EXISTENCE AND APPROXIMATION OF FIXED POINTS OF VICINAL MAPPINGS IN GEODESIC SPACES

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ABSTRACT. We propose the concepts of vicinal mappings and firmly vicinal mappings in metric spaces. We obtain fixed point and convergence theorems for these mappings in complete geodesic spaces with curvature bounded above by one and apply our results to convex optimization in such spaces.

1. INTRODUCTION

In this paper, we first introduce the classes of vicinal mappings and firmly vicinal mappings in metric spaces. We next obtain fixed point and convergence theorems for such mappings in complete CAT(1) spaces such that the distance of two arbitrary points in the space is less than $\pi/2$. Since the resolvents of convex functions proposed by Kimura and Kohsaka [15] are firmly vicinal, we can apply our results to convex optimization in such spaces.

The problem of finding fixed points of nonexpansive mappings is strongly related to convex optimization in Hadamard spaces, i.e., complete CAT(0) spaces. In fact, it is known [3, 4, 13, 21] that if X is an Hadamard space and f is a proper lower semicontinuous convex function of X into $(-\infty, \infty]$, then the resolvent J_f of f, which is given by

$$(1.1) \qquad J_f x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{2} d(y, x)^2 \right\}$$

for all $x \in X$, is a well-defined nonexpansive mapping of X into itself such that the fixed point set $\mathcal{F}(J_f)$ of J_f coincides with the set $\operatorname{argmin}_X f$ of all minimizers of f. It is also known [1, Proposition 3.3] that J_f is firmly nonexpansive, i.e.,

$$d\left(J_f x, J_f y\right) \le d\big(\alpha x \oplus (1-\alpha) J_f x, \alpha y \oplus (1-\alpha) J_f y\big)$$

whenever $x, y \in X$ and $\alpha \in (0, 1)$. Thus we can apply the fixed point theory for nonexpansive mappings to the problem of minimizing convex functions in the space. See also [6, 24] and [5] on the resolvents of convex functions in Hilbert and Banach spaces, respectively.

In 2013, using the resolvent J_f given by (1.1), Bačák [2, Theorem 1.4] obtained a Δ-convergence theorem on the proximal point algorithm for convex functions in Hadamard spaces, which generalizes the corresponding result by Brézis and Lions [7, Théorème 9] in Hilbert spaces to more general Hadamard spaces. This algorithm was first introduced by Martinet [20] for variational inequality problems and generally studied by Rockafellar [23] for maximal monotone operators in Hilbert spaces. See also Bruck and Reich [9] on some related results for strongly nonexpansive mappings in Banach spaces. Recently, Kimura and Kohsaka [16] obtained existence and convergence theorems on two modified proximal point algorithms for convex functions in Hadamard spaces.

On the other hand, Ohta and Pálfia [22, Definition 4.1 and Lemma 4.2] showed that the resolvent J_f in (1.1) is well defined also in a complete CAT(1) space such that $\operatorname{diam}(X) < \pi/2$, where $\operatorname{diam}(X)$ denotes the diameter of X. Using this result, they [22, Theorem 5.1] studied the proximal point algorithm for convex functions in such spaces. We note that if X is a complete CAT(1) space such that $\operatorname{diam}(X) < \pi/2$, then every sequence in X has a Δ-convergent subsequence and every proper lower semicontinuous convex function of X into $(-\infty, \infty]$ has a minimizer; see [10, Corollary 4.4] and [15, Corollary 3.3], respectively. Thus the condition $\operatorname{diam}(X) < \pi/2$ for a complete CAT(1) space X can be seen as a counterpart of the boundedness condition for an Hadamard space X.

Considering the geometric difference between Hadamard spaces and complete CAT(1) spaces, Kimura and Kohsaka [15, Definition 4.3] recently introduced the concept of resolvents of convex functions in complete CAT(1) spaces as follows. Let X be a complete CAT(1) space which is admissible, i.e.,

$$(1.2) \qquad\qquad d(v, v') < \frac{\pi}{2}$$

for all $v, v' \in X$ and f a proper lower semicontinuous convex function of X into $(-\infty, \infty]$. It is known [15, Theorem 4.2] that the resolvent R_f of f, which is given by

$$(1.3) \qquad\qquad R_f x = \operatorname*{argmin}_{y \in X}\big\{ f(y) + \tan d(y, x) \sin d(y, x) \big\}$$

for all $x \in X$, is a well-defined mapping of X into itself. It is also known [15, Theorem 4.6] that $\mathcal{F}(R_f)$ coincides with $\operatorname{argmin}_X f$, the inequality

$$(1.4) \qquad \begin{aligned} &\big(C_x^2(1 + C_y^2)C_y + C_y^2(1 + C_x^2)C_x \big) \cos d(R_f x, R_f y) \\ &\qquad \geq C_x^2(1 + C_y^2) \cos d(R_f x, y) + C_y^2(1 + C_x^2) \cos d(R_f y, x) \end{aligned}$$

holds for all $x, y \in X$, and R_f is firmly spherically nonspreading, i.e.,

$$(1.5) \qquad (C_x + C_y) \cos^2 d(R_f x, R_f y) \geq 2 \cos d(R_f x, y) \cos d(R_f y, x)$$

for all $x, y \in X$, where $C_z = \cos d(R_f z, z)$ for all $z \in X$. The inequality (1.4) means that R_f is firmly vicinal in the sense of this paper.

Moreover, Kimura and Kohsaka [15] obtained fixed point theorems and Δ-convergence theorems for firmly spherically nonspreading mappings and applied them to convex optimization in complete CAT(1) spaces. However, they did not study the fixed point problem for mappings satisfying (1.4). Since (1.4) is stronger

than (1.5), we can obtain fixed point and Δ-convergence theorems which are independent of the results in [15].

