Pure and Applied Functional Analysis
Volume 3, Number 2, 2018, 271–285

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

SECOND-ORDER STATE DEPENDENT SWEEPING PROCESS WITH UNBOUNDED AND NONCONVEX CONSTRAINTS

SAMIR ADLY AND BA KHIET LE

Dedicated to Professor Stephen M. Robinson on the occasion of his 75th birthday.

Abstract. In this paper, by using an implicit discrete scheme, we prove the existence of solutions for a class of second-order sweeping processes with possibly unbounded, closed, prox-regular moving sets depending on both the time and the state in a Hilbert space. We use the local excess (local one-sided Lipschitz continuity property) instead of the global Hausdorff distance to describe the way the set of constraints is moving.

1. Introduction

In 1971, J. J. Moreau introduced and thoroughly studied the notion of the so-called "sweeping process" in a series of seminal papers [22, 23, 24, 25] with an original motivation coming from quasi-static evolution in elastoplasticity in unilateral mechanics . Initially, Moreau considered the case where a point is swept by a moving closed convex set $C(t), t \in [0,T]$, in a Hilbert space H which can be formulated in the form of a differential inclusion as follows

$$\begin{cases} \dot{u}(t) \in -\mathrm{N}_{C(t)}(u(t)) \text{ a.e. } t \in [0,T], \\ u(0) = u_0 \in C(0), \end{cases} \tag{1.1}$$

where $\mathrm{N}_{C(t)}(\cdot)$ denotes the normal cone of $C(t)$ in the sense of convex analysis. To take into account possible perturbations, it is natural to study the following variant

$$\begin{cases} \dot{u}(t) \in -\mathrm{N}_{C(t)}(u(t)) + \mathcal{F}(t, u(t)) \text{ a.e. } t \in [0,T], \\ u(0) = u_0 \in C(0), \end{cases} \tag{1.2}$$

where $\mathcal{F} : [0,T] \times H \to 2^H$ is a set-valued mapping with nonempty weakly compact convex values in H. The case of nonmoving sets $C(t) \equiv C$ in finite-dimensional spaces was considered by C. Henry [17] and later by B. Cornet [13] for the study of planning procedures in mathematical economy. It was recently applied for the simulation of crowd motion in [20, 21]. The existence of solutions for the second-order sweeping process have been also considered by many authors (see, e.g., [6, 7,

2010 *Mathematics Subject Classification.* 34A60, 49J52, 49J53.
Key words and phrases. Moreau's sweeping process, second-order dynamic, nonsmooth dynamic, differential inclusion, quasi-variational inequality, prox-regular set.

8, 9, 11, 18]). In [9], Castaing studied for the first time the second-order sweeping process of the following form:

$$\begin{cases} \ddot{u}(t) \in -\mathrm{N}_{C(u(t))}(\dot{u}(t)) \quad a.e.\ t \in [0,T], \\ \\ u(0) = u_0 \in C(0), \dot{u}(0) = v_0 \in C(u_0), \end{cases} \tag{1.3}$$

where $C : H \to 2^H$ is a set-valued mapping with convex and compact values. In a recent paper [1], the authors considered the (possibly) unbounded convex second-order sweeping processes with perturbation in a Hilbert space of the following form

$$(\mathcal{S}) \begin{cases} \ddot{u}(t) \in -\mathrm{N}_{C(t,u(t))}(\dot{u}(t)) - \mathcal{F}(t, u(t), \dot{u}(t)) \quad a.e.\ t \in [0,T], \\ u(0) = u_0, \dot{u}(0) = v_0 \in C(0, u_0), \end{cases}$$

where $C : [0,T] \times H \to 2^H,\ (t,x) \mapsto C(t,x) \subset H$ has closed convex and possibly unbounded values, and $\mathcal{F} : [0,T] \times H \times H \to 2^H,\ (t,x,y) \mapsto \mathcal{F}(t,x,y) \subset H$ is a given set-valued map. In addition, the set $C(t,.)$ is assumed to satisfy a Lipschitz variation of intersection with some particular ball, a property that is verified by many unbounded sets. The set-valued perturbation $\mathcal{F}$ is supposed to be upper semicontinuous with convex and weakly compact values, and only need to satisfy the weak linear growth condition (i.e. the intersection between the ball with linear growth and the perturbation is nonempty). To go beyond the convexity assumption of the moving set, which could too restrictive in some applications, the notion of prox-regularity seems to be suitable to handle first and second-order sweeping processes in the infinite dimensional Hilbert spaces settings (see, e.g. [2, 3] and the references therein). The prox-regular second-order state-dependent case in a Hilbert space was firstly studied in [12] by using a discretization technique. Recently it was also considered in [19] by using Schauder's fixed point theorem and in [2] when the moving set varies in a bounded variation way which allows discontinuous solutions. However, these last works use the global Hausdorff distance that may not allow the moving set to be unbounded. We propose to analyze in the current paper the prox-regular and unbounded moving set associated to the dynamic $(\mathcal{S})$, which could be of great interest in many applications in unilateral mechanical systems and nonregular electrical circuits. Note that we only use a kind of local one-sided Lipschitz continuity property (local excess) rather than the usual Hausdorff distance to describe the way the set of constraints is moving.

The paper is organized as follows. In Section 2, we recall some basic notations, definitions and results that will be used throughout the paper. The existence of solutions are thoroughly analyzed in Section 3 by using an implicit discretization scheme which allows the unboundedness of the moving set and adapted to the prox-regular settings. Some conclusions and perspectives end the paper in Section 4.

2. Definitions and preliminaries

We begin with some notations, definitions and mathematical backgrounds that will be used later. Let H be a real Hilbert space. We denote by $\langle \cdot, \cdot \rangle$, $\| \cdot \|$ the scalar product and the corresponding norm in H. Denote by I the identity operator, by

$\mathbb{B}$ the unit ball in H and $\mathbb{B}_r := r\mathbb{B}$, $\mathbb{B}_r(x) := x + r\mathbb{B}$. The distance from a point $s \in H$ to a set $C \subset H$, denoted by $d(s, C)$ or $d_C(s)$, is defined by

$$d(s, C) := \inf_{x \in C} \|s - x\|.$$

The excess of C_1 over C_2 is defined by

$$e(C_1, C_2) := \sup_{x_1 \in C_1} d(x_1, C_2).$$

The set of all points in C that are nearest to $s \in H$ is denoted by

$$\mathrm{Proj}(C, s) := \{x \in C : \|s - x\| = d(s, C)\}.$$

When $\mathrm{Proj}(C, s) = \{x\}$, we can write $x = \mathrm{proj}(C, s)$ to emphasize the single-valuedness property. Let $x \in \mathrm{Proj}(C, s)$ and $t \geq 0$, then the vector $t(s - x)$ is called *proximal normal* to C at x. The set of all such vectors is a cone, called *proximal normal cone* of C at x and denoted by $\mathrm{N}^P(C, x)$. It is a known result [14] that $\xi \in \mathrm{N}^P(C, x)$ if and only if there exists some $\sigma > 0$ and $\delta > 0$ such that

$$\langle \xi, y - x \rangle \leq \delta \|y - x\|^2 \text{ for all } y \in C \cap \mathbb{B}_\sigma(x).$$

