Pure and Applied Functional Analysis
Volume 3, Number 2, 2018, 297–308

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

CHARACTERIZATION OF DIFFERENCES OF SUBLINEAR FUNCTIONS

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Abstract. In this paper, we present necessary and sufficient conditions for a positively homogenous function defined on a plane to be a difference of sublinear (convex) functions. In the case of such a function we give a formula for producing two inclusion-minimal compact convex sets such that given function is equal to the difference of support functions of these sets. We also show several examples of application of our results.

1. Introduction

Differences of convex functions or dc-functions were studied by many authors (e.g. [6, 8, 13] and [18]) in particular for generalized differentiation. Directional derivatives of dc-functions are differences of sublinear functions or ds-functions. Due to generalized Minkowski duality these derivatives are represented by pairs of closed bounded convex sets (e.g. [7, 9] and [11]). The mentioned pairs of sets are called quasidifferentials and the convex sets are called respectively sub- and superdifferential. Quasidifferential calculus, an important part of nonsmooth analysis, was developed by many authors, especially Demyanov and Rubinov [1,2]. Positively homogenous or ph-function in $\mathbb{R}^n$ is a ds-function if and only if it is a dc-function. Moreover, a ph-function in $\mathbb{R}^n$ is a ds-function if and only if its restriction to any tangent hyperplane to the unit sphere S^{n-1} is a dc-function (see Theorem 1 in [14]).

A ds-function is uniformly Lipschitz. However a ph-function which is a Lipschitz function does not have to be a ds-function (see [3]). Gorokhovik at the 'Symposium on Functional Analysis and Optimization: Stefan Rolewicz in memoriam' in Warsaw September 2016 posed a question how to recognize a ds-function. A dc-function of one variable can be recognized with the help of total convexity (Theorem 3 in [12] states that a function on an interval of a line which is a function of total finite convexity is a dc-function). Recognizing a dc-function of two variables is extremely difficult. Zalgaller [17] proved that a dc-function of two variables defined on a compact convex set has a unique maximal representation as a difference of two convex functions. In Theorem 3.1, we characterize a ds-function of two variables as a ph-function whose restriction to the unit circle is a function with finite total convexity. Characterization of a ds-function of n variables is probably as difficult as a characterization of a dc-function of $n-1$ variables.

In Theorem 2.1, we present a method (an algorithm) of constructing two compact convex sets A and B corresponding to a given ds-function h, which are suband superdifferentials of this function i.e. $A = \underline{\partial} h|_0$, $B = \overline{\partial} h|_0$ and $h = h_A - h_B$. As we know a pair (A, B) is not unique. However, by [4] or [15] a pair (A, B) of inclusion-minimal sub- and superdifferentials in the plane is unique up to translation. Inclusion-minimal means that if $h = h_A - h_B = h'_A - h'_B$, $A' \subset A$ and $B' \subset B$ then $A = A'$ and $B = B'$. Our method produces minimal pair of sets (minimal quasidifferential). Because of Minkowski duality, minimal pair (A, B) of sub- and superdifferentials of a given ds-function h gives us a pointwise-minimal representation of h as a difference $h_1 - h_2$ of sublinear functions.

2. Minimal representation of ds-function as a difference of two support functions

In this section we present a specific construction for a given ds-function h satisfying certain assumptions of two convex sets A and B such that h is a difference of support functions $h_A - h_B$ of mentioned sets. The constructed pair of convex sets appears to be inclusion-minimal and as inclusion-minimal it is unique up to translation.

Let $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a positively homogenous (ph-) function. Let $\varphi : [0, 2\pi] \longrightarrow \mathbb{R}$ be defined by $\varphi(t) := h(e^{it})$. Here we write $e^{it} := \cos t + i \sin t$ instead of $(\cos t, \sin t)$ for the sake of brevity. In this paper we identify the plane $\mathbb{R}^2$ with the plane of complex numbers $\mathbb{C}$ whenever it is convenient. On the other hand $h(x) = \|x\|\varphi(\mathrm{Arg} x) = \|x\|\varphi(-i \log \frac{x}{\|x\|})$. Let us notice that continuity of φ is equivalent to the continuity of the ph-function h.

We consider right derivative $\varphi'(t) = \lim_{s\to 0^+} \frac{\varphi(t+s)-\varphi(t)}{s}$. We assume that $\varphi'(2\pi) = \varphi'(0)$ so that the domain of φ' is the interval $[0, 2\pi]$. The existence of all directional derivatives of h is equivalent to the existence of right and left derivatives of φ. Namely,

$$h'(x; v) = \lim_{t\to 0^+} \frac{h(x + tv) - h(x)}{t} = \frac{1}{\|x\|}(\langle v, x\rangle \varphi(\mathrm{Arg} x) + \langle v, ix\rangle \varphi'(\mathrm{Arg} x))$$

for $\langle v, ix\rangle > 0$, where $\langle v, x\rangle$ is the inner product of vectors $(v_1, v_2), (x_1, x_2)$ and $\langle v, ix\rangle$ is the inner product of vectors $(v_1, v_2), (-x_2, x_1)$. Our considerations are limited to the right derivative. Analogous results can be obtained for the left derivative.