More recently, applying the resolvent R_f given by (1.3), Kimura and Kohsaka [17] and Espínola and Nicolae [11] independently studied the proximal point algorithm for convex functions in complete CAT(κ) spaces with a positive real number κ.

This paper is organized as follows. In Section 2, we recall some definitions and results needed in this paper. In Section 3, we give the definitions of vicinal mappings and firmly vicinal mappings in metric spaces such that the distance of two arbitrary points is less than or equal to $\pi/2$; see (3.2) and (3.3). In Section 4, we obtain a fixed point theorem for vicinal mappings and a Δ-convergence theorem for firmly vicinal mappings in admissible complete CAT(1) spaces; see Theorems 4.1 and 4.5, respectively. We also apply our results to convex optimization in such spaces; see Corollary 4.8. In Section 5, we define the concepts of κ-vicinal mappings and firmly κ-vicinal mappings and obtain two corollaries of our results in complete CAT(κ) spaces with a positive real number κ; see Corollaries 5.1 and 5.2.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of all positive integers and all real numbers, respectively. We denote by X a metric space with metric d. The diameter of X is denoted by $\mathrm{diam}(X)$. The closed ball with radius $r \geq 0$ centered at $p \in X$ is denoted by $S_r[p]$. For a mapping T of X into itself, we denote by $\mathcal{F}(T)$ the set of all $u \in X$ such that $Tu = u$. For a function f of X into $(-\infty, \infty]$, we denote by $\mathrm{argmin}_X f$ or $\mathrm{argmin}_{y \in X} f(y)$ the set of all $u \in X$ such that $f(u) = \inf f(X)$. In the case where $\mathrm{argmin}_X f = \{p\}$ for some $p \in X$, we identify $\mathrm{argmin}_X f$ with p.

A mapping T of X into itself is said to be asymptotically regular if

$$\lim_{n \to \infty} d(T^{n+1}x, T^n x) = 0$$

for all $x \in X$. For a sequence $\{x_n\}$ in X, the asymptotic center $\mathcal{A}(\{x_n\})$ of $\{x_n\}$ is defined by

$$\mathcal{A}(\{x_n\}) = \left\{ z \in X : \limsup_{n \to \infty} d(x_n, z) = \inf_{y \in X} \limsup_{n \to \infty} d(x_n, y) \right\}.$$

The sequence $\{x_n\}$ is said to be Δ-convergent to a point $p \in X$ if

$$\mathcal{A}(\{x_{n_i}\}) = \{p\}$$

for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$. If X is a Hilbert space, then the sequence $\{x_n\}$ is Δ-convergent to p if and only if it is weakly convergent to the point. For a sequence $\{x_n\}$ in X, we denote by $\omega_\Delta(\{x_n\})$ the set of all $z \in X$ such that there exists a subsequence of $\{x_n\}$ which is Δ-convergent to z. See [3, 10, 19] for more details on the concept of Δ-convergence.

Let λ be a positive real number. A metric space X is said to be λ-geodesic if for each $x, y \in X$ with $d(x, y) < \lambda$, there exists a mapping $c \colon [0, l] \to X$ such that $c(0) = x$, $c(l) = y$, and

$$d(c(t_1), c(t_2)) = |t_1 - t_2|$$

for all $t_1, t_2 \in [0, l]$, where $l = d(x, y)$. The mapping c is called a geodesic from x to y. In this case, the geodesic segment $[x, y]$ is defined by

$$[x, y] = \{c(t) : 0 \le t \le l\}$$

and the point $\alpha x \oplus (1 - \alpha)y$ is defined by

$$\alpha x \oplus (1 - \alpha)y = c\big((1 - \alpha)l\big)$$

for all $\alpha \in [0, 1]$. A subset C of a λ-geodesic space X such that $d(v, v') < \lambda$ for all $v, v' \in C$ is said to be convex if

$$\alpha x \oplus (1 - \alpha)y \in C$$

whenever $x, y \in C$, c is a geodesic from x to y, and $\alpha \in [0, 1]$. We note that the set $[x, y]$ and the point $\alpha x \oplus (1 - \alpha)y$ depend on the choice of a geodesic c from x to y. However, they are determined uniquely if the space X is uniquely λ-geodesic, i.e., there exists a unique geodesic from x to y for each $x, y \in X$ with $d(x, y) < \lambda$. See Bačák [3] and Bridson and Haefliger [8] for more details on geodesic spaces.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$ and S_H the unit sphere of H. The spherical metric ρ_{S_H} on S_H is defined by

$$\rho_{S_H}(x, y) = \arccos \langle x, y \rangle$$

for all $x, y \in S_H$. It is known that (S_H, ρ_{S_H}) is a uniquely π-geodesic complete metric space whose metric topology coincides with the relative norm topology on S_H. If $x, y \in S_H$ and $0 < \rho_{S_H}(x, y) < \pi$, then the unique geodesic c from x to y is given by

$$c(t) = (\cos t)x + (\sin t) \cdot \frac{y - \langle x, y \rangle x}{\|y - \langle x, y \rangle x\|}$$

for all $t \in [0, \rho_{S_H}(x, y)]$. The space (S_H, ρ_{S_H}) is called a Hilbert sphere. We denote by $\mathbb{S}^2$ the unit sphere of the three dimensional Euclidean space $\mathbb{R}^3$ with the spherical metric $\rho_{\mathbb{S}^2}$ on $\mathbb{S}^2$.