The Fréchet normal cone $\mathrm{N}^F(\cdot)$, the limiting (Mordukhovich) normal cone $\mathrm{N}^L(\cdot)$, the Clarke normal cone $\mathrm{N}^C(\cdot)$ are defined respectively as follows

$$\mathrm{N}^F(C, x) := \big\{\xi \in H : \forall \delta > 0, \exists \sigma > 0 \text{ s.t. } \langle \xi, y - x \rangle \leq \delta \|y - x\| \ \forall y \in C \cap \mathbb{B}_\sigma(x)\big\},$$

$$\begin{aligned} \mathrm{N}^L(C, x) &:= \big\{\xi \in H : \exists\, \xi_n \to \xi \text{ weakly and } \xi_n \in \mathrm{N}^P(C, x_n),\ x_n \to x \text{ in } C\big\} \\ &= \big\{\xi \in H : \exists\, \xi_n \to \xi \text{ weakly and } \xi_n \in \mathrm{N}^F(C, x_n),\ x_n \to x \text{ in } C\big\}, \end{aligned}$$

$$\mathrm{N}^C(C, x) := \overline{\mathrm{co}}\, \mathrm{N}^L(C, x).$$

If $x \notin C$, one has $\mathrm{N}^P(C, x) = \mathrm{N}^F(C, x) = \mathrm{N}^L(C, x) = \mathrm{N}^C(C, x) = \emptyset$ and for all $x \in C$:

$$\mathrm{N}^P(C, x) \subset \mathrm{N}^F(C, x) \subset \mathrm{N}^L(C, x) \subset \mathrm{N}^C(C, x).$$

If C is a closed and convex subset of H, then these normal cones coincide. The equality of these four normal cones is also valid for prox-regular sets (see Definition 1). In this case, we write only $\mathrm{N}(C, x)$ for simplicity. For more details, we refer to [26, 28].

Definition 2.1. Let $r \in]0, +\infty]$. The closed set $C \subset H$ is called r-prox-regular iff each point in the r-enlargement of C defined by

$$U_r(C) := \{w \in H : d(w, C) < r\},$$

has a unique nearest point in C and the mapping $\mathrm{proj}(C, \cdot)$ is continuous in $U_r(C)$.

Definition 2.2. Let $A : H \to 2^H$ be a set-valued mapping. It is said to be *hypomonotone* if there exists a constant $k > 0$ such that for all $x, y \in H$ and $x' \in A(x), y' \in A(y)$, we have

$$\langle x - y, x' - y' \rangle \geq -k\|x - y\|^2.$$

If $k = 0$, then A is called *monotone.*

Proposition 2.3 ([27]). *Let C be a closed set in H and $r \in]0,+\infty]$. The followings are equivalent:*
(i) *C is r-prox-regular.*
(ii) *For all $s \in C$ and $v \in \mathrm{N}^L(C,s)$, we have*

$$\langle v, s'-s\rangle \le \frac{\|v\|}{2r}\|s'-s\|^2 \ \ \forall\, s' \in C.$$

(iii) *(Hypomonotonicity) For all $s,s' \in C$, $v \in \mathrm{N}^L(C,s)\cap \mathbb{B}$, $v' \in \mathrm{N}^L(C,s')\cap\mathbb{B}$, we have*

$$\langle v-v', s-s'\rangle \ge -\frac{1}{r}\|s-s'\|^2.$$

If the closed set C is r-prox-regular with $r=+\infty$, then C is convex [15]. Some familiar examples of prox-regular sets [4]:

(1) The finite union of disjoint intervals is nonconvex but r-prox-regular where r depends on the distances between the intervals.
(2) More generally, any finite union of disjoint convex subsets in H is nonconvex but r-prox-regular where r depends on the distances between the sets.

Definition 2.4. Let $\varphi : H \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous function and $x \in H$ such that $\varphi(x)$ is finite. One say that ξ is a proximal (Fréchet, limiting, Clarke, resp.) subgradient of φ at x, written by $\xi \in \partial_P\varphi(x)$ ($\xi\in\partial_F\varphi(x)$, $\xi\in\partial_L\varphi(x)$, $\xi\in\partial_C\varphi(x)$, resp.) iff $(\xi,-1)\in \mathrm{N}^P_{\mathrm{epi}\varphi}(x,\varphi(x))$ ($(\xi,-1)\in N^F_{\mathrm{epi}\varphi}(x,\varphi(x))$,$(\xi,-1)\in\mathrm{N}^L_{\mathrm{epi}\varphi}(x,\varphi(x))$,$(\xi,-1)\in\mathrm{N}^C_{\mathrm{epi}\varphi}(x,\varphi(x))$, resp.).

Proposition 2.5 ([4]). *Let C be a nonempty closed and r-prox-regular subset in H. Then*
(i) $\partial_P d_C(x) = N^P_C(x)\cap\mathbb{B},\ x\in C$.
(ii) If $d_C(x) < r$, then $\partial_P d_C(x) = \partial_C d_C(x)$ is a closed and convex set. In this case, one can write for simplicity $\partial d_C(x)$.

Proposition 2.6 ([8]). *Let C be a nonempty closed and r-prox-regular subset in H. Then for all $x\in C$ and for all $\xi \in \partial d_C(x)$, one has*

$$\langle \xi, y-x\rangle \le \frac{2}{r}\|y-x\|^2 + d_C(y),$$

for all $y\in U_r(C)$ (the r-enlargement of C).

Proposition 2.7 ([4, 8]). *Let $0<r'<r$ and the Hausdorff-continuous set-valued mapping $C:\Omega\to 2^H$ with uniformly r-prox-regular values, where $\Omega\subset[0,T]\times H$. Then the set-valued mapping $(z,x)\mapsto\partial d_{C(z)}(x)$ is upper semicontinuous from $\{(z,x)\in\Omega\times H : x\in C(z)+(r-r')\mathbb{B}\}$ to H endowed with the weak topology, which is equivalent to the upper semicontinuity of the function $(z,x)\mapsto\sigma(\partial d_{C(z)}(x);\xi)$ on $\{(z,x)\in\Omega\times H : x\in C(z)+(r-r')\mathbb{B}\}$, for any $\xi\in H$, where*

$$\sigma(\partial d_{C(z)}(x);\xi) = \sup_{\eta\in\partial d_{C(z)}(x)}\langle \eta,\xi\rangle,$$

denotes the support function of $\partial d_{C(z)}(x)$ at ξ.

Let us recall a continuous and a discrete versions of Gronwall's inequality (see, e.g., Lemma 4.1 in [29]).