Theorem 2.1. *Let $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a positively homogenous (ph-) function. Let $\varphi : [0, 2\pi] \longrightarrow \mathbb{R}$, $\varphi(t) := h(e^{it})$. If the function h is continuous and the right derivative φ' of φ exists and has bounded variation then h is a difference of sublinear functions, namely $h = h_A - h_B$, where h_A and h_B are support functions of compact convex sets A and B described as follows. Let*

$$f(t) := \int_0^t \varphi(s) ds + \varphi'(t) - \varphi'(0),$$

$$f^+(t) := \frac{1}{2}(V_0^t(f) + f(t)), f^-(t) := \frac{1}{2}(V_0^t(f) - f(t)),$$

where $V_0^t(f)$ is the variation of f on the interval $[0, t]$. Denote $b := \int_0^{2\pi} ie^{is} df^+(s)$. Let

$$g(t) := \begin{cases} 0 & 0 \leqslant t < \mathrm{Arg}(ib) \\ \|b\| & \mathrm{Arg}(ib) \leqslant t \leqslant 2\pi \end{cases},$$

$$F^+(t) := \int_0^t ie^{is}d(f^+ + g)(s), F^-(t) := \int_0^t ie^{is}d(f^- + g)(s)$$

and

$$A := \varphi(0) + i\varphi'(0) + \overline{\mathrm{conv}}F^+([0, 2\pi]), B := \overline{\mathrm{conv}}F^-([0, 2\pi]).$$

Moreover, (h_A, h_B) *is a minimal pair of sublinear functions such that* $h = h_A - h_B$*. If also* (h_1, h_2) *is any other minimal pair with* $h_1 - h_2 = h$ *then* $h_1 = h_A + l$*,* $h_2 = h_B + l$*, where* l *is a linear functional.*

Proof. Let positively homogenous function h satisfy the assumptions of the theorem. First, notice that for the function $\bar{f}(t) := \int_0^t \varphi(s)ds$ its variation is equal to $V_0^{2\pi}(\bar{f}) = \int_0^{2\pi} |\bar{f}\,'(s)|ds = \int_0^{2\pi} |\varphi(s)|ds$. Since φ is continuous, the last integral is finite. Therefore the function f is a function of bounded variation.

The functions $f^+, f^- : [0, 2\pi] \longrightarrow \mathbb{R}$ are nondecreasing, the function f^- is nonnegative and $f = f^+ - f^-$. Moreover, they are the smallest of such functions. Notice that $f^+(0) = (0)$ and $f^-(0) = 0$.

The point $b \in \mathbb{R}^2$ is defined by Stieltjes integral as

$$b := \int_0^{2\pi} ie^{is}df^+(s) = \Big(- \int_0^{2\pi} \sin s df^+(s), \int_0^{2\pi} \cos s df^+(s)\Big).$$

Since

$$\begin{aligned}
\int_0^t ie^{is}df(s) &= \int_0^t ie^{is}d(\int_0^s \varphi(u)) + \int_0^t ie^{is}d\varphi'(s) \\
&= \int_0^t ie^{is}\varphi(s)ds + \int_0^t ie^{is}d\varphi'(s) \\
&= \Big[\int_0^t ie^{is}\varphi(s)ds + \int_0^t e^{is}\varphi'(s)ds\Big] \\
&\quad + \Big[- \int_0^t e^{is}\varphi'(s)ds + \int_0^t ie^{is}d\varphi'(s)\Big] \\
&= \int_0^t (ie^{is}\varphi(s) + e^{is}\varphi'(s))ds \\
&\quad + \int_0^t (\varphi'(s)d(ie^{is}) + ie^{is}d\varphi'(s)) \\
&= e^{is}\varphi(s)|_{s=0}^t + ie^{is}\varphi'(s)|_{s=0}^t \\
&= e^{it}\varphi(t) + ie^{it}\varphi'(t) - \varphi(0) - i\varphi'(0),
\end{aligned}$$

we obtain

$$\int_0^{2\pi} ie^{is}df(s) = e^{2\pi i}\varphi(2\pi) + ie^{2\pi i}\varphi'(2\pi) - \varphi(0) - i\varphi'(0) = 0.$$

Hence the point $b \in \mathbb{R}^2$ also satisfies the equality $b = \int_0^{2\pi} ie^{is}df^-(s)$.

The functions $g : [0, 2\pi] \longrightarrow \mathbb{R}$, $F^+, F^- : [0, 2\pi] \longrightarrow \mathbb{R}$ are well defined. Let us notice, that if $b \neq 0$ then $F^+(2\pi) = \int_0^{2\pi} ie^{is}d(f^+ + g)(s) = \int_0^{2\pi} ie^{is}df^+(s) + \int_0^{2\pi} ie^{is}dg(s) = b + ie^{i\mathrm{Arg}(ib)}(g(2\pi) - g(0)) = b + i\|b\|\frac{ib}{\|ib\|} = b - b = 0$. If $b = 0$ then

$g = 0$, and $F^+(2\pi) = \int_0^{2\pi} ie^{is} d(f^+ + g)(s) = \int_0^{2\pi} ie^{is} df^+(s) = b = 0$. In a similar way $F^-(2\pi) = 0$.

Now, we calculate the value of the support function h_A at points from the unit circle $\{e^{it} | t \in [0, 2\pi]\}$. For a fixed $t \in [0, 2\pi]$ denote $I(t_1, t_2) := \int_{t_1}^{t_2} \sin(t - s) d(f^+ + g)(s)$, where $0 \leqslant t_1 \leqslant t_2 \leqslant 2\pi$. In fact $I(0, t_1) = \langle e^{it}, F^+(t_1)\rangle$. Notice that $I(0, 2\pi) = 0$. First, we prove that $I(0, t_1) \leqslant I(0, t)$.

(i) If $0 \leqslant t_1 \leqslant t - \pi$ then $I(0, t_1) = I(0, t_1) + I(0, 2\pi) = I(0, t) + (I(0, t_1) + I(t, 2\pi))$. Since $\sin(t-s) < 0$ for $s \in (0, t_1) \cup (t, 2\pi)$, we have $I(0, t_1) \leqslant I(0, t)$.
(ii) If $t - \pi \leqslant t_1 \leqslant t$ then $I(0, t_1) = I(0, t) - I(t_1, t)$, where $I(t_1, t) \geqslant 0$.
(iii) If $t \leqslant t_1 \leqslant t + \pi$ then $I(0, t_1) = I(0, t) + I(t, t_1)$, where $I(t, t_1) \leqslant 0$.
(iv) If $t + \pi \leqslant t_1 \leqslant 2\pi$ then $I(0, t_1) = I(0, 2\pi) - I(t_1, 2\pi) \leqslant 0 \leqslant I(0, t)$.