Let X be a π-geodesic metric space and x_1, x_2, x_3 points in X satisfying

$$(2.1) \qquad\qquad d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < 2\pi.$$

According to [8, Lemma 2.14 in Chapter I.2], there exist $\bar{x}_1, \bar{x}_2, \bar{x}_3 \in \mathbb{S}^2$ such that $d(x_i, x_{i+1}) = \rho_{\mathbb{S}^2}(\bar{x}_i, \bar{x}_{i+1})$ for all $i \in \{1, 2, 3\}$, where $x_4 = x_1$ and $\bar{x}_4 = \bar{x}_1$. The sets Δ and $\bar{\Delta}$ given by

$$\Delta = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1] \quad \text{and} \quad \bar{\Delta} = [\bar{x}_1, \bar{x}_2] \cup [\bar{x}_2, \bar{x}_3] \cup [\bar{x}_3, \bar{x}_1]$$

are called a geodesic triangle with vertices x_1, x_2, x_3 and a comparison triangle for Δ in $\mathbb{S}^2$, respectively. A point $\bar{p} \in \bar{\Delta}$ is called a comparison point for $p \in \Delta$ if

$$p \in [x_i, x_j], \quad \bar{p} \in [\bar{x}_i, \bar{x}_j], \quad \text{and} \quad d(x_i, p) = \rho_{\mathbb{S}^2}(\bar{x}_i, \bar{p})$$

for some distinct $i, j \in \{1, 2, 3\}$.

A metric space X is said to be a CAT(1) space if it is π-geodesic and

$$d(p, q) \le \rho_{\mathbb{S}^2}(\bar{p}, \bar{q})$$

whenever Δ is a geodesic triangle with vertices $x_1, x_2, x_3 \in X$ satisfying (2.1), $\bar{\Delta}$ is a comparison triangle for Δ in $\mathbb{S}^2$, and $\bar{p}, \bar{q} \in \bar{\Delta}$ are comparison points for $p, q \in \Delta$,

respectively. In this case, X is uniquely π-geodesic. It is known that Hilbert spaces, Hilbert spheres, and Hadamard spaces are complete CAT(1) spaces. See Bačák [3], Bridson and Haefliger [8], and Goebel and Reich [12] for more details on Hadamard spaces, CAT(κ) spaces with a real number κ, and Hilbert spheres, respectively.

A CAT(1) space X is said to be admissible if (1.2) holds for all $v, v' \in X$. A sequence $\{x_n\}$ in X is said to be spherically bounded if

$$\inf_{y \in X} \limsup_{n \to \infty} d(x_n, y) < \frac{\pi}{2}.$$

In particular, if $\mathrm{diam}(X) < \pi/2$, then the space X is admissible and every sequence in X is spherically bounded.

We know the following fundamental lemmas.

Lemma 2.1 ([10, Proposition 4.1 and Corollary 4.4]). *Let X be a complete* CAT(1) *space and $\{x_n\}$ a spherically bounded sequence in X. Then $\mathcal{A}(\{x_n\})$ is a singleton and $\{x_n\}$ has a Δ-convergent subsequence.*

Lemma 2.2 ([18, Proposition 3.1]). *Let X be a complete* CAT(1) *space and $\{x_n\}$ a spherically bounded sequence in X such that $\{d(x_n, z)\}$ is convergent for each z in $\omega_\Delta(\{x_n\})$. Then $\{x_n\}$ is Δ-convergent to an element of X.*

Lemma 2.3 (See, for instance, [15, Lemma 2.3]). *Let X be a* CAT(1) *space and x_1, x_2, x_3 points in X such that (2.1) holds. If $d(x_1, x_3) \leq \pi/2$, $d(x_2, x_3) \leq \pi/2$, and $\alpha \in [0, 1]$, then*

$$\cos d(\alpha x_1 \oplus (1 - \alpha)x_2, x_3) \geq \alpha \cos d(x_1, x_3) + (1 - \alpha) \cos d(x_2, x_3).$$

Let X be an admissible CAT(1) space and f a function of X into $(-\infty, \infty]$. Then f is said to be proper if $f(a) \in \mathbb{R}$ for some $a \in X$. It is also said to be convex if

$$f(\alpha x \oplus (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

whenever $x, y \in X$ and $\alpha \in (0, 1)$. If C is a nonempty closed convex subset of X, then the indicator function i_C for C, which is defined by $i_C(x) = 0$ if $x \in C$ and ∞ if $x \in X \setminus C$, is a proper lower semicontinuous convex function of X into $(-\infty, \infty]$. A function g of X into $[-\infty, \infty)$ is said to be concave if $-g$ is convex. See [14, 25] on some examples of convex functions in CAT(1) spaces.

It is known [15, Theorem 4.2] that if X is an admissible complete CAT(1) space, f is a proper lower semicontinuous convex function of X into $(-\infty, \infty]$, and $x \in X$, then there exists a unique $\hat{x} \in X$ such that

$$f(\hat{x}) + \tan d(\hat{x}, x) \sin d(\hat{x}, x) = \inf_{y \in X} \{f(y) + \tan d(y, x) \sin d(y, x)\}.$$

Following [15, Definition 4.3], we define the resolvent R_f of f by $R_f x = \hat{x}$ for all $x \in X$. In other words, R_f is given by (1.3) for all $x \in X$. The resolvent of the indicator function i_C for a nonempty closed convex subset C of X coincides with the metric projection P_C of X onto C, i.e.,

$$R_{i_C}(x) = \operatorname*{argmin}_{y \in X} \{i_C(y) + \tan d(y, x) \sin d(y, x)\}$$
$$= \operatorname*{argmin}_{y \in C} \tan d(y, x) \sin d(y, x) = \operatorname*{argmin}_{y \in C} d(y, x) = P_C x$$

for all $x \in X$.

We recently obtained the following maximization theorem.