Lemma 2.8. *Let $T > 0$ be given and $a(\cdot), b(\cdot) \in L^1([0,T];\mathbb{R})$ with $b(t) \geq 0$ for almost all $t \in [0,T]$. Let the absolutely continuous function $w : [0,T] \mapsto \mathbb{R}_+$ satisfy*

$$(1-\alpha)w'(t) \leq a(t)w(t) + b(t)w^{\alpha}(t), \ \ a.e.\ t \in [0,T], \tag{2.1}$$

where $0 \leq \alpha < 1$. Then for all $t \in [0,T]$:

$$w^{1-\alpha}(t) \leq w^{1-\alpha}(0)\exp\Big(\int_0^t a(\tau)d\tau\Big) + \int_0^t \exp\Big(\int_s^t a(\tau)d\tau\Big)b(s)ds. \tag{2.2}$$

Lemma 2.9. *Let $\alpha > 0$ and (u_n), (β_n) be nonnegative sequences satisfying*

$$u_n \leq \alpha + \sum_{k=0}^{n-1} \beta_k u_k, \ \ \forall n = 0,1,2,\dots \ \ (\text{with } \beta_{-1} = 0). \tag{2.3}$$

Then for all n, we have

$$u_n \leq \alpha \exp\Big(\sum_{k=0}^{n-1} \beta_k\Big).$$

Finally, we recall the definition of the Kuratowski measure of non-compactness and some of its properties (see, e.g., Proposition 9.1 in [16]).

Definition 2.10. The Kuratowski measure of non-compactness of a bounded set B in H is defined by

$$\gamma(B) := \inf\Big\{r > 0 : B = \bigcup_{i=1}^{n} B_i \ \text{ for some } n \text{ and } B_i \text{ with } \operatorname{diam}(B_i) \leq r\Big\}.$$

We collect some properties of the Kuratowski measure in the following lemma.

Lemma 2.11. [16] *Let B_1 and B_2 be two bounded sets of an infinite dimensional Hilbert space H. Then*

(1) $\gamma(B_1) = 0 \Leftrightarrow B_1$ *is relatively compact.*
(2) *If $B_1 \subset B_2$, then $\gamma(B_1) \leq \gamma(B_2)$.*
(3) $\gamma(B_1 + B_2) \leq \gamma(B_1) + \gamma(B_2)$.
(4) $\gamma(x_0 + r\mathbb{B}) = 2r$ *for some $x_0 \in H$ and $r > 0$.*

3. Main results

In this section, we use a discretization technique by giving an implicit scheme to analyze the existence property of the sweeping process $(\mathcal{S})$. First let us assume the following assumptions:

Assumption 1. (i) For all $t \in [0,T]$ and $x \in H$, $C(t,x)$ is a nonempty closed, r-prox-regular subset of H and there exists $L_C > 0$ such that

$$e\big(C(t,x) \cap M_1\mathbb{B}; C(s,y)\big) \leq L_C(|t-s| + \|x-y\|), \quad \forall s,t \in [0,T] \text{ and } x,y \in M_1\mathbb{B}. \tag{3.1}$$

(ii) For all $t \in [0,T]$, $C(t, M_1\mathbb{B}) \cap 2M_1\mathbb{B}$ is relatively compact in H, or equivalently

$$\gamma(C(t, M_1\mathbb{B}) \cap 2M_1\mathbb{B}) = 0,$$

where γ is the Kuratowski measure of non-compactness and

$$M_1 := \|u_0\| + \|v_0\| + (L_C + 2L_F)T + e^{(L_C+2L_F+1)T}. \tag{3.2}$$

Assumption 2. The set-valued function $\mathcal{F} : \mathrm{gph}(C) \to 2^H$ is upper semicontinuous with nonempty convex, weakly compact values in H and satisfies the weak linear growth condition, *i.e.* there exists $L_{\mathcal{F}} > 0$ such that for all $t \in [0, T]$, $x \in H$ and $y \in C(t, x)$, we have

$$\mathcal{F}(t, x, y) \cap L_{\mathcal{F}}\big(1 + \|x\| + \|y\|\big)\mathbb{B} \neq \emptyset. \tag{3.3}$$

We are now in position to state and prove the main result of the paper.

Theorem 3.1. *Let H be a real Hilbert space and let Assumptions 1, 2 hold. Then for each initial condition $u_0 \in H$ and $v_0 \in C(0, u_0)$, there exists a solution $u : [0, T] \to H$ of the sweeping process $(\mathcal{S})$ in the following sense:*

(i) *$(\mathcal{S})$ is satisfied for a.e. $t \in [0, T]$;*
(ii) *$u(0) = u_0, \dot{u}(0) = v_0$;*
(iii) *$u \in \mathcal{C}^1([0, T]; H)$ and $\ddot{u} \in L^\infty([0, T]; H)$.*

Proof. By setting $\alpha := \|u_0\| + \|v_0\| + (L_C + 2L_{\mathcal{F}})T$ and $\beta := L_C + 2L_{\mathcal{F}} + 1$, the constant M_1 defined in (3.2) can be rewritten as $M_1 = \alpha + e^{\beta T}$. We choose some positive integer n_0 such that

$$n_0 > \frac{T}{r}(L_C + L_{\mathcal{F}})(1 + M_1). \tag{3.4}$$

Let be given some positive integer $n \geq n_0$, define $h_n := T/n$ and $t_i^n := ih$ for $0 \leq i \leq n$. For $0 \leq i \leq n-1$, given u_i^n and v_i^n, our aim is to find u_{i+1}^n, v_{i+1}^n such that

$$\begin{cases} \frac{v_{i+1}^n - v_i^n}{h_n} + f_i^n \in -\mathrm{N}_{C(t_{i+1}^n, u_{i+1}^n)}(v_{i+1}^n), \\ \\ u_{i+1}^n = u_i^n + h_n v_i^n, \end{cases} \tag{3.5}$$

where $f_i^n \in \mathcal{F}(t_i^n, u_i^n, v_i^n)$. Clearly we can compute u_{i+1}^n in terms of u_i^n and v_i^n. The inclusion in (3.5) can be rewritten as

$$v_i^n - h_n f_i^n \in v_{i+1}^n + \mathrm{N}_{C(t_{i+1}^n, u_{i+1}^n)}(v_{i+1}^n), \tag{3.6}$$

which is equivalent to

$$v_{i+1}^n = \mathrm{proj}\big(C(t_{i+1}^n, u_{i+1}^n); v_i^n - h_n f_i^n\big),$$

provided $d\big(C(t_{i+1}^n, u_{i+1}^n); v_i^n - h_n f_i^n\big) < r$. We have the following algorithm to construct the sequences $(u_i^n)_{i=0}^n$, $(v_i^n)_{i=0}^n$ and $(f_i^n)_{i=0}^n$.

Algorithm

- $u_0^n = u_0, v_0^n = v_0 \in C(0, u_0), f_0^n = f_0 \in \mathcal{F}(0, u_0, v_0) \cap L_{\mathcal{F}}(1 + \|u_0\| + \|v_0\|)\mathbb{B}$.