Hence we obtain

$$\sup_{t_1 \in [0,2\pi]} \langle e^{it}, F^+(t_1)\rangle = \sup_{t_1 \in [0,2\pi]} I(0, t_1) = I(0, t) = \langle e^{it}, F^+(t)\rangle.$$

Therefore,

$$\begin{aligned} h_A(e^{it}) &= \max_{a \in A} \langle e^{it}, a\rangle = \langle e^{it}, \varphi(0) + i\varphi'(0)\rangle + \sup_{t_1 \in [0,2\pi]} \langle e^{it}, F^+(t_1)\rangle \\ &= \langle e^{it}, \varphi(0) + i\varphi'(0)\rangle + \langle e^{it}, F^+(t)\rangle. \end{aligned}$$

In a similar way, $h_B(e^{it}) = \langle e^{it}, F^-(t)\rangle$.

Then

$$\begin{aligned} h_A(e^{it}) - h_B(e^{it}) &= \langle e^{it}, \varphi(0) + i\varphi'(0) + F^+(t)\rangle - \langle e^{it}, F^-(t)\rangle \\ &= \Big\langle e^{it}, \varphi(0) + i\varphi'(0) + \int_0^t ie^{is} d(f^+ + g)(s) - \int_0^t ie^{is} d(f^- + g)(s)\Big\rangle \\ &= \Big\langle e^{it}, \varphi(0) + i\varphi'(0) + \int_0^t ie^{is} df(s)\Big\rangle. \end{aligned}$$

Since

$$\int_0^t ie^{is} df(s) = e^{it}\varphi(t) + ie^{it}\varphi'(t) - \varphi(0) - i\varphi'(0),$$

we obtain

$$\begin{aligned} h_A(e^{it}) - h_B(e^{it}) &= \langle e^{it}, e^{it}\varphi(t) + ie^{it}\varphi'(t)\rangle = \langle e^{i0}, e^{i0}(\varphi(t) + i\varphi'(t))\rangle \\ &= \langle (1, 0), (\varphi(t), \varphi'(t))\rangle = \varphi(t) = h(e^{it}). \end{aligned}$$

The last assertion of the theorem follows from the fact that the pair of compact convex sets (A, B) is uniquely-up-to-translation minimal with respect to inclusion in the class of pairs (C, D) of compact convex sets such that $A + D = B + C$ (see Section 3 in [5]). $\square$

In section 4 we show how to use the construction from Theorem 2.1 of the pair of sets in finding a minimal sub- and superdifferential of specific ph-functions.

3. Necessary and sufficient conditions for a difference of two sublinear functions

The purpose of this section is to give sufficient and necessary conditions for a ph-function to be a ds-function. The idea is based on the notion of bounded convexity of functions of one variable [12].

Let $f : [a, b] \longrightarrow \mathbb{R}$. The quantity

$$K_a^b(f) := \sup_{P=\{a=t_0<t_1<\ldots<t_n=b\}} \sum_{k=1}^{n-1} \left| \frac{f(t_{k+1}) - f(t_k)}{t_{k+1} - t_k} - \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}} \right|$$

is called the *total convexity of the function* f *on* $[a, b]$. A function f with finite $K_a^b(f)$ is called a *function of finite total convexity on* $[a, b]$.

Theorem 3.1. *Let* $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be a positively homogenous (ph-) function. Let* $\varphi : [0, 2\pi] \longrightarrow \mathbb{R}$, $\varphi(t) := h(e^{it})$. *The following statements are equivalent*:

(a) *The function* h *is a* ds-*function.*

(b) *The function* h *is continuous and the right derivative* φ' *of* φ *exists and is a function of bounded variation.*

(c) *The function* φ *is a function of finite total convexity on* $[0, 2\pi]$.

(d) $\lim\limits_{n\to\infty} n \sum\limits_{k=1}^{n-1} \left|\varphi\left(\frac{2\pi(k+1)}{n}\right) - 2\varphi\left(\frac{2\pi k}{n}\right) + \varphi\left(\frac{2\pi(k-1)}{n}\right)\right| < \infty.$

Proof. (a)$\Leftrightarrow$(b). The implication (b)$\Rightarrow$(a) follows from Theorem 2.1. Let the function h be a difference of sublinear functions $h_1 - h_2$. In order to prove that h is continuous and φ' exists and is of bounded variation it is enough to prove that h_1, h_2 are continuous and that corresponding derivatives φ_1', φ_2' exist and are of bounded variation. Hence proving the condition (b) for sublinear function h is all we need to do.

Since the sublinear function h is convex, it is continuous. First, we are going to prove that φ' exists and is of bounded variation on any interval $[a, b]$ for $0 \leqslant a < b \leqslant 2\pi, b - a < \pi$. Denote $\exp(i[a, b]) := \{e^{it} | a \leqslant t \leqslant b\}$. The set $\exp(i[a, b])$ is a compact arc of the unit circle S^1 of the length less then π. Since the origin does not belong to the convex hull of $\exp(i[a, b])$, the arc can be separated by a streight line from the origin. In consequence, there exists a linear functional l in $\mathbb{R}^2$ such that $\max\limits_{\exp(i[a,b])} l < -\max\limits_{\exp(i[a,b])} h$. Hence the sublinear function $h + l$ takes only negative values on the open neighbourhood of the set $\exp(i[a - \varepsilon, b + \varepsilon])$ for some $0 < \varepsilon < \pi - b + a$. Denote $\varphi_l(t) := l(e^{it})$. Since for any $s, t \in [a - \varepsilon, b + \varepsilon]$ we have $e^{i\frac{s+t}{2}} = \frac{2}{\|e^{is}+e^{it}\|} \frac{e^{is}+e^{it}}{2}$, we obtain