Theorem 2.4 ([17, Theorem 4.1]). *Let X be an admissible complete* CAT(1) *space,* $\{z_n\}$ *a spherically bounded sequence in X, $\{\beta_n\}$ a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \beta_n = \infty$, and g the real function on X defined by*

$$(2.2) \qquad g(y) = \liminf_{n \to \infty} \frac{1}{\sum_{l=1}^{n} \beta_l} \sum_{k=1}^{n} \beta_k \cos d(y, z_k)$$

for all $y \in X$. Then g is a concave and nonexpansive function of X into $[0, 1]$ and g has a unique maximizer.

We say that a real function f on a nonempty subset I of $\mathbb{R}$ is nondecreasing if $f(s_1) \leq f(s_2)$ whenever $s_1, s_2 \in I$ and $s_1 \leq s_2$. We also say that f is nonincreasing if $-f$ is nondecreasing. It is clear that if A is a nonempty bounded subset of $\mathbb{R}$, I is a closed subset of $\mathbb{R}$ which contains A, and f is a continuous and nondecreasing real function on I, then $f(\sup A) = \sup f(A)$ and $f(\inf A) = \inf f(A)$. In fact, setting $\alpha = \sup A$, we have $\alpha \in I$ by the closedness of I. Since $s \leq \alpha$ for all $s \in A$ and f is nondecreasing, we have $f(s) \leq f(\alpha)$ for all $s \in A$. Thus we obtain $\sup f(A) \leq f(\alpha)$. On the other hand, the definition of α implies that there exists a sequence $\{s_n\}$ in A converging to α. Since f is continuous, we have $f(\alpha) = \lim_n f(s_n) \leq \sup f(A)$. Thus we have $f(\alpha) = \sup f(A)$. The second equality can be shown similarly. Using these two equalities, we can show the following.

Lemma 2.5. *Let I be a nonempty closed subset of $\mathbb{R}$, $\{t_n\}$ a bounded sequence in I, and f a continuous real function on I. Then the following hold.*

(i) *If f is nondecreasing, then $f(\limsup_n t_n) = \limsup_n f(t_n)$;*
(ii) *if f is nonincreasing, then $f(\limsup_n t_n) = \liminf_n f(t_n)$.*

3. Vicinal mappings and firmly vicinal mappings

In this section, motivated by the fact that the resolvent R_f defined by (1.3) satisfies the inequality (1.4), we give the definition of vicinal mappings and firmly vicinal mappings. We also study some fundamental properties of these mappings.

Let X be a metric space such that $d(v, v') \leq \pi/2$ for all $v, v' \in X$, T a mapping of X into itself, and C_z the real number given by

$$(3.1) \qquad\qquad\qquad C_z = \cos d(Tz, z)$$

for all $z \in X$.

The mapping T is said to be

- vicinal if

$$(3.2) \qquad \begin{aligned} &\left(C_x^2(1 + C_y^2) + C_y^2(1 + C_x^2)\right) \cos d(Tx, Ty) \\ &\qquad \geq C_x^2(1 + C_y^2) \cos d(Tx, y) + C_y^2(1 + C_x^2) \cos d(Ty, x) \end{aligned}$$

for all $x, y \in X$;

- firmly vicinal if

(3.3)
$$\left(C_x^2(1 + C_y^2)C_y + C_y^2(1 + C_x^2)C_x\right) \cos d(Tx, Ty)$$
$$\geq C_x^2(1 + C_y^2) \cos d(Tx, y) + C_y^2(1 + C_x^2) \cos d(Ty, x)$$

for all $x, y \in X$.

Recall that T is said to be

- spherically nonspreading [15] if

$$\cos^2 d(Tx, Ty) \geq \cos d(Tx, y) \cos d(Ty, x)$$

for all $x, y \in X$;
- firmly spherically nonspreading [15] if

$$(C_x + C_y) \cos^2 d(Tx, Ty) \geq 2 \cos d(Tx, y) \cos d(Ty, x)$$

for all $x, y \in X$;
- quasi-nonexpansive if $\mathcal{F}(T)$ is nonempty and $d(Tx, y) \leq d(x, y)$ for all $x \in X$ and $y \in \mathcal{F}(T)$.

Since $C_z \leq 1$ for all $z \in X$, every firmly spherically nonspreading mapping is spherically nonspreading. We know the following result.

Lemma 3.1 ([15, Theorem 4.6])**.** *Let X be an admissible complete* CAT(1) *space, f a proper lower semicontinuous convex function of X into $(-\infty, \infty]$, and R_f the resolvent of f. Then R_f is a firmly vicinal mapping of X into itself such that $\mathcal{F}(R_f)$ coincides with $\operatorname{argmin}_X f$.*

We first show the following fundamental lemma.

Lemma 3.2. *Let X be a metric space such that $d(v, v') \leq \pi/2$ for all $v, v' \in X$ and T a mapping of X into itself. Then the following hold.*

(i) *Suppose that T is firmly vicinal. Then T is vicinal. Further, if $d(v, v') < \pi/2$ for all $v, v' \in X$, then T is firmly spherically nonspreading;*
(ii) *if T is firmly vicinal and $\mathcal{F}(T)$ is nonempty, then*

$$\cos d(Tx, x) \cos d(Tx, y) \geq \cos d(x, y)$$

for all $x \in X$ and $y \in \mathcal{F}(T)$;
(iii) *if T is vicinal and $\mathcal{F}(T)$ is nonempty, then it is quasi-nonexpansive;*
(iv) *if $d(v, v') < \pi/2$ for all $v, v' \in X$, T is firmly vicinal, and $\mathcal{F}(T)$ is nonempty, then T is asymptotically regular.*