For $0 \leq i \leq n-1$:
- $u_{i+1}^n = u_i^n + h_n v_i^n$ and

$$v_{i+1}^n = \mathrm{proj}\big(C(t_{i+1}^n, u_{i+1}^n); v_i^n - h_n f_i^n\big). \tag{3.7}$$

- Choose $f_{i+1}^n \in \mathcal{F}(t_{i+1}^n, u_{i+1}^n, v_{i+1}^n) \cap L_{\mathcal{F}}(1 + \|u_{i+1}^n\| + \|v_{i+1}^n\|)\mathbb{B}$.

We will show by induction that for all $i = 0, 1, \ldots, n-1$ one has

$$\|u_i^n\| + \|v_i^n\| \le M_1 \text{ and } d\big(C(t_{i+1}^n, u_{i+1}^n); v_i^n - h_n f_i^n\big) < r, \tag{3.8}$$

and then the algorithm is well-defined. Indeed when $i = 0$, we have $\|u_0^n\| + \|v_0^n\| \le M_1$ and

$$\begin{aligned} d\big(C(t_1^n, u_1^n); v_0 - h_n f_0\big) &\le e\big(C(0, u_0) \cap M_1\mathbb{B}; C(t_1^n, u_1^n)\big) + h_n\|f_0\| \\ &\le h_n L_C(1 + \|v_0\|) + h_n L_{\mathcal{F}}(1 + \|u_0\| + \|v_0\|) \\ &\le h_n(L_C + L_{\mathcal{F}})(1 + M_1) < r, \end{aligned}$$

thanks to (3.4). Suppose that (3.8) holds for $i = 0, \ldots, k-1$ for some positive $k \le n$. One has

$$v_{i+1}^n = \operatorname{proj}\big(C(t_{i+1}^n, u_{i+1}^n); v_i^n - h_n f_i^n\big),$$

for all $i = 0, \ldots, k-1$. Hence

$$\begin{aligned} \|v_{i+1}^n - v_i^n + h_n f_i^n\| &= d\big(C(t_{i+1}^n, u_{i+1}^n); v_i^n - h_n f_i^n\big) \\ &\le e\big(C(t_i^n, u_i^n) \cap M_1\mathbb{B}; C(t_{i+1}^n, u_{i+1}^n)\big) + h_n\|f_i^n\| \\ &\le h_n L_C(1 + \|v_i^n\|) + h_n L_{\mathcal{F}}(1 + \|u_i^n\| + \|v_i^n\|), \end{aligned} \tag{3.9}$$

which implies that

$$\|v_{i+1}^n\| \le \|v_i^n\| + h_n\big(L_C + 2L_{\mathcal{F}} + 2L_{\mathcal{F}}\|u_i^n\| + (L_C + 2L_{\mathcal{F}})\|v_i^n\|\big).$$

Consequently

$$\|v_{i+1}^n\| \le \|v_0\| + (i+1)h_n(L_C + 2L_{\mathcal{F}}) + h_n\big(2L_{\mathcal{F}}\sum_{j=0}^{i}\|u_j^n\| + (L_C + 2L_{\mathcal{F}})\sum_{j=0}^{i}\|v_j^n\|\big). \tag{3.10}$$

On the other hand

$$\|u_{i+1}^n\| \le \|u_i^n\| + h_n\|v_i^n\| \le \ldots \le \|u_0\| + h_n\sum_{j=0}^{i}\|v_j^n\|. \tag{3.11}$$

From (3.10) and (3.11), one has

$$\|u_{i+1}^n\| + \|v_{i+1}^n\| \le \alpha + \beta h_n\sum_{j=0}^{i}(\|u_j^n\| + \|v_j^n\|), \quad i = 0, \ldots, k-1,$$

where $\alpha = \|u_0\| + \|v_0\| + (L_C + 2L_{\mathcal{F}})T$ and $\beta = L_C + 2L_{\mathcal{F}} + 1$. Using Lemma 2.9, we obtain

$$\|u_i^n\| + \|v_i^n\| \le \alpha + e^{\beta i h_n} \le \alpha + e^{\beta T} = M_1, i = 0, \ldots, k. \tag{3.12}$$

Then

$$\begin{aligned} d\big(C(t_{k+1}^n, u_{k+1}^n); v_k^n - h_n f_k^n\big) &\le e\big(C(t_k^n, u_k^n) \cap M_1\mathbb{B}; C(t_{k+1}^n, u_{k+1}^n)\big) + h_n\|f_i^n\| \\ &\le h_n L_C(1 + \|v_k^n\|) + h_n L_{\mathcal{F}}(1 + \|u_k^n\| + \|v_k^n\|) \\ &\le h_n(L_C + L_{\mathcal{F}})(1 + M_1) < r, \end{aligned}$$

due to (3.4). As a consequence, (3.8) holds for $i = k$ and thus the algorithm is well-defined. Furthermore, from the arguments above, the sequences $(u_i^n)_{i=0}^n, (v_i^n)_{i=0}^n$

constructed by this algorithm are bounded by M_1 and the sequence $(f_i^n)_{i=0}^n$ is bounded by $L_{\mathcal{F}}(1+M_1)$ since $\mathcal{F}$ satisfies the weak linear-growth condition. In addition, the sequence $(\frac{v_{i+1}^n - v_i^n}{h_n})_{i=0}^n$ is also bounded by $M_2 := (L_C + 2L_{\mathcal{F}})(1+M_1)$ thanks to (3.9) and (3.12).

We construct the sequences of functions $(u_n(\cdot))_n, (v_n(\cdot))_n, (f_n(\cdot))_n$ from $[0,T]$ to H as follows: $u_n(0) = u_0, v_n(0) = v_0$ and on $I_{n,i} := [t_i^n, t_{i+1}^n)$ for $0 \le i \le n-1$, we set

$$u_n(t) = u_i^n + \frac{u_{i+1}^n - u_i^n}{h_n}(t - t_i^n) \ , v_n(t) = v_i^n + \frac{v_{i+1}^n - v_i^n}{h_n}(t - t_i^n), \ \ f_n(t) = f_i^n,$$

and

$$\theta_n(t) = t_i^n \ \text{ and } \ \eta_n(t) = t_{i+1}^n. \tag{3.13}$$

Then for all $t \in (t_i^n, t_{i+1}^n)$

$$\dot{u}_n(t) = \frac{u_{i+1}^n - u_i^n}{h_n} = v_i^n \in C(t_i^n, u_i^n), \quad \dot{v}_n(t) = \frac{v_{i+1}^n - v_i^n}{h_n},$$

and

$$\max\Big\{ \sup_{t\in[0,T]} |\theta_n(t) - t|, \sup_{t\in[0,T]} |\eta_n(t) - t|\Big\} \le h_n \to 0 \text{ as } \ n \to +\infty. \tag{3.14}$$

