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STEKLOV-TYPE EIGENVALUES OF $\Delta_p + \Delta_q$

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Abstract. The Steklov-like eigenvalue problem associated with the equation $\Delta_p u + \Delta_q u = 0$ in Ω is investigated, where $p \in [2, \infty)$, $q \in (1, \infty)$, $p \neq q$, and Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary. A complete description of the set of eigenvalues is provided in this nonhomogeneous case ($p \neq q$). Note that this case is complementary to the homogeneous case $p = q$ for which a full description of the set of eigenvalues is known only if $p = q = 2$.

1. Introduction

Throughout this paper Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz boundary $\partial\Omega$. Consider the eigenvalue problem

$$\begin{cases} Au = 0, & \text{in } \Omega\,, \\ \frac{\partial u}{\partial \nu_A} = \lambda |u|^{p-2}u, & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where $Au := \Delta_p u + \Delta_q u$, $p \in [2, \infty)$, $q \in (1, \infty)$, $p \neq q$, and

$$\frac{\partial u}{\partial \nu_A} := \left(|\nabla u|^{p-2} + |\nabla u|^{q-2}\right) \frac{\partial u}{\partial \nu}\,,$$

with ν being the unit outward normal to $\partial\Omega$. The solutions u will be sought in the space $V := W^{1,\max\{p,q\}}(\Omega)$, so that the normal derivative $\frac{\partial u}{\partial \nu_A}$ (associated with operator A) exists in a trace sense (see [2]), and the above problem is satisfied in the distribution sense. Using a Green formula (see [2, Corollary 2, p. 71]) one can define the eigenvalues of our problem in terms of weak solutions $u \in V$ as follows: $\lambda \in \mathbb{R}$ *is an eigenvalue of problem* (1.1) *if there exists* $u_\lambda \in V \setminus \{0\}$ *such that*

$$\int_\Omega \left(|\nabla u_\lambda|^{p-2} + |\nabla u_\lambda|^{q-2}\right) \nabla u_\lambda \cdot \nabla v \, dx = \lambda \int_{\partial\Omega} |u_\lambda|^{p-2} u_\lambda v \, ds, \quad \forall\, v \in V\,. \tag{1.2}$$

Conversely, by virtue of the same Green formula, if λ is an eigenvalue then any eigenfunction $u \in V \setminus \{0\}$ corresponding to it satisfies problem (1.1) in the distribution sense.

Note that the usual Steklov problem (i.e., the case when Δ_q is missing), including the classic Steklov's case ($p = 2$), has received considerable attention since 1902

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when W. Steklov published his famous pioneering paper [13]. See, e.g., [6, 8, 16, 19] and the references therein.

In the present paper we are able to find the full set of eigenvalues of problem (1.1). The fact that problem (1.1) is nonhomogeneous (i.e., $p \neq q$) is the key condition that insures a complete description of the set of eigenvalues. Recall that in the homogeneous case $q = p$ a full description of the set of eigenvalues is known only if $q = p = 2$; otherwise (i.e., if $q = p \neq 2$) it is only known that, in view of the infinite dimensional Ljusternik-Schnilerman theory, there exists a sequence of positive eigenvalues converging to $+\infty$, but this sequence may not constitute the whole spectrum [16] (see also [1], [6, Introduction]).

The Neumann eigenvalue problem associated with the negative A has been solved in [5, 10, 11]. The Dirichlet case can be adressed in a similar manner. As far as the Steklov problem (1.1) is concerned, a separate analysis is needed since some specific situations have to be addressed, including those related to the trace on $\partial\Omega$.

The Steklov eigenvalue problem (1.1) is of mathematical interest, since it is a model problem for which the full description of the eigenvalue set is possible. It might also be of interest in fluid mechanics. In the case $q = p = 2$ problem (1.1) is a model for an elastic membrane whose mass is concentrated on the boundary.

2. Some comments and preliminary results

Choosing $v = u_\lambda$ in (1.2) yields

$$\int_\Omega |\nabla u_\lambda|^p \, dx + \int_\Omega |\nabla u_\lambda|^q \, dx = \lambda \int_{\partial\Omega} |u_\lambda|^p \, ds. \tag{2.1}$$

In particular (2.1) shows that no negative λ can be an eigenvalue of problem (1.1). It is also readily seen that $\lambda_0 = 0$ is an eigenvalue of this problem (the corresponding eigenfunctions being the nonzero constant functions). Therefore any other eigenvalue belongs to $(0, +\infty)$.

If $\lambda > 0$ is an eigenvalue of problem (1.1), then choosing $v \equiv 1$ in (1.2) we obtain

$$\int_{\partial\Omega} |u_\lambda|^{p-2} u_\lambda \, ds = 0. \tag{2.2}$$

Therefore all eigenfunctions corresponding to positive eigenvalues necessarily belong to the set

$$D := \left\{ v \in V = W^{1,\max\{p,q\}}(\Omega) : \int_{\partial\Omega} |v|^{p-2} v \, ds = 0 \right\}. \tag{2.3}$$

Obviously, D is a nonempty symmetric cone. Moreover, using the continuity of the trace operator from V to $L^p(\partial\Omega)$ and Lebesgue's Dominated Convergence Theorem, one can easily check that D is weakly closed in V. In addition, the following example shows that D contains nonzero elements.