Proof. Let C_z be the real number given by (3.1) for all $z \in C$. We first prove (i). Suppose that T is firmly vicinal. It is obvious that T is vicinal since $C_z \leq 1$ for all $z \in X$. Further, suppose that $d(v, v') < \pi/2$ for all $v, v' \in X$. Then, using an idea in [15, Theorem 4.6], we show that T is firmly spherically nonspreading. Let $x, y \in X$ be given. By the definition of firm vicinality and the inequality of

arithmetic and geometric means, we have

$$
\begin{aligned}
\big(C_x^2(1 + C_y^2)C_y &+ C_y^2(1 + C_x^2)C_x\big) \cos d(Tx, Ty) \\
&\geq C_x^2 \cos d(Tx, y) + C_y^2 \cos d(Ty, x) \\
&\quad + C_x^2 C_y^2\big(\cos d(Tx, y) + \cos d(Ty, x)\big) \\
&\geq 2 C_x C_y(1 + C_x C_y)\sqrt{\cos d(Tx, y)\cos d(Ty, x)}.
\end{aligned}
$$
(3.4)

On the other hand, we have

$$
(3.5) \qquad C_x^2(1 + C_y^2)C_y + C_y^2(1 + C_x^2)C_x = C_x C_y(C_x + C_y)(1 + C_x C_y).
$$

Noting that $C_x C_y > 0$, we have from (3.4) and (3.5) that

$$
(C_x + C_y)^2 \cos^2 d(Tx, Ty) \geq 4 \cos d(Tx, y) \cos d(Ty, x).
$$

Since $2 \geq C_x + C_y$, we know that T is firmly spherically nonspreading.

We next prove (ii). Suppose that T is firmly vicinal and $\mathcal{F}(T)$ is nonempty. Let $x \in X$ and $y \in \mathcal{F}(T)$ be given. Since $Ty = y$ and $C_y = 1$, we have

$$
\big(2C_x^2 + (1 + C_x^2)C_x\big) \cos d(Tx, y) \geq 2C_x^2 \cos d(Tx, y) + (1 + C_x^2) \cos d(y, x)
$$

and hence

$$
(1 + C_x^2)C_x \cos d(Tx, y) \geq (1 + C_x^2) \cos d(x, y).
$$

Thus we obtain the conclusion.

We next prove (iii). Suppose that T is vicinal and $\mathcal{F}(T)$ is nonempty. Let $x \in X$ and $y \in \mathcal{F}(T)$ be given. Then we have

$$
\big(2C_x^2 + (1 + C_x^2)\big) \cos d(Tx, y) \geq 2C_x^2 \cos d(Tx, y) + (1 + C_x^2) \cos d(y, x)
$$

and hence

$$
(1 + C_x^2) \cos d(Tx, y) \geq (1 + C_x^2) \cos d(x, y).
$$

This implies that

$$
\cos d(Tx, y) \geq \cos d(x, y)
$$

and hence we obtain the conclusion.

We finally prove (iv). Suppose that $d(v, v') < \pi/2$ for all $v, v' \in X$, T is firmly vicinal, and $\mathcal{F}(T)$ is nonempty. Let $x \in X$ and $y \in \mathcal{F}(T)$ be given. Then it follows from (i) and (iii) that T is quasi-nonexpansive. This implies that

$$
d(T^{n+1}x, y) \leq d(T^n x, y) \leq d(x, y) < \frac{\pi}{2}
$$

for all $n \in \mathbb{N}$ and hence $\{d(T^n x, y)\}$ converges to some $l \in [0, \pi/2)$. Then it follows from (ii) that

$$
1 \geq \cos d(T^{n+1}x, T^n x) \geq \frac{\cos d(T^n x, y)}{\cos d(T^{n+1}x, y)} \to \frac{\cos l}{\cos l} = 1.
$$

This yields that $\cos d(T^{n+1}x, T^n x) \to 1$. Therefore we obtain the conclusion. $\square$

The following example shows that there exists a discontinuous spherically non-spreading mapping in an admissible complete CAT(1) space.

Example 3.3. *Let (S_H, ρ_{S_H}) be a Hilbert sphere, both r and δ real numbers such that*

$$\frac{\pi}{8} < r < \frac{\pi}{4}, \quad 0 < \delta < 1, \quad and \quad \cos\frac{\pi}{8} \le \cos^2\frac{\delta\pi}{8},$$

p an element of S_H, $A = \{p\}$, $B = S_{\delta\pi/8}[p]$, $C = S_{\pi/8}[p]$, $X = S_r[p]$, and both P_A and P_B the metric projections of X onto A and B, respectively. Then the mapping T given by

$$Tx = \begin{cases} P_A x & (x \in C); \\ P_B x & (x \in X \setminus C) \end{cases}$$

is a spherically nonspreading mapping of X into itself.

Proof. We denote by d the restriction of ρ_{S_H} on $X \times X$. Since

$$d(x, y) \le d(x, p) + d(p, y) \le 2r < \frac{\pi}{2}$$

for all $x, y \in X$, the space X is admissible. We can see that X is a convex subset of S_H. In fact, if $x, y \in X$ and $\alpha \in [0, 1]$, then Lemma 2.3 implies that

$$\cos d\big(\alpha x \oplus (1 - \alpha)y, p\big) \ge \alpha \cos d(x, p) + (1 - \alpha) \cos d(y, p)$$
$$\ge \alpha \cos r + (1 - \alpha) \cos r = \cos r.$$

This implies that $d(\alpha x \oplus (1 - \alpha)y, p) \le r$ and hence $\alpha x \oplus (1 - \alpha)y \in X$. Since X is a nonempty closed convex subset of the complete CAT(1) space S_H, the space X is also a complete CAT(1) space. We can also see that B is a convex subset of X.