$$\begin{aligned}
(\varphi + \varphi_l)\Big(\frac{s+t}{2}\Big) &= (h+l)(e^{i\frac{s+t}{2}}) = (h+l)\Big(\frac{2}{\|e^{is}+e^{it}\|}\frac{e^{is}+e^{it}}{2}\Big) \\
&= \frac{2}{\|e^{is}+e^{it}\|}(h+l)\Big(\frac{e^{is}+e^{it}}{2}\Big) \leqslant (h+l)\Big(\frac{e^{is}+e^{it}}{2}\Big) \\
&\leqslant \frac{(h+l)(e^{is}) + (h+l)(e^{it})}{2} = \frac{(\varphi+\varphi_l)(s) + (\varphi+\varphi_l)(t)}{2}.
\end{aligned}$$

These inequalities and the continuity of the function $\varphi + \varphi_l$ imply convexity of $\varphi + \varphi_l$ on the interval $[a - \varepsilon, b + \varepsilon]$. Then the convex function $\varphi + \varphi_l$ is directionaly differentiable and the right derivative $(\varphi + \varphi_l)'$ is bounded on the interval $[a, b]$ and nondecreasing, hence it is a function of bounded variation. Since φ_l is a linear combination of sine and cosine functions, φ_l' exists and it is of bounded variation. Therefore, φ' exists and is of bounded variation on the interval $[a, b]$. We have just proved that φ' exists and is of bounded variation on any interval shorter than π. This obviously implies the existence and bounded variation of φ' on all domain of φ.

(b)⇔(c). By Theorems 1 and 3 in [12], the condition (b) follows from $K_0^{2\pi}(\varphi) < \infty$. Moreover, $K_0^{2\pi}(\varphi)$ is equal to the total variation $V_0^{2\pi}(\varphi')$ of φ' on $[0, 2\pi]$. The implication (b)⇒(c) is obvious.

(c)⇔(d). It is easy to see that

$$K_0^{2\pi}(\varphi) = \frac{1}{2\pi} \lim_{n\to\infty} n \sum_{k=1}^{n-1} \left| \varphi\left(\frac{2\pi(k+1)}{n}\right) - 2\varphi\left(\frac{2\pi k}{n}\right) + \varphi\left(\frac{2\pi(k-1)}{n}\right) \right|.$$

□

In the next section we give examples where Theorem 3.1 helps us to decide whether or not a given ph-function is a ds-function.

4. Examples of application

Max-min functions were first ds-functions studied in quasidifferential calculus [1, 2]. Our examples are not of this type. Several examples are rational functions which are ds-functions by the fact that a product and a quotient of dc-functions are dc-functions [6] and since a ph-function which is a dc-function is also a ds-function (Section 8.1, p. 413 in [9]).

Example 4.1. Let

$$h(x, y) := \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$

Using notations from Theorem 2.1 we obtain $\varphi(t) = \frac{1}{2}\sin 2t$, $\varphi'(t) = \cos 2t$, $\int_0^t \varphi(s)ds = -\frac{1}{4}\cos 2t + \frac{1}{4}$ and $f(t) = \frac{3}{4}\cos 2t - \frac{3}{4}$. Since the function $f(t)$ is increasing in the intervals $[\frac{\pi}{2}, \pi]$, $[\frac{3\pi}{2}, 2\pi]$ and decreasing in the intervals $[0, \frac{\pi}{2}]$, $[\pi, \frac{3\pi}{2}]$, we obtain

$$f^+(t) = \begin{cases} 0 & t \in [0, \frac{\pi}{2}] \\ \frac{3}{4}\cos 2t + \frac{3}{4} & t \in (\frac{\pi}{2}, \pi] \\ 3/2 & t \in (\pi, \frac{3\pi}{2}] \\ \frac{3}{4}\cos 2t + \frac{9}{4} & t \in (\frac{3\pi}{2}, 2\pi] \end{cases}, \quad f^-(t) = \begin{cases} \frac{3}{4} - \frac{3}{4}\cos 2t & t \in [0, \frac{\pi}{2}] \\ 3/2 & t \in (\frac{\pi}{2}, \pi] \\ \frac{9}{4} - \frac{3}{4}\cos 2t & t \in (\pi, \frac{3\pi}{2}] \\ 3 & t \in (\frac{3\pi}{2}, 2\pi] \end{cases}.$$

We define and calculate the function $F(t) := \int_0^t ie^{is}df(s) = \int_0^t ie^{is}d(\frac{3}{4}\cos 2s - \frac{3}{4}) = (\frac{3}{4}\sin t - \frac{1}{4}\sin 3t, \frac{3}{4}\cos t + \frac{1}{4}\cos 3t - 1) = \frac{3}{4}ie^{-it} + \frac{1}{4}ie^{3it} - i$. Then we can calculate $b = \int_0^{2\pi} ie^{it}df^+(t) = F(2\pi) - F(\frac{3\pi}{2}) + F(\pi) - F(\frac{\pi}{2}) = 0$. Hence $F^+(t) =$