Let $x_1, x_2 \in \partial\Omega$ be such that $x_1 \neq x_2$, fix $r \in (0, |x_1 - x_2|/3)$ and define the functions $w_k : \mathbb{R}^N \to \mathbb{R}$ as follows

$$w_k(x) = \begin{cases} e^{-\frac{1}{r^2 - |x - x_k|^2}}, & \text{if } x \in B_r(x_k), \\ 0, & \text{otherwise.} \end{cases}$$

Obviously $v_k = w_k|_\Omega$ belongs to V for $k = 1, 2$. Now let $\Gamma_k = \partial\Omega \cap B_r(x_k)$ and define

$$a_k = \int_{\Gamma_k} |v_k|^{p-2} v_k \, ds > 0, \quad k = 1, 2.$$

One can easily show that the function

$$v = \left(\frac{1}{a_1}\right)^{1/(p-1)} v_1 - \left(\frac{1}{a_2}\right)^{1/(p-1)} v_2,$$

is a nonzero element of D (and obviously so is tv for all $t \in \mathbb{R} \setminus \{0\}$).

We also point out the fact that $\lambda = 0$ is the only eigenvalue whose corresponding eigenvalues do not change sign on $\partial\Omega$. Arguing by contradiction, assume that u_λ is an eigenfunction corresponding to some eigenvalue $\lambda > 0$ and $u_\lambda \geq 0$ on Ω. Then

$$0 = \int_{\partial\Omega} |u_\lambda|^{p-2} u_\lambda \, ds = \int_{\partial\Omega} |u_\lambda|^{p-1} \, ds,$$

hence $u_\lambda = 0$ a.e. on $\partial\Omega$. On the other hand, (2.2) implies

$$\int_\Omega |\nabla u_\lambda|^p \, dx = \int_\Omega |\nabla u_\lambda|^q \, dx = 0,$$

so, having in mind Weyl's regularity lemma (see Stroock [14, p. 2]), we infer that u_λ is a constant function in Ω. Since its trace on $\partial\Omega$ vanishes, it follows that u_λ is the null function, a contradiction.

Here and hereafter, for a given $u \in V$, we adopt the notation

$$\Gamma(u) := \{x \in \partial\Omega : \ u(x) \neq 0\}.$$

According (2.1), any eigenfunction u_λ corresponding to a positive eigenvalue λ satisfies

$$\text{meas}\,(\Gamma(u_\lambda)) > 0, \tag{2.4}$$

since this condition is equivalent to

$$\int_{\partial\Omega} |u_\lambda|^p \, ds \neq 0. \tag{2.5}$$

Summarizing, we see that any eigenfunction corresponding to a positive eigenvalue necessarily satisfies two restrictions: (2.1) and (2.5), respectively.

Now, consider the Steklov-type eigenvalue problem

$$\begin{cases} \Delta_p u = 0, & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \mu |u|^{p-2} u, & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

As usual, the number $\mu \in \mathbb{R}$ is said to be an *eigenvalue of problem* (2.6) if there exists a function $u_\mu \in W^{1,p}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u_\mu|^{p-2} \nabla u_\mu \cdot \nabla v \, dx = \mu \int_{\partial\Omega} |u_\mu|^{p-2} u_\mu v \, ds, \quad \forall\, v \in W^{1,p}(\Omega). \tag{2.7}$$

Obviously, $\mu_0 = 0$ is an eigenvalue of problem (2.6) and any other eigenvalue belongs to $(0, +\infty)$.

Following an idea of Véron [17], Torné [16] established an interesting variational characterization of the least (first) positive eigenvalue of (2.6). If $\omega \subseteq \Omega$ is an open subset denote by $W_*^{1,p}(\omega)$ the subset of $W^{1,p}(\Omega)$ consisting of functions which are zero a.e. in $\Omega \setminus \overline{\omega}$ and let A be the family of pairs $(\omega, \tilde{\omega})$ such that ω and $\tilde{\omega}$ are disjoint nonempty open subsets of Ω. Then, according to [16, Theorem 1.3], the least positive eigenvalue of (2.6) satisfies

$$\mu_1 = \inf_{(\omega,\tilde{\omega})\in A} \max\{\nu(\omega), \nu(\tilde{\omega})\}, \tag{2.8}$$

where

$$\nu(\omega) := \inf_{u\in W_*^{1,p}(\omega)} \left\{ \frac{1}{p}\int_\Omega |\nabla u|^p \, dx : \ \frac{1}{p}\int_{\partial\Omega} |u|^p \, ds = 1 \right\},$$

if the quantity in the right-hand side of the equality is well defined and $\nu(\omega) := +\infty$ otherwise.

If $p \geq 2$ we use the Lagrange multiplier rule to give a different variational characterization for the least (first) positive eigenvalue of problem (2.6) as follows. Keeping in mind that problem (2.6) is a particular case of problem (1.1) (corresponding to $q = p$ and $\mu = 2\lambda$), we are led to consider the constraint sets

$$E := \left\{ v \in W^{1,p}(\Omega) : \ \int_{\partial\Omega} |v|^{p-2} v \, ds = 0 \right\},$$

and

$$E_1 := \left\{ v \in W^{1,p}(\Omega) : \ \int_{\partial\Omega} |v|^{p-2} v \, ds = 0, \ \int_{\partial\Omega} |v|^p \, ds = 1 \right\}.$$

Note that $V = W^{1,p}(\Omega)$ if $q \leq p$, so $D = E$. On the other hand, if $q > p$, then $V = W^{1,q}(\Omega)$, hence D is a proper subset of E. Consequently E contains nonzero elements and E_1 is nonempty.

Consider now the minimization problem

$$\sigma_1 := \inf_{v\in E_1} F(v), \tag{2.9}$$

where $F : W^{1,p}(\Omega) \to \mathbb{R}$ is the convex, lower semicontinuous functional

$$F(v) := \int_\Omega |\nabla v|^p \, dx.$$

Lemma 2.1. *If $p \in (1, +\infty)$, then there exists $u_0 \in E_1$ such that*

$$0 < F(u_0) = \sigma_1.$$

Moreover, if $p \geq 2$ then $\mu_1 = \sigma_1$.