By Lemmas 3.1 and 3.2, we know that P_A and P_B are spherically nonspreading and hence

$$\cos^2 d(Tx, Ty) \ge \cos d(Tx, y) \cos d(Ty, x)$$

whenever $(x, y) \in C^2$ or $(x, y) \in (X \setminus C)^2$. Suppose that $x \in X \setminus C$ and $y \in C$. Then we have $d(Tx, Ty) = d(P_B x, p) \le \delta\pi/8$ and hence

$$(3.6) \qquad\qquad \cos^2 d(Tx, Ty) \ge \cos^2 \frac{\delta\pi}{8}.$$

On the other hand, we have $d(Ty, x) = d(p, x) > \pi/8$ and hence

$$(3.7) \qquad\qquad \cos d(Ty, x) < \cos\frac{\pi}{8}.$$

By (3.6) and (3.7), we have

$$\cos d(Tx, y) \cos d(Ty, x) \le \cos d(Ty, x) < \cos\frac{\pi}{8} \le \cos^2\frac{\delta\pi}{8} \le \cos^2 d(Tx, Ty).$$

Therefore T is spherically nonspreading. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

4. Existence and approximation of fixed points

In this section, we study the existence and approximation of fixed points of vicinal mappings and firmly vicinal mappings, respectively.

Using Theorem 2.4, we obtain the following fixed point theorem for vicinal mappings in admissible complete CAT(1) spaces.

Theorem 4.1. *Let X be an admissible complete* CAT(1) *space and T a vicinal mapping of X into itself. Then $\mathcal{F}(T)$ is nonempty if and only if there exists $x \in X$ such that $\{T^n x\}$ is spherically bounded and $\sup_n d(T^n x, T^{n-1} x) < \pi/2$.*

Proof. Let C_z be the real number given by (3.1) for all $z \in C$. The only if part is obvious. In fact, if $\mathcal{F}(T)$ is nonempty and $x \in \mathcal{F}(T)$, then we have

$$\sup_n d(T^n x, T^{n-1} x) = 0 < \frac{\pi}{2}$$

and

$$\inf_{y \in X} \limsup_{n \to \infty} d(T^n x, y) = \inf_{y \in X} d(x, y) < \frac{\pi}{2},$$

where the last inequality follows from the admissibility of X.

We next prove the if part. Suppose that there exists $x \in X$ such that $\{T^n x\}$ is spherically bounded and $\sup_n d(T^n x, T^{n-1} x) < \pi/2$. Set

$$x_n = T^{n-1} x, \quad \beta_n = \frac{C_{x_n}^2}{1 + C_{x_n}^2}, \quad \text{and} \quad \sigma_n = \sum_{k=1}^n \beta_k$$

for all $n \in \mathbb{N}$ and let g be the real function on X defined by

$$g(y) = \liminf_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=1}^n \beta_k \cos d(y, x_{k+1})$$

for all $y \in X$. Since

$$\inf_n C_{x_n} = \inf_n \cos d(T^n x, T^{n-1} x) = \cos \left(\sup_n d(T^n x, T^{n-1} x) \right) > \cos \frac{\pi}{2} = 0,$$

we have

$$\sigma_n = \sum_{k=1}^n \beta_k \geq \frac{1}{2} \sum_{k=1}^n C_{x_k}^2 \geq \left(\inf_m C_{x_m} \right)^2 \frac{n}{2} \to \infty$$

as $n \to \infty$. Hence we have $\sum_{n=1}^\infty \beta_n = \infty$. Thus Theorem 2.4 ensures that g has a unique maximizer $p \in X$.

On the other hand, the vicinality of T implies that

$$\left(C_{x_k}^2 (1 + C_p^2) + C_p^2 (1 + C_{x_k}^2) \right) \cos d(x_{k+1}, Tp)$$
$$\geq C_{x_k}^2 (1 + C_p^2) \cos d(x_{k+1}, p) + C_p^2 (1 + C_{x_k}^2) \cos d(Tp, x_k)$$

and hence

$$\frac{C_{x_k}^2}{1 + C_{x_k}^2} \cos d(x_{k+1}, Tp)$$

$$\geq \frac{C_{x_k}^2}{1 + C_{x_k}^2} \cos d(x_{k+1}, p) + \frac{C_p^2}{1 + C_p^2} \big(\cos d(Tp, x_k) - \cos d(Tp, x_{k+1})\big)$$

for all $k \in \mathbb{N}$. This inequality yields

$$\frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k \cos d(x_{k+1}, Tp)$$

$$\geq \frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k \cos d(x_{k+1}, p) + \frac{1}{\sigma_n} \cdot \frac{C_p^2}{1 + C_p^2} \big(\cos d(Tp, x_1) - \cos d(Tp, x_{n+1})\big).$$

Taking the lower limit in this inequality, we obtain $g(Tp) \geq g(p)$. Since p is the unique maximizer of g, we conclude that $Tp = p$. Therefore T has a fixed point. $\square$

Remark 4.2. In [15, Theorem 5.2], it was shown that if X is an admissible complete CAT(1) space and T is a spherically nonspreading mapping of X into itself, then $\mathcal{F}(T)$ is nonempty if and only if there exists $x \in X$ such that

$$\limsup_{n \to \infty} d(T^n x, Ty) < \frac{\pi}{2}$$

for all $y \in X$. Note that Theorem 4.1 is independent of this result.

As a direct consequence of Theorem 4.1, we obtain the following corollary.

Corollary 4.3. *Let X be a complete* CAT(1) *space such that* $\mathrm{diam}(X) < \pi/2$. *Then every vicinal mapping T of X into itself has a fixed point.*

Before obtaining a Δ-convergence theorem, we show the following demiclosedness principle for vicinal mappings.