Consequently, the sequence $(v_n(\cdot))_n$ is equi-Lipschitz with constant M_2. We will prove that the set $\Omega(t) = \{v_n(t), n \ge n_0\}$ is relatively compact for all $t \in [0,T]$. Indeed, suppose that there exists $t_0 \in [0,T]$ such that $\Omega(t_0)$ is not relatively compact. Then let $3\sigma := \gamma(\Omega(t_0)) > 0$. Note that $\Omega(t_0) \subset M_1\mathbb{B}$, hence $3\sigma = \gamma(\Omega(t_0)) \le \gamma(M_1\mathbb{B}) = 2M_1$, which implies that $\sigma \le M_1$. For each n, we can find i such that $t_0 \in [t_i^n, t_{i+1}^n)$. Then

$$\|u_n(t_0) - u_i^n\| = \|\frac{u_{i+1}^n - u_i^n}{h_n}\|\|(t_0 - t_i^n)\| \le M_1 h_n, \tag{3.15}$$

$$\|v_n(t_0) - v_i^n\| = \|\frac{v_{i+1}^n - v_i^n}{h_n}\|\|(t_0 - t_i^n)\| \le M_2 h_n. \tag{3.16}$$

On the other hand, because of Assumption 1 (i), one has

$$\begin{aligned} v_i^n \in C(t_i^n, u_i^n) \cap M_1\mathbb{B} \quad &\subset \quad C(t_0, u_n(t_0)) + L_C(M_1+1)h_n\mathbb{B} \\ &\subset \quad C(t_0, M_1\mathbb{B}) + L_C(M_1+1)h_n\mathbb{B}. \end{aligned}$$

Thus

$$v_n(t_0) \in C(t_0, M_1\mathbb{B}) + (L_C M_1 + L_C + M_2)h_n\mathbb{B}.$$

We can find $n_1 \ge n_0$ large enough such that for all $n \ge n_1$,

$$(L_C M_1 + L_C + M_2)h_n = (L_C M_1 + L_C + M_2)T/n \le \sigma.$$

Hence for all $n \ge n_1$, we have

$$v_n(t_0) \in (C(t_0, M_1\mathbb{B}) \cap (M_1 + \sigma)\mathbb{B}) + \sigma\mathbb{B} \subset (C(t_0, M_1\mathbb{B}) \cap 2M_1\mathbb{B}) + \sigma\mathbb{B}.$$

Note that the set $C(t_0, M_1\mathbb{B}) \cap 2M_1\mathbb{B}$ is relatively compact (Assumption 1) hence $\gamma(C(t_0, M_1\mathbb{B}) \cap 2M_1\mathbb{B}) = 0$. Then by using Lemma 2.11, one has

$$\begin{aligned} 3\sigma &= \gamma(\Omega(t_0)) = \gamma(\{v_n(t_0) : n \geq n_1\}) \leq \gamma((C(t_0, M_1\mathbb{B}) \cap 2M_1\mathbb{B}) + \sigma\mathbb{B}) \\ &\leq \gamma((C(t_0, M_1\mathbb{B}) \cap 2M_1\mathbb{B})) + \gamma(\sigma\mathbb{B}) = 2\sigma, \end{aligned}$$

which is a contradiction. Hence the set $\Omega(t) = \{v_n(t), n \geq 2\}$ is relatively compact for all $t \in [0,T]$. By applying the Arzela–Ascoli theorem, there exists a Lipschitz function $v(\cdot) : [0,T] \mapsto H$ with ratio M_2 and

- (v_n) converges strongly to $v(\cdot)$ in $\mathcal{C}([0,T]; H)$;
- $(\dot{v}_n)$ converges weakly to $\dot{v}(\cdot)$ in $L^1([0,T]; H)$.

In particular $v(0) = v_0$. Let $u : [0,T] \to H, t \mapsto u(t) = u_0 + \int_0^t v(s)ds$. Then $u(0) = u_0$, $\dot{u} = v$ and $\ddot{u} \in L^\infty([0,T]; H)$. Let us show that $u_n(\cdot)$ converges strongly in $\mathcal{C}([0,T]; H)$ to $u(\cdot)$. Indeed, we have

$$\begin{aligned} \max_{t\in[0,T]} \|u_n(t) - u(t)\| &= \max_{t\in[0,T]} \Big\|u_n(0) + \int_0^t v_n(\theta_n(s))ds - u(0) - \int_0^t v(s)ds\Big\| \\ &= \max_{t\in[0,T]} \Big\| \int_0^t (v_n(\theta_n(s)) - v_n(s) + v_n(s) - v(s))ds\Big\| \\ &\leq \max_{t\in[0,T]} \int_0^t (M_2|\theta_n(s) - s| + \|v_n(s) - v(s)\|ds \\ &\leq \int_0^T (M_2|\theta_n(s) - s| + \|v_n(s) - v(s)\|)ds \to 0, \end{aligned}$$

as $n \to +\infty$ which is due to (3.14) and the strong convergence of $v_n(\cdot)$ to $v(\cdot)$ in $\mathcal{C}([0,T]; H)$.

In the next step, we prove that $\dot{u}(t) = v(t) \in C(t, u(t))$ for every $t \in [0,T]$. From the fact that $v_i^n \in C(t_i^n, u_i^n)$ for all i and by using Assumption 1 (i), we deduce that for all $t \in [0,T]$

$$\begin{aligned} v_n(\theta_n(t)) &\in C(\theta_n(t), u_n(\theta_n(t))) \cap M_1\mathbb{B} \\ &\subset C(t, u(t)) + L_C\big(|\theta_n(t) - t| + \|u_n(\theta_n(t)) - u(t)\|\big)\mathbb{B}. \end{aligned}$$

It is easy to see that for every $t \in [0,T]$, $v_n(\theta_n(t)) \to v(t) = \dot{u}(t)$ and

$$|\theta_n(t) - t| + \|u_n(\theta_n(t)) - u(t)\| \to 0 \text{ as } n \to +\infty$$

due to (3.14) and the strong convergence of $v_n(\cdot)$ to $v(t)$, $u_n(\cdot)$ to $u(\cdot)$ in $\mathcal{C}([0,T]; H)$. Since $C(t, u(t))$ is closed, we obtain that $\dot{u}(t) \in C(t, u(t))$ for every $t \in [0,T]$. It remains to prove that

$$\ddot{u}(t) \in -\mathrm{N}_{C(t,u(t))}(\dot{u}(t)) - \mathcal{F}(t, u(t), \dot{u}(t)), \quad a.e. \ t \in [0,T]. \tag{3.17}$$

From (3.5), we have for almost every $t \in [0,T]$ that

$$\dot{v}_n(t) + f_n(t) \in -\mathrm{N}_{C\left(\eta_n(t), u_n(\eta_n(t))\right)}\big(v_n(\eta_n(t))\big). \tag{3.18}$$

Let us recall that $\dot{v}_n(\cdot)$, $f_n(\cdot)$ are bounded on $[0,T]$ by M_2. Thus

$$\|\dot{v}_n(t) + f_n(t)\| \leq 2M_2. \tag{3.19}$$

Thus by Proposition 2.5, one has

$$-\dot{v}_n(t) - f_n(t) \in 2M_2 \partial d_{C\left(\eta_n(t), u_n(\eta_n(t))\right)}\big(v_n(\eta_n(t))\big) \ a.e. \ t \in [0, T]. \tag{3.20}$$