This result is probably known, however, for the convenience of the reader, we provide a proof of it which makes use of the following version of the Lagrange multiplier rule (see, e.g., [7, Theorem 5.5.26, p. 701], or [20, Theorem 3.3.3., p. 179]).

Lemma 2.2. *Let X and Y are two Banach spaces and assume $f : X \to \mathbb{R}$ is Fréchet differentiable at x_0, $g : X \to Y$ is continuously Fréchet differentiable at x_0*

with $g'(x_0) \in \mathcal{L}(X, Y)$ *being surjective and* $x_0 \in C := \{x \in X : \ g(x) = 0\}$ *is a finite local minimizer of the constraint problem*

$$(P_C): \qquad \min_{x \in C} f(x).$$

Then there exist $y^* \in Y^*$ *such that*

$$f'(x_0) + y^* \circ g'(x_0) = 0, \quad \text{in } X^*. \tag{2.10}$$

Proof of Lemma 2.1. Let $\{u_n\} \subset E_1$ be a minimizing sequence for F, i.e. $F(u_n) \to \sigma_1$. We claim that $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$. Arguing by contradiction, let us assume that, up to a subsequence, $\|u_n\|_{L^p(\Omega)} \to +\infty$, as $n \to +\infty$. Then, the sequence $v_n := \frac{u_n}{\|u_n\|_{L^p(\Omega)}}$ is bounded in $W^{1,p}(\Omega)$ and satisfies $\|v_n\|_{L^p(\Omega)} = 1$, for all $n \geq 1$. Consequently, there exists $v_0 \in W^{1,p}(\Omega)$ such that, up to a subsequence,

$$v_n \rightharpoonup v_0, \ \text{in } W^{1,p}(\Omega),$$

and

$$v_n \to v_0, \ \text{in } L^p(\partial\Omega), \ \ v_n \to v_0, \ \text{in } L^p(\Omega)\,.$$

Thus,

$$\int_\Omega |\nabla v_0|^p \, dx \leq \liminf_{n \to +\infty} \int_\Omega |\nabla v_n|^p \, dx = \liminf_{n \to +\infty} \frac{1}{\|u_n\|^p_{L^p(\Omega)}} F(u_n) = 0,$$

and

$$\int_{\partial\Omega} |v_0|^p \, ds = \lim_{n \to +\infty} \int_{\partial\Omega} |v_n|^p \, ds = \lim_{n \to +\infty} \frac{1}{\|u_n\|^p_{L^p(\Omega)}} = 0,$$

which together show that v_0 is the null function. This contradicts the fact that $\|v_0\|_{L^p(\Omega)} = 1$.

Since $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$, then there exists $u_0 \in W^{1,p}(\Omega)$ such that, on a subsequence again denoted $\{u_n\}$,

$$u_n \rightharpoonup u_0, \ \text{in } W^{1,p}(\Omega),$$

and

$$u_n \to u_0, \ \text{in } L^p(\Omega), \quad u_n \to u_0, \ \text{in } L^p(\partial\Omega).$$

The last convergence and Lebesgue's Dominated Convergence Theorem ensure that $u_0 \in E_1$, which combined with the weak lower semicontinuity of F shows that

$$\sigma_1 = F(u_0) = \min_{v \in E_1} F(v).$$

It can be easily seen that $\sigma_1 > 0$.

Now let us asumme $p \geq 2$ and prove that σ_1 is the least (first) positive eigenvalue of problem (2.6). To this purpose we can apply Lemma 2.2 for $X = W^{1,p}(\Omega)$, $Y = \mathbb{R}^2$, $C = E_1$, $x_0 = u_0$, and $g(v) = (g_1(v), g_2(v))$, $g_1(v) = \int_{\partial\Omega} |v|^p \, ds - 1$, $g_2(v) = \int_{\partial\Omega} |v|^{p-2} v \, ds$. Obviously, the dual Y^* can be identified with $\mathbb{R}^2$. It is easily seen that all the conditions from the statement of Lemma 2.2 are fulfilled, including the surjectivity of $g'(u_0)$, which means that: for any pair $(\zeta_1, \zeta_2) \in \mathbb{R}^2$ there exists $w \in X = W^{1,p}(\Omega)$ such that $\langle g_1'(u_0), w\rangle = \zeta_1$ and $\langle g_2'(u_0), w\rangle = \zeta_2$.

Indeed, choosing in these equations $w = au_0 + b$, with a, $b \in \mathbb{R}$ and keeping in mind that $u_0 \in E_1$, we obtain

$$\begin{aligned} ap \int_{\partial\Omega} |u_0|^p \, ds &= \zeta_1, \\ b(p-1) \int_{\partial\Omega} |u_0|^{p-2} \, ds &= \zeta_2, \end{aligned}$$

so a, b can be uniquely determined, hence $g'(u_0)$ is surjective, as asserted. Consequently, Lemma 2.2 is indeed applicable to the minimization problem (2.9), that is, there exists $y^* = (y_1, y_2) \in \mathbb{R}^2$ such that (see equation (2.10))

$$p \int_\Omega |\nabla u_0|^{p-2} \nabla u_0 \nabla v dx + y_1 p \int_{\partial\Omega} |u_0|^{p-2} u_0 v ds + y_2(p-1) \int_{\partial\Omega} |u_0|^{p-2} v ds = 0, \tag{2.11}$$

for all $v \in X$. Testing with $v \equiv 1$ in (2.11) we deduce that $y_2 = 0$. Now, choosing $v = u_0$ in (2.11) we get $y_1 = -\sigma_1$. With this y_1 and $y_2 = 0$, equation (2.11) shows that σ_1 is an eigenvalue of problem (2.6), and u_0 is a corresponding eigenfunction. In particular $\mu_1 \leq \sigma_1$.