$\int_0^t ie^{is}df^+(s)$. Then $F^+(t) = \begin{cases} 0 & t \in [0, \frac{\pi}{2}] \\ F(t) - F(\frac{\pi}{2}) & t \in (\frac{\pi}{2}, \pi] \\ F(\pi) - F(\frac{\pi}{2}) & t \in (\pi, \frac{3\pi}{2}] \\ F(t) - F(\frac{3\pi}{2}) + F(\pi) - F(\frac{\pi}{2}) & t \in (\frac{3\pi}{2}, 2\pi] \end{cases}$. Thus

$$F^+(t) = \begin{cases} (0,0) & t \in [0, \frac{\pi}{2}] \\ (\frac{3}{4}\sin t - \frac{1}{4}\sin 3t - 1, \frac{3}{4}\cos t + \frac{1}{4}\cos 3t) & t \in (\frac{\pi}{2}, \pi] \\ (-1,-1) & t \in (\pi, \frac{3\pi}{2}] \\ (\frac{3}{4}\sin t - \frac{1}{4}\sin 3t, \frac{3}{4}\cos t + \frac{1}{4}\cos 3t - 1) & t \in (\frac{3\pi}{2}, 2\pi] \end{cases}.$$

In a similar way, $F^-(t) = \int_0^t ie^{is}df^-(s)$. Since $f^- = f^+ - f$, we obtain

$$F^-(t) = \begin{cases} -F(t) & t \in [0, \frac{\pi}{2}] \\ -F(\frac{\pi}{2}) & t \in (\frac{\pi}{2}, \pi] \\ -F(\frac{\pi}{2}) + F(\pi) - F(t) & t \in (\pi, \frac{3\pi}{2}] \\ -F(\frac{\pi}{2}) + F(\pi) - F(\frac{3\pi}{2}) & t \in (\frac{3\pi}{2}, 2\pi] \end{cases}, \text{ and}$$

$$F^-(t) = \begin{cases} (-\frac{3}{4}\sin t + \frac{1}{4}\sin 3t, -\frac{3}{4}\cos t - \frac{1}{4}\cos 3t + 1) & t \in [0, \frac{\pi}{2}] \\ (-1,1) & t \in (\frac{\pi}{2}, \pi] \\ (-\frac{3}{4}\sin t + \frac{1}{4}\sin 3t - 1, -\frac{3}{4}\cos t - \frac{1}{4}\cos 3t) & t \in (\pi, \frac{3\pi}{2}] \\ (0,0) & t \in (\frac{3\pi}{2}, 2\pi] \end{cases}.$$

The image the function $F(t)$ is an astroid (see Figure 4.1). The image of the function $F^+(t)$ translated by vector $(\varphi(0), \varphi'(0)) = (0,1)$ and of the function $F^-(t)$ produce boundaries, respectively, of the sets A and B (see Figure 4.1).

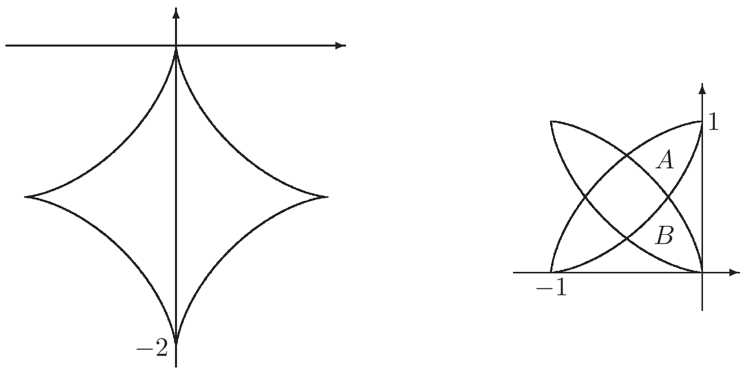

Figure 4.1: Image of the function F (astroid) from Example 4.1. Inclusion minimal pair of sets such that $h_A - h_B = h$ from Example 4.1.

Example 4.2. Let

$$h(x,y) := \begin{cases} \frac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

Then $\varphi(t) = \frac{1}{4}(\sin t + \sin 3t)$, $\varphi'(t) = \frac{1}{4}\cos t + \frac{3}{4}\cos 3t$, $\int_0^t \varphi(s)ds = -\frac{1}{4}\cos t - \frac{1}{12}\cos 3t + \frac{1}{3}$ and $f(t) = \frac{2}{3}\cos 3t - \frac{2}{3}$. Since the function $f(t)$ is increasing in the intervals $[\frac{\pi}{3}, \frac{2\pi}{3}]$, $[\pi, \frac{4\pi}{3}]$ $[\frac{5\pi}{3}, 2\pi]$ and decreasing in the intervals $[0, \frac{\pi}{3}]$, $[\frac{2\pi}{3}, \pi]$, $[\frac{4\pi}{3}, \frac{5\pi}{3}]$, we obtain

$$f^+(t) = \begin{cases} 0 & t \in [0, \frac{\pi}{3}] \\ f(t) - f(\frac{\pi}{3}) & t \in (\frac{\pi}{3}, \frac{2\pi}{3}] \\ f(\frac{2\pi}{3}) - f(\frac{\pi}{3}) & t \in (\frac{2\pi}{3}, \pi] \\ f(t) - f(\pi) + f(\frac{2\pi}{3}) - f(\frac{\pi}{3}) & t \in (\pi, \frac{4\pi}{3}] \\ f(\frac{4\pi}{3}) - f(\pi) + f(\frac{2\pi}{3}) - f(\frac{\pi}{3}) & t \in (\frac{4\pi}{3}, \frac{5\pi}{3}] \\ f(t) - f(\frac{5\pi}{3}) + f(\frac{4\pi}{3}) - f(\pi) + f(\frac{2\pi}{3}) - f(\frac{\pi}{3}) & t \in (\frac{5\pi}{3}, 2\pi] \end{cases},$$

$$f^-(t) = \begin{cases} -f(t) & t \in [0, \frac{\pi}{3}] \\ -f(\frac{\pi}{3}) & t \in (\frac{\pi}{3}, \frac{2\pi}{3}] \\ -f(t) + f(\frac{2\pi}{3}) - f(\frac{\pi}{3}) & t \in (\frac{2\pi}{3}, \pi] \\ -f(\pi) + f(\frac{2\pi}{3}) - f(\frac{\pi}{3}) & t \in (\pi, \frac{4\pi}{3}] \\ -f(t) + f(\frac{4\pi}{3}) - f(\pi) + f(\frac{2\pi}{3}) - f(\frac{\pi}{3}) & t \in (\frac{4\pi}{3}, \frac{5\pi}{3}] \\ -f(\frac{5\pi}{3}) + f(\frac{4\pi}{3}) - f(\pi) + f(\frac{2\pi}{3}) - f(\frac{\pi}{3}) & t \in (\frac{5\pi}{3}, 2\pi] \end{cases}.$$