Lemma 4.4. *Let X be a metric space such that $d(v, v') < \pi/2$ for all $v, v' \in X$, T a vicinal mapping of X into itself, p an element of X, and $\{x_n\}$ a sequence in X such that $\mathcal{A}(\{x_n\}) = \{p\}$ and $d(Tx_n, x_n) \to 0$. Then p is a fixed point of T.*

Proof. Let C_z be the real number given by (3.1) for all $z \in C$. Since $d(Tx_n, x_n) \to 0$, we know that

$$(4.1) \qquad\qquad\qquad \lim_{n \to \infty} C_{x_n} = 1.$$

On the other hand, since $t \mapsto \cos t$ is nonexpansive and $d(Tx_n, x_n) \to 0$, we have

$$|\cos d(x_n, Tp) - \cos d(Tx_n, Tp)| \leq |d(x_n, Tp) - d(Tx_n, Tp)|$$
$$\leq d(x_n, Tx_n) \to 0$$

and hence

$$(4.2) \qquad\qquad \lim_{n \to \infty} \big(d(x_n, Tp) - \cos d(Tx_n, Tp)\big) = 0.$$

The vicinality of T implies that
$$\left(C_{x_n}^2(1+C_p^2) + C_p^2(1+C_{x_n}^2)\right)\cos d(Tx_n, Tp)$$
$$\geq C_{x_n}^2(1+C_p^2)\cos d(Tx_n, p) + C_p^2(1+C_{x_n}^2)\cos d(Tp, x_n)$$

and hence

$$\cos d(Tx_n, Tp)$$

(4.3)
$$\geq \cos d(Tx_n, p) + \frac{C_p^2}{1+C_p^2} \cdot \frac{1+C_{x_n}^2}{C_{x_n}^2}\left(\cos d(x_n, Tp) - \cos d(Tx_n, Tp)\right)$$

for all $n \in \mathbb{N}$.

Using (4.1), (4.2), and (4.3), we have
$$\liminf_{n\to\infty} \cos d(Tx_n, Tp) \geq \liminf_{n\to\infty} \cos d(Tx_n, p).$$

Then it follows from Lemma 2.5 that
$$\cos\left(\limsup_{n\to\infty} d(Tx_n, Tp)\right) = \liminf_{n\to\infty} \cos d(Tx_n, Tp)$$
$$\geq \liminf_{n\to\infty} \cos d(Tx_n, p) = \cos\left(\limsup_{n\to\infty} d(Tx_n, p)\right)$$

and hence
$$\limsup_{n\to\infty} d(Tx_n, Tp) \leq \limsup_{n\to\infty} d(Tx_n, p).$$

It then follows from $d(Tx_n, x_n) \to 0$ that
$$\limsup_{n\to\infty} d(x_n, Tp) = \limsup_{n\to\infty} d(Tx_n, Tp)$$
$$\leq \limsup_{n\to\infty} d(Tx_n, p) = \limsup_{n\to\infty} d(x_n, p).$$

Thus, since $\mathcal{A}(\{x_n\}) = \{p\}$, we conclude that $Tp = p$. $\qquad\square$

We next obtain the following Δ-convergence theorem for firmly vicinal mappings in admissible complete CAT(1) spaces.

Theorem 4.5. *Let X be an admissible complete CAT(1) space and T a firmly vicinal mapping of X into itself. If $\mathcal{F}(T)$ is nonempty, then $\{T^n x\}$ is Δ-convergent to an element of $\mathcal{F}(T)$ for each $x \in X$.*

Proof. Let $x \in X$ be given. By (i) and (iii) of Lemma 3.2, T is quasi-nonexpansive. Combining this property with the admissibility of X, we have
$$d(T^{n+1}x, y) \leq d(T^n x, y) \leq d(x, y) < \frac{\pi}{2}$$

for each $y \in \mathcal{F}(T)$. This gives us that the sequence $\{d(T^n x, y)\}$ converges to an element of $[0, \pi/2)$ for each $y \in \mathcal{F}(T)$. Since
$$\inf_{y\in X} \limsup_{n\to\infty} d(T^n x, y) \leq \inf_{y\in\mathcal{F}(T)} \limsup_{n\to\infty} d(T^n x, y) \leq \inf_{y\in\mathcal{F}(T)} d(x, y) < \frac{\pi}{2},$$

the sequence $\{T^n x\}$ is spherically bounded.

Let z be any element of $\omega_\Delta(\{T^n x\})$. By the definition of $\omega_\Delta(\{T^n x\})$, there exists a subsequence $\{T^{n_i} x\}$ of $\{T^n x\}$ which is Δ-convergent to z. Then we have

$\mathcal{A}(\{T^{n_i}x\}) = \{z\}$. By (iv) of Lemma 3.2, we know that T is asymptotically regular and hence

$$\lim_{i\to\infty} d\big(T(T^{n_i}x), T^{n_i}x\big) = \lim_{n\to\infty} d(T^{n+1}x, T^n x) = 0.$$

According to Lemma 4.4, we have $z \in \mathcal{F}(T)$. Thus $\omega_\Delta\big(\{T^n x\}\big)$ is contained by $\mathcal{F}(T)$. Consequently, the real sequence $\{d(T^n x, z)\}$ is convergent for each z in $\omega_\Delta\big(\{T^n x\}\big)$. Then, it follows from Lemma 2.2 that $\{T^n x\}$ is Δ-convergent to some $p \in X$. Since every subsequence of $\{T^n x\}$ is also Δ-convergent to p, we obtain

$$\{p\} = \omega_\Delta\big(\{T^n x\}\big) \subset \mathcal{F}(T).$$

Therefore we conclude that p is a fixed point of T. $\qquad\square$

Remark 4.6. In [15, Theorem 6.5], it was shown that if X is an admissible complete CAT(1) space and T is a firmly spherically nonspreading mapping of X into itself such that $\mathcal{F}(T)$ is nonempty and

$$(4.4) \qquad\qquad \limsup_{n\to\infty} d(y_n, Ty) < \frac{\pi}{2}$$

whenever $\{y_n\}$ is a sequence in X which is Δ-convergent to $y \in X$, then $\{T^n x\}$ is Δ-convergent to an element of $\mathcal{F}(T)$ for each $x \in X$. By Theorem 4.5, we know that the assumption (4.4) is not needed for the special case where T is firmly vicinal.