To complete the proof we need to show that $\sigma_1 \leq \mu_1$. Assume u_{μ_1} is an eigenfunction corresponding to μ_1. Then $\|u_{\mu_1}\|_{L^p(\partial\Omega)} \neq 0$ (see (2.5)) and the function $v_{\mu_1} = \frac{u_{\mu_1}}{\|u_{\mu_1}\|_{L^p(\partial\Omega)}}$ belongs to E_1, so

$$\sigma_1 \leq F(v_{\mu_1}) = \int_\Omega |\nabla v_{\mu_1}|^p \, dx = \frac{\int_\Omega |\nabla u_{\mu_1}|^p \, dx}{\int_{\partial\Omega} |u_{\mu_1}|^p \, ds} = \mu_1.$$

□

A direct consequence of Lemma 2.1 is that $\sigma_1^{-1/p}$ is the best constant in the following Poincaré-Wirtinger type inequality.

Proposition 2.3. *Assume $p \in (1, +\infty)$. Then, there exists $C > 0$ such that*

$$\|u\|_{L^p(\partial\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)^N}, \quad \forall u \in E.$$

Now, define

$$\lambda_1 := \inf \left\{ \int_\Omega |\nabla v|^p \, dx : \ v \in V, \ \int_{\partial\Omega} |v|^{p-2} v \, ds = 0, \ \int_{\partial\Omega} |v|^p \, ds = 1 \right\}. \tag{2.12}$$

Note that λ_1 can be expressed as

$$\lambda_1 = \inf_{v \in D_1} G(v),$$

where

$$D_1 := \left\{ v \in D : \ \int_{\partial\Omega} |v|^p \, ds = 1 \right\},$$

and $G : V \to \mathbb{R}$ is defined by

$$G(v) = \int_\Omega |\nabla v|^p \, dx,$$

therefore $\lambda_1 = \sigma_1$ if $q \leq p$ and $\lambda_1 \geq \sigma_1$ if $q > p$.

We also point out the fact that λ_1 may be expressed in terms of Rayleigh quotients as follows

$$\lambda_1 = \inf_{v \in D_2} \frac{\int_\Omega |\nabla v|^p \, dx}{\int_{\partial\Omega} |v|^p \, ds}, \tag{2.13}$$

and for $q \neq p$,

$$\lambda_1 = \inf_{v \in D_2} \frac{\frac{1}{p}\int_\Omega |\nabla v|^p \, dx + \frac{1}{q}\int_\Omega |\nabla v|^q \, dx}{\frac{1}{p}\int_{\partial\Omega} |v|^p \, ds}, \tag{2.14}$$

with

$$D_2 := \{v \in D : \ \text{meas}(\Gamma(v)) > 0\}.$$

The fact that (2.13) holds is trivial. The following inequality is also trivial

$$\inf_{v \in D_2} \frac{\frac{1}{p}\int_\Omega |\nabla v|^p \, dx + \frac{1}{q}\int_\Omega |\nabla v|^q \, dx}{\frac{1}{p}\int_{\partial\Omega} |v|^p \, ds} \geq \lambda_1,$$

For the converse inequality fix $w \in D_2$ and $t > 0$. Then $tw \in D_2$ and

$$\inf_{v \in D_2} \frac{\frac{1}{p}\int_\Omega |\nabla v|^p \, dx + \frac{1}{q}\int_\Omega |\nabla v|^q \, dx}{\frac{1}{p}\int_{\partial\Omega} |v|^p \, ds} \quad \leq \quad \frac{\int_\Omega |\nabla w|^p}{\int_{\partial\Omega} |w|^p \, ds} + \frac{p}{q} t^{q-p} \frac{\int_\Omega |\nabla w|^q \, dx}{\int_{\partial\Omega} |w|^p \, ds}$$

Letting t tend to $+\infty$ if $q < p$, respectively to 0 if $q > p$, then taking the infimum over all $w \in D_2$ we get the desired inequality.

3. Main result

The main result of this paper is given by the following theorem.

Theorem 3.1. *Assume $p \in [2, \infty)$, $q \in (1, \infty)$ and $p \neq q$. Then the set of eigenvalues of problem (1.1) is precisely $\{0\} \cup (\lambda_1, +\infty)$, where λ_1 is defined above by (2.12).*

Proof. We already know from the previous section that $\lambda_0 = 0$ is an eigenvalue of problem (1.1) and any other eigenvalue belongs to $(0, +\infty)$. We prove next that no eigenvalue belongs $(0, \lambda_1]$. Arguing by contradiction assume problem (1.1) possesses an eigenvalue $\lambda \in (0, \lambda_1]$ with corresponding eigenfunction u_λ. Then

$$\int_\Omega |\nabla u_\lambda|^p \, dx + \int_\Omega |\nabla u_\lambda|^q \, dx = \lambda \int_{\partial\Omega} |u_\lambda|^p \, ds,$$

and u_λ satisfies

$$\int_{\partial\Omega} |u_\lambda|^{p-2} u_\lambda \, ds = 0 \quad \text{and} \quad \text{meas}\,(\Gamma(\mathrm{u}_\lambda)) > 0,$$

that is, $u_\lambda \in D_2$. Using the characterization of λ_1 given by (2.13) we get

$$\lambda_1 \quad \leq \quad \frac{\int_\Omega |\nabla u_\lambda|^p \, dx}{\int_{\partial\Omega} |u_\lambda|^p \, ds} = \frac{\lambda \int_{\partial\Omega} |u_\lambda|^p \, ds - \int_\Omega |\nabla u_\lambda|^q \, dx}{\int_{\partial\Omega} |u_\lambda|^p \, ds} = \lambda - \frac{\int_\Omega |\nabla u_\lambda|^q \, dx}{\int_{\partial\Omega} |u_\lambda|^p \, ds}.$$

Obviously this is a contradiction if $\lambda < \lambda_1$, while $\lambda = \lambda_1$ forces

$$\int_\Omega |\nabla u_\lambda|^q \, dx = 0,$$

which combined with $u_\lambda \in D$ yields $u_\lambda = 0$, thus contradicting the fact that $u_\lambda \in V \setminus \{0\}$.