Similarly like in Example 4.1, we have $b = (0, 0)$ and

$$F^+(t) = \begin{cases} (0,0) & t \in [0, \frac{\pi}{3}] \\ F(t) - F(\frac{\pi}{3}) & t \in (\frac{\pi}{3}, \frac{2\pi}{3}] \\ F(\frac{2\pi}{3}) - F(\frac{\pi}{3}) & t \in (\frac{2\pi}{3}, \pi] \\ F(t) - F(\pi) + F(\frac{2\pi}{3}) - F(\frac{\pi}{3}) & t \in (\pi, \frac{4\pi}{3}] \\ F(\frac{4\pi}{3}) - F(\pi) + F(\frac{2\pi}{3}) - F(\frac{\pi}{3}) & t \in (\frac{4\pi}{3}, \frac{5\pi}{3}] \\ F(t) - F(\frac{5\pi}{3}) + F(\frac{4\pi}{3}) - F(\pi) + F(\frac{2\pi}{3}) - F(\frac{\pi}{3}) & t \in (\frac{5\pi}{3}, 2\pi] \end{cases},$$

$$F^-(t) = \begin{cases} -F(t) & t \in [0, \frac{\pi}{3}] \\ -F(\frac{\pi}{3}) & t \in (\frac{\pi}{3}, \frac{2\pi}{3}] \\ -F(t) + F(\frac{2\pi}{3}) - F(\frac{\pi}{3}) & t \in (\frac{2\pi}{3}, \pi] \\ -F(\pi) + F(\frac{2\pi}{3}) - F(\frac{\pi}{3}) & t \in (\pi, \frac{4\pi}{3}] \\ -F(t) + F(\frac{4\pi}{3}) - F(\pi) + F(\frac{2\pi}{3}) - F(\frac{\pi}{3}) & t \in (\frac{4\pi}{3}, \frac{5\pi}{3}] \\ -F(\frac{5\pi}{3}) + F(\frac{4\pi}{3}) - F(\pi) + F(\frac{2\pi}{3}) - F(\frac{\pi}{3}) & t \in (\frac{5\pi}{3}, 2\pi] \end{cases},$$

where $F(t) := \int_0^t ie^{is} df(s) = \int_0^t ie^{is} d(\frac{2}{3}\cos 3s - \frac{2}{3}) = (\frac{1}{2}\sin 2t - \frac{1}{4}\sin 4t, \frac{1}{2}\cos 2t + \frac{1}{4}\cos 4t - \frac{3}{4}) = \frac{1}{2}ie^{-2it} + \frac{1}{4}ie^{4it} - \frac{3}{4}i$. The image of the function $F(t)$ is a Steiner curve (see Figure 4.2). The image of the functions $F^+(t)$ translated by the vector $(0, 1)$ and of the function $F^-(t)$ produce boundaries, respectively, of the sets A and B (see Figure 4.2).

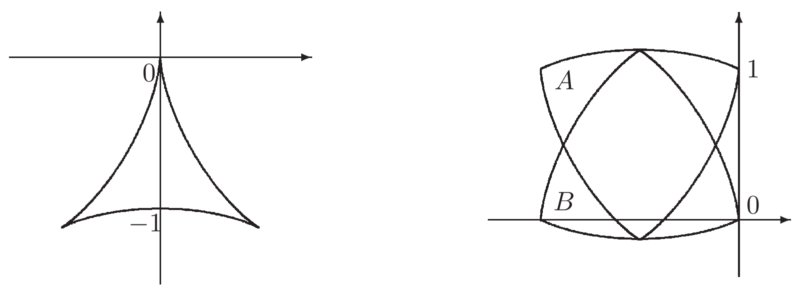

Figure 4.2: Trajectory of the function F (deltoid curve or Steiner curve). Minimal pair of sets such that $h_A - h_B = h$ from Example 4.2.

In [10], Example 10.2.8, the function h was represented as a difference $(h(x, y) + 2\sqrt{x^2 + y^2}) - 2\sqrt{x^2 + y^2}$, where the number $\alpha = 2$ is the smallest number such that the function $h(x,y)+\alpha\sqrt{x^2+y^2}$ is convex. Figure 4.3 shows a pair of subdifferential $A := \underline{\partial} h|_0 = \underline{\partial}(h+2\|\cdot\|_2)|_0$ and superdifferential $B := \overline{\partial} h|_0 = \underline{\partial}(2\|\cdot\|_2)|_0$. Obviously, $h = h_A - h_B$, however, the pair (A, B) is not inclusion-minimal.

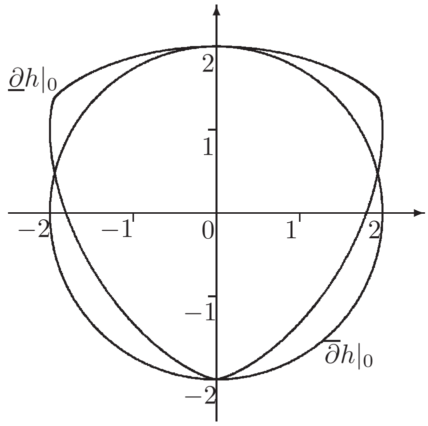

Figure 4.3: Sub- and superdifferential of the function h from Example 10.2.8 in [10].