Since $\|f_n(t)\| \leq M_2$ for all $t \in [0,T]$, there exists a subsequence, without relabelling for simplicity, converging weakly to some mapping f in $L^1([0,T];H)$. So we have $\dot{v}_n + f_n$ converges weakly to $\ddot{u} + f$ in $L^1([0,T];H)$. Applying the Castaing's technique (see, e.g., [11]) and using the Mazur's lemma, one implies for almost every $t \in [0,T]$ that

$$-\ddot{u}(t) - f(t) \in \bigcap_n \overline{\mathrm{co}}\{-\dot{v}_k(t) - f_k(t) : k \geq n\}, \tag{3.21}$$

where $\overline{\mathrm{co}}$ denotes the closed convex hull. Fix some $t \in [0,T]$ such that (3.21) holds and $\xi \in H$. The inclusion (3.21) implies that

$$\langle \xi, -\ddot{u}(t) - f(t) \rangle \leq \inf_n \sup_{k \geq n} \langle \xi, -\dot{v}_k(t) - f_k(t) \rangle. \tag{3.22}$$

Combining with (3.20), we get

$$\begin{aligned} \langle \xi, -\ddot{u}(t) - f(t) \rangle &\leq 2M_2 \limsup_{n \to +\infty} \sigma(\partial d_{C(\eta_n(t), u_n(\eta_n(t)))}(v_n(\eta_n(t))); \xi) \\ &\leq 2M_2 \sigma(\partial d_{C(t,u(t))}(v(t)); \xi), \end{aligned} \tag{3.23}$$

where the last inequality holds due to the upper semicontinuity of the proximal subdifferential (Proposition 2.7) and the fact that $\eta_n(t) \to t, u_n(\eta_n(t)) \to u(t), v_n(\eta_n(t)) \to v(t)$ strongly. Note that the set $\partial d_{C(t,u(t))}(v(t))$ is closed convex and $\dot{u}(t) = v(t) \in C(t, u(t))$. Thus, we get

$$-\ddot{u}(t) - f(t) \in 2M_2 \partial d_{C(t,u(t))}(\dot{u}(t)) \subset N_{C(t,u(t))}(\dot{u}(t)), \quad a.e. \ t \in [0,T]. \tag{3.24}$$

On the other hand, one has $f_n(t) \in \mathcal{F}\big(\theta_n(t), u_n(\theta_n(t)), v_n(\theta_n(t))\big)$ and $\mathcal{F}$ is upper semicontinuous with nonempty convex, weakly-compact values. Using [10, Theorem V-14], we obtain that $f(t) \in \mathcal{F}(t, u(t), \dot{u}(t))$. Thus

$$\ddot{u}(t) \in -\mathrm{N}_{C(t,u(t))}(\dot{u}(t)) - \mathcal{F}(t, u(t), \dot{u}(t)), \quad a.e. \ \ t \in [0,T]. \tag{3.25}$$

The proof of Theorem 3.1 is thereby completed. □

Remark 3.2. Assumption 1-(ii) is satisfied particularly in finite-dimensional spaces. In infinite-dimensional spaces, for each $t \in [0,T]$, we only need to check the relative compactness for a fixed bounded set (see [1] for more comments). In addition, similarly as in [1], the relative compactness assumption can be omitted if for each $t \in [0,T]$, $-C(t,\cdot)$ and $\mathcal{F}$ are both hypomonotone with respect to the third variable on gph(C), i.e., there exists $k > 0$ such that for all $(t_i, x_i, y_i) \in$ gph(C) and $z_i \in \mathcal{F}(t_i, x_i, y_i)$ $(i = 1, 2)$, one has

$$\langle z_1 - z_2, y_1 - y_2 \rangle \geq -k(\|y_1 - y_2\|^2 + \|x_1 - x_2\|^2 + |t_1 - t_2|^2).$$

For simplicity w.l.o.g, we can assume that $-C(t,\cdot)$ is monotone for each $t \in [0,T]$ and F is monotone with respect to the third variable on gph(C) (i.e., $k = 0$). The following theorem can be considered as an extension of [1, 6, 8] in the prox-regular settings by using similar techniques.

Theorem 3.3. *Let the assumptions of Theorem* 3.1 *hold and Assumption* 1-*(ii) is replaced by the monotonicity of* $-C(t,\cdot)$ *for each* $t \in [0,T]$*. Furthermore, assume that* $\mathcal{F}$ *is monotone with respect to the third variable on* $\mathrm{gph}(C)$*. Then for each initial condition, there exists a solution in the sense of Theorem* 3.1.

Proof. We construct the sequences $(u_i^n)_{i=0}^n, (v_i^n)_{i=0}^n, (f_i^n)_{i=0}^n$ and the sequences of functions $(u_n(\cdot))_n$, $(v_n(\cdot))_n$, $(f_n(\cdot))_n, (\theta_n(\cdot))_n, (\eta_n(\cdot))_n$ as in Theorem 3.1. From the proof of Theorem 3.1, it is sufficient to prove the strong convergence of sequence $v_n(\cdot)$ in $\mathcal{C}([0,T];H)$. First we prove the convergence of $u_n(\cdot)$. For all positive integers $m, n \geq n_0$, let

$$\varphi_{m,n}(t) := \frac{1}{2}\|u_m(t) - u_n(t)\|^2.$$

Then $\varphi_{m,n}$ is differentiable almost every $t \in [0,T]$. Let $t \in [0,T)$ be at which $\varphi_{m,n}$ is differentiable. Then there exist i, j such that $t \in [t_i^m, t_{i+1}^m) \cap [t_j^n, t_{j+1}^n)$ and hence for almost all $t \in [0,T]$

$$\frac{d}{dt}\varphi_{m,n}(t) = \langle u_m(t) - u_n(t), \dot{u}_m(t) - \dot{u}_n(t)\rangle = \langle u_m(t) - u_n(t), v_i^m - v_j^n\rangle.$$