For the second part of the proof fix $\lambda \in (\lambda_1, +\infty)$ and define the functional $J : V \to \mathbb{R}$

$$J(u) := \frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{1}{q}\int_\Omega |\nabla u|^q \, dx - \frac{\lambda}{p}\int_{\partial\Omega} |u|^p \, ds.$$

Standard arguments can be employed in order to prove that $J \in C^1(V \setminus \{0\}, \mathbb{R})$ and

$$\langle J'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + \int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla v \, dx - \lambda \int_{\partial\Omega} |u|^{p-2}uv \, ds,$$

for all $u \in V \setminus \{0\}$ and all $v \in V$. Clearly, λ is an eigenvalue of (1.1) with corresponding eigenfunction u_λ if and only if $u_\lambda \in V \setminus \{0\}$ is a critical point of J. We fix $p \in [2, +\infty)$ and consider the following cases.

Case 1: $q \in (p, +\infty)$.

Then $V = W^{1,q}(\Omega)$, D is a proper subset of E and $\lambda_1 \geq \sigma_1 = \mu_1 > 0$.

According to the Theorem on Equivalent Norms in Sobolev Spaces (see, e.g., Denkowski, Migórski and Papageorgiou [3, Cor. 3.9.56], or Nečas[12, Thm. 7.1])

$$\|u\|_1 := \left(\int_\Omega |\nabla u|^q \, dx + \int_{\partial\Omega} |u|^q \, ds\right)^{1/q},$$

and

$$\|u\|_2 := \left(\int_\Omega |\nabla u|^q \, dx + \int_{\partial\Omega} |u|^p \, ds\right)^{1/q},$$

are equivalent norms in $W^{1,q}(\Omega)$, hence there exist positive constants c_1, c_2 such that

$$\|u\|_1 \leq c_1\|u\|_{1,q}, \quad \forall u \in W^{1,q}(\Omega),$$

and

$$\|u\|_2 \geq c_2\|u\|_{1,q}, \quad \forall u \in W^{1,q}(\Omega),$$

where by $\|\cdot\|_{1,q}$ we have denoted the usual norm of $W^{1,q}(\Omega)$.

Using (2.13) we get

$$\begin{aligned}\frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{1}{q}\int_\Omega |\nabla u|^q \, dx &\geq \frac{\lambda_1}{p}\int_{\partial\Omega} |u|^p \, ds + \frac{1}{q}\int_\Omega |\nabla u|^q \, dx \\ &\geq \min\left\{\frac{1}{q}, \frac{\lambda_1}{p}\right\}\|u\|_2^q,\end{aligned}$$

for all $u \in D$.

On the other hand, Hölder's inequality ensures that

$$\int_{\partial\Omega} |u|^p ds \leq (\text{meas }(\partial\Omega))^{(q-p)/q}\left(\int_{\partial\Omega} |u|^q ds\right)^{p/q} \leq (\text{meas }(\partial\Omega))^{(q-p)/q}\|u\|_1^p,$$

for all $u \in W^{1,q}(\Omega)$. Sumarizing, we conclude that for any $u \in D$

$$J(u) \geq c_2^q \min\left\{\frac{1}{q}, \frac{\lambda_1}{p}\right\}\|u\|_{1,q}^q - \frac{\lambda}{p}c_1^p\,(\text{meas }(\partial\Omega))^{(q-p)/q}\,\|u\|_{1,q}^p,$$

which shows that J is coercive on D with respect to V, that is,

$$\lim_{\substack{\|u\|_{1,q}\to+\infty \\ u\in D}} J(u) = +\infty.$$

Recall that J is weakly lower semicontinuous and D is weakly closed in V, thus (see, e.g., [15, Theorem 1.2, p. 4]) J is bounded below on D and there exists $u_* \in D$ such that

$$J(u_*) = \inf_{u\in D} J(u). \tag{3.1}$$

Consequently we can apply again Lemma 2.2 with $X = V$, $Y = \mathbb{R}$, $C = D$, $f = J$, $x_0 = u_*$ and $g(u) = \int_{\partial\Omega} |u|^{p-2}u\, ds$ to deduce that there exists $y^* \in \mathbb{R}$ such that (see (2.10))

$$\langle J'(u_*), v\rangle + y^*(p-1)\int_{\partial\Omega} |u_*|^{p-2}v\, ds = 0, \quad \forall v \in V. \tag{3.2}$$