Example 4.3. Let

$$h(x,y) := \begin{cases} \sqrt{2}|y| - \sqrt{x^2+y^2} & |x| \leqslant |y| \\ \sqrt{x^2+y^2} - \sqrt{2}|x| & |x| > |y| \end{cases}.$$

Then

$$\varphi(t) = \begin{cases} \sqrt{2}|\sin t| - 1 & |\cos t| \leqslant |\sin t| \\ 1 - \sqrt{2}|\cos t| & |\cos t| > |\sin t| \end{cases},$$

$$\int_0^t \varphi(s)ds = \begin{cases} t - \sqrt{2}\sin t & 0 \leqslant t < \frac{\pi}{4} \\ \frac{\pi}{2} - t - \sqrt{2}\cos t & \frac{\pi}{4} \leqslant t < \frac{3\pi}{4} \\ -\pi + t + \sqrt{2}\sin t & \frac{3\pi}{4} \leqslant t < \frac{5\pi}{4} \\ \frac{3\pi}{2} - t + \sqrt{2}\cos t & \frac{5\pi}{4} \leqslant t < \frac{7\pi}{4} \\ -2\pi + t - \sqrt{2}\sin t & \frac{7\pi}{4} \leqslant t \leqslant 2\pi \end{cases},$$

$$\varphi'(t) = \begin{cases} \sqrt{2}\sin t & 0 \leqslant t < \frac{\pi}{4} \\ \sqrt{2}\cos t & \frac{\pi}{4} \leqslant t < \frac{3\pi}{4} \\ -\sqrt{2}\sin t & \frac{3\pi}{4} \leqslant t < \frac{5\pi}{4} \\ -\sqrt{2}\cos t & \frac{5\pi}{4} \leqslant t < \frac{7\pi}{4} \\ \sqrt{2}\sin t & \frac{7\pi}{4} \leqslant t \leqslant 2\pi \end{cases},$$

$$f(t) = \begin{cases} t & 0 \leqslant t < \frac{\pi}{4} \\ \frac{\pi}{2} - t & \frac{\pi}{4} \leqslant t < \frac{3\pi}{4} \\ -\pi + t & \frac{3\pi}{4} \leqslant t < \frac{5\pi}{4} \\ \frac{3\pi}{2} - t & \frac{5\pi}{4} \leqslant t < \frac{7\pi}{4} \\ -2\pi + t & \frac{7\pi}{4} \leqslant t \leqslant 2\pi \end{cases},$$

$$f^+(t) = \begin{cases} t & 0 \leqslant t < \frac{\pi}{4} \\ \frac{\pi}{4} & \frac{\pi}{4} \leqslant t < \frac{3\pi}{4} \\ -\frac{\pi}{2} + t & \frac{3\pi}{4} \leqslant t < \frac{5\pi}{4} \\ \frac{3\pi}{4} & \frac{5\pi}{4} \leqslant t < \frac{7\pi}{4} \\ -\pi + t & \frac{7\pi}{4} \leqslant t \leqslant 2\pi \end{cases}, f^-(t) = \begin{cases} 0 & 0 \leqslant t < \frac{\pi}{4} \\ -\frac{\pi}{4} + t & \frac{\pi}{4} \leqslant t < \frac{3\pi}{4} \\ \frac{\pi}{2} & \frac{3\pi}{4} \leqslant t < \frac{5\pi}{4} \\ -\frac{3\pi}{4} + t & \frac{5\pi}{4} \leqslant t < \frac{7\pi}{4} \\ \pi & \frac{7\pi}{4} \leqslant t \leqslant 2\pi \end{cases}.$$

The image of the function $F^+(t)$ translated by the vector $(1 - \frac{\sqrt{2}}{2}, 0)$ and of the function $F^-(t)$ translated by vector $(\frac{\sqrt{2}}{2}, 0)$ produce boundaries, respectively, of the sets A' and B' (see Figure 4.4), which are lenses. The Minkowski sum $A' + B'$ is a unit disc. We have $h = h_A - h_B = h_{A'} - h_{B'}$, while $h_{A'} + h_{B'} = \| \cdot \|_2$.

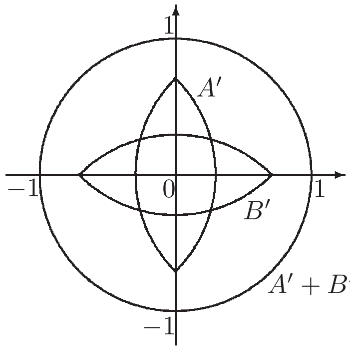

Figure 4.4: The lenses A' and B' from Example 4.3.

The following examples show usefulness of the criterion from Theorem 3.1.

Example 4.4. Let $h(x, y) := \inf_{n \in \mathbb{N}} |y \cos \frac{\pi}{n} - x \sin \frac{\pi}{n}|$. Notice that the function h is positively homogenous, piecewise linear and nonnegative. Then $\varphi(t) = \inf_{n \in \mathbb{N}} | \sin(t - \frac{\pi}{n})|$. For $t \in [\frac{\pi}{n}, \frac{\pi}{n-1}]$, $n \geqslant 3$ we have

$$\varphi(t) \leqslant \sin(t - \frac{\pi}{n}) \leqslant \frac{\pi}{n-1} - \frac{\pi}{n} = \frac{\pi}{(n-1)n}.$$

Since $n \geqslant \frac{\pi}{t}$ and $t \leqslant \pi - 1$, we obtain

$$\varphi(t) \leqslant \frac{\pi}{(\frac{\pi}{t} - 1)\frac{\pi}{t}} = \frac{t^2}{\pi - t} \leqslant t^2.$$

Then for $t \in [0, \frac{\pi}{2}]$ we have $0 \leqslant \varphi(t) \leqslant t^2$, and the right derivative $\varphi'(0)$ exists and $\varphi'(0) = 0$. Since $\varphi(t) = \varphi(t - \pi)$ for $t \in [\pi, 2\pi]$, we obtain $\varphi'(\pi) = 0$. For all $t \in [0, 2\pi]$, $t \neq 0, \pi, 2\pi$ the right derivative $\varphi'(t)$ obviously exists and belongs to $[-1, 1]$. Therefore, the function φ is continous and Lipschitzian with a constant 1. Also the function h is continous and Lipschitzian with a constant 1. Let us notice that for all $n \geqslant 2$, we have $\varphi'(\frac{\pi}{n}) = 1$. Moreover for each n the function φ' is negative in some left neighborhood of $\frac{\pi}{n}$. Hence the variation of φ' is infinite. By Theorem 3.1 the function h is not a difference of sublinear functions. In [3] Gorokhovik and Trafimovich gave another similar function with more complicated definition.