As a direct consequence of Corollary 4.3 and Theorem 4.5, we obtain the following corollary.

Corollary 4.7. *Let X be a complete* CAT(1) *space such that* $\mathrm{diam}(X) < \pi/2$ *and T a firmly vicinal mapping of X into itself. Then $\{T^n x\}$ is Δ-convergent to an element of $\mathcal{F}(T)$ for each $x \in X$.*

As a direct consequence of Lemma 3.1, Theorems 4.1, and 4.5, we obtain the following corollary.

Corollary 4.8. *Let X be an admissible complete* CAT(1) *space, f a proper lower semicontinuous convex function of X into $(-\infty, \infty]$, and R_f the resolvent of f. Then* $\mathrm{argmin}_X f$ *is nonempty if and only if there exists $x \in X$ such that $\{R_f^n x\}$ is spherically bounded and* $\sup_n d(R_f^n x, R_f^{n-1} x) < \pi/2$. *In this case, $\{R_f^n x\}$ is Δ-convergent to an element of* $\mathrm{argmin}_X f$ *for each $x \in X$.*

5. RESULTS IN CAT(κ) SPACES WITH A POSITIVE κ

In this section, we define the concepts of κ-vicinal mappings and firmly κ-vicinal mappings and obtain two corollaries of our results for these mappings in complete CAT(κ) spaces with a positive real number κ.

Let κ be a positive real number, X a metric space such that $d(v, v') \leq \pi/(2\sqrt{\kappa})$ for all $v, v' \in X$, T a mapping of X into itself, and $\tilde{C}_z$ the real number defined by

$$\tilde{C}_z = \cos \sqrt{\kappa} d(Tz, z)$$

for all $z \in X$. The mapping T is said to be

- κ-vicinal if

$$\big(\tilde{C}_x^2(1 + \tilde{C}_y^2) + \tilde{C}_y^2(1 + \tilde{C}_x^2)\big) \cos \sqrt{\kappa}d(Tx, Ty)$$
$$\geq \tilde{C}_x^2(1 + \tilde{C}_y^2) \cos \sqrt{\kappa}d(Tx, y) + \tilde{C}_y^2(1 + \tilde{C}_x^2) \cos \sqrt{\kappa}d(Ty, x)$$

  for all $x, y \in X$;
- firmly κ-vicinal if

$$\big(\tilde{C}_x^2(1 + \tilde{C}_y^2)\tilde{C}_y + \tilde{C}_y^2(1 + \tilde{C}_x^2)\tilde{C}_x\big) \cos \sqrt{\kappa}d(Tx, Ty)$$
$$\geq \tilde{C}_x^2(1 + \tilde{C}_y^2) \cos \sqrt{\kappa}d(Tx, y) + \tilde{C}_y^2(1 + \tilde{C}_x^2) \cos \sqrt{\kappa}d(Ty, x)$$

  for all $x, y \in X$.

Note that 1-vicinal mappings and firmly 1-vicinal mappings are coincident with vicinal mappings and firmly vicinal mappings, respectively.

Let κ be a positive real number, D_κ the real number given by $D_\kappa = \pi/\sqrt{\kappa}$, and (M_κ, ρ_κ) the uniquely D_κ-geodesic space given by

$$(M_\kappa, \rho_\kappa) = \left(\mathbb{S}^2, \frac{1}{\sqrt{\kappa}}\rho_{\mathbb{S}^2}\right).$$

A metric space X is said to be a $\mathrm{CAT}(\kappa)$ space if it is D_κ-geodesic and

$$d(p, q) \leq \rho_\kappa(\bar{p}, \bar{q})$$

whenever Δ is a geodesic triangle with vertices $x_1, x_2, x_3 \in X$ satisfying

$$d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < 2D_\kappa,$$

$\bar{\Delta}$ is a comparison triangle for Δ in M_κ, and $\bar{p}, \bar{q} \in \bar{\Delta}$ are comparison points for $p, q \in \Delta$, respectively. It is obvious that (X, d) is a complete $\mathrm{CAT}(\kappa)$ space such that $d(v, v') < D_\kappa/2$ for all $v, v' \in X$ if and only if $(X, \sqrt{\kappa}d)$ is an admissible complete $\mathrm{CAT}(1)$ space.

As direct consequences of Theorems 4.1 and 4.5, we obtain the following two corollaries, respectively.

Corollary 5.1. *Let κ be a positive real number, X a complete $\mathrm{CAT}(\kappa)$ space such that $d(v, v') < D_\kappa/2$ for all $v, v' \in X$, and T a κ-vicinal mapping of X into itself. Then $\mathcal{F}(T)$ is nonempty if and only if there exists $x \in X$ such that*

$$\inf_{y \in X} \limsup_{n \to \infty} d(T^n x, y) < \frac{D_\kappa}{2} \quad and \quad \sup_n d(T^n x, T^{n-1}x) < \frac{D_\kappa}{2}.$$

Corollary 5.2. *Let κ be a positive real number, X a complete $\mathrm{CAT}(\kappa)$ space such that $d(v, v') < D_\kappa/2$ for all $v, v' \in X$, and T a firmly κ-vicinal mapping of X into itself. If $\mathcal{F}(T)$ is nonempty, then $\{T^n x\}$ is Δ-convergent to an element of $\mathcal{F}(T)$ for each $x \in X$.*

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