Choosing $v \equiv 1$ in (3.2) and keeping in mind that $u_* \in D$ we get

$$y^*(p-1)\int_{\partial\Omega} |u_*|^{p-2}\, ds = 0,$$

which shows that either $y^* = 0$, or $\operatorname{meas}(\Gamma(u_*)) = 0$. Arguing by contradiction assume the latter occurs. Then

$$0 \le \frac{1}{p}\int_\Omega |\nabla u_*|^p\, dx + \frac{1}{q}\int_\Omega |\nabla u_*|^q\, dx = J(u_*) \le J(v), \quad \forall v \in D_2 \subset D,$$

that is,

$$\lambda \le \frac{\frac{1}{p}\int_\Omega |\nabla v|^p\, dx + \frac{1}{q}\int_\Omega |\nabla v|^q\, dx}{\frac{1}{p}\int_{\partial\Omega} |v|^p\, ds}, \quad \forall v \in D_2.$$

Taking the infimum over all $v \in D_2$ and using (2.14) we reach the following contradiction

$$\lambda_1 < \lambda \le \lambda_1.$$

Therefore $y^* = 0$, $u_* \in V \setminus \{0\}$ and (3.2) becomes

$$\langle J'(u_*), v\rangle = 0, \quad \forall v \in V.$$

Case 2: $q \in (1, p)$.

Recall that $V = W^{1,p}(\Omega)$, $D = E$ and $\lambda_1 = \sigma_1 = \mu_1 > 0$ in this case.

A careful analysis shows that the functional J is no longer coercive in this case, hence we cannot use the same arguments as above to prove the existence of a nontrivial critical point. However, not everything is lost. We saw in the previous section that any eigenfunction corresponding to a positive eigenvalue satisfies the (2.2), (2.5), i.e., it belongs to D_2 and satisfies (2.1), hence it makes sense to consider the following set of constraints

$$\mathcal{N} := \left\{ u \in D_2 :\ \int_\Omega |\nabla u|^p\, dx + \int_\Omega |\nabla u|^q\, dx = \lambda \int_{\partial\Omega} |u|^p\, ds \right\}.$$

It is readily seen that $\mathcal{N}$ is the following *Nehari-type manifold*,

$$\mathcal{N} = \left\{ u \in D_2 :\ \langle J'(u), u\rangle = 0 \right\}.$$

Since $D = E$, it follows from Lemma 2.1 that there exists $u_0 \in D$ such that

$$\lambda_1 = \int_\Omega |\nabla u_0|^p \, dx \quad \text{and} \quad \int_{\partial\Omega} |u_0|^p \, ds = 1.$$

For any $t > 0$, $tu_0 \in D$ and

$$\int_\Omega |\nabla(tu_0)|^p dx + \int_\Omega |\nabla(tu_0)|^q dx - \lambda \int_{\partial\Omega} |tu_0|^p ds = t^p(\lambda_1 - \lambda) + t^q \int_\Omega |\nabla u_0|^q dx,$$

which shows that $\mathcal{N}$ is nonempty, as $tu_0 \in \mathcal{N}$ for

$$t = \left(\frac{1}{\lambda - \lambda_1}\right)^{\frac{1}{p-q}} \left(\int_\Omega |\nabla u_0|^q \, dx\right)^{\frac{1}{p-q}}.$$

Consequently, we can consider the following minimization problem

$$m := \inf_{u \in \mathcal{N}} J(u). \tag{3.3}$$

We point out the fact that for any $u \in \mathcal{N}$,

$$\begin{aligned} J(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{1}{q}\int_\Omega |\nabla u|^q \, dx - \frac{\lambda}{p}\int_{\partial\Omega} |u|^p \, ds \\ &= \left(\frac{1}{q} - \frac{1}{p}\right) \int_\Omega |\nabla u|^q \, dx \qquad (3.4) \\ &= \left(\frac{1}{q} - \frac{1}{p}\right) \left(\lambda \int_{\partial\Omega} |u|^p \, ds - \int_\Omega |\nabla u|^p \, dx\right). \qquad (3.5) \end{aligned}$$

In particular $J(u) > 0$, for all $u \in \mathcal{N}$ and $m \geq 0$. We prove next that any minimizing sequence for J is bounded in V. Arguing by contradiction, assume there exists $\{u_n\} \subset \mathcal{N}$ such that

$$J(u_n) \to m \quad \text{and} \quad \|u_n\|_{1,p} \to +\infty, \quad \text{as } n \to +\infty,$$

where by $\|\cdot\|_{1,p}$ we have denoted the usual norm of $W^{1,p}(\Omega)$. Then (3.5) ensures that

$$\int_{\partial\Omega} |u_n|^p \, ds > \frac{1}{\lambda} \int_\Omega |\nabla u_n|^p \, dx. \tag{3.6}$$

On the other hand,

$$\|u\| := \left(\int_\Omega |\nabla u|^p \, dx + \int_{\partial\Omega} |u|^p \, ds\right)^{1/p},$$

is an equivalent norm in $W^{1,p}(\Omega)$ (see, e.g. [3, Corollary 3.9.56, p. 361]). Then (3.6) implies $\|u_n\|_{L^p(\partial\Omega)} \to +\infty$ as $n \to +\infty$. Next, the sequence $w_n := \frac{u_n}{\|u_n\|_{L^p(\partial\Omega)}}$ satisfies $\|w_n\|_{L^p(\partial\Omega)} = 1$, for all $n \in \mathbb{N}$, and (see (3.6))

$$\int_\Omega |\nabla w_n|^p \, dx = \frac{1}{\|u_n\|^p_{L^p(\partial\Omega)}} \int_\Omega |\nabla u_n|^p \, dx \leq \frac{1}{\|u_n\|^p_{L^p(\partial\Omega)}} \lambda \int_{\partial\Omega} |u_n|^p \, ds = \lambda,$$

that is, $\{w_n\}$ is bounded in $W^{1,p}(\Omega)$. Consequently, there exists $w_0 \in W^{1,p}(\Omega)$ such that, up to a subsequence,