Example 4.5. For $t \in [\frac{\pi}{n}, \frac{\pi}{n-1}]$, $n \geqslant 2$ let us define

$$\varphi(t) := \min(\frac{1}{n^2}\sin(t-\frac{\pi}{n}), \frac{1}{(n-1)^2}\sin(\frac{\pi}{n-1}-t)).$$

Let us put $\varphi(t) := \varphi(t-\pi)$ for $t > \pi$ and $\varphi(0) := 0$. We define

$$h(x,y) := \begin{cases} \sqrt{x^2+y^2}\varphi(\operatorname{Arg}(x+iy)) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

Again the function h is positively homogenous, piecewise linear and nonnegative. Moreover, the right derivative $\varphi'(t)$ exists and belongs to the interval $[-1,1]$. Therefore, also the functions φ and h are continous and Lipschitzian with a constant 1. Notice that right derivative φ' is decreasing in each interval $[\frac{\pi}{n}, \frac{\pi}{n-1})$, $n \geqslant 2$. Moreover, $\varphi'(\frac{\pi}{n}) = \frac{1}{n^2}$ and

$$\lim_{t \nearrow \frac{\pi}{n-1}} \varphi'(t) = -\frac{1}{(n-1)^2}.$$

We can calculate that the variation of φ' is finite and equal to

$$8\sum_{n=1}^{\infty}\frac{1}{n^2} - 4 = 8\frac{\pi^2}{6} - 4 \approx 9,1594725.$$

By Theorem 3.1 the function h is a difference of sublinear functions.

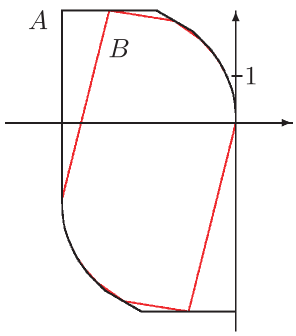

Figure 4.5: The sets A and B are determined by the images of the functions $F^+(t)$ and $F^-(t)$ from Example 4.6.

References

[1] V. F. Demyanov and A. M. Rubinov, *Quasidifferential Calculus*, Optimization Software Inc., Springer-Verlag, New York, 1986.

[2] V. F. Demyanov and A. M. Rubinov, *Quasidifferentiability and Related Topics*, Nonconvex Optimization and Its Applications, vol. 43, Kluwer, Dordrecht, 2000.

[3] V. V. Gorokhovik and M. Trafimovich, *Positively homogeneous functions revisited*, J. Optim. Theory Appl. **171** (2016), 481–503.

[4] J. Grzybowski, *Minimal pairs of convex compact sets*, Archiv der Mathematik,**63** (1994), 173–181.

[5] J. Grzybowski, D. Pallaschke and R. Urbański, *Minimal pairs of bounded closed convex sets as minimal representations of elements of the Minkowski-Rådström-Hörmander spaces*, Banach Center Pub. **84** (2009), 31–55.
[6] Ph. Hartman, *On functions representable as a difference of convex functions*, Pac. J. Math. **9** (1959), 707–713.
[7] L. Hörmander, *Sur la fonction d'appui des ensembles convexes dans un espace localement convexe*, Arkiv för Mathematik **3** (1954), 181–186.
[8] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation. I & 2*, Grundlehren der Mathematischen Wissenschaften **Vol. 330 & 331**, Springer-Verlag, Berlin, 2006.
[9] D. Pallaschke and S. Rolewicz, *Foundations of Mathematical Optimization. Convex analysis without Linearity*, Math. Appl. **388**, Kluwer, Dordrecht–Boston–London, 1997.
[10] D. Pallaschke and R. Urbański, *Pairs of Compact Convex Sets. Fractional Arithmetic with Convex Sets*, Math. Appl. **548**, Kluwer, Dortrecht–Boston–London, 2002.
[11] H. Rådström, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. **3** (1952), 165–169.
[12] A. W/ Roberts and D. E. Varberg, *Functions of bounded convexity*, Bull. Amer. Math. Soc. **75** (1969), 568–572.
[13] S. M. Robinson, *Strongly regular generalized equations*, Math. Oper. Res. **5** (1980), 43– 62.
[14] A. Shapiro, *Quasidifferential calculus and first order optimality conditions in nonsmooth optimization*, Math. Program. Study **29** (1986), 56–68.
[15] S. Scholtes, *Minimal pairs of convex bodies in two dimensions*, Mathematica **39** (1992), 267–273.
[16] R. Urbański, *A generalization of the Minkowski–Rådström–Hörmander Theorem*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **24** (1976), 709–715.
[17] V. A. Zalgaller, *Representation of functions of several variables by differences of convex functions*, J. Math. Sci. **100** (2000), 2209–2227.
[18] A. J. Zaslavski, *Exact penalty in constrained optimization and critical points of Lipschitz functions*, J. Nonlinear Convex Anal. **10** (2009), 149–156.

Manuscript received February 23 2017
revised June 5 2017

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