for all $(\xi_1,\xi_2) \in \mathbb{R}^N \times \mathbb{R}^N$. A similar estimate holds true for f_2.

Case II: Inhomogeneous Initial Values

Here we consider nonnegative, nonzero initial values ψ. Using the same notation regarding the first eigenvalue and eigenfunction of the p_i-Laplacian, we now assume the following conditions on the initial values ψ_i and on the right-hand side vector field (f_1, f_2) of problem (4.1):

(H$_{\psi_i}$) $\psi_i \in L^\infty_+(\Omega)$ and there are $\varrho_i > 0$ such that $\varrho_i \widehat{u}_{1,p_i}(x) \leq \psi_i(x)$ for a.e. $x \in \Omega$.

(H2) There exist continuously differentiable functions $\theta_i : [0,\tau] \to \mathbb{R}$, $i = 1,2$, with $\theta_i(t) > 0$, such that $\delta_i := \min_{t\in[0,\tau]} \theta_i(t) \geq \|\psi_i\|_\infty$, and for a.a. $(x,t) \in Q$

$$f_1(x,t,\theta_1(t),s_2,0,\xi_2) \leq \theta_1'(t) \text{ for all } s_2 \in [0,\theta_2(t)], \text{ all } \xi_2 \in \mathbb{R}^N,$$

$$f_2(x,t,s_1,\theta_2(t),\xi_1,0) \leq \theta_2'(t) \text{ for all } s_1 \in [0,\theta_1(t)], \text{ all } \xi_1 \in \mathbb{R}^N,$$

(H3) There are constants $\beta_1 > \lambda_{1,p_1}$ and $\beta_2 > \lambda_{1,p_2}$ such that

$$\liminf_{s_1\to 0^+,\xi_1\to 0} \frac{f_1(x,t,s_1,s_2,\xi_1,\xi_2)}{s_1^{p_1-1}} \geq \beta_1$$

uniformly for a.a. $(x,t) \in Q$, all $0 \leq s_2 \leq \theta_2(t)$, all $\xi_2 \in \mathbb{R}^N$;

$$\liminf_{s_2\to 0^+,\xi_2\to 0} \frac{f_2(x,t,s_1,s_2,\xi_1,\xi_2)}{s_2^{p_2-1}} \geq \beta_2$$

uniformly for a.a. $(x,t) \in Q$, all $0 \leq s_1 \leq \theta_1(t)$, all $\xi_1 \in \mathbb{R}^N$;

Theorem 4.3. *Assume hypotheses (H$_{\psi_i}$), (H1)–(H3), where the growth conditions (2.1), (2.2) of (H1) are supposed to hold true for all $s = (s_1,s_2)$ within the time-dependent rectangle $[0,\theta_1(t)] \times [0,\theta_2(t)]$ for $t \in [0,\tau]$. Then, for $\varepsilon > 0$ sufficiently small, system (4.1) admits a trapping region of the form $[\varepsilon\widehat{u}_{1,p_1},\theta_1(t)]\times[\varepsilon\widehat{u}_{1,p_2},\theta_2(t)]$, which gives rise to the existence of a positive solution $(u_1,u_2) \in [\varepsilon\widehat{u}_{1,p_1},\theta_1(t)] \times [\varepsilon\widehat{u}_{1,p_2},\theta_2(t)]$.*

Proof. Assumption (H2) ensures that for a.a. $(x,t) \in Q$ one has

(4.9)
$$\begin{cases} \theta_1'(t) - \Delta_{p_1}(\theta_1(t)) - f_1(x,t,\theta_1(t),s_2,0,\xi_2) \geq 0, & s_2 \in [0,\theta_2(t)], \quad \xi_2 \in \mathbb{R}^N, \\ \theta_2'(t) - \Delta_{p_2}(\theta_2(t)) - f_2(x,t,s_1,\theta_2(t),\xi_1,0) \geq 0, & s_1 \in [0,\theta_1(t)], \quad \xi_1 \in \mathbb{R}^N. \end{cases}$$

Assumption (H3) entails that there exists $\delta > 0$ such that for a.a. $(x,t) \in Q$ it holds