We have $v_i^m \in C(t_i^m, u_i^m) \subset C(t, u_i^m) + h_m L_C \mathbb{B}, v_j^n \in C(t_j^n, u_j^n) \subset C(t, u_j^n) + h_n L_C \mathbb{B}$. From the monotonicity of $-C(t,\cdot)$ and the boundedness of u_i^m, u_j^n by M_1, one has

$$\langle v_i^m - v_j^n, u_i^m - u_j^n\rangle \leq 2M_1(h_n + h_m)L_C.$$

Hence,

$$\begin{aligned}
\frac{d}{dt}\varphi_{m,n}(t) &= \langle u_m(t) - u_n(t), v_i^m - v_j^n\rangle \\
&\leq \langle u_m(t) - u_i^m, v_i^m - v_j^n\rangle + \langle u_i^m - u_j^n, v_i^m - v_j^n\rangle + \langle u_j^n - u_n(t), v_i^m - v_j^n\rangle \\
&\leq 2M_1^2 h_m + 2M_1(h_n + h_m)L_C + 2M_1^2 h_n \\
&\leq 2M_1(M_1 + L_C)(h_n + h_m),
\end{aligned}$$

due to the M_1-Lipschitz continuity of $u_m(\cdot), u_n(\cdot)$ and the boundedness by M_1 of $u_i^m, u_j^n, v_i^m, v_j^n$. Consequently,

$$\frac{1}{2}\|u_m(t) - u_n(t)\|^2 = \varphi_{m,n}(t) \leq 2M_1 T(M_1 + L_C)(h_n + h_m) \text{ for all } t \in [0,T], \tag{3.26}$$

which implies that $(u_n(\cdot))_n$ is a Cauchy sequence in $\mathcal{C}([0,T];H)$. Thus, there exists a M_1-Lipschitz function $u(\cdot)$ such that $u_n(\cdot)$ converges to $u(\cdot)$ uniformly and

$$\|u_n(t) - u(t)\| \leq 2\sqrt{M_1 T(M_1 + L_C)h_n}.$$

We choose positive integer $n^* \geq n_0$ such that

$$\big(2L_C + M_2 + 2L_C M_1 + 2\sqrt{M_1 T(M_1 + L_C)})\big)\sqrt{h_{n^*}} < r, \tag{3.27}$$

where $h_{n^*} = T/n^*$. Next, we show the uniform convergence of $(v_n(\cdot))_{n\geq n^*}$. First one has the following estimation

$$\begin{aligned}
& d_{C\left(\eta_n(t),u_n(\eta_n(t))\right)}(v_m(t)) \\
\leq\ & d_H\big(C\big(\eta_n(t),u_n(\eta_n(t))\big);C\big(\eta_m(t),u_m(\eta_m(t))\big)\big) + \|v_m(\eta_m(t)) - v_m(t)\| \\
\leq\ & L_C\big(h_m + h_n + \|u_n(\eta_n(t)) - u_m(\eta_m(t))\|\big) + M_2 h_m \\
\leq\ & (2L_C + M_2)h_{n^*} + L_C(\|u_n(\eta_n(t)) - u_n(\eta_m(t))\| + \|u_n(\eta_m(t)) - u_m(\eta_m(t))\|) \\
\leq\ & (2L_C + M_2)h_{n^*} + 2L_C M_1 h_{n^*} + 2\sqrt{M_1 T(M_1 + L_C)h_{n^*}} \\
\leq\ & \big(2L_C + M_2 + 2L_C M_1 + 2\sqrt{M_1 T(M_1 + L_C)}\big)\big)\sqrt{h_{n^*}} < r,
\end{aligned}$$

due to the Lipschitz continuity of $C, u_n(\cdot), v_m(\cdot)$, (3.14) and (3.26). In particular, we infer that $d_{C\left(\eta_n(t),u_n(\eta_n(t))\right)}(v_m(t)) \to 0$ as $m,n \to +\infty$. Using Proposition 2.6 and the fact that $-\dot{v}_n(t) - f_n(t) \in 2M_2\partial d_{C(\eta_n(t),u_n(\eta_n(t)))}(v_n(\eta_n(t)))$ (see (3.20)), one has

$$\begin{aligned}
(3.28)\quad & \langle \dot{v}_n(t) + f_n(t), v_n(\eta_n(t)) - v_m(t)\rangle \\
& \qquad \leq \frac{4M_2}{r}\|v_n(\eta_n(t)) - v_m(t)\|^2 + 2M_2 d_{C\left(\eta_n(t),u_n(\eta_n(t))\right)}(v_m(t)).
\end{aligned}$$

We have

$$\begin{aligned}
& \langle \dot{v}_n(t), v_n(t) - v_m(t)\rangle \\
\leq\ & \langle \dot{v}_n(t), v_n(t) - v_n(\eta_n(t))\rangle - \langle f_n(t), v_n(\eta_n(t)) - v_m(t)\rangle \\
& + \frac{4M_2}{r}\|v_n(\eta_n(t)) - v_m(t)\|^2 + 2M_2 d_{C\left(\eta_n(t),u_n(\eta_n(t))\right)}(v_m(t)) \\
\leq\ & M_1 M_2 h_n - \langle f_n(t), v_n(t) - v_m(t)\rangle + M_2^2 h_n \\
+\ & \frac{8M_2}{r}\|v_n(t) - v_m(t)\|^2 + \frac{8M_2^3}{r}h_n^2 + 2M_2 d_{C\left(\eta_n(t),u_n(\eta_n(t))\right)}(v_m(t)) \\
=\ & \frac{8M_2}{r}\|v_n(t) - v_m(t)\|^2 - \langle f_n(t), v_n(t) - v_m(t)\rangle + \beta_{n,m}(t),
\end{aligned}$$

where

$$\beta_{n,m}(t) := M_1 M_2 h_n + M_2^2 h_n + \frac{8M_2^3}{r}h_n^2 + 2M_2 d_{C\left(\eta_n(t),u_n(\eta_n(t))\right)}(v_m(t)),$$

satisfying

$$\|\beta_{n,m}\|_\infty \to 0 \text{ as } m,n \to +\infty.$$

Similarly, one has

$$\begin{aligned}
(3.29)\quad & \langle \dot{v}_m(t), v_m(t) - v_n(t)\rangle \\
& \qquad \leq \frac{8M_2}{r}\|v_m(t) - v_n(t)\|^2 - \langle f_m(t), v_m(t) - v_n(t)\rangle + \beta_{m,n}(t).
\end{aligned}$$

As a consequence, we have for almost every $t \in [0,T]$ that

$$\begin{aligned} A &:= \langle \dot{v}_m(t) - \dot{v}_n(t), v_m(t) - v_n(t)\rangle \\ &\leq \frac{16M_2}{r}\|v_m(t) - v_n(t)\|^2 - \langle f_m(t) - f_n(t), v_m(t) - v_n(t)\rangle + \beta_{m,n}(t) + \beta_{n,m}(t) \\ &\leq \frac{16M_2}{r}\|v_m(t) - v_n(t)\|^2 - \langle f_m(t) - f_n(t), v_m(\theta_m(t)) - v_n(\theta_n(t))\rangle + \alpha_{m,n}(t) \\ &\leq \frac{16M_2}{r}\|v_m(t) - v_n(t)\|^2 + \alpha_{m,n}(t), \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} \alpha_{m,n}(t) &= \beta_{m,n}(t) + \beta_{n,m}(t) - \langle f_m(t) - f_n(t), v_m(t) - v_m(\theta_m(t))\rangle \\ &- \langle f_m(t) - f_n(t), v_n(\theta_n(t)) - v_n(t)\rangle. \end{aligned}$$