$$w_n \rightharpoonup w_0, \text{ in } W^{1,p}(\Omega),$$

and

$$w_n \to w_0, \text{ in } L^p(\partial\Omega).$$

It follows that $\|w_0\|_{L^p(\partial\Omega)} = 1$ and $w_0 \in D$. Moreover, the fact that $q < p$ implies

$$w_n \rightharpoonup w_0, \text{ in } W^{1,q}(\Omega).$$

Thus, using (3.4) we obtain

$$\int_\Omega |\nabla w_0|^p \, dx \le \liminf_{n\to+\infty} \int_\Omega |\nabla w_n|^q \, dx = \liminf_{n\to+\infty} \frac{1}{\|u_n\|^q_{L^p(\partial\Omega)}} \frac{J(u_n)}{\frac{1}{q} - \frac{1}{p}} = 0,$$

which combined with $w_0 \in D$ shows that $w_0 \equiv 0$ and this contradicts the fact that $\|w_0\|_{L^p(\partial\Omega)} = 1$. Therefore, any minimizing sequence for J is indeed bounded in V.

Let $\{u_n\} \subset \mathcal{N}$ be such a minimizing sequence for J. Then there exists $u_* \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u_*$ in $W^{1,p}(\Omega)$ and $u_n \to u_*$ in $L^p(\partial\Omega)$. Moreover, $u_* \in D$ and, since $q < p$,

$$u_n \rightharpoonup u_*, \text{ in } W^{1,q}(\Omega).$$

Therefore,

$$\begin{aligned} \lambda \int_{\partial\Omega} |u_*|^p \, ds &= \liminf_{n\to+\infty} \lambda \int_{\partial\Omega} |u_n|^p \, ds \\ &= \liminf_{n\to+\infty} \left(\int_\Omega |\nabla u_n|^p \, dx + \int_\Omega |\nabla u_n|^q \, dx \right) \\ &\ge \liminf_{n\to+\infty} \int_\Omega |\nabla u_n|^p \, dx + \liminf_{n\to+\infty} \int_\Omega |\nabla u_n|^q \, dx \\ &\ge \int_\Omega |\nabla u_*|^p \, dx + \int_\Omega |\nabla u_*|^q \, dx. \end{aligned} \tag{3.7}$$

We claim that $u_* \in \mathcal{N}$. Arguing by contradiction assume this is not the case. Then either

$$\int_{\partial\Omega} |u_*|^p \, ds = 0, \tag{3.8}$$

or

$$\lambda \int_{\partial\Omega} |u_*|^p \, ds > \int_\Omega |\nabla u_*|^p \, dx + \int_\Omega |\nabla u_*|^q \, dx. \tag{3.9}$$

If (3.8) occurs, then (3.7) gives

$$\int_\Omega |\nabla u_*|^p \, dx = \int_\Omega |\nabla u_*|^q \, dx = 0,$$

that is, $u_* = 0$ in $W^{1,p}(\Omega)$. On the other hand, the sequence $w_n := \frac{u_n}{\|u_n\|_{L^p(\partial\Omega)}}$ satisfies $\|w_n\|_{L^p(\partial\Omega)} = 1$ and $\|\nabla w_n\|_{L^p(\Omega)^N} \le \lambda^{1/p}$. Consequently, there exists $w_0 \in W^{1,p}(\Omega)$ such that $w_n \rightharpoonup w_0$ in $W^{1,p}(\Omega)$ and $w_n \to w_0$ in $L^p(\partial\Omega)$. Then $w_0 \in D$ and $\|w_0\|_{L^p(\partial\Omega)} = 1$. Moreover, since $\{u_n\} \subset \mathcal{N}$

$$0 < \int_\Omega |\nabla u_n|^q \, dx = \frac{pq}{p-q} J(u_n) = \frac{pq}{p-q} \left(\lambda \int_{\partial\Omega} |u_n|^p \, ds - \int_\Omega |\nabla u_n|^p \, dx \right).$$

Dividing the above relation by $\|u_n\|^q_{L^p(\partial\Omega)} \neq 0$, we get

$$\int_\Omega |\nabla w_n|^q \, dx \leq \frac{\lambda pq}{p-q} \|u_n\|^{p-q}_{L^p(\partial\Omega)} \to 0, \text{ as } n \to +\infty.$$

Then

$$\int_\Omega |\nabla w_0|^q \, dx \leq \liminf_{n\to+\infty} \int_\Omega |\nabla w_n|^q \, dx = 0,$$

which combined with $w_0 \in D$ shows that w_0 is the null function. This contradicts $\|w_0\|_{L^p(\partial\Omega)} = 1$, hence (3.8) cannot hold. Then $u_* \in D_2$ and (2.13) shows that

$$\int_\Omega |\nabla u_*|^p \, dx \geq \lambda_1 \int_{\partial\Omega} |u_*|^p \, ds > 0,$$

since $\lambda_1 = \mu_1 > 0$ (see Lemma 1). Obviously we also have

$$\int_\Omega |\nabla u_*|^p \, dx > 0, \tag{3.10}$$

so if (3.9) occurs, then the number

$$t := \left(\frac{\int_\Omega |\nabla u_*|^p \, dx}{\lambda \int_{\partial\Omega} |u_*|^p \, ds - \int_\Omega |\nabla u_*|^p \, dx} \right)^{\frac{1}{p-q}} \in (0,1).$$

Moreover, one can easily check that $tu_* \in \mathcal{N}$, therefore

$$0 \leq m \leq J(tu_*) = \left(\frac{1}{q} - \frac{1}{p}\right) t^q \int_\Omega |\nabla u_*|^q \, dx \leq t^q \liminf_{n\to+\infty} J(u_n) = t^q m.$$

This is clearly a contradiction if $m > 0$, while $m = 0$ forces $\int_\Omega |\nabla u_*|^q \, dx = 0$ and this contradicts (3.10).