(4.10)
$$\begin{cases} f_1(x,t,s_1,s_2,\xi_1,\xi_2) > \lambda_{1,p_1} s_1^{p_1-1}, & s_1,|\xi_1| \in (0,\delta),\ s_2 \in [0,\theta_2(t)],\ \xi_2 \in \mathbb{R}^N, \\ f_2(x,t,s_1,s_2,\xi_1,\xi_2) > \lambda_{1,p_2} s_2^{p_2-1}, & s_1 \in [0,\theta_1(t)],\ s_2,|\xi_2| \in (0,\delta),\ \xi_1 \in \mathbb{R}^N. \end{cases}$$

Since the continuous functions $\theta_1(t)$, $\theta_2(t)$ on $[0,\tau]$ are positive, we can choose

$$\delta \in (0, \min\{\delta_1,\delta_2\}).$$

Now we take $\varepsilon > 0$ sufficiently small such that $\varepsilon < \varrho_i$ and

$$0 < \varepsilon\widehat{u}_{1,p_i}(x) \le \delta,\ \varepsilon|\nabla\widehat{u}_{1,p_i}(x)| \le \delta \text{ for all } x \in \Omega,\ i = 1, 2.$$

Due to (4.10), for a.a. $(x,t) \in Q$, for all $s_2 \in [0, \theta_2(t)]$, $\xi_2 \in \mathbb{R}^N$, $s_1 \in [0, \theta_1(t)]$, and $\xi_1 \in \mathbb{R}^N$ we get the inequalities

$$\begin{cases} (\varepsilon\widehat{u}_{1,p_1})' - \Delta_{p_1}(\varepsilon\widehat{u}_{1,p_1}) - f_1(x,t,\varepsilon\widehat{u}_{1,p_1}, s_2, \varepsilon\nabla\widehat{u}_{1,p_1}, \xi_2) \le 0, \\ (\varepsilon\widehat{u}_{1,p_2})' - \Delta_{p_2}(\varepsilon\widehat{u}_{1,p_2}) - f_2(x,t,s_1,\varepsilon\widehat{u}_{1,p_2}, \xi_1, \varepsilon\nabla\widehat{u}_{1,p_2}) \le 0. \end{cases} \tag{4.11}$$

Notice that (4.9) and (4.11) are fulfilled with $w_1 \in [\varepsilon\widehat{u}_{1,p_1}, \theta_1]$ and ∇w_1 in place of s_1 and ξ_1, respectively, as well as with $w_2 \in [\varepsilon\widehat{u}_{1,p_2}, \theta_2]$ and ∇w_2 in place of s_2 and ξ_2, respectively. It follows that $\underline{u} = (\underline{u}_1, \underline{u}_2) := (\varepsilon\widehat{u}_{1,p_1}, \varepsilon\widehat{u}_{1,p_2})$ and $\overline{u} = (\overline{u}_1, \overline{u}_2) := (\theta_1, \theta_2)$ constitute a pair of sub-supersolution for problem (4.1) in the sense of Definition 1.3. Therefore, Theorem 2.4 along with Remark 3.2 can be applied, which guarantees the existence of a (weak) positive solution $u = (u_1, u_2)$ of problem (4.1) within the trapping region $[\underline{u}, \overline{u}]$, so there hold

$$\varepsilon\widehat{u}_{1,p_1} \le u_1 \le \theta_1,\quad \varepsilon\widehat{u}_{1,p_2} \le u_2 \le \theta_2.$$

This completes the proof. □

As an immediate consequence of Theorem 3.1 and applying a regularity result given by [8, Theorem 1, Theorem 2] we get the following corollary.

Corollary 4.4. *Assume the hypotheses of Theorem 4.3. Then problem (4.1) has positive extremal solutions in the trapping region* $[\varepsilon\widehat{u}_{1,p_1}, \theta_1] \times [\varepsilon\widehat{u}_{1,p_2}, \theta_2]$. *Moreover,* $u_i \in C^{1,\gamma}(\overline{\Omega} \times (0,\tau])$, *where* $\gamma \in (0,1)$.