The last inequality holds since

$$f_m(t) \in \mathcal{F}\big(\theta_m(t), u_m(\theta_m(t)), v_m(\theta_m(t))\big), f_n(t) \in \mathcal{F}\big(\theta_n(t), u_n(\theta_n(t)), v_n(\theta_n(t))\big),$$

and $\mathcal{F}$ is monotone with respect to the third variable. Note that

$$\begin{aligned} &\|\langle f_m(t) - f_n(t), v_m(t) - v_m(\theta_m(t))\rangle + \langle f_m(t) - f_n(t), v_n(\theta_n(t)) - v_n(t)\rangle\| \\ &\leq 2M_2^2(h_m + h_n). \end{aligned}$$

Hence

$$\|\alpha_{n,m}\|_\infty \to 0 \text{ as } m, n \to +\infty.$$

From (3.30), one has

$$\frac{d}{dt}\|v_m(t) - v_n(t)\|^2 = 2A \leq \frac{32M_2}{r}\|v_m(t) - v_n(t)\|^2 + 2\alpha_{m,n}(t).$$

Using Gronwall's inequality and the fact that $v_m(0) = v_n(0) = v_0$, one obtains for all $t \in [0,T]$ that

$$\|v_m(t) - v_n(t)\|^2 \leq 2\int_0^t e^{32M_2(t-s)/r}\alpha_{m,n}(s)ds \leq 2Te^{32M_3T/r}\|\alpha_{n,m}\|_\infty.$$

Consequently, $(v_n(\cdot))_n$ is a Cauchy sequence in $C([0,T];H)$ which leads to the uniform convergence of $(v_n(\cdot))_n$. The proof of Theorem 3.3 is thereby completed. □

4. Conclusion

In this paper, by using tools from set-valued and variational analysis and by using an implicit discrete scheme, the existence of solutions of a class of unbounded nonconvex second-order sweeping processes under perturbation in Hilbert space has been thoroughly studied. Many concrete problems are formulated outside the hilbertian setting (for example l^p, L^p or $W^{m,p}$, $1 < p < +\infty$). It would be interesting to extend these results to the more general setting of Banach spaces. This is out of the scope of the current work and will be the subject of a future research project.

References

[1] S. Adly and B. K. Le, *Unbounded second order state dependent Moreau's sweeping processes in Hilbert spaces*, J. Optim. Theory Appl. **169** (2016), 407–423.

[2] S. Adly and F. Nacry, *Well-posedness of discontinuous second-order nonconvex state-dependent sweeping processes*, to appear in Applied Mathematics and Optimization.

[3] S. Adly, F. Nacry and L. Thibault, *Discontinuous sweeping process with prox-regular sets*, ESAIM Control. Optim. Calc. Var. **23** (2017), 1293–1329.

[4] M. Bounkhel and L. Thibault, *Nonconvex sweeping process and prox regularity in Hilbert space*, J. Nonlinear Convex Anal. **6** (2005), 359–374.

[5] S. Adly, T. Haddad and L. Thibault, *Convex sweeping process in the framework of measure differential inclusions and evolution variational inequalities*, Math. Program. Ser. B **148** (2014), 5–47.

[6] M. Bounkhel, *General existence results for second order nonconvex sweeping process with unbounded perturbations*, Port. Math. (N.S.) **60** (2003), 269–304.

[7] M. Bounkhel and R. Al-yusof, *First and second order convex sweeping processes in reflexive smooth Banach spaces*, Set-Valued Var. Anal. **18** (2010), 151–182.

[8] M. Bounkhel and D. L. Azzam, *Existence results on the second order nonconvex sweeping processes with perturbations*, Set-Valued Anal. **12** (2004), 291–318.

[9] C. Castaing, *Quelques problèmes d'évolution du second ordre,* Sém. d'Anal. Convexe, Montpellier, 5, 1988.

[10] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Springer, Berlin, 1977.

[11] C. Castaing, T. X. Duc Ha and M. Valadier, *Evolution equations governed by the sweeping process*, Set-Valued Analysis **1** (1993), 109–139.

[12] C. Castaing, A.G. Ibrahim and M. Yarou, *Some contributions to nonconvex sweeping process*, J. Nonlinear Convex Anal. **10** (2009), 1–20.

[13] B. Cornet, *Existence of slow solutions for a class of differential inclusions*, J. Math Anal. Appl. **96** (1983), 130–147.

[14] F. H. Clarke, Y. S. Ledyaev, R. J. Stern and P. R. Wolenski , *Nonsmooth Analysis and Control Theory*, vol. 178 of Graduate Texts in Mathematics. Springer, New York, 1998.

[15] R. Correa, A. Jofre and L. Thibault, *Subdifferential monotonicity as characterization of convexity*, Numer. Funct. Anal. Optim. **15** (1994), 531–536.

[16] K. Deimling, *Multivalued Differential Equations*. Walter de Gruyter, Berlin, 1992.

[17] C. Henry, *An existence theorem for a class of differential equations with multivalued righthand side*, J. Math. Anal. Appl. **41** (1973), 179–186.

[18] M. Kunze and M. D. P. Monteiro Marques, *An introduction to Moreau's sweeping process* , in: Impacts in Mechanical Systems. Analysis and Modelling, (B. Brogliato, Ed), Springer, Berlin, 2000, pp. 1–60.

[19] S. Lounis, T. Haddad and M. Sene, *Non-convex second-order Moreau's sweeping processes in Hilbert spaces*, to appear in Journal of Fixed Point Theory and Applications.

[20] B. Maury and J. Venel, *A mathematical framework for a crowd motion model*, C. R. Acad. Sci. Paris, Ser. I **346** (2008), 1245–1250.

[21] B. Maury, J. Venel, *Handling of Contacts in Crowd Motion Simulations*, Traffic and Granular Flow 07, Springer, 2009, pp. 171–180.

[22] J. J. Moreau, *Proximité et dualité dans un espace hilbertien*, Bull. Soc. Math. France **93** (1965), 273–299.

[23] J. J. Moreau, *Sur l'evolution d'un système élastoplastique*, C. R. Acad. Sci. Paris Sér. A-B, **273** (1971), A118–A121.

[24] J. J. Moreau, *Rafle par un convexe variable I*, Sém. Anal. Convexe Montpellier, Exposé 15, 1971.

[25] J. J. Moreau, *Rafle par un convexe variable II*, Sém. Anal. Convexe Montpellier, Exposé 3, 1972.

[26] B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation I*, Grundlehren Series Vol. 330, Springer, 2006.

[27] R. A. Poliquin, R.T. Rockafellar and L. Thibault, *Local Differentiability of distance functions*, Trans. Amer. Math. Soc. **352** (2000), 5231–5249.

[28] R. T. Rockafellar and R. B. Wets, *Variational Analysis*. Springer, Berlin, 1998.

[29] R. E. Showalter, *Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations*, Amer. Math. Society, Providence, RI, 1997.

Manuscript received October 21 2017
revised November 30 2017

S. Adly
Institut de recherche XLIM. UMR CNRS 7252. Université de Limoges, France
E-mail address: `samir.adly@unilim.fr`

B. K. Le
University of O'Higgins, Rancagua, Chile
E-mail address: `lebakhiet@gmail.com`