In conclusion, $u_* \in \mathcal{N}$ and

$$m \leq J(u_*) \leq \liminf_{n\to+\infty} J(u_n) = m,$$

that is, u_* is a solution of the minimization problem (3.3). Consequently, we can apply Lemma 2.2 with $X = V$, $Y = \mathbb{R}^2$, $f = J$, $x_0 = u_*$ and $g(u) = (g_1(u), g_2(u))$ defined by

$$g_1(u) = \int_{\partial\Omega} |u|^{p-2} u \, ds, \quad g_2(u) = \int_\Omega |\nabla u|^p \, dx + \int_\Omega |\nabla u|^q \, dx - \lambda \int_{\partial\Omega} |u|^p \, ds,$$

to deduce that there exists $y^* = (y_1, y_2) \in \mathbb{R}^2$ such that

$$\langle J'(u_*), v\rangle + y_1(p-1) \int_{\partial\Omega} |u_*|^{p-2} v \, ds + y_2 p \int_\Omega |\nabla u_*|^{p-2} \nabla u_* \cdot \nabla v \, dx$$
$$+ y_2 q \int_\Omega |\nabla u_*|^{q-2} \nabla u_* \cdot \nabla v \, dx - y_2 \lambda p \int_{\partial\Omega} |u_*|^{p-2} u_* v \, ds = 0, \ \forall v \in V.$$

Choosing in the above equation $v \equiv 1$ and keeping in mind that $u_* \in \mathcal{N}$ we get

$$y_1(p-1) \int_{\partial\Omega} |u_*|^{p-2} \, ds = 0,$$

so $y_1 = 0$, while for $v = u_*$ we get

$$y_2(q-p) \int_\Omega |\nabla u_*|^q \, dx = 0.$$

Therefore $y_1 = y_2 = 0$ and (3.11) becomes

$$\langle J'(u_*), v\rangle = 0, \quad \forall v \in V,$$

concluding the proof. □

We show next that any eigenfunction of (1.1) is essentially bounded.

Theorem 3.2. *Assume u_λ is an eigenfunction of problem* (1.1) *corresponding to some $\lambda \in \{0\} \cup (\lambda_1, +\infty)$. Then $u_\lambda \in L^\infty(\Omega)$.*

Proof. Following the ideas of Drábek, Kufner and Nicolosi [4], Winkert [18] used the Moser iteration technique to prove global a priori bounds for a class of variational inequalities involving second-order elliptic differential operators. In particular, Winkert's results (see [18, Corrolary 3.1 and Example a)]) ensure that any weak solution of the following problem

$$\begin{cases} \operatorname{div}(a(x,u,\nabla u)) = 0, & \text{in } \Omega\,, \\ a(x,u,\nabla u)\cdot\nu + j'(x,u) = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.11}$$

belongs to $L^\infty(\Omega)$, provided that $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and $a : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$, $j : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ are Carathéodory functions which satisfy the following structure conditions:

There exist positive constants a_i, b_j, c_k $(i, j \in \{1, 2, 3\}, k \in \{1, 2\})$ and fixed numbers r, s_1, s_2 with $1 < r < \infty$, $r \le s_1 < r^*$, $r \le s_2 < r_*$ such that

$h1)$ $a(x,t,\xi)\cdot\xi \ge a_1|\xi|^r - a_2|t|^{s_1} - a_3$, for a.e. $x \in \Omega$, all $t \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$;

$h2)$ $|a(x,t,\xi)| \le b_1|\xi|^{r-1} + b_2|t|^{s_1\frac{r-1}{r}} + b_3$, for a.e. $x \in \Omega$, all $t \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$;

$h3)$ j is differentiable with respect to the second variable and

$$|j'(x,t)| \le c_1|t|^{s_2-1} + c_2,$$

for a.e. $x \in \partial\Omega$ and all $t \in \mathbb{R}$.

Here, r^* and r_* stand for the critical exponents,

$$r^* = \begin{cases} \frac{Nr}{N-r}, & \text{if } r < N, \\ +\infty, & \text{otherwise,} \end{cases} \qquad r_* = \begin{cases} \frac{(N-1)r}{N-r}, & \text{if } r < N, \\ +\infty, & \text{otherwise.} \end{cases}$$

One can easily check that problem (1.1) can be written equivalently in the form (3.11) simply by choosing $a(x,t,\xi) = (|\xi|^{p-2} + |\xi|^{q-2})\xi$ and $j(x,t) = -\frac{\lambda}{p}|t|^p$. A simple example for which conditions h1)-h3) hold is to take $r = s_1 = s_2 = \max\{p, q\}$, $a_i = 1$ for $i \in \{1, 2, 3\}$, $b_j = 2$ for $j \in \{1, 2, 3\}$ and $c_k = \lambda$ for $k \in \{1, 2\}$, respectively. Thus any weak solution of problem (1.1), i.e. any eigenfunction corresponding to λ, belongs to $L^\infty(\Omega)$. □

We close this paper by stating some open problems as follows.

• As already pointed out, the eigenalue set in the case $q = p$ is not completely known, except for the classical case $q = p = 2$. An idea to solve this open problem would be to use our main result here (Theorem 1) and let q tend to p (either from the left or from the right). It is expected that such a limiting process provide information about the eigenvalue set of Δ_p, $p > 2$.

• Another interesting problem is to investigate the Steklov eigenvalues for Δ_∞ or for $\Delta_p + \Delta_\infty$.

• The case $1 < p < 2$ also remains open.

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