Remark 4.5. Hypothesis (H_{ψ_i}) on the initial data ψ_i can be satisfied by a large class of functions from $L^2_+(\Omega)$ such as follows: If $\psi_i(x) \ge \sigma_i > 0$ then the condition $\varrho_i\widehat{u}_{1,p_i}(x) \le \psi_i(x)$ for a.e. $x \in \Omega$ is obviously satisfied by choosing $\varrho_i > 0$ small enough, since $0 \le \widehat{u}_{1,p_i} \in C^1_0(\overline{\Omega})_+$, where

$$C^1_0(\overline{\Omega})_+ = \left\{u \in C^1_0(\overline{\Omega}) :\ u(x) \ge 0 \text{ for all } x \in \Omega\right\}.$$

Moreover, $\widehat{u}_{1,p_i}$ even belongs to the interior $\mathrm{int}(C^1_0(\overline{\Omega})_+)$ of $C^1_0(\overline{\Omega})_+$ which is known to be nonempty and characterized as follows

$$\mathrm{int}(C^1_0(\overline{\Omega})_+) = \{u \in C^1_0(\overline{\Omega})_+ :\ u(x) > 0 \text{ for all } x \in \Omega \text{ and } \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial\Omega\}.$$

Therefore, any nonnegative initial value $\psi_i \in \mathrm{int}(C^1_0(\overline{\Omega})_+)$ satisfies condition (H_{ψ_i}).

A simple explicit example for the right-hand side vector field (f_1, f_2) that satisfies hypotheses (H1), (H2), (H3) is as follows.

Example 4.6. Let Ω, τ, p_1, p_2 be as in the statement of problem (1.1). Fix constants $\beta_1 > \lambda_{1,p_1}$, $\beta_2 > \lambda_{1,p_2}$, $\delta_1 > 0$, $\delta_2 > 0$, and $r > 0$ with $\delta_i \ge \|\psi_i\|_\infty$, and

$$\frac{2}{r} \le \min\left\{p_1 - 1, p_2 - 1, \frac{p_2(p_1-1)}{p_1}, \frac{p_1(p_2-1)}{p_2}\right\}.$$

We define $(f_1, f_2) : Q \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^2$ by

$$f_1(x,t,s_1,s_2,\xi_1,\xi_2) = \beta_1 \sin(|s_1|^{p_1-2}s_1) + t\arctan(s_1)|\xi_1|^{\frac{1}{r}}|\xi_2|^{\frac{1}{r}},$$

$$f_2(x,t,s_1,s_2,\xi_1,\xi_2) = \beta_2 \sin(|s_2|^{p_2-2}s_2) + t\arctan(s_2)|\xi_1|^{\frac{1}{r}}|\xi_2|^{\frac{1}{r}}$$

for all $(x,t,s_1,s_2,\xi_1,\xi_2) \in Q \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$.

We show that hypotheses (H1), H(2), (H3) of Theorem 4.3 are verified. Indeed, there holds

$$|f_i(x,t,s_1,s_2,\xi_1,\xi_2)| \le \beta_i + \frac{\tau\pi}{4}\left(|\xi_1|^{\frac{2}{r}} + |\xi_2|^{\frac{2}{r}}\right) \le a_i\Big(1 + |\xi_i|^{p_i-1} + |\xi_j|^{\frac{p_j}{q_i}}\Big)$$

with a constant $a_i > 0$, for $i = 1,2$, $j = 1,2$, $i \neq j$, so condition (H1) is satisfied.

Set $\theta_i(t) = \beta_i t + \delta_i$ and for all $t \in [0,\tau]$, $i = 1,2$. It is straightforward to check that condition (H2) is fulfilled. For instance, one has

$$f_1(x,t,\theta_1(t),s_2,0,\xi_2)| = \beta_1 \sin((\beta_1 t + \delta_1)^{p_1-1}) \le \beta_1 = \theta_1'(t).$$

A direct verification shows that condition (H3) is satisfied too. For instance, it is seen that

$$\liminf_{s_1\to 0^+,\xi_1\to 0} \frac{f_1(x,t,s_1,s_2,\xi_1,\xi_2)}{s_1^{p_1-1}} \ge \lim_{s_1\to 0^+} \frac{\beta_1 \sin(s_1^{p_1-1})}{s_1^{p_1-1}} = \beta_1$$

uniformly for a.a. $(x,t) \in Q$, all $s_2 \in \mathbb{R}$, all $\xi_2 \in \mathbb{R}^N$.

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